

Vector and Matrix Algebra

These basic definitions don't need to be done in a live lecture - that's why I'm recording this one.

Since we'll be doing algebra with vectors & matrices as well as numbers, we give "numbers" a new name to distinguish them.

"definition"

Def: A scalar is a real number.

Notation: $c \in \mathbb{R}$
 "is an element of"
 $\mathbb{R}$ ← the set of all real numbers

"example"

Eg: $2, -\pi, e^{\sqrt{3}}, 0 \in \mathbb{R}$

Def: A vector is a finite (ordered) list of numbers

The size of a vector is the length of the list.

The numbers in the list are the coordinates.

Notation: $v \in \mathbb{R}^n$
 "is an element of"
 $\mathbb{R}^n$ ← "the set of all lists of n numbers"
 n = the size of v

Eg: $v = \begin{pmatrix} 2 \\ -\pi \\ e^{\sqrt{3}} \end{pmatrix} \in \mathbb{R}^3$
 (size 3)
 $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4$
 (size 4)
 ← coordinates

"note"

NB:

We will usually write vectors in a column like $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (a "column vector") but this is just notation; $v = (1, 2, 3)$ means the same thing.

NB: Some people decorate vectors with
boldface: $\mathbf{v}$
arrow: $\vec{v}$

but I won't do that since it's annoying and it's usually clear from context which letters represent vectors.

Important Example:

The unit coordinate vectors in $\mathbb{R}^n$ are vectors with one coordinate = 1 and the rest = 0.

Notation: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^n$

This notation is fixed for the whole semester.

The size of e_i must be inferred from context.

In $\mathbb{R}^3$ the unit coordinate vectors are:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eg: The **zero vector** is the vector

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

(again the size must be inferred from context)

Def: Two vectors are **equal** if they have the **same size** and the **same coordinates**.

Eg: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since the sizes are different.

Vector Algebra

You can multiply a vector by a scalar:

Scalar Multiplication:

$$c \in \mathbb{R}, \quad v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \rightsquigarrow c \cdot v = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \in \mathbb{R}^n$$

(scalar) $\times$ (vector) = (vector)

$$\begin{aligned} \text{Eg: } 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 2 \cdot 1 \\ 2 \cdot 2 \\ 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} & \quad 0 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 \\ 0 \cdot 2 \\ 0 \cdot 3 \end{pmatrix} = \mathbf{0} \\ - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} &= (-1) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \end{aligned}$$

You can add & subtract vectors componentwise:

Vector Addition & Subtraction:

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\leadsto u \pm v = \begin{pmatrix} x_1 \pm y_1 \\ \vdots \\ x_n \pm y_n \end{pmatrix} \in \mathbb{R}^n$$

$$(\text{vector}) \pm (\text{vector}) = (\text{vector})$$

NB: You can only add/subtract vectors of the same size.

$$\text{Eg: } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} \pi \\ e \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 + \pi \\ 2 + e \\ 3 + \sqrt{2} \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \text{X}$$

You can "multiply" two vectors, but you get a scalar.

Dot Product / Inner Product:

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\leadsto u \cdot v = x_1 y_1 + \dots + x_n y_n \in \mathbb{R}$$

$$(\text{vector}) \cdot (\text{vector}) = (\text{scalar})$$

NB: You can only dot vectors of the same size.

Eg: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 4 = 12$

Eg: $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = 1(-2) + 2(-1) + 2(2) = 0$

Eg: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14$

NB: If $v = (x_1, \dots, x_n)$ then $v \cdot v = x_1^2 + \dots + x_n^2$.

This is a nonnegative number; it is $= 0$

$\iff v = 0$.
"if and only if"

$\rightarrow$ Do not write ~~$v^2 = v \cdot v$~~ !

(You can't take any other "powers" of a vector: $v^3 = v \cdot v \cdot v$ doesn't make sense because $v \in \mathbb{R}^n$ and $v \cdot v \in \mathbb{R}$.)

Rules for Vector Algebra: $c \in \mathbb{R}$ $u, v, w \in \mathbb{R}^n$

(1) $c(u \pm v) = cu \pm cv$ (distributivity over scalar $\times$)

(2) $u \cdot (cv) = c(u \cdot v) = (cu) \cdot v$ (associativity of $\cdot$)

(3) $u \cdot v = v \cdot u$ (commutativity of $\cdot$)

(4) $u \cdot (v \pm w) = u \cdot v \pm u \cdot w$ (distributivity over $\cdot$)

Eg: if $u, v, x, y \in \mathbb{R}^n$ then

$$\begin{aligned}(u+v) \cdot (x+y) &\stackrel{(4)}{=} (u+v) \cdot x + (u+v) \cdot y \\ &\stackrel{(3,4)}{=} u \cdot x + v \cdot x + u \cdot y + v \cdot y\end{aligned}$$

Upshot: FOIL works fine.

