The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory

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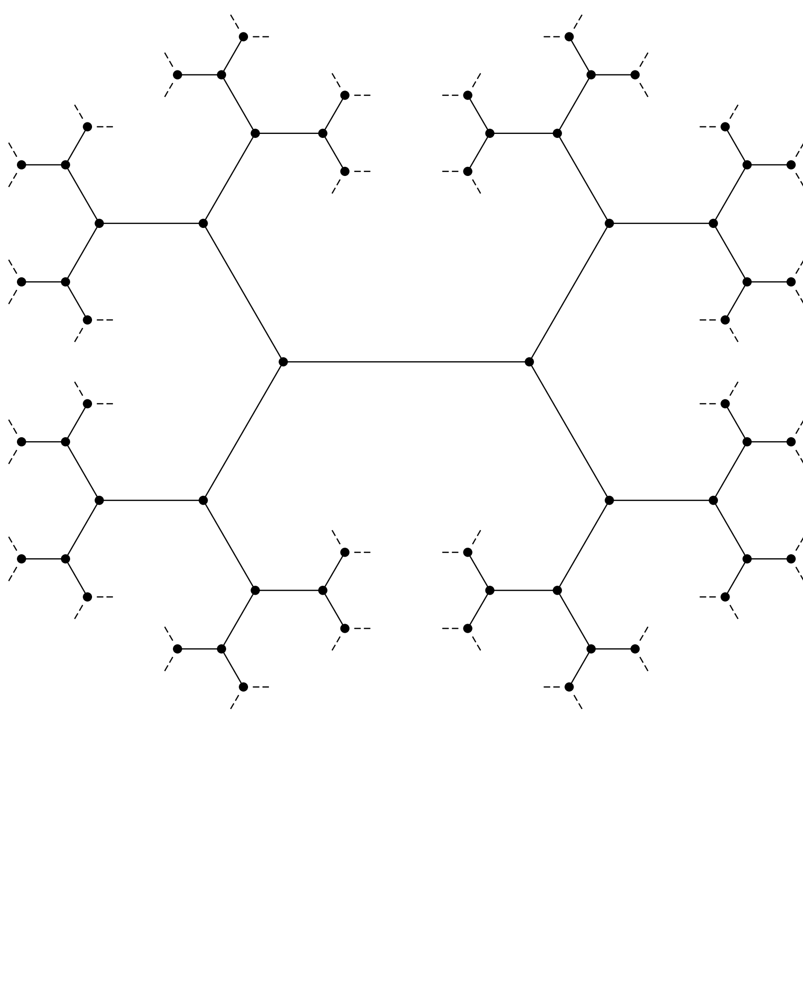

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Abstract

This thesis is concerned with the structure theory of generalized Levi subgroups G of simply-connected Chevalley groups defined over a finite extension of a p-adic field. We present a geometric parameterization of this structure known as the Bruhat-Tits building $\mathcal{B}(G)$. The building facilitates visualizing and reasoning about the structure of G, and therefore has applications to all things related to such groups. We present Moy and Prasad's classification of depth-zero super-cuspidal representations of G using the building. Such representations are obtained by induction from cuspidal representations of finite Chevalley groups — this is therefore an important connection between p-adic representation theory and the representation theory of finite groups of Lie type.

1. Introduction

Much of the research done in the field of algebra in the past century has been the analysis of certain well-behaved matrix groups defined over various fields, and the representations thereof. Real and complex matrix groups such as $\mathrm{SO}_n(\mathbf{R})$, the rotations of n-space, and $\mathrm{GL}_n(\mathbf{R})$, the group of invertible $n \times n$ matrices, are vital to physicists: for instance, the atomic orbitals of the hydrogen atom come from representations of the Lie algebra of the matrix group SU_2, and particles can be thought of as irreducible representations of various symmetry groups such as SU_3. Matrix groups defined over the finite fields $\mathbf{F}_q$ comprise *all* finite simple groups except for the alternating groups A_n and a finite set of exceptions, and are therefore of primary importance to finite group theory. The p-adic fields have many characteristics of continuous fields such as $\mathbf{R}$, but are also closely related to finite fields such as $\mathbf{F}_q$. Matrix groups over p-adic fields turn out to be very important to number theorists. It is this latter class of groups which will concern us in this paper: we will study generalized Levi subgroups G of simply-connected Chevalley groups defined over finite extensions of p-adic fields. Specifically, we will investigate the structure theory and representation theory of these groups.

Review of the p-adics. First we review the concept of a p-adic field, where p denotes a fixed prime number. Such fields were originally introduced by number theorists to facilitate calculations which involve an infinite number of congruences modulo p^n (it is therefore not surprising that p-adic representation theory has applications to number theory). In short, the p-adic field $\mathbf{Q}_p$ is obtained from the rational numbers as the completion of $\mathbf{Q}$ with respect to the p-adic norm, in the same way that $\mathbf{R}$ is obtained as the completion of $\mathbf{Q}$ with respect to the standard absolute value $|\cdot|$. To wit: the sequence $3, 3.1, 3.14, 3.141, \ldots$ is a nice sequence of rational numbers, called a Cauchy sequence, whose elements get arbitrarily close together, in the sense that $|3.1415 - 3.141|$ is small. However, this sequence converges to π, which is not rational. One can realize the real numbers by declaring that all such sequences converge to something, in this way obtaining all infinite nonrepeating decimals trailing off to the right.

The exact same construction works if we replace the standard absolute value by the following size function. Let $a \in \mathbf{Z}$, and write $a = p^n \cdot b$, where $b \in \mathbf{Z}$ and p does not divide b — the number n is well-defined because of the uniqueness of prime factorizations. Then we can set the p-adic absolute value of a to be $|a|_p = p^{-n}$ for $a \neq 0$ and $|0|_p = 0$. For example, if $p = 2$, then $48 = 2^4 \cdot 3$ has norm 2^{-4}; if we write 48 in its binary expansion 110000, we can see that its norm is completely dependent on its *least* significant nonzero digit. This is opposite from the standard absolute value, where a number's size depends on its most significant digit. One can extend the p-adic norm to all rational numbers by setting $|a/b|_p = |a|_p/|b|_p$ for $b \neq 0$, and using this, one can again declare that all Cauchy sequences (i.e., sequences $\{x_n\}$ of rational numbers such that $|x_n - x_m|_p$ gets arbitrarily small) must converge. One thus obtains the complete p-adic field $\mathbf{Q}_p$ as the set of all infinite, nonrepeating decimals (written in base p) which trail off to the *left*, instead of to the right — note that numbers are "small" if their *least* significant digits are zero.

One should note that if $a \in \mathbf{Z}$ and p does not divide a, then $|a|_p = 1$, whether a is large or small — in this sense, the p-adic norm obliterates all information about any prime other than p. This is one reason that fields such as $\mathbf{Q}_p$ are useful to number theorists. These fields also have the interesting property that if we set $R = \{a \in \mathbf{Q}_p \ : \ |a|_p \leq 1\}$ and $\wp = \{a \in \mathbf{Q}_p \ : \ |a|_p < 1\}$, then $R/\wp$ is naturally isomorphic to the finite field $\mathbf{F}_p$ of p elements, called the ***residue field***. This fact is vital.

The basic objects. Claude Chevalley [Che55] has developed a method, later extended by Steinberg [Ste68], by which one can take a complex semisimple Lie group, such as $\mathrm{SL}_n(\mathbf{C})$, and an arbitrary field k, and produce the analogous group ***defined over*** k — $\mathrm{SL}_n(k)$ in this example. The resulting groups are called Chevalley groups. It is these objects with which we will be concerned, where k is a finite extension of some field $\mathbf{Q}_p$. We should note that all of the analysis that we will present is usually done in the more general context of reductive algebraic groups, but that the latter approach is more difficult and is beyond the scope of this paper. We give a full example of Chevalley's construction in Section 1.1; at this point, one can gain a good intuition for the structure theory of our groups by analyzing the simple example of $G = \mathrm{GL}_3(\mathbf{Q}_p)$.

The first thing to notice is that the group T of diagonal matrices in G is an abelian group, called a ***maximal torus***, which is isomorphic to $(\mathbf{Q}_p^\times)^3$. The torus T acts nicely on the off-diagonal matrix entries by conjugation: for example,

$$\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \begin{bmatrix} 1 & & x \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & x \cdot a/c \\ & 1 & \\ & & 1 \end{bmatrix}.$$

The group B of all invertible upper-diagonal matrices forms a large subgroup of G, with the nice property that $B = TU$ as a semidirect product, where U is the normal subgroup of B with ones on the diagonal. That is,

$$\begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix} = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

We call B a ***Borel Subgroup***, U its ***unipotent radical***, and the decomposition $B = TU$ a ***Levi Decomposition***; in this context, T is a ***Levi subgroup*** of B.

We define any subgroup P containing B to be a ***standard parabolic subgroup***. It turns out that

$$\begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * \\ & * & * \\ & * & * \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

are the only standard parabolic subgroups in G. Any parabolic subgroup has a Levi decomposition into a semidirect product of a Levi subgroup and a unipotent radical; for instance, we have

$$P = \begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix} = \begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} = M_P U_P.$$

The Levi subgroup M_P above is isomorphic to $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_1(\mathbf{Q}_p)$, whose structure theory is similar to that of $G = \mathrm{GL}_3(\mathbf{Q}_p)$ itself. This is vital to the structure theory and the representation theory of the type of group we are concerned with: that is, all (generalized) Levi subgroups of a simply-connected p-adic Chevalley group can be analyzed in a uniform way. This suggests using inductive arguments in our analysis.

So far we have ignored the fact that $\mathbf{Q}_p$ is a p-adic field, as opposed to an arbitrary field; indeed, the above definitions hold upon replacing $\mathbf{Q}_p$ with any field. There is, however, much structure theory which depends on the p-adic nature of $\mathbf{Q}_p$. For $n \in \mathbf{Z}$, define

$$\wp^n = \{a \in \mathbf{Q}_p \ : \ |a| \le p^{-n}\}.$$

Then

$$\{0\} \subset \cdots \subset \wp^3 \subset \wp^2 \subset \wp \subset R \subset \wp^{-1} \subset \wp^{-2} \subset \cdots \subset \mathbf{Q}_p$$

is an exhaustive filtration of $\mathbf{Q}_p$ by compact open subgroups (the topology of $\mathbf{Q}_p$ is given by the norm $|\cdot|_p$). Using this, we can define many interesting compact open subgroups of our group G (where the topology on G is induced from the natural topology on the space of 3×3 p-adic matrices). For instance, if we set

$$G_y = \begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix} \qquad \text{and} \qquad G_y^+ = \begin{bmatrix} 1+\wp & \wp & \wp \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix}$$

(where all sets of matrices are of course intersected with G), then these are compact open subgroups. These groups have the vital property that

$$G_y/G_y^+ \cong \begin{bmatrix} \mathbf{F}_p & \mathbf{F}_p & \mathbf{F}_p \\ \mathbf{F}_p & \mathbf{F}_p & \mathbf{F}_p \\ \mathbf{F}_p & \mathbf{F}_p & \mathbf{F}_p \end{bmatrix} \cong \mathrm{GL}_3(R/\wp) = \mathrm{GL}_3(\mathbf{F}_p)$$

which is the "same" group over the finite field. (Note that this is a generalization of the fact that $\mathbf{F}_p = R/\wp$.) This fact suggests that we will be able to use the theory of finite Chevalley groups to enhance our understanding of the p-adic versions

All of the objects treated in this section can be defined in any generalized Levi subgroup of a simply-connected p-adic Chevalley group; this is precisely the structure theory we will investigate in the majority of this thesis.

Representation theory. In Chapter 5, we will give an important application of the structure theory of our groups (as developed up to that point) to p-adic representation theory. It is therefore useful at this point to give a summary of some of the basic results in that area. The results mentioned here are the result of the invaluable work done by Jacquet, Langlands, Kottwitz, Mautner, Bernstein, Steinberg, Casselman, Iwahori, Moy, Prasad, and many others.

Let (π, V) be a ***representation*** of a group G of the type that we are considering — that is, V is some complex vector space, and π is a homomorphism from G into $\mathrm{GL}(V)$ (in other words, each $g \in G$ acts on V by an invertible linear map, such that $\pi(g) \circ \pi(h) = \pi(gh)$). We say that (π, V) is ***smooth*** if, for all $v \in V$, the subgroup of G fixing v contains a compact open subgroup. In other words, the action of G on V is continuous with respect to the discrete topology on V. All representations we will encounter are smooth. We write $\mathfrak{R}(G)$ for the category of smooth representations of G. If, in addition,

$$V^K = \{v \in V \ : \ kv = v \text{ for all } k \in K\}$$

is finite-dimensional for every compact open subgroup K of G, then we say that V is ***admissible***.

If V has no G-invariant subspace — that is, V has no vector subspace U such that $\pi(g) \cdot u \in U$ for every $u \in U$ and $g \in G$ — then we say that V is ***irreducible***. Equivalently, irreducible representations are simple G-modules. One can show that the only finite-dimensional smooth irreducible representations of G are one-dimensional, so that all interesting smooth representations are infinite-dimensional.

A fundamental problem in p-adic representation theory is to find all smooth irreducible representations of G (all such representations are automatically admissible). One of the most basic objects used to accomplish this goal, for reasons that will become clear, is the super-cuspidal representation.

Let $P \subset G$ be a proper parabolic subgroup with Levi decomposition $P = MU$ as above, and let $V(U)$ be the subspace of V generated by $\{\pi(u)v - v \ : \ u \in U, \ v \in V\}$. Then $V_U := V/V(U)$ is the largest quotient of V on which U acts trivially, and is therefore a natural representation of M because $P = MU$.

Definition 1.1. We say that the smooth representation (π, V) is ***super-cuspidal*** if for every (standard) proper parabolic subgroup P of G with Levi decomposition MU, the space V_U is trivial.

Intuitively, then, a super-cuspidal representation has nothing to do with any proper Levi subgroup — it is determined only by the data that differentiates the full group G from its Levi subgroups. One should note that this definition is almost identical to the definition of a cuspidal representation of a simply-connected Chevalley group over a finite field (over the finite field, one usually requires that cuspidal representations be irreducible):

Definition 1.2. A finite-dimensional irreducible representation (τ, W) of a simply-connected Chevalley group G defined over a finite field is ***cuspidal*** if for every (standard) proper parabolic subgroup P of G with Levi decomposition MU, the space W_{U} is trivial.

If $H \subset G$ is a subgroup and (σ, W) is a smooth representation of H, then we define $\mathrm{Ind}_H^G \sigma$ to be the space of all maps $f : G \to W$ such that f is locally constant on the right (that is, there is some compact open subgroup $K_f \subset G$ such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K_f$), and for any $g \in G$ and $h \in H$, we have $f(hg) = \sigma(h)f(g)$. For $g \in G$ and $f \in \mathrm{Ind}_H^G \sigma$, we define $g \cdot f$ by the right-regular action, that is, $(g \cdot f)(x) = f(xg)$ for $x \in G$. With this action, $\mathrm{Ind}_H^G \sigma$ is a smooth representation of G, called an ***induced representation***. We also define $\mathrm{ind}_H^G \sigma$ to be the subspace of $\mathrm{Ind}_H^G \sigma$ consisting of functions whose support is compact modulo H (on the left). Intuitively, the functors ind and Ind are methods by which one can take a representation of a subgroup $H \subset G$, and obtain a representation of all of G, which, in some sense, does not contain any more information.

With these definitions, one can show that if $(\pi, V) \in \mathfrak{R}(G)$ is irreducible, then there exists a (not necessarily proper) parabolic subgroup P of G with Levi decomposition MU and an irreducible super-cuspidal representation σ of M such that π is a subrepresentation of $\mathrm{Ind}_P^G \widetilde{\sigma}$, where $\widetilde{\sigma}$ is σ extended to $P = MU$ by declaring that U acts trivially (recall that U is normal in P). Therefore we start to see that the smooth irreducible representations of G divide into those that can be induced from smaller well-behaved subgroups of G, and those that arise from the data that is unique to G and independent of its proper Levi subgroups. This division is in fact a splitting of the category $\mathfrak{R}(G)$ into the full subcategories of super-cuspidal representations and those representations found in representations induced from super-cuspidal representations of proper Levi subgroups.

Therefore, with the inductive argument hinted at before, we see that in a sense, we can understand all of the smooth irreducible representations of G if we can understand its super-cuspidal representations. Thus the super-cuspidal representations are the fundamental objects in p-adic representation theory. That said, the only class of groups for which all of the super-cuspidal representations are known is $\mathrm{GL}_n(k)$ (and some of its close relatives like $\mathrm{SL}_n(k)$, under some extra conditions on k). Thus finding the super-cuspidal representations is a major outstanding problem in p-adic representation theory.

The general strategy for finding super-cuspidal representations is by compact induction from subgroups K of G which are compact mod the center of G. The justification for this dates back to Mautner [Mau64]. He showed that that if σ is a smooth irreducible representation of K and $\mathrm{ind}_K^G \sigma$ is irreducible, then it is super-cuspidal. Many, many people have tried to find all super-cuspidal representations in this fashion, but none have been successful. However, if (π, V) is an irreducible super-cuspidal representation of G such that the space of G_y^+-fixed vectors is nonzero (in the example above), then Moy and Prasad showed that the natural representation of G_y/G_y^+ (which is $\mathrm{GL}_3(\mathbf{F}_p)$ in our example above) on $V^{G_y^+}$ contains a cuspidal representation, and that (π, V) can be obtained by compact induction from this representation. This is a rather surprising connection between the representation theory of G and the representation theory of its analog over the finite residue field — somehow, cuspidal and super-cuspidal representations are related. In this way, Moy and Prasad classify all super-cuspidal representations of G with vectors fixed under certain subgroups similar to G_y^+ (which is a nontrivial class of representations), using information about the cuspidal representations of finite groups of Lie type, which are well understood (in the sense that all finite simple groups are classified). We will carry out this classification in Chapter 5.

Motivation. The structure theory of our groups is rich and beautiful, and deserves to be studied for that reason alone. There are, however, applications to other areas of mathematics. For instance, Moy and Prasad [MP94] devised a method of characterizing a representation (π, V) of G by its ***depth***, that is, how "deep" one must look into a certain filtration of the compact open subgroups G_y in order to find a compact open subgroup K such that the space V^K of K-fixed vectors is nonzero. This increases our understanding of the representation theory of G — see Chapter 5. As another example, DeBacker [DeB02] has found a way to parameterize nilpotent orbits in the Lie algebra of G via the structure theory presented in this paper, which aids in the study of harmonic analysis on G. Harmonic analysis on reductive p-adic groups and p-adic representation theory in turn have some deep applications to number theory.

Our focus. We will present the construction of the Bruhat-Tits building $\mathcal{B}(G)$, which is a geometric parameterization of the structure of G. The building is a huge CW-complex with a natural G action, which is formed by gluing together many apartments $\mathcal{A}$, which are all copies of affine ℓ-space $\mathbf{R}^\ell$, cut up by hyperplanes in a regular fashion (see Figure 4.1). For instance, an apartment of SL_3 is a plane, tiled by equilateral triangles. The building also has a natural metric, with respect to which the group G acts by isometries.

Much of the structure theory of G can be realized using the building and its G-action: for example, if $y \in \mathcal{B}(G)$ is any point, then we can form the stabilizer subgroup $G_y = \{g \in G \ : \ g \cdot y = y\}$, justifying the notation above. This subgroup always turns out to have a natural quotient which is isomorphic to a generalized Levi subgroup of the analogous Chevalley group defined over the residue field. In general, many important structural properties translate directly into the geometric language of the building, and vice-versa; thus the building makes it much easier to reason about and visualize the structure of G.

The building was invented by Bruhat and Tits in [BT72] the 1970's, but its power was only exploited to a limited extent until the mid-1990's, when Moy and Prasad defined certain filtration subgroups $G_{x,r}$ and $\mathfrak{g}_{x,r}$ for $x \in \mathcal{B}(G)$. Using this new structure, Moy and Prasad were able to prove, among other things, the results already mentioned, which we present in Chapter 5. Since then, an intensive study has been made of the building and its applications to all things related to p-adic Chevalley groups.

Roadmap. In Section 1.1, we will present Chevalley's construction of $\mathrm{Sp}_4(k)$; if the reader is not familiar with Chevalley groups, this should be a sufficient introduction.

In Chapter 2, we present the basic unit of the Bruhat-Tits building: the affine apartment. Since much of the "affine" structure of G (that is, the structure of a p-adic Chevalley group G which depends on the fact that it is defined over a p-adic field) is analogous to the "spherical" structure of any Chevalley group (that is, the structure which does not depend on the field k), we also give an introduction to the spherical apartment of an arbitrary Chevalley group G, and show how the spherical apartment gives rise to much of the standard Chevalley group structure theory of G. Since the spherical structure theory may be more familiar to the reader, Section 2.1 will hopefully be helpful in understanding Section 2.2.

