# Welcome to Math 218D-1!

L1

## Introduction to Linear Algebra

What is linear algebra?

→ The study of (systems of) linear equations

Like: $y = 3x + 2$ (rearrange) $-3x + y = 2$

(usually put the variables on the left & constants on the right)

Several equations: $\begin{cases} x + y + z = 1 \\ y - z = 3 \end{cases}$ : solve both at once

(arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number)·(variable) or (number)

Not: $xy + z = 1$ ($xy$: variable × variable)

$x + 3 = y^2$ ($y^2$: power of a variable)

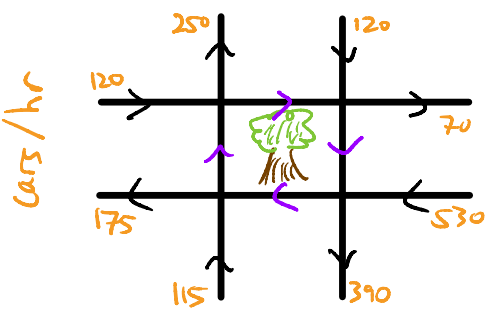
$e^x = \cos(y)$ ($e^x$, $\cos(y)$: complicated functions)

# Why learn linear algebra?

- It's simple enough to understand very well.
- Algorithmic parts can be implemented efficiently on computers.
- It's powerful enough to have a huge range of applications!

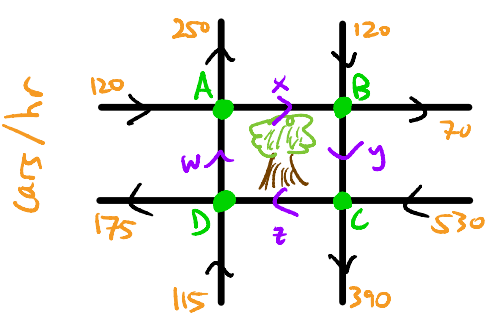
"example"

**Eg:** Here's a map of roads in the town square:




**Question:** How many cars/hr travel on the unlabeled roads?

When you have an unknown quantity, name it!




**Observation:**

# cars entering each intersection
= # cars leaving it

A: $120 + w = 250 + x$

B: $120 + x = 70 + y$

C: $530 + y = 390 + z$

D: $115 + z = 175 + w$

rearrange →

variables! ← → constants!

$$\begin{cases} -x + w = 130 \\ x - y = -50 \\ y - z = -140 \\ z - w = 60 \end{cases}$$

columns!

This is a system of 4 linear equations in 4 unknowns.

Question: You know a priori that there are infinitely many solutions. How?

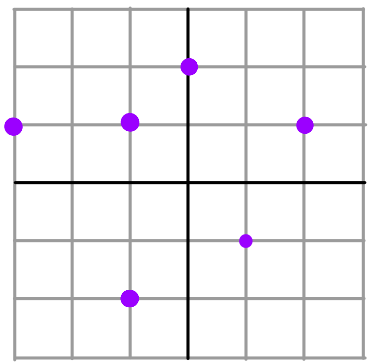
Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations.

Your job is to formulate the problem in terms of linear equations & to interpret the answer in the context of the original problem.

**Eg:** An asteroid has been observed at coordinates:

$(0,2)$ $(2,1)$ $(1,-1)$ $(-1,-2)$ $(-3,1)$ $(-1,1)$



**Question:** What is the most likely orbit? Will the asteroid crash into Earth?

**Fact (Kepler):** The orbit is an **ellipse**.

## Equation for an Ellipse:

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

**Wait!** This equation isn't linear...

If our **data points** lay on the ellipse, then this equation would be satisfied if we substitute the $(x,y)$ values of our data points:

$x$ $y$

$(0,2)$: $0 + 4B + 0 + 0 + 2E + F = 0$

$(2,1)$: $4 + B + 2C + 2D + E + F = 0$

$(1,-1)$: $1 + B - C + D - E + F = 0$

$(-1,-2)$: $1 + 4B + 2C - D - 2E + F = 0$

$(-3,1)$: $9 + B - 3C + D - 3E + F = 0$

$(-1,1)$: $1 + B - C - D + E + F = 0$

This is a system of 6 linear equations in 5 variables.

