Math 218D-1: Homework #1

due Wednesday, January 14, at 11:59pm

The homework assignments in this class will require *a lot of work*. You should expect to spend around 10 hours per week solving them—more for some people, less for others—so allocate your time accordingly. There are good reasons for this: on the one hand, I have to make sure that you've had practice with any concept or computation that you'll see on an exam, but more importantly, it simply takes time to get good at math (or anything else). You may be tempted to take shortcuts to reduce the amount of time you spend, but you'll be shooting yourself in the foot come exam time: you will get out of the homework what you put into it, and you will need to be fluent in the material to do well on the exams. Also note that use of AI on homework assignments is **not allowed:** you'll learn a lot more by coming to office hours and *asking a human* when you're stuck.

This first homework is relatively light on concepts. The homework will become heavily conceptual starting around week 3.

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u+v+w$ and $u+2v-w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Express $u+3v-w$ as a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to ____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Express the vector $(-1, 2, 3)$ as a linear combination of the unit coordinate vectors e_1, e_2, e_3.
(See the notes for the definition of the standing notation "e_i".)

3. Decide if each statement is true or false, and explain why. In this problem, v and w denote vectors in $\mathbf{R}^n$.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) The vector 0 is a linear combination of v and w.

c) $\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

d) If $v \neq 0$ then $v \cdot v > 0$.

e) $v^T w = v \cdot w$.

4. Let v and w be vectors in $\mathbf{R}^n$. Suppose that $v \cdot v = 1$, $w \cdot w = 2$, and $v \cdot w = 3$. Compute the following quantities using the algebra of dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w)\cdot(v-w)$ **c)** $(v+2w)\cdot(3v)$.

5. **a)** Find a nonzero vector $v \in \mathbf{R}^3$ such that $v \cdot (1,1,1) = 0$.

b) Find a nonzero vector $w \in \mathbf{R}^3$ such that $w \cdot (1,1,1) = 0$ and $w \cdot v = 0$, where v is your vector from **a)**.

6. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why. *Show your work.*

a) $\begin{pmatrix}2\\5\end{pmatrix}\begin{pmatrix}1\\-3\\-1\end{pmatrix}$ **b)** $\begin{pmatrix}1 & -2\\0 & -1\\3 & 2\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}$ **c)** $\begin{pmatrix}7 & 2 & 4\\3 & -3 & 1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$

d) $\begin{pmatrix}7 & 4\\-2 & 2\\4 & 1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}$ **e)** $\begin{pmatrix}2 & 6 & -1\end{pmatrix}\begin{pmatrix}5\\-1\\0\end{pmatrix}$

Check your work using SymPy on the Sage cell on the website, as in:

```
A = Matrix([[7,  2, 4],
            [3, -3, 1]])
  # Shortcut for specifying a column vector:
x = Matrix([1, -1, 1])
pprint(A*x)
```

7. For each pair of vectors u and v, compute $u^T v$ and $u \cdot v$.

$$\textbf{a)}\ u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \textbf{b)}\ u = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix},\ v = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix}$$

Check your work using SymPy on the Sage cell on the website, as in:

```
  # Shortcut for specifying a column vector:
u = Matrix([1, 2, 3])
v = Matrix([4, 5, 6])
pprint(u.T*v)
pprint(u.dot(v))
```

8. Suppose that $u = (a, b, c)$ and $v = (d, e, f)$ are vectors satisfying $2u + 3v = 0$. *Without doing any computations at all,* find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

9. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \qquad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \qquad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Find A, *without doing any computations at all.*

10. Find the matrix A satisfying

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ -x - 2y \end{pmatrix}.$$

[**Hint:** compute $A\binom{1}{0}$ and $A\binom{0}{1}$.]

11. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \\ 9 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \\ 3 \end{pmatrix}.$$

Let x be the vector

$$x = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

Find Ax *without trying to solve for A.*

12. Suppose that A is a 2×3 matrix and that B and C are 2×2 matrices. Which of the following expressions are defined, and which are undefined?

a) $2A$ **b)** $2A - B$ **c)** $2B - C$

d) AB **e)** BA **f)** $A^T B$

g) B^2 **h)** $BA - A$ **i)** $(B - I_2)A$

13. Suppose that A, B, and C are 3×3 matrices. Simplify the following expressions (write them without parentheses or identity matrices):

a) $(A+B)^2$ **b)** $A(2I_3 - B)C$ **c)** $(AB)^T C$

d) $C(A+3B)^T$ **e)** $(A^T + I_3)^T C$ **f)** $(A^T A)^T$

14. Compute the matrix products

a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

in two ways: using the column form, and using the outer product form. *Show your work.*

Check your work using SymPy on the Sage cell on the website, as in:

```
A = Matrix([[1,  2],
            [2, -1]])
B = Matrix([[2,  1, -1],
            [4, -1,  2]])
pprint(A*B)
```

15. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \qquad C = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Note that B and C are diagonal.

a) Compute AB and BA.

b) How do the columns or rows of a matrix change when it is multiplied by a diagonal matrix on the right or left?

c) Compute BC and CB.

d) What happens when you multiply two diagonal matrices together?

16. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

a) For which 2×2 matrices B does $AB = BA$?

b) Find *nonzero* 2×2 matrices B and C such that $B \neq C$ yet $AB = AC$.

17. Recall that a matrix A is *symmetric* if $A^T = A$. Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample.

a) A symmetric matrix is square.

b) If A and B are symmetric of the same size, then AB is symmetric.

c) If A is symmetric, then A^3 is symmetric.

d) If A is any matrix, then $A^T A$ is symmetric.