If the Bruhat-Tits building is made by gluing together many affine apartments, parahoric subgroups comprise the glue. It is these subgroups which make up the structure theory used to build the building, and most of the structure that the building parameterizes is realized in terms of parahorics. We therefore devote all of Chapter 3 to parahoric subgroups. Parahoric subgroups also have the extremely important property that they have a quotient which is isomorphic to a generalized Levi of a Chevalley group defined over the residue field; this is discussed in Section 3.2. Parahoric subgroups play a role in the affine structure theory of G analogous to the role parabolics play in the spherical structure theory of G; some of the results which solidify this analogy are given in Section 3.3.

With the knowledge about parahorics developed in Chapter 3, we define the Bruhat-Tits building in Chapter 4. There is an analogous spherical building associated with any Chevalley group (formed by gluing together many spherical apartments), but, unlike in Chapter 2, we do not present that construction. We give the full example of the building of $\mathrm{SL}_2(k)$ in Section 4.2.

Chapter 5 is devoted to Moy and Prasad's classification of irreducible super-cuspidal representations of depth zero using the Bruhat-Tits building; in particular, we show that all such representations can be obtained from *cuspidal* representations of Chevalley groups over the finite field. It is very useful to understand this relationship between finite group representation theory and p-adic representation theory — for one, the representation theory of finite Chevalley groups is well understood. In Section 5.1, we define the notion of the depth of a representation; in Section 5.2, we present the classification; and in Section 5.3, we show how Section 5.2 is used to find all depth-zero irreducible super-cuspidal representations of $\mathrm{SL}_2(k)$. We make good use of Section 4.2 in Section 5.3.

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1.1. Chevalley group example: Sp_4

The constructive nature of Chevalley groups obliges us to define much notation, which is perhaps easier to absorb if presented in a concrete example. In addition, one who is not familiar with the general construction of Chevalley groups may find this example and the one in Appendix A helpful. We therefore present the Chevalley-Steinberg construction of the group $\mathrm{Sp}_4(k)$, where k is an arbitrary field. All of the notation in this section can be defined in the same way for any Chevalley group; after this section, we will no longer assume that we are working with Sp_4.

The Chevalley Data. The basic data needed to construct $\mathrm{Sp}_4(k)$ is the Dynkin diagram of type C_2, which shows two simple roots, a short one and a long one, at a $135°$ angle. From this data, we can construct the (abstract) root system Φ^c, which is given in Figure 1.1, along with its co-root system (note that the short and long roots are switched). In this section we label the simple roots α and β and the simple root system $\Delta^c = \{\alpha, \beta\}$; the other positive roots are given in the figure. A complex simple Lie algebra which gives rise to this root diagram will be of the form $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \bigoplus_{\gamma\in\Phi^c}(\mathfrak{g}_\gamma)_{\mathbf{C}}$, where $(\mathfrak{g}_\gamma)_{\mathbf{C}}$ is the one-dimensional root space associated with the root γ and $\mathfrak{h}_{\mathbf{C}}$ is the two-dimensional Cartan subalgebra. There are elements $X_{\pm\gamma} \in (\mathfrak{g}_{\pm\gamma})_{\mathbf{C}}$ and $Z_\gamma \in \mathfrak{h}_{\mathbf{C}}$ such that $(X_\gamma, X_{-\gamma}, Z_\gamma)$ is an $\mathfrak{sl}_2$ triple, with the elements Z_γ scaled such that for every root δ,

$$[Z_\gamma, X_\delta] = 2(\delta, \gamma)/(\gamma, \gamma) \cdot X_\delta, \tag{1.1}$$

which is always an integer multiple since Φ^c is a root system.

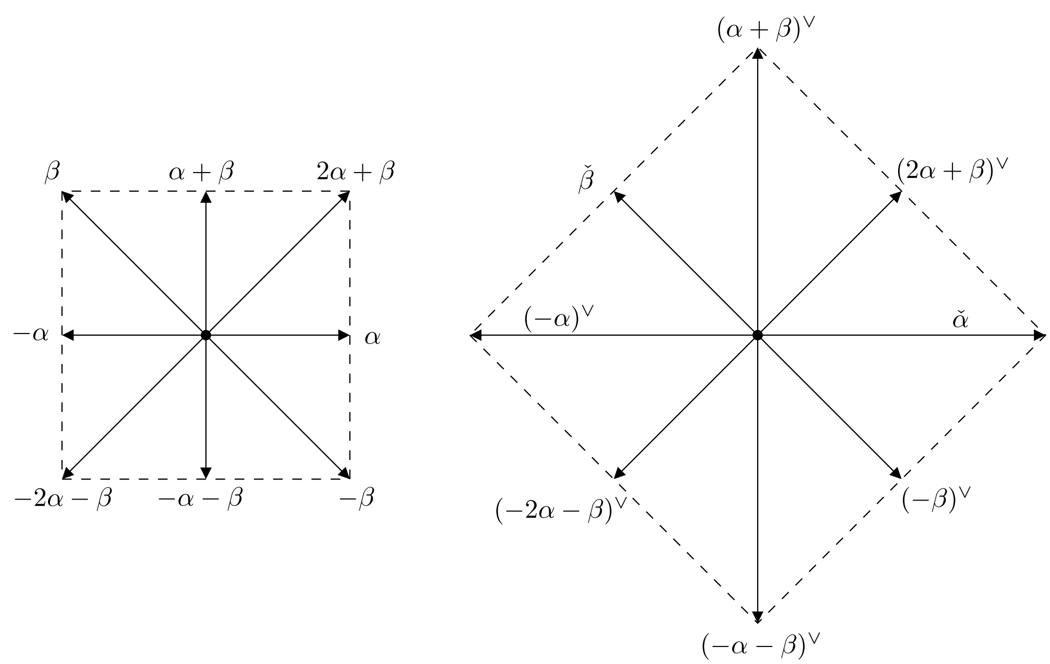

Figure 1.1: The root system and co-root system of type C_2. This co-root system is usually drawn rotated clockwise $45°$ from this drawing; in this representation, the Euclidean space containing the root system and its dual containing the co-root system are identified via the inner product.

We can realize $\mathfrak{g}_{\mathbf{C}}$ as the space of 4×4 matrices which preserve the skew form $M = \left[\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right]$. One can show [FH91, p.239] that $\mathfrak{g}_{\mathbf{C}}$ is the space of block matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that B and C are symmetric and A and D are negative transposes of each other. Therefore a basis for $\mathfrak{g}_{\mathbf{C}}$ is

$$
\begin{gathered}
X_\alpha = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & -1 & 0 \end{bmatrix} \qquad X_{-\alpha} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix} \qquad Z_\alpha = [X_\alpha, X_{-\alpha}] = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \\
X_\beta = \begin{bmatrix} 0 & & 0 & 0 \\ & 0 & 0 & 1 \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad X_{-\beta} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ 0 & 0 & 0 & \\ 0 & 1 & & 0 \end{bmatrix} \qquad Z_\beta = [X_\beta, X_{-\beta}] = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} \\
X_{\alpha+\beta} = [X_\alpha, X_\beta] = \begin{bmatrix} 0 & & 0 & 1 \\ & 0 & 1 & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad X_{-\alpha-\beta} = [X_{-\beta}, X_{-\alpha}] = \begin{bmatrix} 0 & & & \\ & 0 & & \\ 0 & 1 & 0 & \\ 1 & 0 & & 0 \end{bmatrix} \\
X_{2\alpha+\beta} = \frac{1}{2}[X_\alpha, X_{\alpha+\beta}] = \begin{bmatrix} 0 & & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad X_{-2\alpha-\beta} = \frac{1}{2}[X_{-\alpha-\beta}, X_{-\alpha}] = \begin{bmatrix} 0 & & & \\ & 0 & & \\ 1 & 0 & 0 & \\ 0 & 0 & & 0 \end{bmatrix}.
\end{gathered} \tag{1.2}
$$

One can check that the requirements (1.1) are satisfied, so this is indeed a simple Lie algebra with the correct Dynkin diagram. It is important that the integer span

$$\mathfrak{g} := \left(\bigoplus_{\gamma \in \Phi^c} \mathbf{Z} \cdot X_\gamma \right) \oplus \left(\bigoplus_{\delta \in \Delta^c} \mathbf{Z} \cdot H_\delta \right)$$

of the basis elements is stable under the bracket, so that we have in fact chosen a ***Chevalley basis*** for $\mathfrak{g}_{\mathbf{C}}$. This is the first important step in constructing a Chevalley group; Chevalley showed that any complex semisimple Lie algebra has such a basis.

We will use the following notation for our abstract root system and our Chevalley data:

- $\mathfrak{g}$ is our integer Lie algebra, which comes from the complex semisimple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ and a Chevalley basis.
- Φ^c is the (abstract) root system of $\mathfrak{g}$.
- L^c $= |\Phi^c|$ is the number of roots.
- $(\Phi^c)^\pm$ is some choice of positive and negative roots; in our case, $(\Phi^c)^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$.
- Δ^c is the set of simple roots with respect to $(\Phi^c)^+$; in our case, $\Delta^c = \{\alpha, \beta\}$.
- $\mathfrak{g}_\gamma$ $= \mathbf{Z} \cdot X_\gamma$ is the (integer) root space in $\mathfrak{g}$ of the root $\gamma \in \Phi^c$.
- $\mathfrak{h}$ is the (integer) Cartan subalgebra. In our case, $\mathfrak{h} = \mathbf{Z} \cdot Z_\alpha \oplus \mathbf{Z} \cdot Z_\beta$.
- ℓ^c $= |\Delta^c|$ is the rank of $\mathfrak{g}$.
- E is the Euclidean space containing the abstract root system Φ^c.
- E^* is the linear dual of E. (In general, if X is a vector space, then X^* denotes its linear dual.)
- $(\cdot, \cdot)$ is the inner product on E and on E^*.
- Λ_R^c $\subset E$ is the ***root lattice***, i.e., the lattice spanned by Φ^c.
- $\langle \gamma, \delta \rangle$ $= 2(\gamma, \delta)/(\delta, \delta)$ is always an integer for $\gamma, \delta \in \Phi^c$ (by the definition of a root system).
- $\langle \cdot, \cdot \rangle$ is also used to denote the canonical pairing of E with E^* (sorry, there are only so many kinds of brackets). For $\alpha \in E$ and $x \in E^*$ we will often write $\alpha(x)$ for $\langle x, \alpha \rangle$.
- $\check{\Phi}^c$ $\subset E^*$ is the system of co-roots.
- $\check{\gamma}$ $\in E^*$ is the co-root corresponding to $\gamma \in \Phi^c$, defined by the equation $\langle \check{\gamma}, \delta \rangle = \langle \gamma, \delta \rangle$ for all $\delta \in \Phi^c$.

$\check{\Lambda}_R^c$ $\subset E^*$ is the ***co-root lattice***, that is, the lattice generated by $\check{\Phi}^c$. (In general, if $X \subset \Lambda_R^c$ is a subset of the root lattice, then $\check{X} \subset \check{\Lambda}_R$ is the corresponding subset of the co-root lattice.)

Our Chevalley basis is

$$\{X_\gamma,\ Z_\delta\ :\ \gamma \in \Phi^c,\ \delta \in \Delta^c\}$$

where each X_γ spans the root space $\mathfrak{g}_\gamma$, and the Z_δ span the Cartan subalgebra $\mathfrak{h}$. All of the above Chevalley data (except the specific choice of simple and positive roots) will be *fixed* throughout this paper (again, we will not assume that our Chevalley group is $\mathrm{Sp}_4(k)$ after this section).

Before moving on, we must make a note about root systems: if one starts with a complex semisimple Lie algebra, one uses the root space decomposition and the Killing form to realize the root system as a subset of a real subspace of $h^*_{\mathbf{C}}$. We can recover that realization: let $\mathfrak{h}_{\mathbf{R}}$ be the real span of the elements $\{Z_\gamma\ :\ \gamma \in \Delta^c$ (or $\gamma \in \Phi^c)\}$. For each $\delta \in \Phi^c$, Equation (1.1) allows us to define a linear map $\mathfrak{h}_{\mathbf{R}} \to \mathbf{R}$, also denoted by δ, by the formula

$$\delta(Z_\gamma) \cdot X_\delta = [Z_\gamma, X_\delta] = \langle \delta, \gamma \rangle \cdot X_\delta \tag{1.3}$$

for $\gamma \in \Delta^c$; the above formula then holds for any $\gamma \in \Phi^c$. So for instance, one calculates that $\alpha(\mathrm{diag}(a,b,-a,-b)) = a-b$ and $\beta(\mathrm{diag}(a,b,-a,-b)) = 2b$. In a dual manner, any co-root $\check{\gamma}$ defines a linear map $\mathbf{R} \to \mathfrak{h}_{\mathbf{R}}$ simply by $\check{\gamma}(r) = r \cdot Z_\gamma$; we then recover $\delta \circ \check{\gamma} = \langle \delta, \gamma \rangle$. Extending linearly, we can identify E with $\mathfrak{h}^*_{\mathbf{R}}$ and E^* with $\mathfrak{h}_{\mathbf{R}}$ in a natural way. That said, after this section we will never use any realization of the root system Φ^c; rather, we will regard it as a separate object which interacts with our groups.

The Chevalley Group. The strategy is to obtain $\mathrm{Sp}_4(k)$ by formally exponentiating the above Lie algebra. One can obtain all of the isogeny forms of a Lie group via Steinberg's method, so we need one more piece of data in order to construct the simply-connected form Sp_4. In general, *we will assume that all of our Chevalley groups are simply-connected* (see Appendix A). Using our identification of $\mathfrak{h}^*_{\mathbf{R}}$ with E, we can define the ***weight lattice*** $\Lambda_W^c \supset \Lambda_R^c$ to be the set of all $\lambda \in \mathfrak{h}^*_{\mathbf{R}}$ such that $\lambda(Z_\gamma) \in \mathbf{Z}$ for all $\gamma \in \Delta^c$. One can show that the lattice generated by the weights of any representation of $\mathfrak{g}_{\mathbf{C}}$ is contained between Λ_R^c and Λ_W^c. Fix a faithful finite-dimensional representation $V^{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$ whose ***weight lattice*** (i.e., the lattice generated by the weights of $V^{\mathbf{C}}$) is all of Λ_W^c (the particular choice of representation ends up being irrelevant) — this is the property that will allow us to generate the simply-connected form.[1]

The next step is to choose a full-rank lattice V in $V^{\mathbf{C}}$ which is invariant under the set

$$\{X_\gamma^n/n!\ :\ n \in \mathbf{N},\ \gamma \in \Phi^c\}, \tag{1.4}$$

where we are thinking of $X_\gamma^n/n!$ as a member of $\mathrm{End}(V^{\mathbf{C}})$. One can show [Ste68] that such a lattice exists, and that the choice of the lattice is ultimately unimportant. Set V^k to be the vector space $V \otimes k$; note that each $X_\gamma^n/n!$ acts on V^k in a natural way. Since the representation $V^{\mathbf{C}}$ has a finite number of weights, there is some n for each γ such that $X_\gamma^n \in \mathrm{End}(V^k)$ is zero. Therefore for $t \in k$ and $\gamma \in \Phi^c$,

$$\exp(tX_\gamma) = 1 + tX_\gamma + \frac{(tX_\gamma)^2}{2!} + \frac{(tX_\gamma)^3}{3!} + \cdots \in \mathrm{GL}(V^k)$$

[1] E.g., if we had chosen $V^{\mathbf{C}}$ to be the adjoint representation, whose weight lattice is simply the root lattice, then we would generate the adjoint form.

is a well-defined element, with inverse $\exp(-tX_\gamma)$. Define the ***Chevalley group*** G^c to be the group generated by $\{\exp(tX_\gamma) \ : \ t \in k, \ \gamma \in \Phi^c\}$.

In our case, the weight lattice of the standard representation of $\mathrm{Sp}_4(\mathbf{C})$ is all of Λ_W^c. Indeed, one sees that $\Lambda_W^c = \mathbf{Z}(\alpha/2) \oplus \mathbf{Z}\beta$, and that the weight corresponding to the weight space spanned by $(1,0,0,0)$ is $\alpha/2$. We have $X_\gamma^2 = 0$ for all $\gamma \in \Phi^c$, so $\mathbf{Z}^4 \subset \mathbf{C}^4$ is a good choice for a full-rank sublattice. We then have $V^k = k^4$, and X_γ acts on V^k via the matrices (1.2). Since $\exp(tX_\gamma) = 1 + tX_\gamma$, we have

$$
\begin{aligned}
\exp(tX_\alpha) &= \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & \\ & & -t & 1 \end{bmatrix} &
\exp(tX_{-\alpha}) &= \begin{bmatrix} 1 & & & \\ t & 1 & & \\ & & 1 & -t \\ & & & 1 \end{bmatrix} \\
\exp(tX_\beta) &= \begin{bmatrix} 1 & & 0 & 0 \\ & 1 & 0 & t \\ & & 1 & \\ & & & 1 \end{bmatrix} &
\exp(tX_{-\beta}) &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ 0 & 0 & 1 & \\ 0 & t & & 1 \end{bmatrix} \\
\exp(tX_{\alpha+\beta}) &= \begin{bmatrix} 1 & & 0 & t \\ & 1 & t & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix} &
\exp(tX_{-\alpha-\beta}) &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ 0 & t & 1 & \\ t & 0 & & 1 \end{bmatrix} \\
\exp(tX_{2\alpha+\beta}) &= \begin{bmatrix} 1 & & t & 0 \\ & 1 & 0 & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix} &
\exp(tX_{-2\alpha-\beta}) &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ t & 0 & 1 & \\ 0 & 0 & & 1 \end{bmatrix}
\end{aligned}
\tag{1.5}
$$

which are elements of $\mathrm{GL}(V^k)$.

We will use the following notation for the objects defined above:

- V is some lattice contained in a faithful complex representation $V^{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$ which is invariant under the set (1.4). We will require that our Chevalley groups be *simply-connected* (see Appendix A), so the lattice of weights of the representation $V^{\mathbf{C}}$ must be the weight lattice.
- V_μ is the (integer) weight space of V corresponding to the weight μ. In our case, the weight spaces are generated by the coordinate vectors; for instance,

  $$\mathrm{diag}(a, b, -a, -b) \cdot (1,0,0,0) = a \cdot (1,0,0,0),$$

  so the space $V_\mu = (\mathbf{Z}, 0, 0, 0)$ corresponds to the weight $\mu : \mathrm{diag}(a, b, -a, -b) \mapsto a$.
- v_μ generates the weight space V_μ, so the v_μ generate V as an integer lattice. In our case, the v_μ are the unit coordinate vectors.
- V^k $= V \otimes_{\mathbf{Z}} k$, which is k^4 in our case.
- $x_\gamma(t)$ $= \exp(tX_\gamma)$ for $t \in k$. Note that $x_\gamma(t+u) = x_\gamma(t)x_\gamma(u)$ for $u \in k$.
- $\mathfrak{X}_\gamma$ $= \mathfrak{X}_\gamma(k)$ is the root subgroup $\{x_\gamma(t) \ : \ t \in k\} \cong k^+$.
- G^c is the full Chevalley group, defined as the subgroup of $\mathrm{GL}(V^k)$ generated by the $\mathfrak{X}_\gamma$ for $\gamma \in \Phi^c$.

The Cartan subgroup. Since the elements $Z_\gamma \in \mathfrak{h}$ do not necessarily have finite order as elements of $\mathrm{End}(V^k)$, we cannot exponentiate them directly. However, the relation $Z_\gamma = [X_\gamma, X_{-\gamma}]$ gives us a clue as to how to define the Cartan subgroup. Define $w_\gamma(t) = x_\gamma(t)x_{-\gamma}(-t^{-1})x_\gamma(t)$ for $t \in k^\times$ and $\gamma \in \Phi^c$, so for instance,

$$w_\alpha(t) = \begin{bmatrix} 0 & t & & \\ -t^{-1} & 0 & & \\ & & 0 & t^{-1} \\ & & -t & 0 \end{bmatrix} \quad \text{and} \quad w_\beta(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & -s^{-1} & 0 & 0 \end{bmatrix}.$$

Let $h_\gamma(t) = w_\gamma(t)w_\gamma(1)^{-1}$, so, in our case, we have

$$h_\alpha(t) = \begin{bmatrix} t & & & \\ & t^{-1} & & \\ & & t^{-1} & \\ & & & t \end{bmatrix} \quad \text{and} \quad h_\beta(s) = \begin{bmatrix} 1 & & & \\ & s & & \\ & & 1 & \\ & & & s^{-1} \end{bmatrix}.$$

A list of relations satisfied by the $w_\gamma(t)$ and $h_\gamma(t)$ can be found in [Ste68, pp. 27–30]. Define the ***Cartan subgroup*** T of G^c to be the subgroup generated by $\{h_\gamma(t) \ : \ \gamma \in \Phi^c, \ t \in k^\times\}$. This subgroup has many useful properties (cf. [Ste68, Lemma 28]); we point out that T is abelian, and each h_γ defines a homomorphism $k^\times \to T$. It is also true that T is generated by $h_\gamma(k^\times), \gamma \in \Delta^c$, so in our case,

$$T = \left\{ \begin{bmatrix} t & & & \\ & t^{-1}s & & \\ & & t^{-1} & \\ & & & s^{-1}t \end{bmatrix} \ : \ s, t \in k^\times \right\} = \left\{ \begin{bmatrix} t & & & \\ & s & & \\ & & t^{-1} & \\ & & & s^{-1} \end{bmatrix} \ : \ s, t \in k^\times \right\}$$

which is the set of all diagonal elements preserving our skew form M. It is very important to note that any element of T has a *unique* expression of the form $h_\alpha(s)h_\beta(t)$, so that $T \cong (k^\times)^2$. This property is equivalent to the fact that we exponentiated the simply-connected form, as proved in [Ste68] (recall that over $\mathbf{C}$, isogeny forms are obtained from the simply-connected form by quotienting by subgroups of the center of G^c, which must live in T).

