UNIQUE FACTORIZATION AND FERMAT'S LAST THEOREM
HOMEWORK 4

Problem 1. Let $D \geq 5$ be a squarefree integer, and suppose that $D \equiv 1, 2 \pmod 4$. Let $\delta = \sqrt{-D}$ and let $R = \mathbf{Z}[\delta]$ be the imaginary quadratic integer ring for $-D$. Prove that 2 is irreducible but not prime in R. Conclude that R is not a principal ideal domain, and does not have unique factorizations.

Problem 2. Let L be a lattice in $\mathbf{C}$ and let (z, w) be a lattice basis for L. Recall that $\Delta(L)$ is the area of the paralellogram whose vertices are $0, z, w$, and $z + w$. In this problem you will show that $\Delta(L)$ is independent of the choice of lattice basis.

(i) Let (z', w') be another lattice basis for L. Prove that there exist $a, b, c, d \in \mathbf{Z}$ such that

$$z' = az + bw \qquad w' = cz + dw.$$

(ii) Using the fact that (z', w') is also a lattice basis, show that $ad - bc = \pm 1$.

(iii) Show that the area of the paralellogram with vertices $0, z', w', z' + w'$ is equal to $|ad - bc|$ times the area of the paralellogram with vertices $0, z, w, z + w$.

Problem 3. Let R be an imaginary quadratic integer ring and let $I, I' \subset R$ be nonzero ideals. Prove that I and I' are homothetic if and only if there exists a nonzero ideal $J \subset R$ such that IJ and $I'J$ are principal ideals.

Problem 4. Let R be an imaginary quadratic integer ring and let $I \subset R$ be a nonzero ideal. Prove that there is a lattice basis (z, w) for I such that z is a positive integer.

Problem 5. Let $\delta = \sqrt{-6}$ and let $R = \mathbf{Z}[\delta]$, the imaginary quadratic integer ring for -6. Let $I = (2, \delta)$ and $J = (3, \delta)$.

(i) Show that $(2, \delta)$ is a lattice basis for I and that $(3, \delta)$ is a lattice basis for J.

(ii) Find a lattice basis for the product ideal IJ.

(iii) Calculate $\Delta(I)$, $\Delta(J)$, and $\Delta(IJ)$.

(iv) Factor the principal ideal (6) into prime ideals. (See Problem 5(i) on Homework 3.)

(v) Prove that $I \sim J$.