"Note"

NB: The variables/unknowns are the coefficients $B, C, D, E, F$: we're finding the equation of the ellipse!

NB: There is no solution – the points do not lie on an ellipse (measurement error).

Question: What is the best approximate solution?

Answer: the method of least squares (week 8-ish)

[DEMO]

Historical Note: Gauss invented much of the 1st half of this course to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on Demos: Geometric understanding of linear algebra is a core component in this class.

- The pictures are the material (not a supplement).
- Don't turn off your brain when we do geometry!
- Play with the demos!
- I promise you will have to draw a picture on every exam.

Eg: In a population of rabbits,

(1) Half survive their first year. 

(2) Half of those survive their second year.

(3) Maximum life span is 3 years.

(4) Each rabbit produces (on average)

0, 6, 8 offspring in years

0, 1, 2, respectively.

Problem: Understand the long-term behavior of the rabbit population both qualitatively and quantitatively.

Let's give names to the unknowns.

$x_n$: # rabbits aged 0 in year $n$

$y_n$: # rabbits aged 1 in year $n$

$z_n$: # rabbits aged 2 in year $n$

Our rules say:

$$x_{n+1} = 6y_n + 8z_n$$

$$y_{n+1} = \frac{1}{2}x_n$$

$$z_{n+1} = \frac{1}{2}y_n$$

A system of equations of this form is called a difference equation.

We will solve them using eigenvalues & diagonalization (week 10-ish).

[DEMO] It looks like eventually,

- the population approximately doubles each year.
- the ratio of rabbits aged 0, 1, 2 is $\approx 16:4:1$

We'll learn to extract these facts from the equations!

NB: Google's PageRank algorithm is a special kind of difference equation called a stochastic process. We'll learn about these in week 11-ish.

Eg: Netflix knows what movies you'll like using the Singular Value Decomposition (SVD) and the Principal Component Analysis (PCA). (Weeks 14-15)

The SVD & PCA are the most powerful & most technical tools you will learn this term. In some sense, the whole course is leading up to them.

(I just don't know how to present an example in a few minutes on day 1.)

# Geometry of Solutions

**Convention:** In a system of linear equations, variables go on the left, organized in columns = constants go on the right

$$\begin{cases} -x + w = 130 \\ x - y = -50 \\ y - z = -140 \\ z - w = 60 \end{cases}$$

**Def:** The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

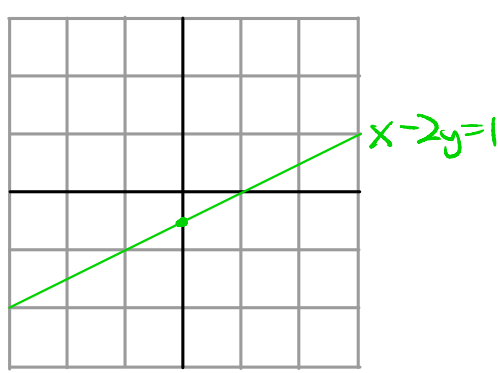
**Question:** What does the solution set of a system of linear equations look like?

**1 equation in 2 variables:**

$x - 2y = 1 \rightsquigarrow y = \frac{1}{2}x - \frac{1}{2}$

$y = mx + b$

line in the plane



## 1 equation in 3 variables:

$x+y+z=1 \rightsquigarrow z=1-x-y$

plane in space

[DEMO]

## 1 equation in 4 variables:

"3-plane in 4-space"

**Note on Dimensions:** Is the fourth dimension time? Physicists use 4-space to model spacetime, but you'll probably use it to model other things — like traffic around the town square...

## Can you visualize 4-space?

Well, I can't. But I use my pictures in the plane and in 3-space to inform my intuition in higher dimensions.