In summary, we have defined:

$w_\gamma(t) \ = x_\gamma(t)x_{-\gamma}(-t^{-1})x_\gamma(t)$ for $t \in k^\times$.

$h_\gamma(t) \ = w_\gamma(t)w_\gamma(1)^{-1}$ for $t \in k^\times$.

T is the Cartan subalgebra, or distinguished maximal torus, which is generated by the $h_\gamma(t)$. By [Ste68, Lemma 28], simply-connectedness implies that any element of T can be written uniquely as

$$h_{\gamma_1}(t_1) \cdots h_{\gamma_n}(t_n)$$

for $\gamma_i \in \Delta^c$, so that $T \cong (k^\times)^{\ell^c}$. Note that we are *fixing* the torus T throughout this paper.

Generalized Levi subgroups. Now that we have defined the basic objects associated with the Chevalley group G^c, we can begin to understand the generalized Levi subgroups G, which comprise the more general class of groups with which we will be concerned.

Definition 1.3. Let $\Phi \subset \Phi^c$ be a subset which is also a root system (we do not require that Φ spans E), and such that if $\gamma, \delta \in \Phi$ and $\gamma + \delta \in \Phi^c$, then $\gamma + \delta \in \Phi$. We call such a Φ a ***closed sub-root system***.

Remark 1.4. One can obtain all closed sub-root systems of Φ^c by deleting vertices from an extended Dynkin diagram — see [BDS49].

Let G be the subgroup of G^c generated by T and the $\mathfrak{X}_\gamma$ for $\gamma \in \Phi$. We call G the ***(standard) generalized Levi subgroup*** associated with the closed sub-root system Φ; conjugates of G are ***generalized Levi subgroups***. In this paper, unless otherwise specified, all generalized Levi subgroups will contain our fixed torus T — that is, they will correspond as above to some closed sub-root system $\Phi \subset \Phi^c$. For a full treatment of generalized Levi subgroups, see [BDS49].

For example, let $\Phi = \{\pm\beta, \pm(2\alpha + \beta)\}$. This is a closed sub-root system, which is of type $A_1 \times A_1$ because β and $2\alpha + \beta$ are orthogonal. Exponentiating the root spaces, we find

$$G = \begin{bmatrix} * & & * & \\ & * & & * \\ * & & * & \\ & * & & * \end{bmatrix}.$$

Although G is a generalized Levi subgroup, it is *not* a Levi subgroup — that is, we cannot choose a system of simple roots for Φ which extend to a set of simple roots for Φ^c (more on this in the next subsection). The group G' generated by $\mathfrak{X}_{\pm\beta}$ and $\mathfrak{X}_{\pm(2\alpha+\beta)}$ (but not T) is a Chevalley group in its own right; it is one of the isogeny forms of $\mathrm{SO}_4(k)$. The group G' need not be simply-connected; in particular, it is not necessarily the case that any element of the Cartan subalgebra T' of G' can be written uniquely as

$$h_{\alpha_1}(t_1)\cdots h_{\alpha_n}(t_n)$$

for $\alpha_i \in \Delta$ (see Appendix A). In other words, T' is not necessarily a torus. In our case, however, we have $T' = T$ by coincidence, and G' happens to be the simply-connected form $\mathrm{Spin}_4(k) \cong \mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$. In general, we have $G = TG'$ — note that T normalizes G' because it acts diagonally on all of the root spaces.

Since generalized Levi subgroups behave so much like full Chevalley groups, some of the notation in place for G^c can also be defined for G. One should also regard most of this notation as fixed:

- Φ $\subset \Phi^c$ is a closed sub-root system.
- L $= |\Phi|$.
- $\Phi^{\pm}$ is some choice of positive and negative roots (we will not necessarily assume that $\Phi^{\pm} \subset (\Phi^c)^{\pm}$).
- E_Φ $\subset E$ is the subspace of E spanned by the root system Φ.
- E^*_Φ is the dual of E_Φ, which can be identified with the subspace of E^* spanned by $\check{\Phi}$.
- Λ_R $\subset E_\Phi$ is the lattice generated by Φ.
- Δ is a set of simple roots for Φ. Note that Δ may not be contained in any Δ^c — in our case, we can take $\Delta = \{\beta, 2\alpha+\beta\}$, which is not a simple root system for Φ^c.
- ℓ $= |\Delta|$.
- G is the generalized Levi subgroup of G^c associated with Φ. Note that the rank of G is still ℓ^c because G contains all of T.

Since we do not use an "intrinsic" definition of a generalized Levi subgroup of a Chevalley group, we must always work in an ambient larger group. For clarity, then, whenever we define an object for a generalized Levi subgroup, the same symbol with a superscript c will denote the same object in the maximal generalized Levi subgroup, i.e., all of G^c. So for instance, we have Φ and Δ defined for G, whereas Φ^c and Δ^c are the maximal versions of Φ and Δ associated with G^c. Also, if $\Phi_0 \subset \Phi$ is a smaller closed sub-root system, then we will regard the generalized Levi subgroup $G_0 \subset G$ associated with Φ_0 as a generalized Levi subgroup of G as well as of G^c.

Parabolic subgroups. The basic structure theory of Chevalley groups is not very difficult. It revolves around parabolic subgroups. Let $\Theta \subset \Delta$ and let Φ_Θ be the set of integer linear combinations of elements of Θ which are also roots in Φ. Then Φ_Θ is a closed sub-root system of Φ^c, so there is a corresponding generalized Levi subgroup M_Θ — generalized Levis which arise in this way are called ***standard Levi subgroups***, and conjugates of the M_Θ are ***Levi subgroups***. Let P_Θ be the subgroup of G generated by T and the root subgroups $\mathfrak{X}_\gamma$ for $\gamma \in \Phi_\Theta \cup \Phi^+$ — that is,

$$P_\Theta = \langle T,\ \mathfrak{X}_\gamma \ :\ \gamma \in \Phi_\Theta \cup \Phi^+ \rangle.$$

Then P_Θ is a ***standard parabolic subgroup*** associated with the subset Θ of Δ; conjugates of P_Θ are called ***parabolic subgroups***.[2] If $\Theta = \emptyset$, i.e., $M_\Theta = T$, then $B = P_\emptyset$ is the minimal standard parabolic subgroup, called the ***(standard) Borel subgroup***. One can show that the subgroups P_Θ for $\Theta \subset \Delta$ are the only subgroups of G containing B. If we set $U_\Theta = \langle \mathfrak{X}_\gamma \ :\ \gamma \in \Phi^+ \setminus \Phi_\Theta \rangle$, then U_Θ is normal in P_Θ, $U_\Theta \cap M_\Theta$ is trivial, and $P_\Theta = M_\Theta U_\Theta$ as a semidirect product. This decomposition is called a ***Levi decomposition*** of P_Θ. Note that $P_\Theta / U_\Theta \cong M_\Theta$. If we set

$$\overline{P}_\Theta = \langle T,\ \mathfrak{X}_\gamma \ :\ \gamma \in \Phi_\Theta \cup \Phi^- \rangle \qquad \overline{U}_\Theta = \langle \mathfrak{X}_\gamma \ :\ \gamma \in \Phi^- \setminus \Phi_\Theta \rangle,$$

then $\overline{P}_\Theta$ is a (not standard) parabolic subgroup with Levi decomposition $\overline{P}_\Theta = M_\Theta \overline{U}_\Theta$ such that $\overline{P}_\Theta \cap P_\Theta = M_\Theta$. We call $\overline{P}_\Theta$ the ***opposite parabolic*** of P_Θ with respect to M_Θ.

In our example of Sp_4, the only subsets of $\Delta^c = \{\alpha, \beta\}$ are $\emptyset, \{\alpha\}, \{\beta\}$, and Δ^c. Referring to the root space equations (1.5), we see that

$$\begin{array}{lllll}
M_\emptyset = \begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} &
P_\emptyset = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{bmatrix} &
U_\emptyset = \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & \\ & & * & 1 \end{bmatrix} &
\overline{P}_\emptyset = \begin{bmatrix} * & & & \\ * & * & & \\ * & * & * & * \\ * & * & & * \end{bmatrix} &
\overline{U}_\emptyset = \begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & * \\ * & * & & 1 \end{bmatrix} \\
M_\alpha = \begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix} &
P_\alpha = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} &
U_\alpha = \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} &
\overline{P}_\alpha = \begin{bmatrix} * & * & & \\ * & * & & \\ * & * & * & * \\ * & * & * & * \end{bmatrix} &
\overline{U}_\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{bmatrix} \\
M_\beta = \begin{bmatrix} * & & & \\ & * & & * \\ & & * & \\ & * & & * \end{bmatrix} &
P_\beta = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{bmatrix} &
U_\beta = \begin{bmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{bmatrix} &
\overline{P}_\beta = \begin{bmatrix} * & & & \\ * & * & & * \\ * & * & * & * \\ * & * & & * \end{bmatrix} &
\overline{U}_\beta = \begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & * \\ * & & & 1 \end{bmatrix} \\
M_{\Delta^c} = \mathrm{Sp}_4(k) & P_{\Delta^c} = \mathrm{Sp}_4(k) & U_{\Delta^c} = \{1\} & \overline{P}_{\Delta^c} = \mathrm{Sp}_4(k) & \overline{U}_{\Delta^c} = \{1\}
\end{array}$$

where we write $M_\alpha = M_{\{\alpha\}}$, etc.

For another example of parabolic subgroups, see Example 2.14.

More about root systems. Much of our work will be done in terms of root systems, so we have some more to say about how Φ interacts with our group G. For every root γ (resp. co-root $\check{\gamma}$), we have defined a linear map $\mathfrak{h}_{\mathbf{R}} \to \mathbf{R}$ (resp. $\mathbf{R} \to \mathfrak{h}_{\mathbf{R}}$). Not surprisingly, we can find "exponential" versions of these maps, obtaining homomorphisms $T \to k^\times$ (resp. $k^\times \to T$). Everything that follows is completely analogous to the discussion starting on page 9.

For $\gamma, \delta \in \Phi$, [Ste68, p.30, (R8)] tells us that

$$h_\gamma(t) x_\delta(s) h_\gamma(t)^{-1} = x_\delta(t^{\langle \delta, \gamma \rangle} s); \tag{1.6}$$

compare this with (1.3) on page 9. Proceeding analogously, since G^c is simply-connected, we may define a homomorphism $\delta : T \to k^\times$ by

$$\delta(h_\gamma(t)) = t^{\langle \delta, \gamma \rangle}$$

for $\delta, \gamma \in \Delta^c$; the above equation is then true for all $\gamma, \delta \in \Phi^c$. Let $X^*(T)$ be the group of homomorphisms $T \to k^\times$ generated by the roots $\gamma \in \Phi^c$;[3] we have defined things in such a way that we have a natural surjective homomorphism from Λ_R^c onto $X^*(T)$.[4] Also, any

[2] Again, unless otherwise specified, all Levis, generalized Levis, and parabolics are assumed to contain T — this means that the Levis and generalized Levis are standard, but the parabolics are only standard with respect to some choice of simple roots.

[3] This notation is nonstandard. In an algebraic group setting, $X^*(T)$ is defined to be the group of algebraic characters of T; this is not a natural definition from the standpoint of Chevalley groups.

[4] This homomorphism is not injective if the field k is finite; therefore in the Chevalley group context, we can not realize our root system as a subset of $X^*(T) \otimes \mathbf{R}$ as is standard.

co-root $\check{\gamma}$ defines a homomorphism $k^\times \to T$ given by h_γ; we then have $\delta \circ \check{\gamma}(t) = t^{\langle \delta, \gamma \rangle}$. Let $X_*(T)$ be the group of homomorphisms $k^\times \to T$ generated by the $\check{\gamma}$ for $\gamma \in \Phi^c$, so, as before, we have a surjective homomorphism from the co-root lattice $\check{\Lambda}_R^c$ onto $X_*(T)$.

In the example of $\mathrm{Sp}_4(k)$, we have calculated that the root α corresponds to the map $\mathfrak{h}_{\mathbf{R}}^* \to \mathbf{R}$ given by $\alpha(\mathrm{diag}(a, b, -a, -b)) = a - b$; the same calculation gives that our "exponentiated" realization is $\alpha(\mathrm{diag}(a, b, a^{-1}, b^{-1})) = ab^{-1}$, as expected. Similarly, $\beta(\mathrm{diag}(a, b, -a, -b)) = 2b$ so $\beta(\mathrm{diag}(a, b, a^{-1}, b^{-1})) = b^2$.

To reiterate, we have defined:

$X^*(T)$ is the group of homomorphisms from T to $k^\times$ generated by the roots. If $\lambda \in \Lambda_R^c$, then we will also write λ for its image in $X^*(T)$; that is, we will regard λ as a homomorphism $T \to k^\times$ when it suits us.

$X_*(T)$ is the group of homomorphisms $k^\times \to T$ generated by the co-roots. If $\check{\lambda} \in \check{\Lambda}_R^c$, then we will again write $\check{\lambda}$ for its image in $X_*(T)$.

We never explicitly refer to the groups $X_*(T)$ and $X^*(T)$, but we will consistently identify co-roots and roots with their images in $X_*(T)$ and $X^*(T)$, respectively. This is also a good place to define:

Λ_C $\subset \Lambda_R^c$ is the set of all $\lambda \in \Lambda_R^c$ with $\langle \lambda, \gamma \rangle = 0$ for all $\gamma \in \Phi$. In other words, $\Lambda_C = \Lambda_R \cap E_\Phi^\perp$.

C $\subset T$ is the ***split center*** of G, which we define to be the subgroup of T generated by the $\check{\lambda}(t)$ for $t \in k^\times$ and $\lambda \in \Lambda_C$. The relations in [Ste68, pp. 27–30] show that C is contained in the center of G.

The Weyl group. Let W be the ***Weyl group*** associated with Φ, that is, the group of isometries of E (or of E^*) generated by the reflections over the hyperplanes orthogonal to the roots in Φ (resp. co-roots in $\check{\Phi}$). Let N be the subgroup of G generated by T and the $w_\gamma(t)$ for $\gamma \in \Phi$ and $t \in k^\times$. One can show [Ste68, Exercise, p.36] that T is normal in N, and that N is the full normalizer in G of T if k has more than three elements. It is a basic fact that the map $N \to W$ which sends $w_\gamma(t)$ to the reflection over the hyperplane orthogonal to γ (or $\check{\gamma}$) induces an isomorphism of N/T with W.

If $G = \mathrm{Sp}_4(k)$, we find that coset representatives for N^c/T are the determinant-1 permutation matrices corresponding to the permutations

$$1, (13), (24), (1234), (1432), (12)(34), (13)(24), (14)(23)$$

which we recognize as the dihedral group D_4 of symmetries of the square. The roots for Sp_4 lie on a square in E (see Figure 1.1), and we readily see that the isomorphism of N^c/T with W^c is the same as realizing W^c as the symmetries of that square. If $G = \mathrm{Spin}_4(k) \subset \mathrm{Sp}_4(k)$ as above, then $N/T \cong \{1, (13), (24), (13)(24)\}$ is the Klein four group.

We define the following objects relating to the Weyl group:

N is the subgroup of G generated by T and the $w_\gamma(t)$ for $\gamma \in \Phi$. If the field in question has more than three elements, then $N = N_G(T)$ by [Ste68, Exercise, p.36].

W $\cong N/T$ is the Weyl group of G, realized as a group of isometries of E or E^*.

r_γ is the reflection in E (resp. E^*) over the hyperplane orthogonal to γ (resp. $\check{\gamma}$). In E, we have

$$r_\gamma(x) = x - \langle x, \gamma \rangle \gamma \tag{1.7}$$

and in E^*,

$$r_\gamma(x^*) = x^* - \langle x^*, \gamma \rangle \check{\gamma}. \tag{1.8}$$

The group N acts on E and E^* via the identification $N/T \cong W$, with $w_\gamma(t)$ acting by r_γ and T acting trivially.

H_γ is the hyperplane in E^* fixed by r_γ, which is also the set of all $\lambda \in E^*$ such that $\langle \lambda, \gamma \rangle = 0$, or alternatively, the hyperplane orthogonal to $\check{\gamma}$. Note that $H_\gamma = H_{-\gamma}$.

p-adic fields. Everything we have done so far holds when k is an arbitrary field, but after all, this thesis is about p-adic Chevalley groups, so from now on we fix the following notation:

k is a fixed non-Archimedean local field of characteristic zero, or equivalently, k is a finite extension of a p-adic field.[5]

ω : $k^\times \to \mathbf{Z}$ is a nontrivial discrete valuation, such that $\omega(k^\times) = \mathbf{Z}$.

R is the ring of integers of k.

ϖ is a uniformizing element.

$\wp$ $= \varpi R$ is the prime ideal of R.

f $= R/\wp$ is the residue field.

q $= |\mathsf{f}|$.

We will encounter generalized Levi subgroups of our fixed simply-connected Chevalley group defined over the field k *and* its residue field f, so we need to define one last bit of notation. The objects associated with Chevalley groups over k will be written in the standard italic computer modern math type as above, and objects associated with Chevalley groups over f will be written in a Serif font. So, for instance, over k we have the groups G, T, N, and $\mathfrak{X}_\alpha(k)$, and the elements $x_\alpha(t), w_\beta(s)$; over f, the analogous objects will be denoted $\mathsf{G}, \mathsf{T}, \mathsf{N}, \mathfrak{X}_\alpha(\mathsf{f}), \mathsf{x}_\alpha(\mathsf{t})$, and $\mathsf{w}_\beta(\mathsf{s})$. The objects that depend only on the Chevalley data and not the field, like $\mathfrak{g}$, Φ, and everything else relating to the root system (including W) will not be written differently when we are thinking about them in relation to the finite field or to k.

Remark 1.5. As mentioned before, everything that we will do in this paper works in the more general context of connected reductive algebraic groups defined over non-Archimedean local fields. It is more difficult to incorporate this added generality — when we work with Chevalley groups, we have a canonical choice of maximal torus (i.e., the Cartan subgroup), but in the algebraic group context, we do not necessarily even have such a torus. For the more general treatment, see Bruhat and Tits [BT84].

2. The spherical and affine apartments

Much of the structure of a generalized Levi subgroup G of a p-adic Chevalley group can be represented geometrically via its affine apartment, denoted $\mathcal{A}(G)$. This structure exists because of the fact that our non-Archimedean local field k possesses a nontrivial discrete valuation. Some of the structure of G, however, arises from the fact that G is a Chevalley group, without respect to which field G is defined over; namely, we can find a Cartan subgroup, a Borel, parabolics, etc. These subgroups are related to the spherical apartment of the Chevalley group G, which is defined independently of the field k.

2.1. The spherical apartment

[5] Everything that we will do also works in the positive characteristic case, but for simplicity we assume that k is a finite extension of a p-adic field.

Since the affine apartment locally looks like a spherical apartment, we will need to understand the latter as well as the former. Also, many of the results for the affine apartment are completely analogous to those for the spherical apartment, so it is important to review them. In this section, f can be replaced with any field, but we will almost always be concerned with the spherical apartment of groups over the residue field. Again we emphasize that the definition and properties of the spherical apartment are completely determined by the Chevalley data and not the field.

Definition 2.1. The ***(standard) spherical apartment*** of G is the Euclidean space $\mathcal{A}_s = \mathcal{A}_s(\mathsf{G}) = E^*$; that is, $\mathcal{A}_s$ is the Euclidean space containing the (abstract) co-root system of G^c.

