Number of Solutions

In the examples last time, our system of equations had only one solution. This is because we could isolate each variable. In terms of pivots:

> **One Solution:** this happens when the augmented matrix has a pivot in every non-augmented column but not the augmented column.

non-augmented columns

one solution

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{aligned} x_1 \quad &= 2 \\ x_2 &= 3 \\ 0 &= 0 \end{aligned}$$

augmented column

What happens if there's a pivot in the augmented column?

Eg:

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \\ -x_1 + x_2 &= 7 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 4 \\ -1 & 2 & 7 \end{array}\right)$$

pivots

$$\xrightarrow[R_3 += R_1]{R_2 -= 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -2 \\ 0 & 3 & 10 \end{array}\right) \xrightarrow{R_3 += R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 8 \end{array}\right)$$

This system of equations is:

$$\begin{aligned} x_1 + x_2 &= 3 \\ -3x_2 &= -2 \\ 0 &= 8 \end{aligned}$$

This system has no solutions — no values of x_1, x_2 can satisfy the 3rd equation!

Logically, if you had a solution (x_1, x_2) of the original equations, then you do row ops $\rightsquigarrow$ get a solution of $0=8$, which is a contradiction. Hence no solution can exist!

> **Zero Solutions:** this happens when there is a pivot in the augmented column.
>
> $$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \underset{\substack{\uparrow \\ \text{pivot!}}}{1} \end{array}\right) \rightsquigarrow \begin{array}{r} x_1 \qquad = 0 \\ x_2 = 0 \\ \underset{\substack{\uparrow \\ \text{no solutions!}}}{0 = 1} \end{array}$$

Def: A system of equations is **consistent** if it has a solution. It is **inconsistent** if it doesn't have any solutions.

If a system is consistent and there's a non-augmented column with no pivot, that corresponds to a variable that cannot be isolated. It turns out there are infinitely many solutions in that case.

> **∞ Solutions:** this happens when there is no pivot in the augmented column and in another column too.
>
> $$\left(\begin{array}{cc|c} 1 & \overset{\substack{\text{no pivot!} \\ \downarrow}}{2} & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \underbrace{\begin{array}{r} x_1 + \overset{\substack{\text{not isolated!} \\ \downarrow}}{2x_2} = 3 \\ 0 = 0 \end{array}}_{\substack{\infty \text{ solutions:} \\ \text{the line } x_1 = 3 - 2x_2}}$$

NB: You have to find the pivots to count solns, which means elimination.

Parametric Vector Form

When there are infinitely many solutions, how do we "list", or parameterize, them all?

Eg:

$$\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \rightsquigarrow \begin{aligned} x \quad +5z &= 1 \\ y+2z &= -1 \end{aligned}$$

Observation: If you substitute any number for z, you get the system

variables

$$\begin{aligned} x &= 1-5z \\ y &= -1-2z \end{aligned}$$

numbers

This has a unique solution!

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$$

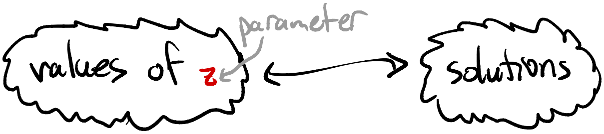
eg $z=1 \rightsquigarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$

Check:

$$\begin{aligned} 2(-4)+(-3)+12(1)&=1 \\ (-4)+2(-3)+9(1)&=-1 \end{aligned}$$

✓

Upshot: For any value of z you get a unique solution. This means the solutions are parameterized by z.



[DEMO]

Shapes: Implicit vs Parameterized Description

- The equations $\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned}$ describe a line (as an intersection of 2 planes). This is an

implicit description

which means it expresses the line as the solutions of a system of equations.

→ It's easy to check if $(-4,-3,1)$ lies on the line: just substitute $x=-4$, $y=-3$, $z=1$ into the equations & see if they're satisfied.

→ It's hard to produce points on the line with this description - can't directly write solutions.

