L8

Last time: We discussed basis & dimension of two subspaces:

- Col(A) : basis = pivot cols; dim = rank = # pivots
- Nul(A) : basis = vectors in PVF; dim = # free vars

Today we'll discuss the other two subspaces (just replace A by A^T).

Why? This is the setup for orthogonality theory
⇝ least squares / approximate solutions of $Ax = b$.

The Row Space

Def: The row space of A is $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

The columns of A^T are the rows of A, so

$$\mathrm{Row}(A) = \mathrm{Span}\{\text{rows of } A\}.$$

This is a subspace of $\mathbb{R}^n$ $\quad n = \#\text{columns of } A$

$\rightsquigarrow$ row picture. $\qquad = \#\text{entries of each row}$

(That's why we call it the "row picture.")

The row space is a span $\rightsquigarrow$ parametric description.

Eg:
$$\mathrm{Row}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \mathrm{Col}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}\right\}$$

Fact: Row operations do not change the row space.

Proof: Say the rows are v_1, v_2, v_3.

- $R_2 \mathrel{+}= 2R_1$: the new rows are $v_1, v_2 + 2v_1, v_3$.

  $$\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$$

  because $v_1 + 2v_2 \in \mathrm{Span}\{v_1, v_2, v_3\}$ and

  $$v_2 = (v_2 + 2v_1) - 2v_1 \in \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$$

- $R_1 \longleftrightarrow R_2$: the new rows are v_2, v_1, v_3, and

  $$\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_2, v_1, v_3\} \quad \checkmark$$

- $R_2 \times = 2$: the new rows are $v_1, 2v_2, v_3$, and

$$\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, 2v_2, v_3\} \checkmark \quad //$$

This is a column space (of A^T), so we know how to find a basis (pivot cols of A^T). Here's a way to find a basis for $\mathrm{Row}(A)$ by doing elimination on A, not A^T.

Thm: The nonzero rows in any REF of A form a basis for $\mathrm{Row}(A)$.

Each nonzero row in an REF of A has exactly one pivot in it, so:

Consequence: $\boxed{\dim \mathrm{Row}(A) = \#\text{pivots} = \mathrm{rank}(A)}$

NB: This implies

$$\mathrm{rank}(A^T) = \dim \mathrm{Col}(A^T) = \dim \mathrm{Row}(A) = \mathrm{rank}(A)$$

which is not obvious! It says A & A^T have the same #pivots, but they may be in different positions! We don't get this without the Thm.

$$\boxed{\begin{array}{c}\text{Row Rank} = \text{Column Rank} \\ \operatorname{rank}(A) = \operatorname{rank}(A^T)\end{array}}$$

Eg: Find a basis for $\operatorname{Row}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightsquigarrow$ basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}$

You can use any ref, eg. the rref:

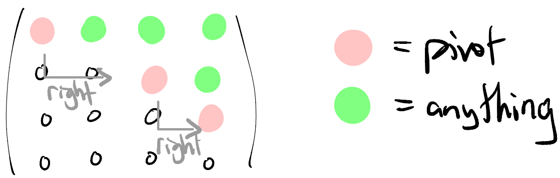
$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightsquigarrow$ another basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Proof of the Thm:

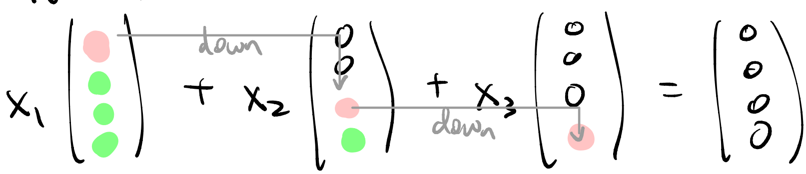
(1) Spans: row ops don't change the row space, and the zero rows don't make the span any bigger.

