A BASIS FOR $\mathrm{Nul}(A^T)$

The following result appears in the lecture notes.

Theorem/Procedure. *Let A be an $m \times n$ matrix. To find a basis for* $\mathrm{Nul}(A^T)$*:*

(1) *Form the augmented matrix* $\left(\, A \mid I_m \,\right)$.

(2) *Eliminate to find a row echelon form of this matrix:* $\left(\, U \mid E \,\right)$.

Then the rows of E to the right of the zero rows of U form a basis for $\mathrm{Nul}(A^T)$.

The purpose of this supplement is to explain why this is true, in case you are curious.

Proof. First note that U is a row echelon form of A (it satisfies both properties involving pivots). If the elementary matrices for the row operations you performed are $E_1, E_2, \ldots, E_r$ then

$$\left(\, U \mid E \,\right) = E_r \cdots E_2 E_1 \left(\, A \mid I_m \,\right) = \left(\, E_r \cdots E_2 E_1 A \mid E_r \cdots E_2 E_1 I_m \,\right),$$

which means that $E = E_r \cdots E_2 E_1$ just keeps track of the row operations you performed. It follows that

$$U = E_r \cdots E_2 E_1 A = EA.$$

Let $r = \mathrm{rank}(A)$. There is a pivot in every nonzero column of U, and the number of pivots is by definition the rank of A, so there are $m - r$ zero columns of U at the bottom. Let $v_{r+1}^T, v_{r+2}^T, \ldots, v_m^T$ be the rows of E to the right of the zero rows of U. We want to show that $\{v_{r+1}, v_{r+2}, \ldots, v_m\}$ forms a basis for A^T.

First let's check that $v_{r+1}, v_{r+2}, \ldots, v_m$ are at least contained in $\mathrm{Nul}(A^T)$. In other words, we want to show that $A^T v_i = 0$ for $i = r+1, \ldots, m$, or equivalently (by taking transposes), that $v_i^T A = 0$. The ith row of EA is just the ith row of E times A, and $EA = U$, so if $i > r$ then the ith row of EA is equal to zero. This justifies why $v_i^T A = 0$ for $i = r+1, \ldots, m$.

We know that $\dim \mathrm{Nul}(A^T) = m - \mathrm{rank}(A^T)$—this is because we could compute a basis for $\mathrm{Nul}(A^T)$ by finding the parametric vector form of the solutions of $A^T x = 0$. We also showed that $\mathrm{rank}(A^T) = \mathrm{rank}(A)$ when we computed a basis for $\mathrm{Row}(A)$, so we already know that $\dim \mathrm{Nul}(A^T) = m - r$. Since we have $m - r$ vectors $v_{r+1}, v_{r+2}, \ldots, v_m$ inside the subspace $\mathrm{Nul}(A^T)$ of dimension $m - r$, the Basis Theorem says that we only have to check that $\{v_{r+1}, v_{r+2}, \ldots, v_m\}$ is *linearly independent*.

Since E is a product of elementary matrices, it is *invertible*, so it has m pivots. It follows that E^T has m pivots as well (because $\mathrm{rank}(E) = \mathrm{rank}(E^T)$), so E^T has a pivot in every column. This implies that the columns of E^T (the rows of E) are linearly independent. Hence the last $m - r$ rows of E are also linearly independent (any subset of a linearly independent set is linearly independent: you showed this on the homework). This completes the proof. □

Note that in the proof, we only used that U is a row echelon form of A, not that $\left(\, U \mid E \,\right)$ is itself in row echelon form.