This definition of $\mathcal{A}_s$ is the same for every generalized Chevalley group of G^c containing T. In a sense, though, there is more structure to $\mathcal{A}_s$ than that of a Euclidean space; that is, we would like to distinguish between $\mathcal{A}_s(\mathsf{G})$ and $\mathcal{A}_s(\mathsf{G}')$ if G and G' are two different generalized Levi subgroups of G^c containing T.

Definition 2.2. The root system Φ gives rise to the hyperplanes $H_\alpha \subset \mathcal{A}_s$ for $\alpha \in \Phi$. We say that these hyperplanes are ***hyperplanes*** of $\mathcal{A}_s(\mathsf{G})$, and that the collection of such hyperplanes is the ***hyperplane structure*** of $\mathcal{A}_s(\mathsf{G})$.

Therefore whereas $\mathcal{A}_s(\mathsf{G})$ and $\mathcal{A}_s(\mathsf{G}')$ may be the same as Euclidean spaces, they will have different hyperplane structures if $\mathsf{G} \neq \mathsf{G}'$. In other words, if $\alpha \in \Phi^c \setminus \Phi$, then we do not consider H_α to be a hyperplane of $\mathcal{A}_s(\mathsf{G})$. We will not define a "hyperplane structure" any more carefully than this, as its purpose is more intuitive than mathematical.

The Weyl group W of G is usually realized as a group of isometries of the Euclidean space E^*_Φ spanned by the co-root system $\check{\Phi}$. This turns out not to be the most natural object to use for our purposes, because the dimension of E^*_Φ may be smaller than the rank of G. However, all of the geometric information about $\mathcal{A}_s$, most importantly the hyperplane structure, is carried by E^*_Φ, so we define:

Definition 2.3. The ***reduced apartment*** $\mathcal{A}_s^{\mathrm{red}}(\mathsf{G}) \subset \mathcal{A}_s(\mathsf{G})$ of G is the vector space E^*_Φ. Let $\pi_{\mathsf{G}} : \mathcal{A}_s \to \mathcal{A}_s^{\mathrm{red}}$ be the orthogonal projection map.

Thus

$$\mathcal{A}_s = \mathcal{A}_s^{\mathrm{red}} \oplus (E^*_\Phi)^\perp.$$

Since $\alpha(x) = \alpha(\pi_{\mathsf{G}}(x))$ for $\alpha \in \Phi$, all hyperplanes H_α for $\alpha \in \Phi$ are parallel to $(E^*_\Phi)^\perp$, in the sense that $H_\alpha = \pi_{\mathsf{G}}(H_\alpha) \times (E^*_\Phi)^\perp$.. Therefore the Weyl group acts trivially on $(E^*_\Phi)^\perp$ in the above decomposition. In other words, $\mathcal{A}_s^{\mathrm{red}}$ really does contain all of the relevant geometric information, and the factor of $(E^*_\Phi)^\perp$ is (for the moment) completely redundant.

Remark 2.4. Proposition B.4 shows that $\check{\Lambda}_C$ spans $(E^*_\Phi)^\perp$ — that is, the decomposition $E^* = E^*_\Phi \oplus (E^*_\Phi)^\perp$ is carried by the co-root lattice $\check{\Lambda}^c_R$.

Definition 2.5. Since there is a linear action of $W \cong \mathsf{N}/\mathsf{T}$ on $\mathcal{A}_s$, we have a natural action of N on $\mathcal{A}_s$, with $\mathsf{w}_\alpha(\mathsf{t})$ acting by r_α and T acting trivially. We will denote this action by $\mathsf{n} \cdot x$, for $\mathsf{n} \in \mathsf{N}$ and $x \in \mathcal{A}_s$.

Recall that if G has root system Φ with simple roots Δ, then the Weyl group is generated by the r_α for $\alpha \in \Delta$, i.e., W is generated by flips over the H_α for $\alpha \in \Delta$ [Car89, Prop. 2.1.8(ii)].

Definition 2.6. A ***Weyl chamber*** $D \subset \mathcal{A}_s$ is a connected component of $A_s \setminus \bigcup_{\alpha\in\Phi} H_\alpha$. The ***fundamental Weyl chamber*** with respect to Δ is the Weyl chamber D such that $\alpha(x) > 0$ for every $x \in D$ and all simple roots α.

Following [Car89, Chapter 2], one can show that:

Proposition 2.7. *(i) A Weyl chamber D uniquely determines a set of simple roots Δ with respect to which D is the fundamental Weyl chamber, and vice-versa. The simple roots are the roots α such that H_α borders D and such that $\alpha(x) > 0$ for $x \in D$. Therefore W is generated by reflections over the hyperplanes bordering D.*

(ii) If $D, D' \subset \mathcal{A}_s$ are two Weyl chambers, then there is a unique element w of W such that $w \cdot D = D'$.

Therefore the number of Weyl chambers is the size of the Weyl group. A system of positive roots gives rise to a Borel subgroup B generated by T and $\mathfrak{X}_\alpha(\mathsf{f})$ for $\alpha \in \Phi^+$, so Proposition 2.7 tells us that the minimal parabolic subgroups containing T correspond to Weyl chambers D. One of the most important facts (for our purposes) about the Borel subgroups is the following:

Theorem 2.8 ([Car89, Prop. 8.2.1]**).** *Let $S = \{r_\alpha : \alpha \in \Delta\}$ be the simple reflections that generate W. The data $(\mathsf{B}, \mathsf{N}, W, S)$ form a BN-pair, in the sense of* [Tit62].[6]

A large amount of structure theory follows.

Corollary 2.9 (Spherical Bruhat decomposition; [Car89, Prop. 8.2.2]**).** $\mathsf{G} = \mathsf{BNB}$.

Notation 2.10. For Θ any subset of Δ, let W_Θ be the subgroup of W generated by the r_α for $\alpha \in \Theta$, let N_Θ be the lift in N of W_Θ, and let $\mathsf{P}_\Theta = \mathsf{BN}_\Theta\mathsf{B}$.

One should compare the above definition to the (equivalent) definition of P_Θ given in Section 1.1. Note that $\mathsf{B} = \mathsf{P}_\emptyset$. The existence of a BN-pair gives:

Corollary 2.11 ([Car89, Theorem 8.3.4]**).** *The sets P_Θ are the subgroups of G containing B. If $\Theta' \subset \Delta$, then $\mathsf{P}_\Theta = \mathsf{P}_{\Theta'}$ if and only if $\Theta = \Theta'$, and $\mathsf{P}_\Theta \cap \mathsf{P}_{\Theta'} = \mathsf{P}_{\Theta\cap\Theta'}$.*

Definition 2.12. Let $x \in \mathcal{A}_s$, and let $W_x^s = \mathrm{stab}_W(x)$. One can show [Car89, §2.6] that if D is a Weyl chamber whose closure contains x and Δ is the set of simple roots associated with D, then there is a subset Θ of Δ such that W_x^s is generated by the reflections r_α for $\alpha \in \Theta$. Let $\mathsf{P}_x = \mathsf{P}_\Theta$ (cf. Corollary 2.11).

Note that $\mathsf{P}_x = \mathsf{G}$ if and only if $\pi_{\mathsf{G}}(x) = 0$. The definition of P_x has a more geometric interpretation.

[6]Carter actually only proves this fact for the adjoint form of a Chevalley group, but the proof holds for any Chevalley group. The theorem is true for a generalized Levi subgroup G as well because $\mathsf{G} = \mathsf{T}\mathsf{G}'$, where G' is a Chevalley group.

Proposition 2.13 ([Car89, Prop. 8.5.1]**).** *We can write* P_x *as*

$$\mathsf{P}_x = \langle \mathsf{T},\ \mathfrak{X}_\alpha(\mathsf{f})\ :\ \alpha \in \Phi,\ \alpha(x) \geq 0 \rangle. \tag{2.1}$$

A Levi subgroup M_x *of* P_x *is*

$$\mathsf{M}_x = \langle \mathsf{T},\ \mathfrak{X}_\alpha(\mathsf{f})\ :\ \alpha \in \Phi,\ \alpha(x) = 0 \rangle. \tag{2.2}$$

Thus the definition of P_x does not depend on the choice of D. In this way, we can associate a parabolic subgroup to any point in $\mathcal{A}_s$; note that every parabolic subgroup containing T arises in this way. Note also that $\mathsf{P}_x = \mathsf{P}_{\pi_{\mathsf{G}}(x)}$. One should keep in mind that the subgroup P_x really depends only on the hyperplanes containing x, and not on the specific choice of the point x — that is, P_x actually depends on the local hyperplane structure around x.

Example 2.14. The group $\mathsf{G} = \mathrm{SL}_3(\mathsf{f})$ has two simple roots α and β of the same length, with an angle of $120°$ between them, so the same is true for the co-roots. Therefore the spherical apartment is given in Figure 2.1. The Levi subgroup with root system $\{\pm\alpha\}$ (which is isomorphic to $\mathrm{GL}_2(\mathsf{f})$) has the same apartment, but the hyperplanes H_β and $H_{\alpha+\beta}$ do not appear. Thus we see that $\mathcal{A}_s(\mathrm{GL}_2(\mathsf{f})) = \mathcal{A}_s(\mathrm{SL}_2(\mathsf{f})) \times \mathbf{R}$ — note that $\mathcal{A}_s^{\mathrm{red}}(\mathrm{GL}_2(\mathsf{f}))$ is spanned by $\check{\alpha}$.[7]

Assume the root spaces in $\mathrm{SL}_3(\mathsf{f})$ are as follows:

$$\mathfrak{X}_\alpha(\mathsf{f}) = \begin{bmatrix} 1 & \mathsf{f} & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \mathfrak{X}_\beta(\mathsf{f}) = \begin{bmatrix} 1 & & \\ & 1 & \mathsf{f} \\ & & 1 \end{bmatrix} \qquad \mathfrak{X}_{\alpha+\beta}(\mathsf{f}) = \begin{bmatrix} 1 & & \mathsf{f} \\ & 1 & \\ & & 1 \end{bmatrix}$$
$$\mathfrak{X}_{-\alpha}(\mathsf{f}) = \begin{bmatrix} 1 & & \\ \mathsf{f} & 1 & \\ & & 1 \end{bmatrix} \qquad \mathfrak{X}_{-\beta}(\mathsf{f}) = \begin{bmatrix} 1 & & \\ & 1 & \\ & \mathsf{f} & 1 \end{bmatrix} \qquad \mathfrak{X}_{-\alpha-\beta}(\mathsf{f}) = \begin{bmatrix} 1 & & \\ & 1 & \\ \mathsf{f} & & 1 \end{bmatrix}.$$

With our choice of simple roots, D labels the fundamental Weyl chamber in Figure 2.1, so if $z \in D$, then $\mathsf{P}_z = \mathsf{P}_\emptyset$ is a Borel subgroup, i.e.,

$$\mathsf{P}_z = \mathsf{P}_\emptyset = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

is the set of all upper-triangular matrices in $\mathrm{SL}_3(\mathsf{f})$. Since $x \in H_\alpha$ and $y \in H_\beta$, we have

$$\mathsf{P}_x = \mathsf{P}_\alpha = \begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix} \quad \text{and} \quad \mathsf{P}_y = \mathsf{P}_\beta = \begin{bmatrix} * & * & * \\ & * & * \\ & * & * \end{bmatrix}.$$

If x' is in the same connected component of $H_\alpha \backslash \{0\}$ as x, then $\mathsf{P}_{x'} = \mathsf{P}_x$. Of course, $\mathsf{P}_0 = \mathsf{P}_\Delta = \mathsf{G}$ (where 0 is the origin) by the spherical Bruhat decomposition (Corollary 2.9).

Example 2.15. As we know from Section 1.1, the Chevalley group Sp_4 has a short root α and a long root β at a $135°$ angle — see Figure 1.1. Therefore the spherical apartment $\mathcal{A}_s(\mathrm{Sp}_4(\mathsf{f}))$ is given in Figure 2.2. If G denotes the generalized Levi subgroup of $\mathrm{Sp}_4(\mathsf{f})$ with root system $\{\pm\beta, \pm(2\alpha+\beta)\}$ (cf. Section 1.1), then $\mathcal{A}_s(\mathsf{G})$ looks the same, except missing the diagonal hyperplanes.

Assume that the root spaces are as in Section 1.1. With our choice of simple roots, D is a fundamental Weyl chamber, so if $z \in D$, then the corresponding Borel subgroup is

$$\mathsf{P}_z = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{bmatrix}$$

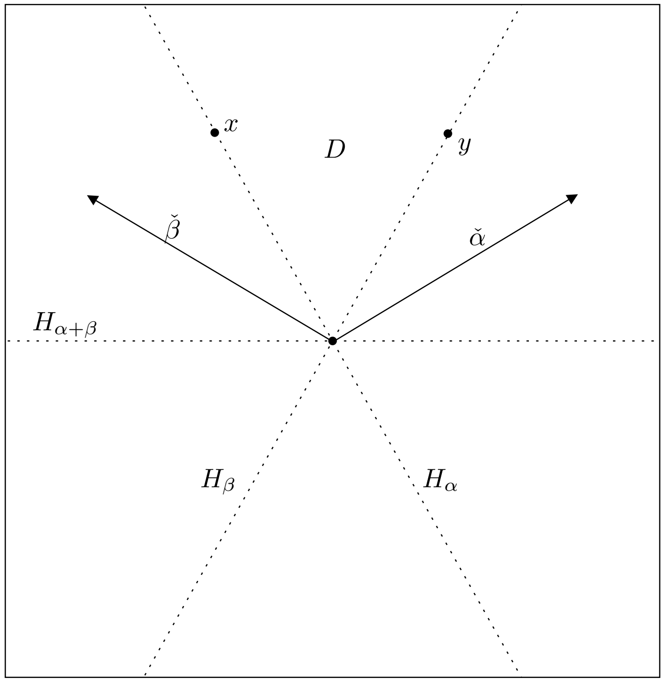

Figure 2.1: The spherical apartment of $\mathrm{SL}_3(\mathfrak{f})$.

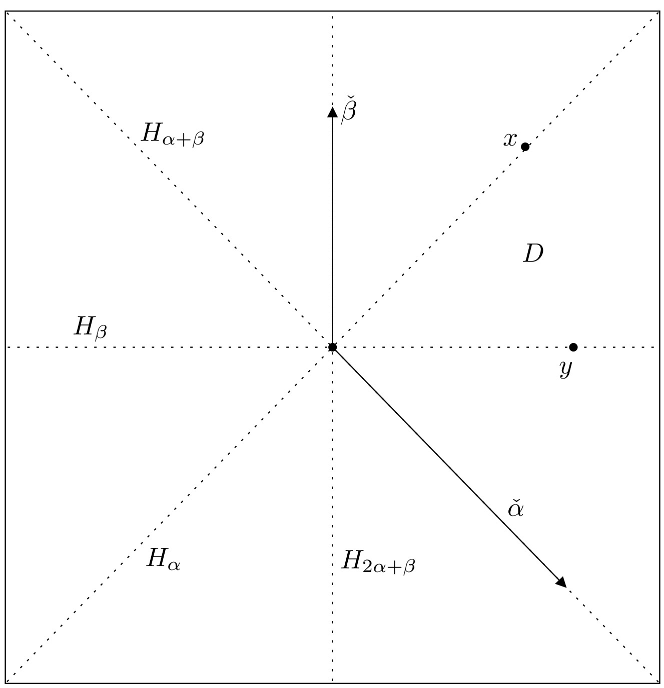

Figure 2.2: The spherical apartment of $\mathrm{Sp}_4(\mathfrak{f})$.

(i.e., all elements of $\mathrm{Sp}_4(\mathsf{f})$ with nonzero entries as indicated). If $x \in H_\alpha$ and $y \in H_\beta$ are as indicated, we have

$$\mathsf{P}_x = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \qquad \mathsf{P}_y = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{bmatrix}.$$

One should refer back to Section 1.1, where we have already found these parabolics.

2.2. The affine apartment

The crucial structure that we will use in order to define the affine apartment is the nontrivial discrete valuation on k. In short, the valuation allows us to define an action of N on the apartment consisting of translations as well as reflections. Since translations are *affine* maps, we need to think of the apartment differently:

Definition 2.16. The ***standard affine apartment***, or ***apartment***, of the group G is the affine space under the vector space $\mathcal{A}_s = E^*$, and is denoted $\mathcal{A} = \mathcal{A}(G)$.

When we say that $\mathcal{A}$ is the affine space under $\mathcal{A}_s$, we simply mean that:

- $\mathcal{A}$ is set-theoretically the same as $\mathcal{A}_s$.
- The natural automorphisms of $\mathcal{A}$ are affine isometries — that is, the maps $\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ generated by the linear isometries and the translations.
- If x, y are two ***points*** in $\mathcal{A}$, then the difference $x - y$ is a *vector* in $\mathcal{A}_s$.
- If $x \in A$ and $v \in \mathcal{A}_s$, then the sum $x + v$ is another point in $\mathcal{A}$.

A general affine space has no origin, or distinguished point, and it is possible to treat $\mathcal{A}$ in such a manner; see [BT72]. We will not take this approach: we allow the origin $0 \in \mathcal{A}$ to remain distinguished, so to every point $p \in \mathcal{A}$ there is naturally associated the vector $p - 0$. (We will often simply write p for $p - 0$.) Thus, with our choice of origin, we are in fact identifying $\mathcal{A}$ with the vector space $\mathcal{A}_s$.

As in the spherical case, the hyperplane structure of $\mathcal{A}$ will play an important conceptual role.

Definition 2.17. Given any $\alpha \in \Phi$ and $n \in \mathbf{Z}$, our choice of origin allows us to define an affine functional $\alpha + n : \mathcal{A} \to \mathbf{R}$ given by

$$(\alpha + n)(x) = \alpha(x - 0) + n.$$

Let $H_{\alpha+n}$ be the codimension one hyperplane killed by $\alpha + n$, and let $r_{\alpha+n}$ be the reflection over $H_{\alpha+n}$. Note that $H_{\alpha+n} = H_{-\alpha-n}$. Define the ***hyperplane structure*** of $\mathcal{A}(G)$ to be the set of hyperplanes $H_{\alpha+n}$ with $\alpha \in \Phi$ and $n \in \mathbf{Z}$.

Note again that $\mathcal{A}(G) = \mathcal{A}(G')$ for any two generalized Levi subgroups G, G' of G^c, but that the hyperplane structure of $\mathcal{A}(G)$ may be different from that of $\mathcal{A}(G')$.

Example 2.18. We again use the example of $G = \mathrm{SL}_3(k)$. Let the notation be as in Example 2.14. Since $\gamma(\check{\gamma}) = 2$ for any root γ, the affine apartment of SL_3 is given in Figure 2.3.

[7] In general, $\mathcal{A}_s(G) \cong \mathcal{A}_s(G^{\mathrm{der}}) \times \mathcal{A}_s(Z)$, where G^{der} is the derived group of G and Z is the center of G. Note that $\mathrm{SL}_2(\mathsf{f})$ is the derived group of $\mathrm{GL}_2(\mathsf{f})$ and that the center of $\mathrm{GL}_2(\mathsf{f})$ is isomorphic to $\mathrm{GL}_1(\mathsf{f})$, whose apartment is $\mathbf{R}$.

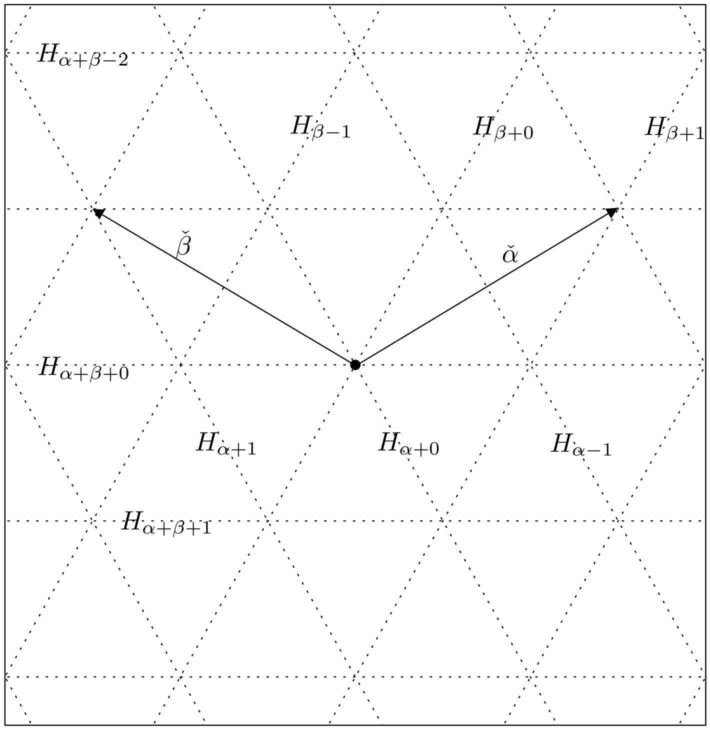

Figure 2.3: The affine apartment of $\mathrm{SL}_3(k)$.