You can add and scalar multiply at the same time:

Def: A linear combination of vectors $v_1, \dots, v_n \in \mathbb{R}^m$ with weights $x_1, \dots, x_n \in \mathbb{R}$ is the vector

$$x_1 v_1 + x_2 v_2 + \dots + x_n v_n \in \mathbb{R}^m$$

Eg: If $v = (x_1, \dots, x_m) \in \mathbb{R}^m$ then

$$\begin{aligned}v &= \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_m \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= x_1 e_1 + x_2 e_2 + \dots + x_m e_m\end{aligned}$$

The coordinates of v became the weights of this linear combination of unit coordinate vectors.

Matrix Algebra

Def: A **matrix** is a box holding a 2D grid of numbers.
The **size** of a matrix is $(\# \text{rows}) \times (\# \text{cols})$.

We usually (but not always) write
 $m = \# \text{rows}$ $n = \# \text{cols}$
so A is an $m \times n$ matrix.

The (i,j) -entry of A is the number in the i^{th} row and j^{th} column.

Eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$ is a 3×4 matrix.

The (i,j) -entry is a_{ij} .

Eg: $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ is a 3×2 matrix.

(sometimes I'll use square brackets - it means the same)

The $(3,2)$ -entry is 6.

Def: The **diagonal** entries of a matrix are the (i,i) -entries for $i=1, \dots, n$:

$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$ $\bullet = \text{diagonal entries}$

Def: A matrix is **diagonal** if all non-diagonal entries are zero:

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \end{pmatrix} \quad \bullet = \text{any number}$$

Def: A matrix is **square** if $(\# \text{ rows}) = (\# \text{ columns})$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad 3 \times 3 \rightsquigarrow \text{square}$$

Eg: The **$n \times n$ identity matrix** is

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \dots & e_n \\ | & | & & | \end{pmatrix}$$

This is a **square, diagonal** matrix.

Its **columns** are $e_1, \dots, e_n$. (So are its rows.)

Eg: The **$m \times n$ zero matrix** is

$$O = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

This is a diagonal matrix because the **non-diagonal entries** are zero.

(so are the diagonal entries)

Scalar Multiplication & Matrix Addition/Subtraction are again done componentwise:

$$c \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \\ ca_{31} & ca_{32} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} \end{pmatrix}$$

They satisfy distributivity: $c(A \pm B) = cA \pm cB$

NB: You can only add/subtract matrices of the same size.

NB: A vector of size n is just an $n \times 1$ matrix

Def: A row vector is a matrix with one row.

$$u = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \quad 1 \times 3 \rightsquigarrow \text{row vector}$$

You can multiply a matrix & a vector:

Matrix $\times$ Vector = Vector?

There are 2 ways to compute this:

(1) By Columns: If A has columns $v_1, \dots, v_n \in \mathbb{R}^m$ then

$$Ax = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + \dots + x_n v_n \in \mathbb{R}^m$$

for $x \in \mathbb{R}^n$.

The **coordinates** of x are the **weights** of the columns of A in a linear combination.

$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is the linear combination of the columns of A with weights $x_1, \dots, x_n$

(2) By Rows: If A has rows $w_1, \dots, w_m \in \mathbb{R}^n$ then

$$Ax = \begin{pmatrix} - & w_1 & - \\ & \vdots & \\ - & w_m & - \end{pmatrix} x = \begin{pmatrix} w_1 \cdot x \\ \vdots \\ w_m \cdot x \end{pmatrix} \quad \left. \begin{matrix} \text{dot} \\ \text{products} \end{matrix} \right\}$$

The i^{th} coordinate of Ax is $(\text{row } i) \cdot x$

NB: You get the same answer either way!
(#2 is probably easier by hand)

$$\text{Eg: } \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \stackrel{\#1}{=} 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \stackrel{\#2}{=} \begin{pmatrix} 1 \cdot 2 + 4 \cdot (-1) \\ 2 \cdot 2 + 5 \cdot (-1) \\ 3 \cdot 2 + 6 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 2-4 \\ 4-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