- The parametric equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$ describe the same line in terms of one parameter z: this is a

parametric description

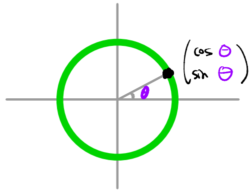
→ It's hard to check if a point lies on the line: does there exist a value of the parameter producing that point?

→ It's easy to produce points on the line - just choose any value for z!

Both $\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$ are different descriptions of the same line! Each has its uses. This theme will recur throughout the semester.

Non-linear example: (just to emphasize the point)

Here is the unit circle:



Here is an implicit equation:

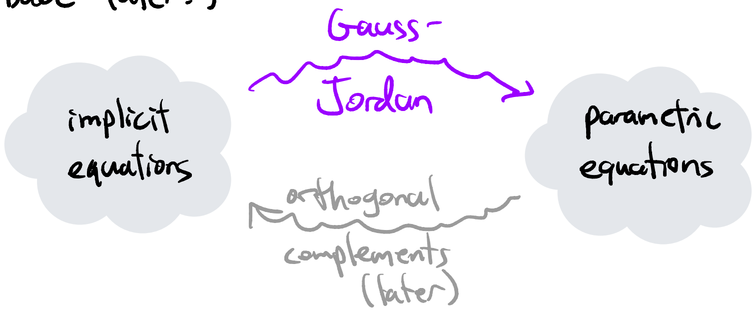
$$x^2+y^2=1$$

Here is a parametric description:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

θ = parameter

In the first example, we used Gauss-Jordan to go from the implicit form to the parametric form. (We'll learn how to go back later.)



Back to our example: We can write our parametric solution using vectors:

$$\begin{aligned} x &= 1-5z \\ y &= -1-2z \\ z &= z \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5z \\ -2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

This is how we will parameterize solutions of systems of linear equations. Here's the procedure.

Def: A pivot column of a matrix is a column with a pivot.

(Remember, pivots are the 1st entry in each row in REF)

Def: A free variable of a coefficient matrix is a variable corresponding to a column with no pivot.

$$\begin{array}{cccc} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

x, y are in pivot columns

z is a free variable

These are exactly the variables that you can't isolate in elimination.

Procedure: To find the parametric vector form of the solutions of a system of equations:

(1) Put $(A \mid b)$ into RREF. Stop if inconsistent.

(Nothing to parameterize in this case.)

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 1 \\ 2x_1 + 4x_2 + x_3 - x_4 = -1 \end{cases} \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 1 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

(2) Write out the corresponding equations.

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \begin{array}{l} x_1 + 2x_2 \qquad - x_4 = -1 \\ \qquad\qquad x_3 + x_4 = 1 \end{array}$$

free

free

(3) Move the free variables to the right side of =.
Keep the free variables in columns, and add trivial equations $x_i = x_i$ for the free variables, in order.

$$\begin{array}{l} x_1 + 2x_2 \qquad - x_4 = -1 \\ \qquad\qquad x_3 + x_4 = 1 \end{array} \rightsquigarrow \begin{array}{rcrrr} x_1 & = & -1 & -2x_2 & + x_4 \\ x_2 & = & & x_2 & \\ x_3 & = & 1 & & - x_4 \\ x_4 & = & & & x_4 \end{array}$$

trivial equations

columns

(4) Gather the columns into vectors. Pull out the free variables as weights.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \boxed{x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}$$

linear combination

Result: $x = \begin{pmatrix} \text{constant} \\ \text{vector} \end{pmatrix} + \begin{pmatrix} \text{a linear combination with} \\ \text{free variables as weights} \end{pmatrix}$

All solutions of the system are obtained by substituting any values for the free variables.

In other words, we obtain a parametric form of the solution set in which the free variables are the parameters.