It is useful to have an explicit formula for r_α:

Definition 2.19. Any co-root $\check{\alpha} \in \check{\Phi}^c$ gives rise to a translation T_α on $\mathcal{A}$, given by

$$T_\alpha(x) = x + \check{\alpha}.$$

One can check that $r_{\alpha+n}$ is given by the formula

$$r_{\alpha+n}(x) = (T_\alpha)^{-n} \circ r_\alpha(x - 0) = (x - \langle x, \alpha \rangle \check{\alpha}) - n\check{\alpha}. \tag{2.3}$$

It is important to note that when we think of the apartment $\mathcal{A}(G)$, we do not consider the hyperplanes H_α for $\alpha \in \Phi^c \setminus \Phi$, as mentioned above (so the hyperplane structure of $\mathcal{A}(G^c)$ may not be the same as that of $\mathcal{A}(G)$), but that we do consider translations T_α for all $\alpha \in \Phi^c$. See Example 2.29.

Like the spherical apartment, the affine apartment has a natural decomposition.

Definition 2.20. Let the ***reduced (affine) apartment*** $\mathcal{A}^{\mathrm{red}} = \mathcal{A}^{\mathrm{red}}(G)$ of G be the affine space under $\mathcal{A}_s^{\mathrm{red}}(G) = E_\Phi^*$, and let $\pi_G : \mathcal{A}(G) \to \mathcal{A}^{\mathrm{red}}(G)$ be the orthogonal projection map.

Remark 2.21. If the Chevalley group G' with root system Φ is also a simply-connected Chevalley group, then the reduced apartment $\mathcal{A}^{\mathrm{red}}(G)$ can be identified with the apartment $\mathcal{A}(G')$, with the same hyperplane structure.

Note that, as before, each $H_{\alpha+n}$ for $\alpha \in \Phi$ and $n \in \mathbf{Z}$ is parallel to $(E_\Phi^*)^\perp$, so all geometric information about $\mathcal{A}$ is carried by $\mathcal{A}^{\mathrm{red}}$. Also, as in Remark 2.4, the lattice $\check{\Lambda}_C$ spans $(E_\Phi^*)^\perp$.

The action of N on the apartment $\mathcal{A}$ will now be given by affine transformations, as opposed to linear transformations. This action is a bit more difficult to define. We will start by defining the action in the case that $G = G^c$ (so that $E^*_{\Phi^c} = E^*$), and then proceed as in the spherical case, first defining an affine analog $\widetilde{W}^c$ of the Weyl group with a natural action on $\mathcal{A}^c = \mathcal{A}(G^c)$, and then showing that $\widetilde{W}^c$ is a quotient of N^c, thus obtaining an action of N^c on $\mathcal{A}^c$.

First we define the affine version of the Weyl group. We can do this for any generalized Levi $G \subset G^c$ containing T.

Definition 2.22. The ***affine Weyl group*** $\widetilde{W}$ is the group of affine transformations of $\mathcal{A}$ generated by the reflections $r_{\alpha+n}$ for $\alpha \in \Phi$ and $n \in \mathbf{Z}$.

Note that this is completely analogous to the (spherical) Weyl group W, which is generated by the reflections r_α. The spherical Weyl group is also generated by the reflections over the hyperplanes bordering a Weyl chamber; there is an analogous result for the affine Weyl group, which we will prove in Proposition 3.32. Note also that if G' is a generalized Levi of G, then its affine Weyl group $\widetilde{W}'$ is a subgroup of $\widetilde{W}$. In order to realize $\widetilde{W}^c$ as a quotient of N^c, we need to define the relevant subgroup:

Definition 2.23. Define

$$T(R) = \langle h_\alpha(t) \ : \ t \in R^\times,\ \alpha \in \Delta^c \rangle \cong (R^\times)^{\ell^c}.$$

This is the maximal compact subgroup of T.

Proposition 2.24. *The group N^c is the normalizer in G^c of $T(R)$.*

Proof.

If $w_\beta(u) \in N^c$ and $h_\alpha(t) \in T(R)$ for $\alpha \in \Delta^c$, then [Ste68, Lemma 20] gives

$$w_\beta(u)h_\alpha(t)w_\beta(u)^{-1} = h_\beta(u)h_{r_\beta(\alpha)}(t)h_\beta(u)^{-1} = h_{r_\beta(\alpha)}(t).$$

If $r_\beta(\alpha) = a_1\alpha_1 + \cdots + a_m\alpha_m$ is an expression of $r_\beta(\alpha)$ as an integer linear combination of simple roots, then

$$h_{r_\beta(\alpha)}(t) = h_{\alpha_1}(t^{a_1}) \cdots h_{\alpha_m}(t^{a_m}) \in T(R).$$

Since the generators of N^c normalize $T(R)$, we have that $T(R)$ is normal in N^c.

In order to prove the converse, we cheat and assume that there is some $t \in T(R)$ such that the centralizer in G of t is T (this is just the statement that there is some regular semisimple element in $T(R)$, which is a standard fact from the theory of linear algebraic groups). If $g \in G$ is such that $gtg^{-1} \in T(R)$, then $gtg^{-1} \in T$, so $t \in g^{-1}Tg$. Thus $g^{-1}Tg$ centralizes t, so $g^{-1}Tg = T$, and therefore $g \in N$.

❐

We will shortly identify $\widetilde{W}^c$ with $N^c/T(R)$, thus defining the action of N^c on $\mathcal{A}^c$. (It is not in general true that $\widetilde{W} = N/T(R)$, but only that $\widetilde{W} \subset N/T(R)$; more on this later.) In the following proposition, the group $\mathbf{Z}^{\ell^c}$ is the group of translations, so this proposition is analogous to the fact that the group of isometries of Euclidean space is a semidirect product of the orthogonal group and the group of translations.

Lemma 2.25. *The groups $\widetilde{W}^c$ and $N^c/T(R)$ are both isomorphic to a semidirect product of $\mathbf{Z}^{\ell^c}$ with W^c. Furthermore, the map $N^c \to \widetilde{W}^c$ that takes $w_\alpha(1) \mapsto r_\alpha$ and $h_\alpha(t) \mapsto (T_\alpha)^{-\omega(t)}$ induces an isomorphism of $N^c/T(R)$ with $\widetilde{W}^c$.*

Proof.

First we show that $\widetilde{W}^c = W^c \ltimes \mathbf{Z}^{\ell^c}$. Let $\Lambda \triangleleft \widetilde{W}^c$ be the subgroup of translations, and note that W^c is the subgroup of $\widetilde{W}^c$ generated by the $r_{\alpha+0}$. The intersection $W^c \cap \Lambda$ is trivial, and $\widetilde{W}^c = \Lambda \cdot W^c$ by (2.3). Thus $\widetilde{W}^c = W^c \ltimes \Lambda$; we would like to show that $\Lambda \cong \mathbf{Z}^{\ell^c}$. We note that for each $\alpha \in \Delta^c$, we have $T_\alpha = r_{\alpha-1} r_\alpha \in \Lambda$, so Λ contains $\Lambda' = \langle T_\alpha \ : \ \alpha \in \Delta^c \rangle \cong \mathbf{Z}^{\ell^c}$. Furthermore, it is easy to see that the lattice generated by $\{\check{\alpha} \ : \ \alpha \in \Delta^c\}$ is stabilized by $\widetilde{W}^c$, so Λ can only take the origin to points in that lattice; thus $\Lambda' = \Lambda$. It is useful to write down the interaction between translations and reflections:

$$r_\alpha T_\beta r_\alpha^{-1}(x) = x + r_\alpha(\check{\beta}) = T_{r_\alpha \beta}(x) \tag{2.4}$$

because $r_\alpha(\check{\beta}) = (r_\alpha \beta)^\vee$.

Now we show that $N^c/T(R) = W^c \ltimes \mathbf{Z}^{\ell^c}$. First note that $T/T(R) \triangleleft N^c/T(R)$ is a normal subgroup isomorphic to $\mathbf{Z}^{\ell^c}$. Let

$$N^c(R) = \langle w_\alpha(t) \ : \ t \in R^\times, \ \alpha \in \Delta^c \rangle \supset T(R).$$

We claim that $W^c \cong N^c(R)/T(R) \subset N^c/T(R)$. The map $n \cdot T(R) \mapsto n \cdot T$ from $N^c(R)/T(R)$ to $W^c = N^c/T$ is well-defined and surjective because the $w_\alpha(1)$ generate the Weyl group. We would like to show that the map is injective, i.e., that $T \cap N^c(R) = T(R)$. This is true because the subgroup of T that preserves the Chevalley lattice $V \otimes R$ is exactly $T(R)$, and $N^c(R)$ preserves $V \otimes R$, so $T \cap N^c(R) \subset T(R)$; the other inclusion is clear. Noting that $N^c(R)/T(R) \cap T/T(R) = (N^c(R) \cap T)/T(R) = \{1\}$ and that $N^c/T(R) = (T/T(R)) \cdot (N^c(R)/T(R))$ because the elements $w_\alpha(1)$ are coset representatives of $T \backslash N^c$, we have $\widetilde{W}^c = \mathbf{Z}^{\ell^c} \ltimes W^c$ as well.

The action of $N^c(R)/T(R)$ on $T/T(R)$ is given by [Ste68, Lemma 20]:

$$w_\alpha(1) h_\beta(t) w_\alpha(1)^{-1} = h_{r_\alpha \beta}(t)$$

which is compatible with the action (2.4); therefore the map that sends $w_\alpha(1) \mapsto r_\alpha$ (both generators of W^c) and $h_\alpha(t) \mapsto (T_\alpha)^{-\omega(t)}$ (both generators of $\mathbf{Z}^{\ell^c}$) is a well-defined isomorphism.

❐

In other words, the valuation on k gives us a notion of the "size" of an element of T, which we can use to define a translation. This in turn allows us to define an action of N on $\mathcal{A}$.

Definition 2.26. Let N^c act on $\mathcal{A}^c$ by setting $n \cdot x = w \cdot x$, where $w \in \widetilde{W}^c$ is the image of n under the projection $N^c \twoheadrightarrow N^c/T(R) = \widetilde{W}^c$. Define the action of N on $\mathcal{A}$ to be the action of N^c restricted to N.

Corollary 2.27 (to Lemma 2.25). *(i) If $w_\alpha(t) = h_\alpha(t) w_\alpha(1)$ is a generator of N, then it acts by $(T_\alpha)^{\omega(t)} r_\alpha = r_{\alpha - \omega(t)}$.*

(ii) If $n \in N$ with $n \cdot x = x$ for every $x \in \mathcal{A}$, then $n \in T(R)$. That is, $\bigcap_{x \in \mathcal{A}} \operatorname{stab}_N(x) = T(R)$.

There are some important things to note about the action of N on $\mathcal{A}$.

- The linear part of the action of any element of N is in the Weyl group W.
- If $(E_\Phi^*)^\perp \neq \{0\}$ then it is no longer true that N acts trivially on $(E_\Phi^*)^\perp$ (so this information is no longer redundant) — if $\lambda \in \Lambda_C$ and $t \in k^\times$, then $\check{\lambda}(t) \in T$ will act as a translation in $(E_\Phi^*)^\perp$. (Since Λ_C spans $E_\Phi^\perp$, there is always some element of T which will act nontrivially on $(E_\Phi^*)^\perp$.)
- However, it is still true that $\widetilde{W}$ acts trivially on $(E_\Phi^*)^\perp$.
- Therefore, $\widetilde{W}$ may be a proper subgroup of $N/T(R)$.

Example 2.28. The group $G = \mathrm{SL}_2(k)$ has one root α, so the affine apartment of SL_2 is one-dimensional and the "hyperplanes" $H_{\alpha+n}$ are just points. See Figure 2.4. Suppose that we choose the simple root α such that $\alpha(\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right]) = a^2$, so $\check{\alpha}(t) = \left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]$ and $\mathfrak{X}_\alpha(k) = \left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]$. Then $h_\alpha(\varpi^n) = \left[\begin{smallmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{smallmatrix}\right]$ is a translation to the right by $n \cdot \check{\alpha}$; that is, $h_\alpha(\varpi^n)$ takes 0 to $H_{\alpha-2n}$. We also have $w_\alpha(1) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ which is a flip over $H_{\alpha+0}$, so $h_\alpha(\varpi^n) w_\alpha(1) = \left[\begin{smallmatrix} 0 & \varpi^n \\ -\varpi^{-n} & 0 \end{smallmatrix}\right]$ is a flip over $H_{\alpha-n}$. Note that the action of N stabilizes the lattice $\mathbf{Z} \cdot \check{\alpha}$ — for instance, there is no $n \in N$ such that $nH_{\alpha+0} = H_{\alpha+1}$.

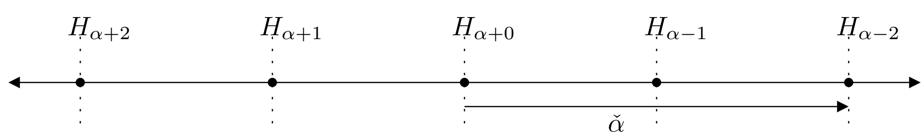

Figure 2.4: The affine apartment of $\mathrm{SL}_2(k)$.

Example 2.29. Let $G \subset \mathrm{SL}_3(k)$ be the Levi subgroup with root system $\Phi = \{\pm\alpha\}$ (cf. Example 2.18), so we have

$$G \cong \mathrm{GL}_2(k) = \begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \subset \mathrm{SL}_3(k).$$

Here E_Φ^* is spanned by $\check{\alpha}$; see Figure 2.5. Note the (conceptual) absence of the hyperplanes $H_{\pm\beta+n}$ and $H_{\pm\alpha\pm\beta+n}$ in the apartment of $\mathrm{GL}_2(k)$. We have

$$h_\alpha(t) = \begin{bmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{bmatrix} \quad \text{and} \quad h_\beta(s) = \begin{bmatrix} 1 & & \\ & s & \\ & & s^{-1} \end{bmatrix},$$

so

$$h_\alpha(t) h_\beta(t)^2 = \begin{bmatrix} t & & \\ & t & \\ & & t^{-2} \end{bmatrix} \in C.$$

This corresponds to the fact that $\check{\alpha} + 2\check{\beta}$ is orthogonal to $\mathcal{A}^{\mathrm{red}}(\mathrm{GL}_2(k))$, as in Figure 2.5. Therefore $h_\alpha(t) h_\beta(t)^2$ acts as a translation in the $(E_\Phi^*)^\perp$ direction — note that $(E_\Phi^*)^\perp$ is generated by $\check{\alpha} + 2\check{\beta}$.

In this case, $\widetilde{W}$ is generated by the reflections over the $H_{\alpha+n}$, which is a subgroup of $\widetilde{W}^c$ which fixes $(E_\Phi^*)^\perp$.

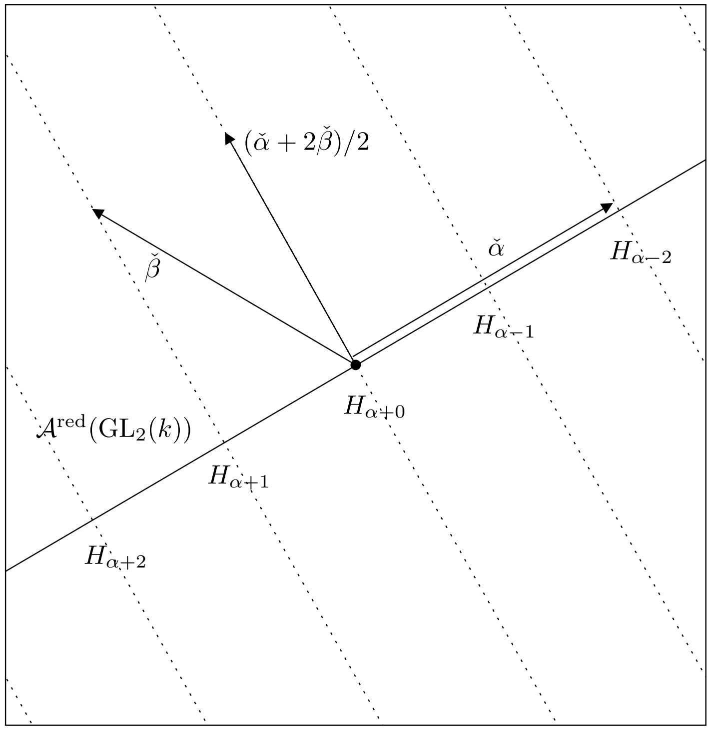

Figure 2.5: The apartment of $\mathrm{GL}_2(k) \subset \mathrm{SL}_3(k)$. Compare this with Figure 2.3.

Before finishing this section, we spend some time investigating the hyperplane structure of the apartment $\mathcal{A}$. The fact that $\mathcal{A}$ is cut out by hyperplanes makes it look like a simplicial complex (although in general it is not one) whose structure is preserved by the action of N. This result follows from the following lemma, whose proof follows from the facts that each map $r_\beta : \mathcal{A}_s \to \mathcal{A}_s$ is an isometry, and that $\langle \check{\beta}, \alpha \rangle \in \mathbf{Z}$ for all roots α and β.

Lemma 2.30. *Let $\alpha \in \Phi$, and let $x \in \mathcal{A}$ be any point. Then*

- *if $\beta \in \Phi$, then $\langle x, \alpha \rangle \in \mathbf{Z}$ if and only if $\langle r_{\beta+0}x, r_\beta\alpha \rangle \in \mathbf{Z}$ and*
- *if $\beta \in \Phi^c$, then $\langle x, \alpha \rangle \in \mathbf{Z}$ if and only if $\langle T_\beta x, \alpha \rangle \in \mathbf{Z}$.*

Now we can define the "simplicial decomposition" of $\mathcal{A}$ and show that N preserves this structure.

Definition 2.31. For $x \in \mathcal{A}$, let

$$\Phi_x = \{\alpha \in \Phi \ : \ \alpha(x) \in \mathbf{Z}\} \tag{2.5}$$

and

$$\mathcal{H}_n = \{x \in \mathcal{A} \ : \ |\Phi_x| = n\}.$$

Define a ***facet*** or n***-facet*** to be a connected component of $\mathcal{H}_n$.[8] If F is such a facet, then $F = \pi_G(F) \times (E_\Phi^*)^\perp$ (see Figure 2.5); therefore it is much more natural to speak

[8] Since the dimension of a facet is not determined by the number of hyperplanes that facet is contained in, it is inconvenient to label facets by their dimension.

about geometric properties of $\pi_G(F)$ than of F. Thus we define the ***dimension*** of a facet F to be the dimension of $\pi_G(F)$ (as a manifold). Since $\mathcal{H}_0$ consists of elements that are not contained in any hyperplane and $\mathcal{H}_{2\ell}$ contains elements in the intersection of ℓ hyperplanes, one would expect that the dimension of an $2n$-facet would be $\ell - n$. We define a 0-facet to be a ***maximal facet*** and a L-facet to be a ***special facet*** (where $L = |\Phi|$). More generally, if $\pi_G(F) = \{v\}$ is one point, or equivalently, $\overline{F} = F$, then we say that F is a ***minimal facet***, and any point in F is a ***vertex***; if F is a special facet, we call the points in F ***special vertices*** (note that all special facets are minimal).

It may be helpful at this point to refer back to Example 2.29 (where all vertices are special), and to glance at Example 2.34.

First note that $\mathcal{A} = \coprod_{n=0}^{L} \mathcal{H}_n$, so the facet structure is somewhat like a simplicial decomposition. Note also that a 2ℓ-facet can be a minimal facet without being a special facet; this can happen because the roots that assume integer values at a point may not contain a set of simple roots (consider the vertex v in Figure 2.6). Lemma 2.30 immediately implies that:

Corollary 2.32 (to Lemma 2.30). *The action of N preserves each $\mathcal{H}_n$. In particular, N preserves the hyperplane structure of $\mathcal{A}$.*

In Chapter 3, we will associate a subgroup G_x of G to any point $x \in \mathcal{A}$, in analogy to Section 2.1. At this point, we may associate a generalized Levi subgroup $M(x)$ of G to a point $x \in \mathcal{A}$, as follows.

Definition 2.33. Define $M(x)$ to be the generalized Levi subgroup of G containing T with root system Φ_x.

Lemma 2.30 easily implies that Φ_x is a closed sub-root system of Φ, so the above definition makes sense. Note that x is a special vertex of $\mathcal{A}(M(x))$.