NB: Ax only makes sense when the **size** of x equals the number of **columns** of A :

$$A \text{ is } m \times n \rightsquigarrow x \in \mathbb{R}^n \quad Ax \in \mathbb{R}^m$$

$$\begin{matrix} A & \cdot & x & \rightsquigarrow & Ax \\ (m \times n) & & (n \times 1) & & (m \times 1) \end{matrix}$$

$$\text{Eg: } A \cdot e_i \stackrel{\#1}{=} 0 \cdot (\text{col } 1) + \dots + 1 \cdot (\text{col } i) + \dots + 0 \cdot (\text{col } n) \\ = \text{col } i$$

$$A e_i = \text{the } i^{\text{th}} \text{ column of } A$$

$$\text{Eg: } \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{e_2} = 0 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 0 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

identity matrix: p. 8

Eg: $I_n x = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\#1}{=} x_1 e_1 + \dots + x_n e_n \stackrel{\text{p.6}}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$

$$I_n x = x \text{ for all } x$$

Matrix $\times$ Matrix = Matrix: Column Form

Let A be an $m \times n$ matrix.

Let B be an $n \times p$ matrix

with columns $u_1, \dots, u_p \in \mathbb{R}^n$

The product AB is the $m \times p$ matrix
with columns $Au_1, \dots, Au_p \in \mathbb{R}^m$:

$$AB = A \begin{pmatrix} u_1 & \dots & u_p \end{pmatrix} = \begin{pmatrix} Au_1 & \dots & Au_p \end{pmatrix}$$

NB: This only makes sense if $(\# \text{cols of } A) = (\# \text{rows of } B)$

$$\begin{array}{ccc} A & B & AB \\ (m \times n) & \times & (n \times p) \rightarrow m \times p \end{array}$$

NB: You can compute the Au_i using #1 or #2.

Using #2: if A has rows $w_1, \dots, w_m$ then

$$Au_i = (w_1 \cdot u_i, \dots, w_m \cdot u_i) \quad \text{so}$$

$$\begin{pmatrix} -w_1- \\ -w_2- \\ \vdots \\ -w_m- \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 \\ u_1 \end{pmatrix} \begin{pmatrix} 1 \\ u_2 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ u_p \end{pmatrix} \end{pmatrix} = \begin{pmatrix} w_1 \cdot u_1 & w_1 \cdot u_2 & \cdots & w_1 \cdot u_p \\ w_2 \cdot u_1 & w_2 \cdot u_2 & \cdots & w_2 \cdot u_p \\ \vdots & \vdots & \ddots & \vdots \\ w_m \cdot u_1 & w_m \cdot u_2 & \cdots & w_m \cdot u_p \end{pmatrix}$$

So (i,j) entry of $AB = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$

Eg: Compute $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$.

$2 \times 3 \qquad 3 \times 2 \qquad \rightsquigarrow 2 \times 2$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix} \stackrel{\#1}{=} \begin{pmatrix} 1\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} & 1\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} \end{pmatrix} \\ = \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix} \stackrel{\#2}{=} \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 4 & 1 \cdot 3 + 2 \cdot 1 + 3 \cdot (-1) \\ -1 \cdot 1 + 2 \cdot 2 + (-4) \cdot 4 & -1 \cdot 3 + 2 \cdot 1 + (-4) \cdot (-1) \end{pmatrix} \\ = \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}$$

identity matrix: p. 8

Eg: $A \overset{\uparrow}{I_n} = A \begin{pmatrix} \begin{smallmatrix} | \\ e_1 \\ | \end{smallmatrix} & \dots & \begin{smallmatrix} | \\ e_n \\ | \end{smallmatrix} \end{pmatrix} = \begin{pmatrix} A \begin{smallmatrix} | \\ e_1 \\ | \end{smallmatrix} & \dots & A \begin{smallmatrix} | \\ e_n \\ | \end{smallmatrix} \end{pmatrix}$
 $= \begin{pmatrix} (1^{st} \text{ col of } A) & \dots & (n^{th} \text{ col of } A) \end{pmatrix} = A$

Eg: $I_m A = I_m \begin{pmatrix} \begin{smallmatrix} | \\ v_1 \\ | \end{smallmatrix} & \dots & \begin{smallmatrix} | \\ v_n \\ | \end{smallmatrix} \end{pmatrix} = \begin{pmatrix} I_m v_1 & \dots & I_m v_n \end{pmatrix} \stackrel{\text{p. 12}}{=} \begin{pmatrix} \begin{smallmatrix} | \\ v_1 \\ | \end{smallmatrix} & \dots & \begin{smallmatrix} | \\ v_n \\ | \end{smallmatrix} \end{pmatrix} = A$

$$I_m A = A = A I_n$$

Def: A column vector times a row vector is called an outer product:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot (y_1 \ y_2 \ y_3) = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \end{pmatrix}$$

