Math 218D-1: Homework #2

due Wednesday, January 21, at 11:59pm

Once you are comfortable doing the Gauss–Jordan elimination algorithm(s) by hand, *please* start using SymPy on the Sage cell on the course webpage to do the computations! Just write "used SymPy" so that you don't confuse the graders. This class is about formulating linear algebra problems that a computer can solve, not mastering computations that a computer can do better than you. (You will still need to do computations by hand on exams.)

Sage cell tips:

```
  # Specify a matrix
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
  # Shorthand for specifying a column vector
b = Matrix([1, 2, 3])
  # Solve Ax=b (only works when there's a unique solution)
pprint(A.solve(b))
  # Or, augment [A|b] and find the rref:
pprint(A.row_join(b).rref(pivots=False))
```

1. In the table below, a linear system is expressed as a system of equations, as a matrix equation, as a vector equation, or as an augmented matrix. Fill in the rows of the table with the other three equivalent ways of writing each system of equations.

System of Equations	Matrix Equation	Vector Equation	Augmented Matrix
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$			
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$		
		$x_1 \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	
			$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

2. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Rewrite the system as an augmented matrix.

b) Use row replacements to eliminate x_1 from the second and third equations.

c) Use a row replacement to eliminate x_2 from the third equation (with x_1 still only appearing in the first).

d) Translate your augmented matrix back into a system of equations.

e) Solve for x_3, then for x_2, then for x_1. What is the solution?

Show your work.

3. **(Internalizing a Definition)** Which of the following matrices are *not* in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

4. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

5. **(Internalizing a Definition)** Which of the following matrices are *not* in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

6. **(Practicing a Procedure)** Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer, but *write out all row operations you perform.*

a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b)** $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$ **c)** $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ **d)** $\left(\begin{array}{ccc|c} 0 & 0 & 2 & 2 \\ 1 & -1 & 3 & -1 \\ -1 & 1 & 1 & 8 \end{array}\right)$

Check your work using SymPy. For instance, in **a)** you would do something like:

```
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
pprint(A.echelon_form())
pprint(A.rref(pivots=False))
```

(Omitting `pivots=False` would cause Sympy to print the pivot locations as well. Note that Sympy may produce a different REF than you.)

By the way, Sympy has no notion of an augmented matrix—the augmentation line only exists to help a human remember that it came from a system of equations. To solve **b)** in Sympy, you would do something like:

```
A = Matrix([[1, 1, 0, 1],
            [1, 2, 1, 1],
            [0, 1, 2, 2]])
  # or even fancier:
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]]).row_join(Matrix([1, 1, 2]))
```

7. **(Practicing a Procedure)** Solve each of the following systems of equations (they all have a unique solution).

a)
$$\begin{aligned} -x_1 \quad\quad\quad + 4x_3 &= 2 \\ 3x_1 + 2x_2 + 4x_3 &= 8 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

b)
$$\begin{pmatrix} 0 & 3 & 2 \\ 1 & 3 & -3 \\ 4 & 9 & -16 \end{pmatrix} x = \begin{pmatrix} 11 \\ -7 \\ -47 \end{pmatrix}$$

c)
$$x_1 \begin{pmatrix} 2 \\ 6 \\ 4 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 6 \\ -4 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -5 \\ -5 \\ 5 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ -7 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 7 \\ -12 \end{pmatrix}$$

8. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

9. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

[Find the relevant criterion involving pivots in the notes.]

10. Give examples of matrices A in *reduced row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

11. **(Practicing a Procedure)** For each matrix A and vector b, decide if the system $Ax = b$ is consistent. If so, find the parametric vector form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Also answer the following questions (for the systems that have solutions): Which variables are free? How many solutions does the system have? What is the dimension of the solution set?

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

e) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

You can check your work again using SymPy. When $Ax = b$ has infinitely many solutions, `A.solve(b)` will throw an error; instead, try this:

```
A = Matrix([[2, 1, 1, 4],
            [4, 2, 1, 7]])
b = Matrix([1, 1])
  # Find the parametric form (free variables are labelled
  # tau0, tau1, ...)
pprint(A.gauss_jordan_solve(b))
  # Or, form the augmented matrix (A|b) and find its rref,
  # then do the rest by hand:
pprint(A.row_join(b).rref(pivots=False))
```

12. Is $\begin{pmatrix}3\\3\\3\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}4\\5\\6\end{pmatrix}$, $\begin{pmatrix}7\\8\\9\end{pmatrix}$? If so, what are the weights?

[Translate the problem into a linear algebra problem that you can solve.]

13. **(Foreshadowing)** Find the parametric vector form of the solution sets of the following systems of equations:

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases}$$

How are the solution sets related to each other geometrically?

14. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix}1+t\\2-t\\t\end{pmatrix} \qquad t \in \mathbf{R}.$$

15. Suppose that A is a 3×3 matrix and b is a vector such that the solution set of $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

16. **(Examples Problem)** In each part, find an example of a matrix with the stated property, or explain why no such matrix exists.

a) A 3×3 matrix with one free variable.

b) An invertible 3×3 matrix with one free variable.

c) A 2×3 matrix with 3 pivots.

d) A 2×3 matrix with no free variables.

e) A 3×2 matrix A such that $Ax = (1, 1, 1)$ has infinitely many solutions.

f) An invertible 2×2 matrix A such that A^3 is not invertible.

17. **(Practicing a Procedure)** Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

a) $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ **b)** $\begin{pmatrix}3&7\\2&4\end{pmatrix}$ **c)** $\begin{pmatrix}1&2\\2&4\end{pmatrix}$

18. **(Practicing a Procedure)** Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why. You're welcome to use Rabinoff's Reliable Row Reducer, but *write out all row operations you perform.*

$$\textbf{a)}\begin{pmatrix}1&1&0\\1&2&1\\0&1&2\end{pmatrix}\qquad \textbf{b)}\begin{pmatrix}1&0&-2\\2&-3&4\\-3&1&4\end{pmatrix}\qquad \textbf{c)}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix}6&-4&-7&-1\\7&0&1&3\\-1&2&3&1\\2&0&1&1\end{pmatrix}$$

Check your answers in SymPy, as in:

```
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
pprint(A.inv())
```

19. Consider the linear system

$$\begin{aligned} x_1 + x_2 &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 18(a) to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3. Do *not* use Gauss–Jordan elimination!

[Find the relevant big red box in the notes.]

20. Suppose that

$$A\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\qquad A\begin{pmatrix}-1\\3\\2\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad A\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

What is A^{-1}?

[**Hint:** multiply both sides by A^{-1}. This requires no computations.]

21. Suppose that A, B, and C are invertible 3×3 matrices. Simplify the following expressions (write them without parentheses or unnecessary identity matrices):

a) $(ABC)^{-1}$ **b)** $C(A-2I_3)C^{-1}$ **c)** $A^T(A^{-1})^T$ **d)** $A^3(A^{-1})^2$