Example 2.34. Consider the group $G = \mathrm{Sp}_4(k)$; let the notation be as in Example 2.15. The facet structure of $\mathrm{Sp}_4(k)$ is especially interesting because there are two types of vertices. We have $0 \in \mathcal{H}_8$, so 0 is special (note that there are eight roots), but the vertex v in Figure 2.6 is only in $\mathcal{H}_4$ because $\Phi_v = \{\pm\beta, \pm(2\alpha+\beta)\}$. This corresponds to the fact that $M(v) \cong \mathrm{SL}_2(k) \times \mathrm{SL}_2(k) = \mathrm{Spin}_4(k)$ is a generalized Levi subgroup of $\mathrm{Sp}_4(k)$, but is not an actual Levi subgroup (cf. Section 1.1). All one-dimensional facets are in $\mathcal{H}_2$; for instance, if x is in the facet F' in the figure, then $\Phi_x = \{\pm\alpha\}$ and $M(x) = \mathrm{GL}_2(k)$. The maximal facets are the regions which look like F; if $x \in F$, then $M(x) = T$.

3. Parahoric subgroups

Parahoric subgroups are the affine analogs of parabolic subgroups. They are associated with points in the apartment (and later the building) of G, in analogy to Section 2.1. As mentioned before, the building is constructed by gluing affine apartments together; the parahoric subgroups are the glue. They are therefore very important to the building, so we devote this entire chapter to them.

To give a preliminary idea of what parahoric subgroups are, let us consider the case

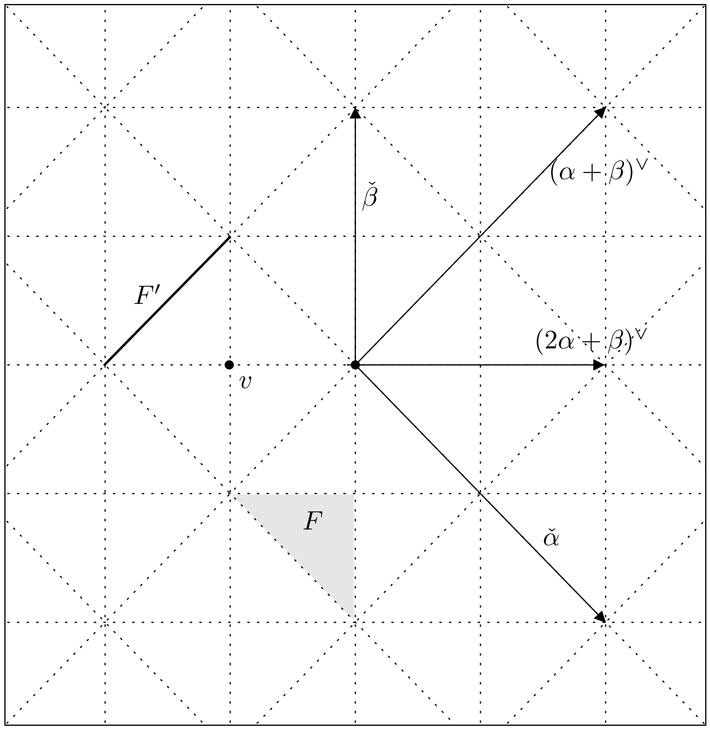

Figure 2.6: The affine apartment of $\mathrm{Sp}_4(k)$, with facet structure and co-roots of positive roots.

$G = \mathrm{SL}_3(k)$. Some standard parahoric subgroups of G are

$$G_x = \begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix} \qquad G_y = \begin{bmatrix} R & R & R \\ R & R & R \\ \wp & \wp & R \end{bmatrix} \qquad G_z = \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{bmatrix}$$

(where all sets of matrices are implicitly intersected with $\mathrm{SL}_3(k)$). The above matrix groups, and parahoric subgroups in general, have the following important properties:

(i) They are compact and open with respect to the induced topology on G as a group of p-adic matrices. This is because R and $\wp$ are compact and open. This property is important to the representation theory of G; see Chapter 5.

(ii) Parahoric subgroups have a natural quotient which is isomorphic to a generalized Levi subgroup of G^c, which is the Chevalley group defined over the residue field by the same Chevalley data as G^c. To wit:

$$\begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix} \Big/ \begin{bmatrix} 1+\wp & \wp & \wp \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix} = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \end{bmatrix} \cong \mathrm{SL}_3(\mathsf{f})$$

$$\begin{bmatrix} R & R & R \\ R & R & R \\ \wp & \wp & R \end{bmatrix} \Big/ \begin{bmatrix} 1+\wp & \wp & R \\ \wp & 1+\wp & R \\ \wp & \wp & 1+\wp \end{bmatrix} = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \\ \mathsf{f} & \mathsf{f} & \\ & & \mathsf{f} \end{bmatrix} \cong \mathrm{GL}_2(\mathsf{f})$$

$$\begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{bmatrix} \Big/ \begin{bmatrix} 1+\wp & R & R \\ \wp & 1+\wp & R \\ \wp & \wp & 1+\wp \end{bmatrix} = \begin{bmatrix} \mathsf{f} & & \\ & \mathsf{f} & \\ & & \mathsf{f} \end{bmatrix} \cong \mathrm{GL}_1(\mathsf{f}) \times \mathrm{GL}_1(\mathsf{f}) \times \mathrm{GL}_1(\mathsf{f}).$$

(iii) A minimal parahoric and the subgroup N form a generalized BN-pair. The BN-pair machinery can thus be used to reveal some of the structure theory of G. For instance,

G_z above is a minimal parahoric, and $N \cap G_z = T(R)$, so $N/(N \cap G_z) = N/T(R) \cong \widetilde{W}$ (this is in general true only when $G = G^c$ however). This is analogous to the theorem that $N/(N \cap B) = N/T \cong W$, where B is a minimal parabolic.

The sections in this chapter roughly correspond to the entries in the list above.

3.1. Definition and basic properties

Without further ado, we define the parahoric subgroups.

Definition 3.1. Let Ψ be the subset of affine functionals on $\mathcal{A}$ given by

$$\Psi = \{\alpha + n \ : \ \alpha \in \Phi, \ n \in \mathbf{Z}\};$$

we call Ψ the set of ***affine roots***. For $\psi = \alpha + n \in \Psi$, write

$$\mathfrak{X}_\psi = \mathfrak{X}_\alpha(\wp^n) = \{x_\alpha(t) \ : \ t \in \wp^n\}.$$

For $x \in \mathcal{A}$, define the groups

$$\begin{aligned} G_x &= \langle T(R), \ \mathfrak{X}_\psi \ : \ \psi \in \Psi, \ \psi(x) \geq 0 \rangle \\ G_x^+ &= \langle T(1+\wp), \ \mathfrak{X}_\psi \ : \ \psi \in \Psi, \ \psi(x) > 0 \rangle \end{aligned}$$

where

$$T(1+\wp) = \langle h_\alpha(t) \ : \ \alpha \in \Delta_0, \ t \in 1+\wp \rangle \subset T(R)$$

is defined as before.[9] Written more explicitly, we have

$$\begin{aligned} G_x &= \langle T(R), \ \mathfrak{X}_\alpha(\wp^{-\lfloor \alpha(x) \rfloor}) \ : \ \alpha \in \Phi \rangle \\ G_x^+ &= \langle T(1+\wp), \ \mathfrak{X}_\alpha(\wp^{1-\lceil \alpha(x) \rceil}) \ : \ \alpha \in \Phi \rangle. \end{aligned} \tag{3.1}$$

The subgroups G_x are called ***parahoric subgroups***, and the subgroup G_x^+ is the ***pro-unipotent radical*** of G_x.[10] A minimal G_x (i.e., $x \in \mathcal{H}_0$) is called an ***Iwahori subgroup***, and if x is a vertex, then G_x is a ***maximal parahoric*** (we will see later that, in this case, G_x really is not contained in any larger parahoric). Note that G_x and G_x^+ only depend on the facet that contains x, since they change exactly when some $\alpha(x)$ is an integer; therefore if F is a facet, we will often write G_F and G_F^+. Note that $G_{\pi_G(x)} = G_x$ and $G_{\pi_G(x)}^+ = G_x^+$.

We should note that the G in G_x symbolizes the group G, so that if M is another generalized Levi subgroup, then its associated parahorics will be denoted M_x, etc.

Parahoric subgroups are the affine analogs of parabolic subgroups, and Iwahori subgroups are the affine analogs of Borels. It will become clear later (cf. Theorem 3.34) exactly how and why these behave like parabolics and Borels.

[9] The definitions of G_x and G_x^+ depend on the choice of Chevalley basis. One can show that a different choice of Chevalley basis is equivalent to choosing a $y \in \mathcal{A}_s$ such that $\alpha(y) \in \mathbf{Z}$ for all $\alpha \in \Phi$. With this choice of Chevalley basis, the parahoric subgroups G_x' are given by G_{x+y}. This is another reason that $\mathcal{A}$ is naturally an affine space.

[10] The subgroup G_x^+ is the minimal normal subgroup of G_x with the property that G_x/G_x^+ is the group of $\mathfrak{f}$ points of a connected reductive group defined over $\mathfrak{f}$.

Example 3.2. Consider $G = \mathrm{SL}_3(k)$ again (see Example 2.18), and take the root spaces to be as in Example 2.14. We will use the example of $\mathrm{SL}_3(k)$ throughout this chapter. The minimal parahorics in a neighborhood of 0 are given in Figure 3.1 — of course, all sets of matrices are implicitly intersected with $\mathrm{SL}_3(k)$ (and will always be so unless otherwise mentioned). Note how the generating subgroups correspond roughly to matrix entries. The only generators of a parahoric that are different for its pro-unipotent radical are the torus and the root subgroups $\mathfrak{X}_\alpha$ where $\alpha(x) \in \mathbf{Z}$. For a minimal parahoric, there are no roots that satisfy this property, so no off-diagonal matrix entries change: for instance, if y is as in the figure, then the parahoric at y and its pro-unipotent radical are given by

$$G_y = \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{bmatrix} \qquad G_y^+ = \begin{bmatrix} 1+\wp & R & R \\ \wp & 1+\wp & R \\ \wp & \wp & 1+\wp \end{bmatrix}.$$

On the other hand, 0 is a special vertex, so every matrix entry will change; the parahoric of the point 0 and its pro-unipotent radical are therefore

$$G_0 = \begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix} \qquad G_0^+ = \begin{bmatrix} 1+\wp & \wp & \wp \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix}.$$

The point x in Figure 3.1 has $\Phi_x = \{\pm\beta\}$, so its parahoric and pro-unipotent radical are

$$G_x = \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & R & R \end{bmatrix} \qquad G_x^+ = \begin{bmatrix} 1+\wp & R & R \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix}.$$

We will now present the basic properties of these groups. We shall not go into the low-level machinery used to prove these results, so we refer the reader to [BT72] for the details. The presentation given there is very good.

Definition 3.3. We define the topology on G to be that induced by the natural topology on the k-vector space $\mathrm{End}(V^k)$ — in other words, the induced topology as a subset of matrices with entries in k.

By explicitly determining the possible matrix entries for the groups G_x and G_x^+, one finds that:

Proposition 3.4. *Each group G_x and G_x^+ is compact, open, and closed.*

Using the Chevalley commutator relations, one can show that:

Proposition 3.5. *Each G_x^+ is normal in G_x.*

If $n \in N$, then it is easy to show that $n\mathfrak{X}_\alpha(\wp^m)n^{-1} = \mathfrak{X}_{n\cdot\alpha}(\wp^m)$. This immediately implies that:

Proposition 3.6. *If $n \in N$, then $nG_xn^{-1} = G_{n\cdot x}$ and $nG_x^+n^{-1} = G_{n\cdot x}^+$.*

The following fact can easily be proved using information about the action of $x_\alpha(t)$ and $h_\beta(t)$ on V^k.

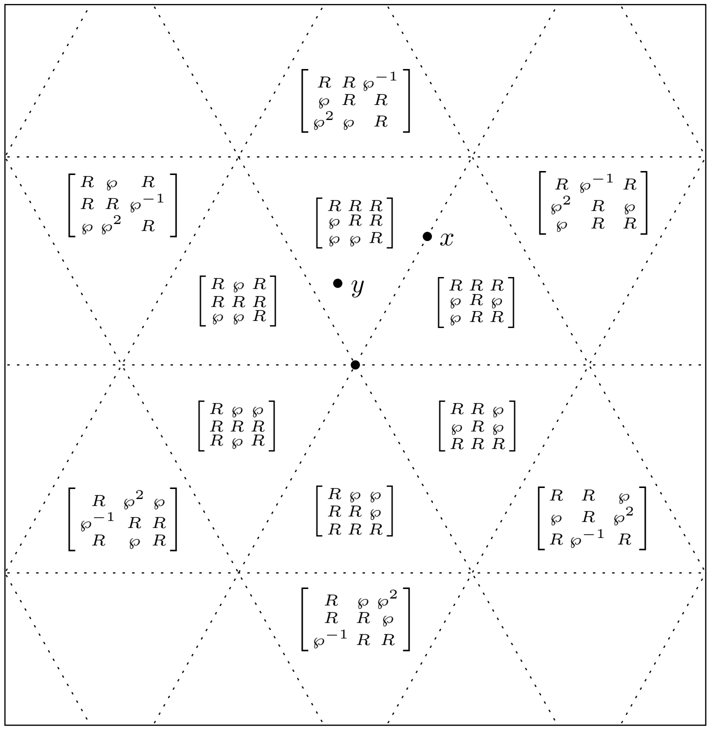

Figure 3.1: Some minimal parahoric subgroups of $\mathrm{SL}_3(k)$.

Proposition 3.7. *For $x \in \mathcal{A}$, define lattices $V_x, V_x^+ \subset V^k$ by*

$$V_x = \bigoplus_{\mu} V_\mu \otimes_{\mathbf{Z}} \wp^{-\lfloor \mu(x) \rfloor}$$

$$V_x^+ = \bigoplus_{\mu} V_\mu \otimes_{\mathbf{Z}} \wp^{1-\lfloor \mu(x) \rfloor}$$

where these are to be interpreted as tensor products of additive abelian groups. The groups G_x (and G_x^+) preserve V_x and V_x^+. If x is a special vertex, then G_x^+ fixes V_x/V_x^+.

It turns out that if x is a special vertex, then G_x is the largest subgroup that preserves V_x.

Proposition 3.8. *We have $G_x \cap T = T(R)$ for any $x \in \mathcal{A}$.*

Proof.

By definition, $T(R) \subset G_x \cap T$. Conversely, we know that G_x preserves the lattice V_x, so since $T(R)$ is the largest subgroup of T that has this property, we must have the opposite inclusion. (Cf. the proofs of Lemma 2.25 and Theorem 3.17.)

❐

Proposition 3.9. *The intersection $\bigcap_{x \in \mathcal{A}} G_x = T(R)$.*

Proof.

We will only provide a sketch, as this result should be intuitively obvious from the definitions. It suffices to prove the result in the full Chevalley group. The strategy is to show that $P = \bigcap_x G_x$ acts diagonally on the root spaces V_μ, so that P is contained in the centralizer in G^c of T, which is T itself (cf. Proposition B.3). Indeed, consider how a $p \in P$ acts on V_μ: p takes a generator v_μ to some multiple of v_μ plus a sum of other root spaces. Moving in any direction orthogonal to μ, one finds from Proposition 3.7 that the components of $p \cdot v_\mu$ that are not in the V_μ root space must be arbitrarily small. Therefore P acts diagonally.

❒

Parahoric subgroups have very nice (but not surprising) intersection properties and decompositions. We refer the reader to [IM65] for the proofs in some specific cases and to [BT72] for proofs in much greater generality.

Proposition 3.10. *(i) Let $M \subset G$ be a generalized Levi subgroup, and let $x \in \mathcal{A}$. Then $G_x \cap M \supset M_x$ and $G_x^+ \cap M = G_x^+ \cap M_x = M_x^+$.*

(ii) Let P be a standard parabolic subgroup with Levi decomposition MU. Let Θ be the root system of M, so that P is generated by T and the root subgroups $\mathfrak{X}_\alpha$ for $\alpha \in \Phi^+ \cup \Theta$. Then

$$U \cap G_x = \prod_{\alpha \in \Phi^+ \setminus \Theta} \mathfrak{X}_\alpha(\wp^{-\lfloor \alpha(x) \rfloor})$$
$$U \cap G_x^+ = \prod_{\alpha \in \Phi^+ \setminus \Theta} \mathfrak{X}_\alpha(\wp^{1-\lceil \alpha(x) \rceil})$$

where the product is taken in any fixed order, and the expression is unique with respect to that order.

Remark 3.11. Recall [Ste68, Lemmas 17 and 18] that in the notation of Proposition 3.10(ii),

$$U = \prod_{\alpha \in \Phi^+ \setminus \Theta} \mathfrak{X}_\alpha \tag{3.2}$$

with uniqueness of expression, where the product is taken in any fixed order.

Proposition 3.12. *Let P be a standard parabolic subgroup with Levi decomposition MU and opposite parabolic $\overline{P} = M\overline{U}$.*

(i) For any $x \in \mathcal{A}$, the product map

$$(\overline{u}, m, u) \mapsto \overline{u}mu : \ (G_x^+ \cap \overline{U}) \times (G_x^+ \cap M) \times (G_x^+ \cap U) \xrightarrow{\sim} G_x^+$$

is a bijection. The product can be taken in any order.

(ii) Let Φ_M be the root system of M. If $x \in \mathcal{A}$ with $\Phi_x \subset \Phi_M$, then we have a decomposition

$$G_x = (G_x \cap \overline{U})(G_x \cap M)(G_x \cap U).$$

(iii) Suppose that P is a minimal parabolic. Then

$$\begin{aligned} G_x &= (G_x \cap N)(G_x \cap U)(G_x \cap \overline{U}) = (G_x \cap N)(G_x \cap \overline{U})(G_x \cap U) \\ &= (G_x \cap U)(G_x \cap \overline{U})(G_x \cap N) = (G_x \cap \overline{U})(G_x \cap U)(G_x \cap N). \end{aligned}$$

Proposition 3.12(i) says that each G_x^+ has an ***Iwahori decomposition*** with respect to any $P = MU$. Proposition 3.12(iii) can be seen as a kind of Jordan decomposition: any element of G_x can be written as $nu\overline{u}$, where n is semisimple, u and $\overline{u}$ are unipotent, and all three are contained in G_x.

Example 3.13. To illustrate Propositions 3.10 and 3.12, we give the example of $G = \mathrm{SL}_3(k)$. Let the notation be as in Example 3.2. Let $M \subset \mathrm{SL}_3(k)$ be a Levi subgroup isomorphic to $\mathrm{GL}_2(k)$, and let $P = MU$ be the standard parabolic subgroup containing M; we will take

$$M = \begin{bmatrix} * & & \\ & * & * \\ & * & * \end{bmatrix} \qquad U = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P = \begin{bmatrix} * & * & * \\ & * & * \\ & * & * \end{bmatrix}.$$

For the point x in Figure 3.1, Proposition 3.10(i) tells us that

$$M \cap G_x = M \cap \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & R & R \end{bmatrix} = \begin{bmatrix} R & & \\ & R & R \\ & R & R \end{bmatrix} = M_x$$
$$M \cap G_x^+ = M \cap \begin{bmatrix} 1+\wp & R & R \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix} = \begin{bmatrix} 1+\wp & & \\ & 1+\wp & \wp \\ & \wp & 1+\wp \end{bmatrix} = M_x^+.$$

Proposition 3.10(ii) affirms the unsurprising fact that

$$U \cap G_x = U \cap G_x^+ = \begin{bmatrix} 1 & R & R \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & R & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & R \\ & 1 & \\ & & 1 \end{bmatrix}.$$

The parabolic $\overline{P} = M\overline{U}$ opposite to P with respect to M, and its unipotent radical $\overline{U}$, are given by

$$\overline{P} = \begin{bmatrix} * & & \\ * & * & * \\ * & * & * \end{bmatrix} \qquad \overline{U} = \begin{bmatrix} 1 & & \\ * & 1 & \\ * & & 1 \end{bmatrix}.$$

The Iwahori decomposition (Proposition 3.12(i)) of G_x^+ with respect to $P = MU$ is:

$$\begin{bmatrix} 1+\wp & R & R \\ \wp & 1+\wp & \wp \\ \wp & \wp & 1+\wp \end{bmatrix} = \begin{bmatrix} 1 & & \\ \wp & 1 & \\ \wp & & 1 \end{bmatrix} \begin{bmatrix} 1+\wp & & \\ & 1+\wp & \wp \\ & \wp & 1+\wp \end{bmatrix} \begin{bmatrix} 1 & R & R \\ & 1 & \\ & & 1 \end{bmatrix}$$

where the product can be taken in any order. In this case, the analogous decomposition (Proposition 3.12(ii)) holds for G_x as well:

$$\begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & R & R \end{bmatrix} = \begin{bmatrix} 1 & & \\ \wp & 1 & \\ \wp & & 1 \end{bmatrix} \begin{bmatrix} R & & \\ & R & R \\ & R & R \end{bmatrix} \begin{bmatrix} 1 & R & R \\ & 1 & \\ & & 1 \end{bmatrix}.$$

This last decomposition works because $\Phi_x = \{\pm\beta\}$, which is the root system for M. It is *not* true that

$$\begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & R & R \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ \wp & R & 1 \end{bmatrix} \begin{bmatrix} R & R & \\ \wp & R & \\ & & R \end{bmatrix} \begin{bmatrix} 1 & & R \\ & 1 & R \\ & & 1 \end{bmatrix}.$$

For any nonempty $\Omega \subset \mathcal{A}$, we can define subgroups G_Ω which generalize parahoric subgroups.