$2 \times 1 \quad 1 \times 3 \quad \rightarrow \quad 2 \times 3$

Eg: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} (3 \ 4 \ 5) \stackrel{\text{p. 12}}{=} \left(3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ 4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$
 $= \begin{pmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{pmatrix}$

Matrix $\times$ Matrix = Matrix: Outer Product Form

Let A be an $m \times n$ matrix.

with columns $v_1, \dots, v_n \in \mathbb{R}^m$

Let B be an $n \times p$ matrix

with rows $w_1, \dots, w_n \in \mathbb{R}^p$

The product AB is the $m \times p$ matrix

$$AB = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} -w_1- \\ \vdots \\ -w_n- \end{pmatrix}$$

$$= v_1 w_1^T + v_2 w_2^T + \dots + v_n w_n^T$$

(w_i^T means write w_i as a row vector)

Eg: $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1 \ 3) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} (2 \ -1) + \begin{pmatrix} 3 \\ -4 \end{pmatrix} (4 \ -1)$$

$$= \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 12 & -3 \\ -16 & 4 \end{pmatrix} = \begin{pmatrix} 17 & 2 \\ -18 & 3 \end{pmatrix} \checkmark$$

There is one more algebraic operation on matrices:

Def: Let A be an $m \times n$ matrix. Its transpose is the matrix A^T whose rows are the columns of A (& vice-versa)

The (i,j) -entry of A is the (j,i) -entry of A^T :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$(1,2)$ -entry $(2,1)$ -entry

You can think of transposing as "reflecting over the diagonal"

Eg: $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Eg: If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ then

$$v^T w = (x_1 \dots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (x_1 y_1 + \dots + x_n y_n) = v \cdot w$$

(a 1×1 matrix is a scalar)

$$v^T w = v \cdot w \quad \text{for } v, w \in \mathbb{R}^n$$

Compare:

$v^T w =$ inner product (scalar)

$v w^T =$ outer product (matrix)

Def: A matrix is symmetric if it equals its transpose: $A = A^T$

NB: symmetric matrices are square!

Eg: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$ is not $A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

Eg: Let A be an $m \times n$ matrix with columns $v_1, \dots, v_n \in \mathbb{R}^m$. Then A^T is $n \times m \rightsquigarrow A^T A$ is $n \times n$.

$$A = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} -v_1- \\ \vdots \\ -v_n- \end{pmatrix}$$

$$A^T A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_n- \end{pmatrix} \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \dots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \dots & v_n \cdot v_n \end{pmatrix}$$

Since $v_i \cdot v_j = v_j \cdot v_i$, this is symmetric.

$A^T A$ is the matrix of column dot products
It is symmetric.

The matrix $A^T A$ will play a huge role in the 2nd half of the course.

Rules for Matrix Algebra:

Let A, B, C be matrices. Assume all sizes are compatible in what follows.

(1) $A(BC) = (AB)C$ (associativity)

→ Thus it makes sense to write ABC
(evaluate AB or BC first → same thing)

→ If $C = v$ is a vector then $A(Bv) = (AB)v$

(2) $A(B \pm C) = AB \pm AC$
 $(A \pm B)C = AC \pm BC$ (distributivity)

(3) $I_m A = A = A I_n$ (identity)

(4) $A(cB) = c(AB) = (cA)B$ for $c \in \mathbb{R}$. (scalars)

(5) $(A^T)^T = A$ (double transpose)

(6) $(A \pm B)^T = A^T \pm B^T$ (transpose & sums)

(7) $(AB)^T = \cancel{A^T B^T} B^T A^T$ (transpose & products)

NB: $A^T B^T$ may not even make sense:

$$A: m \times n \quad B: n \times p$$

$$\rightarrow AB: (m \times n) \cdot (n \times p) = (m \times p)$$

$$A^T B^T: (n \times m) \cdot (p \times n) = ??$$

$$B^T A^T: (p \times n) \cdot (n \times m) = (p \times m) \quad \checkmark$$

$$\text{Eg: } (A^T A)^T \stackrel{(6)}{=} A^T (A^T)^T \stackrel{(4)}{=} A^T A$$

$\Rightarrow A^T A$ is symmetric (again)

Eg: If A is square then AA makes sense, as do $A \cdot A \cdot A$, $A \cdot A \cdot A \cdot A$, etc.

Def: If A is square then its n th power ($n > 0$) is

$$A^n = A \cdots A \quad (n \text{ times})$$

This only makes sense because of associativity.

Question: What about A^{-1} ? (week 3)

Caveats:

Commutativity fails in general:

$$AB \neq BA$$

$$\text{Eg: } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

Cancellation fails in general:

$$A \neq 0, AB = AC \not\Rightarrow B = C$$

$$\text{Eg: } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$