(2) LI: A matrix in REF looks like this:

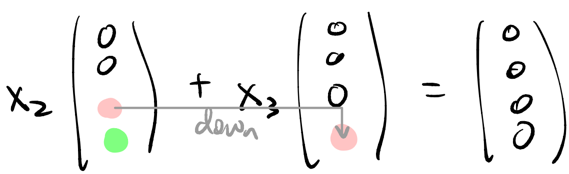


Let's solve the vector equation

$x_1(\text{row } 1) + x_2(\text{row } 2) + x_3(\text{row } 3) = 0$:



The first coordinate is $\bullet\, x_1 = 0 \Rightarrow x_1 = 0$ because $\bullet \neq 0$. So this simplifies to



Now the 3rd coordinate is $\bullet\, x_2 = 0 \Rightarrow x_2 = 0$.

Then $\bullet\, x_3 = 0 \Rightarrow x_3 = 0$. So the only solution is $x_1 = x_2 = x_3 = 0$

//

The Left Null Space

(no new notation)

Def: The left null space of A is $\mathrm{Nul}(A^T)$.
This is the solution set of $A^T x = 0$.
This is a subspace of $\mathbb{R}^m$ $\quad m = \#\text{rows of } A$
$= \#\text{cols of } A^T$
$\rightsquigarrow$ column picture.

The left null space is a solution set $\rightsquigarrow$ implicit description.

Why is it called the "left null space"?

$$A^T x = 0 \iff 0 = (A^T x)^T = x^T A, \text{ so}$$

$$\mathrm{Nul}(A^T) = \{\text{row vectors } x \in \mathbb{R}^m : x^T A = 0\}$$

$\mathrm{Nul}(A^T)$ is a null space, so you know how to compute a basis (PVF of $A^T x = 0$). You can also find a basis by doing elimination on A, not A^T.

Thm / Procedure: To find a basis of $\mathrm{Nul}(A^T)$:

(1) Form the augmented matrix $(A \mid I_m)$.

(2) Eliminate to REF $(U \mid E)$.

The rows of E to the right of the zero rows of U form a basis for $\mathrm{Nul}(A^T)$.

In fact, you can stop elimination once U is in REF; you don't need all of $(U \mid E)$ to be in REF.

There's a slick proof in the supplement.

Eg: Find a basis for $\mathrm{Nul}\begin{pmatrix}1&2&2&1\\2&4&1&-1\\1&2&-1&-2\end{pmatrix}^T$.

Form the augmented matrix & eliminate:

$$\left(\begin{array}{cccc|ccc}1&2&2&1&1&0&0\\2&4&1&-1&0&1&0\\1&2&-1&-2&0&0&1\end{array}\right) \overset{\text{ref}}{\rightsquigarrow} \left(\begin{array}{cccc|ccc}1&2&2&1&1&0&0\\0&0&-3&-3&-2&1&0\\0&0&0&0&1&-1&1\end{array}\right)$$

zero

$\rightsquigarrow$ basis $\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$

Check: $$\begin{pmatrix}1&-1&1\end{pmatrix}\begin{pmatrix}1&2&2&1\\2&4&1&-1\\1&2&-1&-2\end{pmatrix} = \begin{pmatrix}0&0&0\end{pmatrix} \checkmark$$

NB: if you just want a basis for $\mathrm{Nul}(A^T)$ and don't have any other reason to do elimination on A, it's easier to just find the PVF of $A^Tx=0$.

Since U is an REF of A, the #nonzero rows is the # pivots of A = the rank of A.

Consequence: $$\boxed{\dim \mathrm{Nul}(A^T) = \#\text{rows} - \mathrm{rank}(A)}$$

NB: row operations change $\mathrm{Nul}(A^T)$.

eg: According to the Thm, to find a basis for

$$\mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T,$$ we put this matrix in REF:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

It's already in REF, so a basis is $\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

But we computed in a previous example that

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix}^T$ has basis $\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$

so $$\mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{pmatrix}^T \neq \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Summary: The Four Subspaces