Definition 3.14. Let $\Omega \subset \mathcal{A}$ be any nonempty region. Define

$$G_\Omega = \langle T(R),\ \mathfrak{X}_\alpha(\wp^{-\lfloor \inf_{x\in\Omega} \alpha(x)\rfloor}) \ :\ \alpha \in \Phi \rangle.$$

We will use the following facts about the groups G_Ω:

Proposition 3.15. *(i)* $G_\Omega = \bigcap_{x\in\Omega} G_x$.

(ii) Proposition 3.12(iii) holds for any G_Ω.

3.2. Passage to the residue field

The goal of this section is to demonstrate that for any x in the apartment, G_x/G_x^+ is a generalized Levi of a Chevalley group over the residue field f, and that the hyperplane structure of the spherical apartment of G_x/G_x^+ is the same as the local hyperplane structure of $\mathcal{A}$ at x. This allows us to use the theory of Chevalley groups over a finite field to analyze G. With this in mind, we put:

Notation 3.16.

$$\mathsf{G}_x = G_x/G_x^+.$$

Again, the G in G_x is associated with the symbol G, so if M is another generalized Levi subgroup of G^c, then we put $\mathsf{M}_x = M_x/M_x^+$.

Theorem 3.17. *Suppose that $x \in \mathcal{A}$. Then G_x is isomorphic to the generalized Levi of G with root system Φ_x — in particular, if x is a special vertex of $\mathcal{A}(G)$, then $\mathsf{G}_x = \mathsf{G}$. The generators $x_\alpha(t), h_\beta(s) \in G_x$ ($\alpha \in \Phi_x, \beta \in \Delta, t \in \wp^{-\alpha(x)}, s \in R^\times$) correspond to the generators $\mathsf{x}_\alpha(\mathsf{t}), \mathsf{h}_\beta(\mathsf{s}) \in \mathsf{G}$ ($\mathsf{t} \in \mathsf{f} = \wp^{-\alpha(x)}/\wp^{1-\alpha(x)}, \mathsf{s} \in \mathsf{f}^\times = R^\times/(1+\wp)$) under this isomorphism.*

Proof.

Suppose first that x is a special vertex. By Proposition 3.7, there is a natural homomorphism $\pi : G_x/G_x^+ \to \mathrm{GL}(V_x/V_x^+) = \mathrm{GL}(V^{\mathsf{f}})$. Writing out the definitions, we find that image of this homomorphism is generated by $\pi(x_\alpha(t)) = \mathsf{x}_\alpha(\mathsf{t})$ for $\alpha \in \Phi$ and $\pi(h_\beta(s)) = \mathsf{h}_\beta(\mathsf{s})$ for $\beta \in \Delta^c$, where t is the image of t in the residue field f, which is identified with $\wp^{-\alpha(x)}/\wp^{1-\alpha(x)}$ via our uniformizer ϖ, and and s is the image of s in $R^\times/(1+\wp) = \mathsf{f}^\times$. Therefore $\operatorname{im}\pi = \mathsf{G}$. It remains to show that π is injective, so that $G_x/G_x^+ = \mathsf{G}_x$ is G.

Define a map $i : \mathsf{G} \to \mathsf{G}_x$ by $i(\mathsf{x}_\alpha(\mathsf{t})) = x_\alpha(t)$ and $i(\mathsf{h}_\alpha(\mathsf{s})) = h_\alpha(s)$, where $t \in \wp^{-\alpha(x)}$ (resp. $s \in R^\times$) is any lift of t (resp. s), as before. Note that this map is well-defined on the generators because if $t \in \wp^{1-\alpha(x)}$, then $x_\alpha(t) \in G_x^+$, so $x_\alpha(t+u) = x_\alpha(t)x_\alpha(u) = x_\alpha(u)$ in G_x; similarly for $h_\alpha(s)$. In order to show that i indeed extends to a homomorphism, we must show that all relations in G are satisfied in G_x as well. This is not so hard in our case — we have a list of relations in [Ste68, Corollary 3]. These are a defining set of relations for G as well as G, so they must hold in both groups, at least formally; for instance,

$$i(\mathsf{x}_\alpha(\mathsf{t}+\mathsf{u})) = x_\alpha(t+u) = x_\alpha(t)x_\alpha(u) = i(\mathsf{x}_\alpha(\mathsf{t})\mathsf{x}_\alpha(\mathsf{u})).$$

The other relations are satisfied similarly. Therefore i extends to a homomorphism, which must be surjective; thus we have a two-sided inverse of π, so π is an isomorphism.

Now suppose that $x \in \mathcal{A}$ is any point, and let $M = M(x)$ as in Definition 2.33, so x projects onto a special vertex of M. We therefore want to show that $\mathsf{G}_x \cong M_x/M_x^+$.

By checking generators, we find $M_x \subset G_x$; inclusion composed with projection gives a map $M_x \to G_x/G_x^+$ whose kernel is

$$M_x \cap G_x^+ = (M \cap G_x) \cap G_x^+ = M \cap G_x^+ = M_x^+$$

by Proposition 3.10. The generators for G_x have lifts in M_x, so this homomorphism is surjective; therefore we have $M_x/M_x^+ \cong G_x/G_x^+$ as required.

❒

Corollary 3.18. *If $x \in \mathcal{A}$ and $M = M(x)$, then inclusion $M_x = M \cap G_x \hookrightarrow G_x$ gives a natural isomorphism $\mathsf{G}_x = \mathsf{M}_x$.*

Now we can begin to see how $\mathcal{A}_s(\mathsf{G}_y)$ looks like a neighborhood of y in $\mathcal{A}$:

Remark 3.19. For $y \in \mathcal{A}$, the hyperplanes containing y are exactly the hyperplanes in the spherical apartment of G_y, so that, locally, the hyperplane structure of $\mathcal{A}$ at y is the same as that of $\mathcal{A}_s(\mathsf{G}_y)$, if we think of y as the origin. In other words, we have an identification of $\mathcal{A}_s(\mathsf{G}_y)$ with $\mathcal{A}$ such that for any $x \in \mathcal{A}$, if $\alpha(x) = \alpha(y) \in \mathbf{Z}$ (i.e., x and y are both contained in $H_{\alpha-\alpha(y)}$), then x is identified with a point in $H_\alpha \subset \mathcal{A}_s(\mathsf{G}_y)$. We will often be sloppy and think of any $x \in \mathcal{A}$ as an element of $\mathcal{A}_s(\mathsf{G}_y)$; this is acceptable because identifying $x \in \mathcal{A}$ with $x_0 \in \mathcal{A}_s(\mathsf{G}_y)$, we have $\Phi_x = \{\alpha \in \Phi \,:\, \alpha(x_0) = 0\}$ when x is in a small enough neighborhood of y.

Example 3.20. Let the notation be as in Example 3.2. Theorem 3.17 tells us that

$$G_0/G_0^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \end{bmatrix} \qquad G_x/G_x^+ = \begin{bmatrix} \mathsf{f} & & \\ & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \end{bmatrix} \qquad G_y/G_y^+ = \begin{bmatrix} \mathsf{f} & & \\ & \mathsf{f} & \\ & & \mathsf{f} \end{bmatrix}$$

as one would expect. The local hyperplane structure of $\mathcal{A}$ at the point x in Figure 3.1 consists only of the hyperplane $H_{\beta+0}$ in a two-dimensional space; compare this with the hyperplane structure of $\mathcal{A}_s(\mathsf{G}_x)$, which is a two-dimensional vector space with one hyperplane H_β. See Figure 3.2.

The identification above will be made much stronger in Theorem 3.28. Before proceeding along that line, we first prove a proposition relating parabolics over f with parahorics.

Remark 3.21. If x is in some facet $F \subset \mathcal{A}$ and y is in the closure of F, then we can see from (3.1) that $G_x \subset G_y$ and $G_x^+ \supset G_y^+$ because $\Phi_x \subset \Phi_y$. In fact, one can show the converse, i.e., that $G_x \subset G_y$ implies that y is in the closure of the facet containing x, which justifies calling the parahorics corresponding to vertices maximal. (Cf. the proof of Proposition 4.22(iii).)

The above remarks motivate the following result:

Proposition 3.22. *Let x be in some facet $F \subset \mathcal{A}$ and y in the closure $\overline{F}$; suppose that x is identified with $x_0 \in \mathcal{A}_s(\mathsf{G}_y)$ as in Remark 3.19. Then G_x/G_y^+ is isomorphic to the parabolic subgroup P_{x_0} of G_y defined in Proposition 2.13. Furthermore, the image of G_x^+ under this isomorphism is the unipotent radical of P_{x_0}, so G_x is isomorphic to a standard Levi of G_y.*

Remark 3.23. It is not true that G_x can always be naturally identified with a standard Levi subgroup of G — this is only the case when the closure of the facet containing x contains a special vertex. Consider the vertex v in Example 2.34, Figure 2.6.

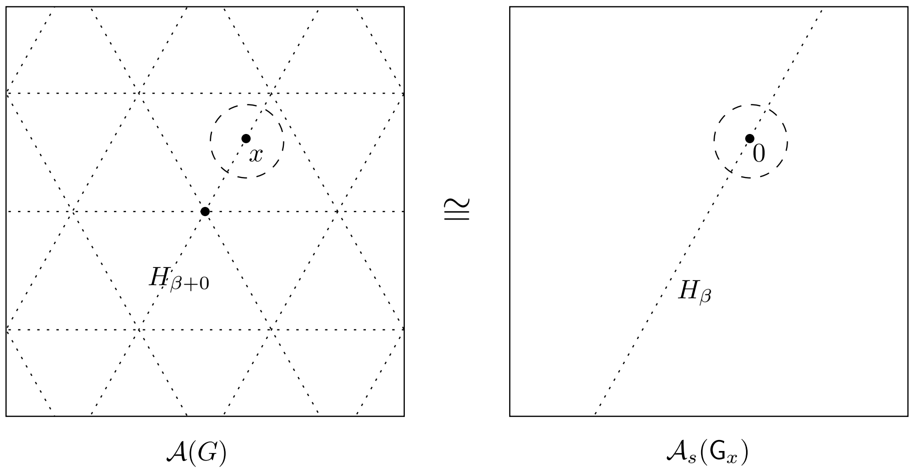

Figure 3.2: The affine apartment of $\mathrm{SL}_3(k)$, as compared to the spherical apartment of G_x, which is isomorphic to $\mathrm{GL}_2(\mathfrak{f})$.

Proof of Proposition 3.22.

Inclusion of G_x into G_y composed with projection onto G_y/G_y^+ gives a map $i : G_x \to G_y/G_y^+$ whose kernel is $G_x \cap G_y^+ = G_y^+$. We would like to show that $\mathsf{P}_x := G_x/G_y^+ \subset G_y/G_y^+ = \mathsf{G}_y$ is a parabolic subgroup. We lose nothing by assuming that y is a special vertex because we can replace G with $M(y)$.

Let

$$\Theta = \{\alpha \in \Phi_y \ : \ \alpha(y) \leq \alpha(x)\}.$$

The group G_x/G_y^+ is generated by the root spaces of roots α such that $\lfloor \alpha(x) \rfloor = \lceil \alpha(y) \rceil$, i.e., G_x/G_y^+ is generated by the root spaces Θ. Comparing this with (2.1) and thinking of y as the origin, we see that P_x is the parabolic subgroup of G_y corresponding to x, as claimed.

By definition, the group G_x^+/G_y^+ will be generated by all roots α with $\alpha(y) < \alpha(x)$, which are exactly the roots α in Θ such that $-\alpha \notin \Theta$. Therefore G_x^+/G_y^+ is the unipotent radical of P_x. By the third isomorphism theorem, then, $\mathsf{G}_x = (G_x/G_y^+)/(G_x^+/G_y^+)$ is isomorphic to a standard Levi of G_y.

❒

Corollary 3.24. *If $x, y \in \mathcal{A}$ are any points, then $G_x \cap G_y$ projects onto a parabolic in both G_x and G_y. The unipotent radicals are the images of $G_x \cap G_y^+$ and $G_x^+ \cap G_y$, respectively.*

Proof.

If we move along the line in $\mathcal{A}$ from x to y and apply Proposition 3.22 every time we enter a new facet, the result follows.

❒

Example 3.25. Let the notation be as in Example 3.2. By Proposition 3.22, G_y and G_y^+

project onto G_0 and G_x as follows:

$$G_y/G_0^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \\ & & \mathsf{f} \end{bmatrix} \qquad G_y^+/G_0^+ = \begin{bmatrix} 1 & \mathsf{f} & \mathsf{f} \\ & 1 & \mathsf{f} \\ & & 1 \end{bmatrix}$$
$$G_y/G_x^+ = \begin{bmatrix} \mathsf{f} & & \\ & \mathsf{f} & \mathsf{f} \\ & & \mathsf{f} \end{bmatrix} \qquad G_y^+/G_x^+ = \begin{bmatrix} 1 & & \\ & 1 & \mathsf{f} \\ & & 1 \end{bmatrix}$$

so G_y projects onto the standard Borel and G_y^+ projects onto its unipotent radical. We also have

$$G_x/G_0^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \end{bmatrix} \qquad G_x^+/G_0^+ = \begin{bmatrix} 1 & \mathsf{f} & \mathsf{f} \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Identifying a neighborhood of 0 in $\mathcal{A}$ with the spherical apartment of $\mathrm{SL}_3(\mathsf{f})$, the point x in our example can be identified with the point y in Example 2.14, which explains why G_x/G_0^+ here is the same as P_y there.

In the rest of this section, we will show that not only does $\mathcal{A}_s(\mathsf{G}_x)$ have the same hyperplane structure as $\mathcal{A}$ does in a neighborhood of x, but that the actions of N and $\mathsf{N} \subset \mathsf{G}$ correspond.

Notation 3.26. Define $\mathsf{N}_x \subset \mathsf{G}_x$ to be the subgroup generated by T and the $\mathsf{w}_\alpha(\mathsf{t})$ for $\alpha \in \Phi_x$ (so $\mathsf{N}_x = N_{\mathsf{G}_x}(\mathsf{T})$ if $|\mathsf{f}| > 3$). Put $W_x = \mathsf{N}_x/\mathsf{T}$, the Weyl group of G_x, realized as a group of isometries of $\mathcal{A}_s(\mathsf{G}_x)$. Let N_x be the group $N \cap G_x$.

Lemma 3.27. *The image of N_x in G_x is $\mathsf{N}_x \subset \mathsf{G}_x$.*

Proof.

We will assume that $|\mathsf{f}| > 3$; we cheat and refer to standard facts from the theory of linear algebraic groups for the other two cases. By [Ste68, Exercise, p.36] we know that $N_{\mathsf{G}_x}(\mathsf{T}) = \mathsf{N}_x$. If $n \in N_x$, then n normalizes $T(R)$, so the image of n in G_x certainly normalizes T; therefore the image of N_x is contained in N_x. Conversely, if $\mathsf{w}_\alpha(\mathsf{t}) \in \mathsf{N}_x$, then if $t \in \wp^{-\alpha(x)}$ is any lift of $\mathsf{t} \in \mathsf{f}$, then $w_\alpha(t) \in N_x$ is a lift of $\mathsf{w}_\alpha(\mathsf{t})$, so the image of N_x is all of N_x.

❒

Recall from Remark 3.19 the identification of $\mathcal{A}$ in a neighborhood of y with $\mathcal{A}_s(\mathsf{G}_y)$.

Theorem 3.28. *For any $x \in \mathcal{A}$, the group N_x fixes x, and the action of N_x on $\mathcal{A}$ is generated by reflections over the hyperplanes containing x. Identifying $\mathcal{A}$ with $\mathcal{A}_s(\mathsf{G}_x)$, the action of $n \in N_x$ on $\mathcal{A}$ is the same as the action of its image $\mathsf{n} \in \mathsf{N}_x$ on $\mathcal{A}_s(\mathsf{G}_x)$.*

Proof.

Let $\mathsf{n} \in \mathsf{N}_x$. First we show that any two lifts of n in N_x act on $\mathcal{A}$ in the same way, i.e., that $N \cap G_x^+$ acts trivially on $\mathcal{A}$. Indeed, let $n \in N \cap G_x^+$. Since G_x^+ has an Iwahori decomposition (Proposition 3.12(i)) with respect to a Borel subgroup B with Levi decomposition TU, we can write $n = tu\overline{u}$ with $t \in T, u \in U$, and $\overline{u} \in \overline{U}$. By Corollary B.2, $nt^{-1} \in N \cap (U\overline{U}) = \{1\}$ so $n = t \in T \cap G_x^+$. Thinking of T as a generalized Levi subgroup of G, we have $T(1+\wp) = T_x^+ = T \cap G_x^+$, so $n \in T(R)$ acts trivially on $\mathcal{A}$.

Let $\mathsf{n} = \mathsf{w}_\alpha(\mathsf{t}) \in \mathsf{N}_x$ be a generator. If t is a lift of $\mathsf{t} \in \mathsf{f} \cong \wp^{-\alpha(x)}/\wp^{1-\alpha(x)}$ in $\wp^{-\alpha(x)}$ (cf. the proof of Theorem 3.17), then $n = w_\alpha(t)$ is a lift of n. Since the valuation $\omega(t) = -\alpha(x)$, Corollary 2.27(i) tells us that $w_\alpha(t)$ acts as $r_{\alpha-\alpha(x)}$. Therefore any lift of a generator of N_x fixes x, so N_x fixes x as well. This also proves the last claim.

❒

The above theorem tells us that there is a natural surjective homomorphism of N_x into W_x given by the action of N_x on $\mathcal{A}_s(\mathsf{G}_x)$, and that this is compatible with the projection of N_x onto N_x:

Corollary 3.29. *The group $N_x/T(R)$ is isomorphic to W_x. This isomorphism can be obtained either by projection of N_x onto N_x, or by the action of N_x on $\mathcal{A}_s(\mathsf{G}_x)$.*

In a sense, then, we have an "N_x-isomorphism" of the spherical apartment of G_x with a neighborhood of x in $\mathcal{A}$ for any $x \in \mathcal{A}$, which preserves hyperplane structure.

Corollary 3.30. *For any $x \in \mathcal{A}$, we have $\mathrm{stab}_N(x) = N_x$.*

Proof.

We have $N = N_G(T) = G \cap N_{G^c}(T) = G \cap N^c$ and $\mathrm{stab}_N(x) = G \cap \mathrm{stab}_{N^c}(x)$, so it suffices to prove the claim when $G = G^c$. Theorem 3.28 gives $N_x^c \subset \mathrm{stab}_{N^c}(x)$.

Let $F \subset \mathcal{A}^c$ be a maximal facet such that $x \in \overline{F}$, and let $n \in \mathrm{stab}_N(x)$, so $F' := n \cdot F$ is also a maximal facet containing x in its closure. Since W_x acts transitively on the Weyl chambers in $\mathcal{A}_s(\mathsf{G}_x)$, which correspond to the maximal facets in $\mathcal{A}$ containing x in their closure, we can find an $n' \in N_x^c$ such that $n' \cdot F = n \cdot F$, i.e., $n^{-1}n' \cdot F = F$. We will show in Proposition 3.32(ii) that if $w \in \widetilde{W}^c$ and $w \cdot F = F$, then $w = 1$, so since $N^c/T(R) = \widetilde{W}^c$, we must have $n^{-1}n' \in T(R)$, so $n \in T(R) \cdot N_x = N_x$.

❒

Example 3.31. Let us continue with the example of $G = \mathrm{SL}_3(k)$ (see Examples 3.2 and 3.25). Since $G_0 = \mathrm{SL}_3(R)$, we can see immediately that we can find representatives for the determinant-one permutation matrices in G_0, so N_0 really does project onto N_0. A neighborhood of the origin in the spherical apartment of $\mathrm{SL}_3(\mathsf{f})$ looks like a neighborhood of 0 in $\mathcal{A}$ (cf. Example 2.14). By Corollary 2.27(i), the element $w_\alpha(1) \in G_0$, for instance, acts as a reflection over $H_{\alpha+0} = H_\alpha$, as expected.