IMPLICIT PARAMETRIC

$$\begin{cases} x_1+2x_2+2x_3+x_4=1 \\ 2x_1+4x_2+x_3-x_4=-1 \end{cases} \overset{\text{PVF}}{\rightsquigarrow} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Eg: Setting $x_2=1$ and $x_3=2$ gives one solution:

$$\begin{pmatrix} -5 \\ 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

NB: The $\begin{pmatrix}\text{constant} \\ \text{vector}\end{pmatrix}$ is the solution obtained by substituting 0 for all the free variables:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The $\begin{pmatrix}\text{constant} \\ \text{vector}\end{pmatrix}$ is a solution of the system!

Def: This vector is called a particular solution.

Eg: $x+y+z=1 \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$ is already in RREF

$$\rightsquigarrow x+\underbrace{y+z}_{\text{free}}=1 \rightsquigarrow \begin{aligned} x &= 1 - y - z \\ y &= \phantom{1 - {}} y \\ z &= \phantom{1 - y - {}} z \end{aligned}$$

$$\xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boxed{y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \leftarrow \text{linear combination}$$

This is a parameterized plane.

[DEMO]

Eg: $\begin{aligned} x+y&=2 \\ x-y&=0 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$

In this case there are no free variables!

$$\begin{aligned} x &= 1 \\ y &= 1 \end{aligned} \xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is a point.

Observation:

- 2 free variables / parameters
  $\rightsquigarrow$ solution set is a plane.
- 1 free variable / parameter
  $\rightsquigarrow$ solution set is a line.
- 0 free variables / parameters
  $\rightsquigarrow$ solution set is a point.

Provisional Definition: The dimension of the solution set of a consistent system of equations is the number of free variables.

We'll develop a more robust notion of dimension later.

Inverse Matrices

In an equation involving numbers, you can solve

$ax=b$ by dividing both sides by a: $x=\frac{b}{a}$.

Does this work with matrices? $Ax=b \Leftrightarrow x=\frac{b}{A}$??
Yes, but only for certain matrices.

NB: $b=a^{-1}$ (numbers) $\Leftrightarrow ab=1$
and $c=1 \Leftrightarrow cd=d$ for every number d.

Consider the identity matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.
Since $I_n D = D$ for every matrix D, this plays the role of the number 1.

Def: An $n\times n$ (square!) matrix A is invertible if there exists an $n\times n$ matrix A^{-1} such that $A^{-1}A=I_n$.
Otherwise (if A^{-1} does not exist), A is called singular.

Fact: $A^{-1}A=I_n \Leftrightarrow AA^{-1}=I_n$: there's no difference between a "left inverse" & a "right inverse".

NB: Non-square matrices have no inverse by definition.
There's good reason for this, as you'll see on the HW.

It's easy to compute the inverse of a 2×2 matrix:

Fact: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\iff ad-bc \neq 0$, in which case

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Check: $$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix}$$

$$\frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$ ✓

Eg: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is nonzero but not invertible:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so no inverse exists.

Fact: $(A^{-1})^{-1} = A$

Fact: If A & B are invertible then so is AB, and

$$(AB)^{-1} = \cancel{A^{-1}B^{-1}} \quad B^{-1}A^{-1}$$

Check: $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$ ✓

Wait – what was wrong with $A^{-1}B^{-1}$?

$$(A^{-1}B^{-1})(AB) = A^{-1}B^{-1}AB$$

nothing to cancel!

Here's how to compute A^{-1} in general.

Matrix Inversion: Let A be an $n \times n$ matrix.

(1) Form the augmented matrix $(A \mid I_n)$.

(2) Find its RREF:

$$(A \mid I_n) \rightsquigarrow (B \mid E)$$

If $B = I_n$ then $E = A^{-1}$. Otherwise A is not invertible.

Eg: Compute $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & \\ 1 & 1 & 0 & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \\ -1 & 2 & 1 & \end{array}\right) \overset{R_2 -= R_1}{\underset{R_3 += R_1}{\rightsquigarrow}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\overset{R_3 += 4R_2}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 4 & 1 \end{array}\right)$$

$$\overset{R_2 \times= -1}{\rightsquigarrow} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 & 4 & 1 \end{array}\right)$$

$$\overset{R_1 -= 2R_2}{\rightsquigarrow} \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} & \underbrace{\begin{matrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -3 & 4 & 1 \end{matrix}}_{A^{-1}} \end{array}\right)$$

So $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$.