A: $m \times n$ matrix of rank r

Subspace	of	row/col	dim	basis	implicit/parametric
$\mathrm{Col}(A)$	$\mathbb{R}^m$	col	r	pivot cols of A	parametric
$\mathrm{Nul}(A)$	$\mathbb{R}^n$	row	$n-r$	vectors in PVF	implicit
$\mathrm{Row}(A)$	$\mathbb{R}^n$	row	r	nonzero rows in $\mathrm{REF}(A)$	parametric
$\mathrm{Nul}(A^T)$	$\mathbb{R}^m$	col	$m-r$	last $m-r$ rows of E	implicit

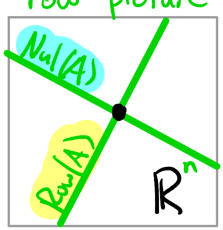
The row picture subspaces $\mathrm{Nul}(A)$, $\mathrm{Row}(A)$ are unchanged by row operations

The column picture subspaces $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$ are changed by row operations

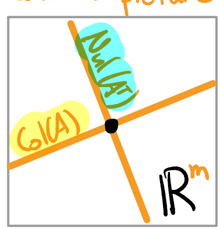
The row & column pictures each have one parametric & one implicit form subspace.

$$\dim \mathrm{Nul}(A) + \dim \mathrm{Row}(A) = n$$

row picture



column picture



$$\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A^T) = m$$

♫ The row space lives in the... row picture!
The null space lives in the... row picture!
The other two live in the... column picture!
That's how you keep them straight! ♫

Here's an important numerical consequence. It relates the dimension ("size") of the column space to the dimension of the null space.

Rank-Nullity Theorem

$$\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n = \#\text{columns of } A$$

[DEMO 1] [DEMO 2] [DEMO 3] [DEMO 4]

NB: You can compute bases for all four subspaces by doing elimination once on $(A \mid I_m) \rightsquigarrow (U \mid E)$

(1) An REF of $A \rightsquigarrow$ pivots $\rightsquigarrow$ basis of $\mathrm{Col}(A)$

(2) The RREF of $A \rightsquigarrow$ PVF $\rightsquigarrow$ basis of $\mathrm{Nul}(A)$

(3) An REF of $A \rightsquigarrow$ nonzero rows $\rightsquigarrow$ basis of $\mathrm{Row}(A)$

(4) Last $m-r$ rows of $E \rightsquigarrow$ basis of $\mathrm{Nul}(A^T)$.

(However, if you only need a basis for $\mathrm{Nul}(A^T)$, just find the RREF of A^T instead.)

Full-Rank Matrices

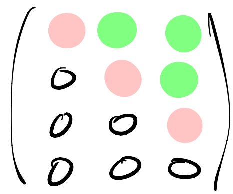
Since each row and each col has at most one pivot,

$$\boxed{\#\text{pivots} = \text{rank}(A) \leq \min\{\#\text{rows}, \#\text{cols}\}}$$

A "random" matrix will have the largest rank possible: ie, this will be an equality. This is a very important special case.

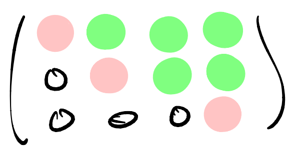
Def: An $m \times n$ matrix of rank r has:

- full column rank (FCR) if $r = n$
  (every column has a pivot)



$3=3$

- full row rank (FRR) if $r = m$
  (every row has a pivot)



$3=3$

NB:

- If A has FCR then $n = r \leq m$
  $\Rightarrow A$ is tall or square (#rows $\geq$ #cols)
- If A has FRR then $m = r \leq n$
  $\Rightarrow A$ is wide or square (#cols $\geq$ #rows)

- If A has FCR and FRR then $n = r = m$
  $\Rightarrow A$ is square.

Here is a list of properties of matrices that translate into "pivot in every column":

Thm: Let A be an $m \times n$ matrix.

The Following Are Equivalent: ← (for a given matrix, either they're all true or they're all false.)