## 2 equations in 2 variables:

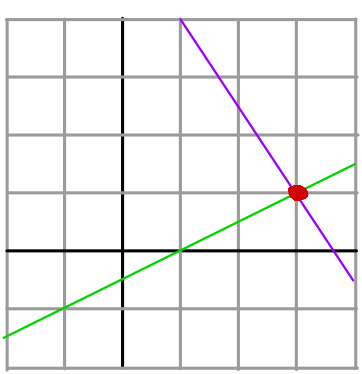
When are both true?

$x-2y=1$

$3x+2y=11$

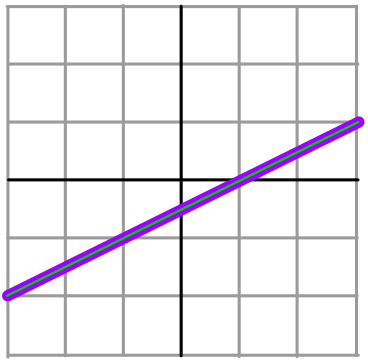
Intersection of 2 lines.

Answer: $(3,1)$



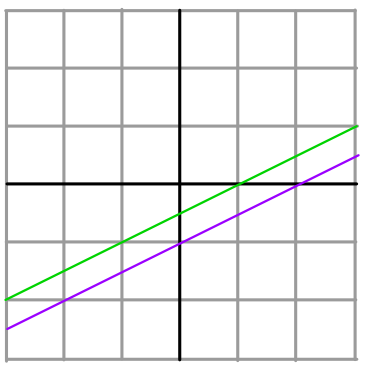
## What else can happen?

$x - 2y = 1$
$3x - 6y = 3$



Same line! $\infty$ solutions.

$x - 2y = 1$
$3x - 6y = 6$



Parallel lines! $0$ solutions.

## 2 equations in 3 variables:

$x + y + z = 1$
$x \quad - z = 0$

[DEMO]

Intersection of 2 planes in space:
in this case, it's a line: $\infty$ solutions.

## 3 equations in 3 variables:

$x + y + z = 1$
$x \quad - z = 0$
$y \quad = 0$

Intersection of 3 planes in space: in this case it's a point: one solution.

Question: How many "ways" can 3 planes in space intersect / be arranged?

Answer: I count 8.

# Syllabus / Course Information

See the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:

  https://services.math.duke.edu/~jdr/2526s-218/index.html

- We will use Canvas for:
  - → Announcements
  - → Gradebook
  - → Gradescope (use school credentials to log in)
  - → Ed Discussion: for asking questions

    > Don't email us with math questions!
    > Post it on Ed — then everyone can see the question & answer.

  - → Warpwire: there are 2 recorded lectures (total). The first is on matrix algebra. Watch it before L2!

Textbook: Officially it's

Strang, "Introduction to Linear Algebra", 5th Ed.

In reality, Joe's Notes are the textbook.

**Quizzes:**

A very basic, 10-minute small-group quiz will be held at the beginning of each discussion section.

**Homework:** due Wednesday, 11:59pm every week.

- Scan & submit on Gradescope.
  Use a scanning app.
- Tag the pages on Gradescope with the problems on that page.
- Expect to spend 8-10 hours per week.
- I won't test you on any concept you haven't seen on the homework.
- See the syllabus about late HW tokens
- Graded for correctness.

**Midterms:** 2 of them, during discussion sections

- Schedule is on the course webpage

**Final:** as scheduled by the registrar, in lecture room.

**ADAPTS:** reserve a room at the testing center ASAP for the midterms and the final.

Advice:

- Read the notes! Joe's Notes are the textbook.
- Engage with the homework!
  - → You will get out of it what you put into it. You will be tested on what you got out of it.
  - → This is your responsibility.
  - → If ChatGPT did most of your HW, you'll do poorly on exams. (Also, AI is not allowed.)
  - → You'll learn more by asking real people, like me!
  - → Find a study group!
  - → More HW tips on Ed
- Don't fall behind!

  The material is highly cumulative and very fast-paced, so it's very difficult to catch up.
- Come to office hours!
- If this is your first college math class, you may notice some differences from high school.
  - → There's too much material to spoon-feed you every detail.
  - → Instead, you'll be drinking from the fire hose.

  Showing up & paying attention in lecture is only the first step to mastery.