Check: $\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

Why does this work? Elementary matrices (next time).

Eg: Compute $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -18 & -7 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 -= 3R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 5 & -3 & 1 \end{array}\right)$$

There's a zero row on the left, so the RREF can't be $(I_3 \mid *)$. Hence the matrix is not invertible.

In fact, the only way you'll get $(I_n \mid *)$ is if there are n pivots on the left, ie, n pivots in A.

Thm: Let A be an $n \times n$ matrix.

The Following Are Equivalent (TFAE):

for a given matrix, either they're all true or they're all false

(1) A is invertible

(2) The RREF of A is I_n

(3) A has n pivots.

Back to solving $Ax=b$ by "dividing by A":

Suppose A is invertible.

$$\begin{aligned} Ax=b &\iff A^{-1}(Ax)=A^{-1}b \\ &\iff (A^{-1}A)x = A^{-1}b \\ &\iff I_n x = A^{-1}b \iff x = A^{-1}b. \end{aligned}$$

Solving $Ax=b$ by Dividing by A

If A is invertible then

$$Ax=b \iff x=A^{-1}b$$

In particular, $Ax=b$ has exactly one solution for every value of b, and we get a formula for x in terms of b.

Eg: Solve
$$\begin{aligned} 2x_1+3x_2&=b_1 \\ x_1+2x_2&=b_2. \end{aligned}$$

This is $Ax=b$ for $A=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

$$\rightsquigarrow A^{-1} = \frac{1}{4-3}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}b = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2b_1-3b_2 \\ -b_1+2b_2 \end{pmatrix}$$

ie $\quad \boxed{\begin{aligned} x_1&=2b_1-3b_2 \\ x_2&=-b_1+2b_2 \end{aligned}} \leftarrow$ formula for x in terms of b

Computational Complexity of Gauss & Jordan

In real life you won't be running these algorithms by hand — you'll ask a computer! So it's important to know how much processing time is required (approximately).

Def: A floating point operation (flop) refers to the computer instructions $+, -, \times, \div$ for floating point numbers (the computer's internal representation of decimals).

You can think of one flop as a unit of computer time.

Matrix × Vector takes how many flops?

Let's multiply an $n\times n$ matrix times an $n\times 1$ vector.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ & & & \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ \vdots \\ \end{pmatrix}$$

For the first coordinate:

1. $a_{11}\times b_1$ 2. $a_{12}\times b_2$ ...   n. $a_{1n}\times b_n$

n+1. $a_{11}b_1 + a_{12}b_2$ n+2. $(a_{11}b_1 + a_{12}b_2) + a_{13}b_3$ ...

2n-1: $(a_{11}b_1 + \cdots + a_{1,n-1}b_{n-1}) + a_{1n}b_n$

Total is $2n-1$ flops per coordinate × n coordinates

$$= 2n^2 - n \approx \boxed{2n^2 \text{ flops}}$$

Matrix × Matrix takes how many flops?

Let's multiply an $n \times n$ matrix times an $n \times n$ matrix.

$$A\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix}$$

This is just n matrix×vector products

so $\approx 2n^3$ flops.

... Or Is It? Amazingly, there are more clever procedures that can multiply matrices faster!

Nobody knows exactly how quickly matrix products can be computed, but the current record is

$$O(n^p), \quad p = 2.371552 .$$

Gaussian Elimination takes how many flops?

Let's use an $n \times n$ (square) matrix for simplicity.

(1a) Row swaps require 0 flops (no $+, -, \div, \times$)

(1b) How many flops for a row replacement?