(1) A has full column rank

 (1′) A has a pivot in every column

 (1″) A has no free columns

(2) $\boxed{\mathrm{Nul}(A) = \{0\}}$ ← simple description of $\mathrm{Nul}(A)$

 (2′) $Ax=0$ only has the trivial solution

(3) $Ax=b$ has 0 or 1 solution for every $b \in \mathbb{R}^m$.

(4) A has linearly independent columns.

 (4′) $\dim \mathrm{Col}(A) = n$

 (4″) $\dim \mathrm{Row}(A) = n$

(5) $\boxed{\mathrm{Row}(A) = \mathbb{R}^n}$ ← simple description of $\mathrm{Row}(A)$

(6) A^T has full row rank.

Notes:
- (1) $\Leftrightarrow$ (6) because $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.
- (2), (3), (4) all mean "no free variables" = (1″)
- (4′), (4″) are because $\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A) = \mathrm{rank} = n$.
- (4″) $\Leftrightarrow$ (5) because the only n-dimensional subspace of $\mathbb{R}^n$ is all of $\mathbb{R}^n$ (L7).

Here is a list of properties of matrices that translate into "pivot in every row":

Thm: Let A be an $m \times n$ matrix.

The Following Are Equivalent:

(1) A has full row rank

(1') A has a pivot in every row

(2) $\boxed{\mathrm{Col}(A) = \mathbb{R}^m}$ ← simple description of $\mathrm{Col}(A)$

(3) $Ax=b$ is consistent for every $b \in \mathbb{R}^m$

(4) A has linearly independent rows

(4') $\dim \mathrm{Col}(A) = m$

(4'') $\dim \mathrm{Row}(A) = m$

(5) $\boxed{\mathrm{Nul}(A^T) = \{0\}}$ ← simple description of $\mathrm{Nul}(A^T)$

(6) A^T has full column rank.

Notes:

- (2) $\Leftrightarrow$ (3) is the column picture criterion for consistency.
- (5) $\Leftrightarrow$ (1) because $\dim \mathrm{Nul}(A^T) = m - \mathrm{rank} = m - m = 0$
- (5) $\Leftrightarrow$ (4) because the rows of A are the cols of A^T.
- (4') $\Leftrightarrow$ (2) because the only m-dimensional subspace of $\mathbb{R}^m$ is all of $\mathbb{R}^m$ (L7).

If A has full column rank and full row rank then

$r = m = n \implies$ square and invertible (n pivots)

Thm: Let A be an $n \times n$ (square) matrix.

The Following Are Equivalent:

(1) A is invertible.

(1') A has n pivots.

(1'') A has full column rank.

↳ all of the equivalent conditions for FCR

(1''') A has full row rank.

↳ all of the equivalent conditions for FRR

(2) There is a matrix A^{-1} such that $A^{-1}A = I_n$

(2') There is a matrix A^{-1} such that $AA^{-1} = I_n$

(3) $Ax = b$ has exactly 1 solution for every $b \in \mathbb{R}^n$

↳ namely, $x = A^{-1}b$

(4) A^T is invertible.

Notes:

- We discussed (1) $\iff$ (1') $\iff$ (2) $\iff$ (2') $\iff$ (3) in L3.
- (1) $\iff$ (4) because $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Consequence: Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$

$\rightsquigarrow$ $n \times n$ matrix $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

(1) $\operatorname{Span}\{v_1, v_2, \ldots, v_n\} = \mathbb{R}^n \iff \operatorname{Col}(A) = \mathbb{R}^n$

$\iff$ A has FRR $\iff$ A is invertible

(2) $\{v_1, v_2, \ldots, v_n\}$ is LI $\iff \operatorname{Nul}(A) = \{0\}$

$\iff$ A has FCR $\iff$ A is invertible

Recall: $\operatorname{Span}\{v_1, v_2, \ldots, v_n\} = \mathbb{R}^n$ and $\{v_1, v_2, \ldots, v_n\}$ is LI mean $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$.

Upshot:

bases for $\mathbb{R}^n$ $\equiv$ columns of invertible $n \times n$ matrices.