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \xrightarrow{R_2 -= \frac{b_1}{a_1}R_1} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & b_2 - \frac{b_1}{a_1}a_2 & \cdots & b_n - \frac{b_1}{a_1}a_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

Count:

1. $b_1 \div a_1$
2. $\frac{b_1}{a_1} \times a_2$
3. $b_2 - \frac{b_1}{a_1}a_2$
4. $\frac{b_1}{a_1} \times a_3$
5. $b_3 - \frac{b_1}{a_1}a_3$

$\vdots$

$2n-2$. $\frac{b_n}{a_n} \times a_n$

$2n-1$. $b_n - \frac{b_1}{a_1}a_n$

Total flops for each row replacement: $2n-1 = 2(n-1)+1$

Total # row replacements in (1b): $n-1$

Total flops in (1b): $[2(n-1)+1](n-1) = 2(n-1)^2+(n-1)$

Step (2) is just step (1) applied to an $(n-1)\times(n-1)$ matrix.

(2) Total flops: $2(n-2)^2+(n-2)$

(3) Total flops: $2(n-3)^2+(n-3)$

$\vdots$

(n-1) Total flops: $2\cdot 1+1$

Total flops for Gaussian elimination:

(pyramidal number)

$$2(n-1)^2+2(n-2)^2+\cdots+2 = 2\frac{n(n-1)(2n-1)}{6}$$

$$+(n-1)+(n-2)+\cdots+1 = +\frac{n(n-1)}{2}$$

(triangular number)

$$= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n \approx \boxed{\frac{2}{3}n^3 \text{ flops}}$$

Jordan Substitution takes how many flops?

Start with an n×n matrix in REF. Worst case is n pivots:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{array}\right)$$

(1a)

$$R_n \div= a_{nn} \rightsquigarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n/a_{nn} \end{array}\right)$$

1. $b_n \div a_{nn}$ (1 flop)

(1b)

$R_1 -= a_{1n} R_n$

$R_2 -= a_{2n} R_n$

$\vdots$

$R_{n-1} -= a_{n-1,n} R_n$

$$\rightsquigarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & 0 & b_1 - a_{1n}\frac{b_n}{a_{nn}} \\ 0 & a_{21} & \cdots & 0 & b_2 - a_{2n}\frac{b_n}{a_{nn}} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n/a_{nn} \end{array}\right)$$

Count: 2. $a_{1n} \times \frac{b_n}{a_{nn}}$ 3. $b_1 - a_{1n}\frac{b_n}{a_{nn}}$

4. $a_{2n} \times \frac{b_n}{a_{nn}}$ 5. $b_2 - a_{2n}\frac{b_n}{a_{nn}}$

$\vdots$

$2(n-1)$. $a_{n-1,n} \times \frac{b_n}{a_{nn}}$ $2(n-1)+1$. $b_{n-1} - a_{n-1,n}\frac{b_n}{a_{nn}}$

Total flops in (1): $2(n-1)+1$

Step (2) is just step (1) applied to an $(n-1)\times(n-1)$ matrix.

(2) Total flops: $2(n-2)+1$

(3) Total flops: $2(n-3)+1$

$\vdots$

$(n-1)$ Total flops: $2 \cdot 1 + 1$

(n) Total flops: $2 \cdot 0 + 1$

Total flops for Jordan substitution:

$$2[1+2+\cdots+(n-1)]+n = n(n-1)+n$$

$$= \boxed{n^2 \text{ flops}}$$

Summary:

- Multiply $(n\times n)$ matrix & $(n\times 1)$ vector: $\simeq 2n^2$ flops
- Multiply two $(n\times n)$ matrices: $O(2^p)$ $2<p<3$ flops
- Gaussian Elimination $(n\times n)$: $\simeq \frac{2}{3}n^3$ flops
- Jordan Substitution $(n\times n)$: n^2 flops
- Inverting an $(n\times n)$ matrix: $\simeq \frac{4}{3}n^3$ flops

I'll expect you to know which are $O(n^2)$ algorithms & which are $O(n^3)$.

Important Observation:

$\frac{2}{3}n^3$ is an order of magnitude larger than n^2

Eg: If $n=1{,}000$ then $n^2 = 10^6$, $\frac{2}{3}n^3 = 666\times 10^6$, which is 666 times slower!

$$\boxed{\text{Elimination is much slower than Substitution}}$$