At the point x, we have $\mathsf{G}_x = \mathrm{GL}_2(\mathsf{f}) \subset \mathrm{SL}_3(\mathsf{f})$ (cf. Example 2.29). The element $w_\beta(1) \in G_x$ acts as a reflection over $H_{\beta+0} = H_\beta$, which is the only nontrivial reflection in $\mathcal{A}_s(\mathsf{G}_x)$.

Lastly, at the point y we have $\mathsf{G}_y = \mathsf{T}$, which has trivial Weyl group; this corresponds to the fact that there are no hyperplanes intersecting y.

3.3. Structure theory

The first goal of this section is to find an *affine* BN-pair contained in G, where "B" is a minimal parahoric (instead of parabolic) subgroup. In order to do so, we first need to know what the "simple" reflections are that generate the affine Weyl group. Proposition 3.32(iii) (resp. 3.32(i,ii)) is the analog of Proposition 2.7(i) (resp. 2.7(ii)).

Let F be any maximal facet in $\mathcal{A}$.

Proposition 3.32. *(i) All maximal facets are conjugate to F under $\widetilde{W}$.*

(ii) If $w(F) = F$, then $w = 1$ in $\widetilde{W}$.

(iii) The affine Weyl group $\widetilde{W}$ of G is generated by the reflections over the hyperplanes bordering F.

Proof.

We will assume for convenience that $0 \in \overline{F}$, and that Φ spans E^*, so that vertices really are points — we can do this because $\widetilde{W}$ fixes $(E^*_\Phi)^\perp$. Throughout this proof, one should refer to Figure 3.3.

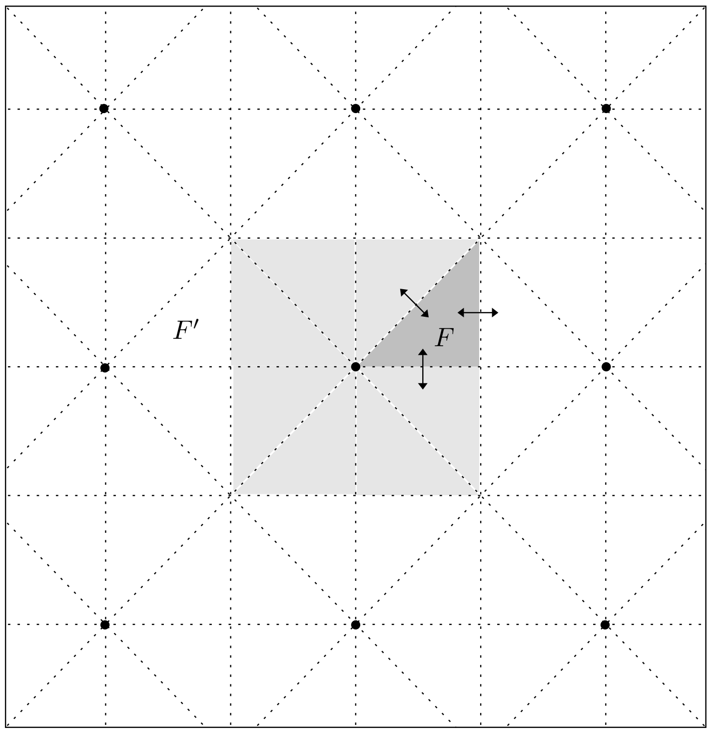

Figure 3.3: An illustration of Proposition 3.32 via the example of $\mathrm{Sp}_4(k)$. The dark triangle is our chosen maximal facet F, and the shaded region is the neighborhood $N(0)$ of 0. The dots indicate the special vertices at which all roots take integer values; this lattice is invariant under $\widetilde{W}$. The arrows indicate the reflections which generate $\widetilde{W}$.

(i) Suppose that there were some facet F' not conjugate to F under $\widetilde{W}$. We may assume that F' shares a hyperplane with a facet F'' which is conjugate to F under $\widetilde{W}$ — see Figure 3.3. The reflection over that hyperplane takes F'' to F', so F' cannot exist.

(ii) Let $\mathcal{O}$ be the orbit of the origin under $\widetilde{W}$. It is easy to see that $\alpha(x)$ is even for every root α and every $x \in \mathcal{O}$ — in other words, the lattice of points x such that $\alpha(x)$ is even for every α is invariant under $\widetilde{W}$. Every special vertex other than 0 contained in $\overline{F}$ has $\alpha(x)$ odd for some α, so if $w(F) = F$, we must have $w(0) = 0$. Therefore w can be thought of as an element of the spherical Weyl group of G, so using Proposition 2.7(ii), we can conclude that $w = 1$.

(iii) Let $\widetilde{W}'$ be the subgroup of $\widetilde{W}$ generated by the reflections over the hyperplanes bordering F. It suffices to show that the orbit $\mathcal{C}$ of F under $\widetilde{W}'$ contains all maximal facets. Reflections over the hyperplanes bordering a Weyl chamber D generate the Weyl group, and by Proposition 2.7(ii), the action of the Weyl group on such chambers is transitive, so $\mathcal{C}$ contains all maximal facets whose closure contains 0. Call this set the ***neighborhood*** $N(0)$ of the special vertex 0, and define $N(x)$ for any $x \in \mathcal{A}$ in the same way. Since we can reflect over all hyperplanes bordering F, $\mathcal{C}$ must contain all neighborhoods bordering $N(0)$ (i.e., all neighborhoods sharing a hyperplane with $N(0)$). Given any neighborhood $N(x)$ of a special vertex x, there is a sequence of neighborhoods $N_0, N_1, \ldots, N_n$, each bordering the next, such that $N_0 = N(x)$ and $N_n = N(0)$ (consider the line from x to 0 in $\mathcal{A}$). Since all maximal facets contain a special vertex in their closure, $\mathcal{C}$ contains all maximal facets.

❒

Remark 3.33. It is not hard to show (cf. [IM65, Prop. 1.2]) that there is a maximal facet F in $\mathcal{A}^c$ such that the hyperplanes bordering F are

$$\{H_{\alpha+0},\ H_{1-\gamma}\ :\ \alpha \in \Delta,\ \gamma \text{ is the highest root of a simple component of } \mathfrak{g}\}.$$

In particular, if G^c comes from a simple Lie algebra, then there is only one other hyperplane bordering F.

The proof of this next theorem, which states that G^c has an affine BN-pair, is quite long and is very similar in spirit to the proof that all Chevalley groups contain a (spherical) BN-pair — that is, it is mostly Coxeter group theory and root space decompositions. A proof can be found in [IM65]. It is true that any generalized Levi of G^c has a *generalized* BN-pair (see [Iwa66]), but we will not state that version of the theorem. Recall (Definition 3.1) that an Iwahori subgroup is a minimal parahoric subgroup, that is, a parahoric subgroup G_F corresponding to a maximal facet F.

Theorem 3.34. *Let $F \subset \mathcal{A}$ be a maximal facet, let $\mathcal{I}$ be the Iwahori subgroup G^c_F, and let S be the set of generators given in Proposition 3.32(iii). Then the data $(\mathcal{I}, N^c, \widetilde{W}^c, S)$ form a BN-pair, or Tits system, in the sense of* [Tit62].

Note that this theorem justifies calling the subgroups G_x parahoric subgroups, and explains in what sense parahoric (resp. Iwahori) subgroups correspond to parabolic (resp. Borel) subgroups. The existence of a BN-pair implies several corollaries, all of which are completely analogous to the spherical case and can be found in [Car89]. The following is the analog of Corollary 2.11:

Corollary 3.35. *Let F be a maximal facet and let $x \in \overline{F}$. Then $G^c_x = G^c_F N^c_x G^c_F$.*

Also, for any generalized Levi subgroup G, we have:

Theorem 3.36 (Affine Bruhat Decomposition). *If G_x and G_y are any two parahoric subgroups, then $G = G_x N G_y$.*

Proof.

Theorem 3.34 tells us that $G = G_F N G_F$ when F is any maximal facet and $G = G^c$; the same is also true when G has a generalized BN-pair, so we can drop the latter condition (cf. [Iwa66]). Let F (resp. F') be a maximal facet such that $x \in \overline{F}$ (resp. $y \in \overline{F}'$), and let $w \in \widetilde{W}$ be the element that takes F to F'. Let $n \in N$ be a lift of w, so that $nG_F n^{-1} = G_{F'}$. Since $G_F \subset G_x$ and $G_{F'} \subset G_y$, we have

$$G = G_F N G_F \cdot n^{-1} = G_F N(nG_F n^{-1}) = G_F N G_{F'} \subset G_x N G_y.$$

❒

Example 3.37. We return to our example of $G = \mathrm{SL}_3(k)$; see Examples 3.2, 3.25, and 3.31. Consider the point x in Figure 3.1. The group N_x is

$$\left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \cdot T(R)\right) \cup \left(\begin{bmatrix} 1 & & \\ & & 1 \\ & -1 & \end{bmatrix} \cdot T(R)\right)$$

so it is not surprising that

$$G_x = \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & R & R \end{bmatrix} = \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{bmatrix} \cdot N_x \cdot \begin{bmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{bmatrix}.$$

This is completely analogous to the fact that

$$G_x/G_x^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \end{bmatrix} = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \\ & & \mathsf{f} \end{bmatrix} \cdot \mathsf{N}_x \cdot \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & \mathsf{f} & \mathsf{f} \\ & & \mathsf{f} \end{bmatrix}.$$

All of this is due to the fact that x lies on $H_{\alpha+0}$ (resp. H_α), which borders a maximal facet (resp. Weyl chamber).

We will use the rest of this section to establish an important technical result used in the building, as well as present some very clever reasoning about structure theory. First we need some new notation.

Notation 3.38. The following notation will be used for the rest of this section.

Let $\Omega \subset \mathcal{A}$ be a nonempty set, and set $N_\Omega = N \cap G_\Omega$. If $D \subset \mathcal{A}_s$ is a Weyl chamber, we write the corresponding Borel subgroup B_D and unipotent radical U_D. If $-D$ is the opposite Weyl chamber, then the Borel subgroup B_{-D} is opposite to B_D (with respect to T). Set $U_{\Omega\pm D} = U_{\pm D} \cap G_\Omega$, or in the case that $\Omega = \{x\}$, write $U_{x\pm D} = U_{\{x\}\pm D}$.

Lemma 3.39 ([BT72]**).** *For any $x \in \mathcal{A}$ we have $N \cap (G_\Omega \cdot G_x) = N_\Omega \cdot N_x$.*

Proof.

We may assume that $x \notin \Omega$ (otherwise $G_\Omega \subset G_x$ and $N_\Omega \subset N_x$; cf. Proposition 3.15(i)). Fix $y \in \Omega$, and find a Weyl chamber D such that x is in $y + \overline{D}$. By Proposition 3.12(iii), we have $U_{\Omega+D} \subset U_{y+D} \subset U_{x+D}$; note that $U_{x-D}, U_{\Omega-D} \subset U_D^-$ and $U_{x+D} \subset U_D^+$. Proposition 3.15(ii) therefore gives

$$\begin{aligned} G_\Omega \cdot G_x &= N_\Omega U_{\Omega-D} U_{\Omega+D} U_{x+D} U_{x-D} N_x \\ &= N_\Omega U_{\Omega-D} U_{x+D} U_{x-D} N_x \\ &= N_\Omega U_{\Omega-D} U_{x-D} U_{x+D} N_x \\ &\subset N_\Omega U_D^- U_D^+ N_x. \end{aligned}$$

If $n \in N \cap (G_\Omega \cdot G_x)$, the above decomposition tells us that there are $n_\Omega \in N_\Omega, u_\pm \in U_D^\pm$, and $n_x \in N_x$ such that $n = n_\Omega u_- u_+ n_x$. By Corollary B.2, this is true if and only if $n_\Omega^{-1} n n_x^{-1} = u_- u_+ = 1$, because the left-hand side is in N and the right-hand side is in $U_D^- U_D^+$. Therefore $n = n_\Omega n_x \in N_\Omega \cdot N_x$, as required.

❒

The following result will imply a fundamental property of the building (Proposition 4.10). The beautiful proof is due to Bruhat and Tits.

Theorem 3.40 ([BT72, Prop. 7.4.8]**).** $\bigcap_{x\in\Omega}(G_x \cdot N) = G_\Omega \cdot N$ *and* $\bigcap_{x\in\Omega}(N \cdot G_x) = N \cdot G_\Omega$.

Proof.

The second statement follows from the first by taking inverses. First we will inductively prove the theorem for all finite subsets Ω of $\mathcal{A}$. The case $\Omega = \{y\}$ is trivial. Suppose that $\Omega = \Omega_0 \cup \{y\}$ where the theorem is true for the finite subset $\Omega_0 \subset \mathcal{A}$. Let

$$g \in (G_y \cdot N) \cap \bigcap_{x \in \Omega_0} (G_x \cdot N) = (G_y \cdot N) \cap (G_{\Omega_0} \cdot N)$$

so there exist $n_0, n_y \in N$ such that $gn_0 \in G_{\Omega_0}$ and $gn_y \in G_y$. Therefore $n_0^{-1} n_y \in N \cap (G_{\Omega_0} G_y)$, so by Corollary 3.39, there exist $n_0' \in N_{\Omega_0}$ and $n_y' \in N_y$ such that $n_0^{-1} n_y = n_0' (n_y')^{-1}$, i.e., $n_0 n_0' = n_y n_y'$. Let n be this common value, so we have $gn = gn_0 n_0' \in G_{\Omega_0} \cdot N_{\Omega_0} = G_{\Omega_0}$, and similarly, $gn \in G_y$. Therefore $g \in (G_{\Omega_0} \cap G_y) \cdot N = G_\Omega \cdot N$, so by induction, the theorem is true for all finite Ω.

Now we move on to the general case. Since $\mathcal{A}$ contains countably many facets, we may assume that $\Omega = \{x_0, x_1, \ldots\}$ is countable. Set $\Omega_k = \{x_0, x_1, \ldots, x_k\}$, so by the above, the theorem is true for each Ω_k. Therefore if $g \in (\bigcap_{x \in \Omega} G_x \cdot N)$, we can find an $n_m \in N$ for each m such that $gn_m \in G_{\Omega_m} \subset G_{x_0}$, i.e., each $n_m^{-1} n_0 \in N \cap G_{x_0} = N_{x_0}$. By Proposition 3.8, the natural homomorphism $N_{x_0}/T(R) \hookrightarrow N/T = W$ is injective, so the group $N_{x_0}/T(R)$ is finite. Therefore there must be an infinite set $\{m_0, m_1, m_2, \ldots\}$ for which there exist $t_i \in T(R)$ with $t_i = (n_{m_0}^{-1} n_0)(n_{m_i}^{-1} n_0)^{-1} = n_{m_0}^{-1} n_{m_i}$. Therefore $gn_{m_0} = (gn_{m_i}) t_i^{-1} \in G_{\Omega_{m_i}} \cdot T(R) = G_{\Omega_{m_i}}$, i.e.,

$$g \in \left(\bigcap_{i=0}^{\infty} G_{\Omega_{m_i}} \right) \cdot N = G_\Omega \cdot N$$

because $\bigcup_{i=0}^{\infty} \Omega_{m_i} = \Omega$.

❐

4. The Bruhat-Tits building

The weakness of the theory of the affine apartment and its parahoric subgroups is that it is tied to the choice of the torus T. We would like to understand the representation theory of G by induction from compact open subgroups, and we want to determine these subgroups *up to conjugacy*. From our perspective, then, the building is a way to view the affine structure of G without being restricted to using only one torus.

Remark 4.1. We are only concerned with the Bruhat-Tits building, which represents the affine structure of G; however, one can analogously define a *spherical* building of G which parameterizes much of the spherical structure of G, up to conjugacy. Indeed, for any point x in the affine building, the spherical building of G_x is isomorphic to a neighborhood of x in the affine building, in the same sense as in Section 3.2.

4.1. Definition and basic properties

We are now in a position to define the full building and prove some of its basic properties. We give the full example of $\mathrm{SL}_2(k)$ in Section 4.2. Almost all of this material is based on [BT72] and [BT84].

Definition 4.2. Put the following equivalence relation on $G \times \mathcal{A}$: $(g, x) \sim (h, y)$ if there exists an $n \in N$ such that $nx = y$ and $g^{-1}hn \in G_x$. Define the Bruhat-Tits building of G to be $\mathcal{B} = \mathcal{B}(G) = G \times \mathcal{A} / \sim$, and write $[g, x]$ for the equivalence class of (g, x). Define an action of G on $\mathcal{B}$ by $g \cdot [h, x] = [gh, x]$.

The relation $nG_xn^{-1} = G_{n \cdot x}$ for $n \in N$ implies that $\sim$ is indeed an equivalence relation, and it is immediate that the action is well-defined. Perhaps the most basic fact about the building is that the apartment remains intact. This fact is an immediate consequence of Theorem 3.28:

Proposition 4.3. *The map $x \mapsto [g, x]$ from $\mathcal{A} \to \mathcal{B}$ is an injection for any $g \in G$.*

In this section we will implicitly identify $\mathcal{A}$ with its image $1 \times \mathcal{A} \subset \mathcal{B}$. Note that the action of N on the building is compatible with its action on the apartment, i.e., if $n \in N$, then $n[1, x] = [1, nx]$. Note also that since $[g, x] = g[1, x]$, the orbit of the apartment $\mathcal{A}$ under G is all of $\mathcal{B}(G)$.

Notation 4.4. In this section, letters like $x, y, \ldots$ will denote both elements of $\mathcal{A}(G)$ and elements of $\mathcal{B}(G)$; we will always specify which. Note that if $x \in \mathcal{A}$ and $g \in G$, then $g \cdot x = [g, x]$ which is not necessarily in $\mathcal{A}$.

Our next task is to define G_x for any $x \in \mathcal{B}(G)$. In order to do this, we first need two lemmas.

Lemma 4.5. *For $x \in \mathcal{A}$, we have $G_x = \mathrm{stab}_G(x)$.*

Proof.

It is immediate from the definitions that G_x fixes x. If $g[1, x] = [g, x] = [1, x]$, then, by definition, there is some $n \in N$ such that $nx = x$ and $g^{-1}n \in G_x$, so $g \in \mathrm{stab}_N(x) \cdot G_x$, which is $N_x \cdot G_x = G_x$ by Corollary 3.30.

❐

Lemma 4.6. *If $g \in G$, $x \in \mathcal{A}$, and $gx \in \mathcal{A}$, then $gG_xg^{-1} = G_{gx}$.*

Proof.

By definition, we have an $n \in N$ such that $nx = gx$, i.e., $n^{-1}gx = x$, so by Lemma 4.5, $n^{-1}g \in G_x$, so $(n^{-1}g)G_x(n^{-1}g)^{-1} = G_x$, i.e.,

$$gG_xg^{-1} = nG_xn^{-1} = G_{nx} = G_{gx}.$$

❐

Now we can define G_x for any $x \in \mathcal{B}(G)$.

Definition 4.7. If $g \in G$ and $x \in \mathcal{A}$, then we define $G_{g \cdot x} := gG_xg^{-1}$ and $G^+_{g \cdot x} := gG^+_xg^{-1}$.

The above two lemmas imply that if $x \in \mathcal{A}$ and $gx = hx$, then $G_{gx} = G_{hx}$, and if $gx \in \mathcal{A}$, then the above definition of G_{gx} agrees with the one given previously. Note also that we have defined G_y for any $y \in \mathcal{B}(G)$.

Remark 4.8. Note that Lemma 4.5 holds true for any $x \in \mathcal{B}(G)$. Also, any $x \in \mathcal{B}(G)$ is conjugate to a $y \in \mathcal{A}$, so that $G_x/G^+_x \cong G_y/G^+_y = \mathsf{G}_y$; in particular, Theorem 3.17, Proposition 3.22, and Corollary 3.24 hold, in a sense. However, all of these isomorphisms are dependent on the element g such that $gx = y$; in other words, the isomorphism $G_x/G_x^+ \cong \mathsf{G}_y$ is only defined up to an inner (by Lemma 4.5) automorphism.

From the definitions it is easy to see that:

Lemma 4.9. *The element $[g, x] \in \mathcal{A}$ if and only if $g \in N \cdot G_x$.*

The following proposition is an important sanity check. It says that G acts on $\mathcal{A}$ by affine isometries, when this makes sense.

Proposition 4.10. *For all $g \in G$, there is an $n \in N$ such that $gx = nx$ for all $x \in \mathcal{A} \cap g^{-1}\mathcal{A}$.*

Proof.

Set $\Omega = \mathcal{A} \cap g^{-1}\mathcal{A}$. By Lemma 4.9, we have $g^{-1} \in \bigcap_{x\in\Omega}(G_x \cdot N)$, which is $G_\Omega \cdot N$ by Theorem 3.40. Thus there is an $n \in N$ such that $g^{-1}n \in G_x$ for every $x \in \Omega$. By definition, then, $gx = nx$ for every $x \in \Omega$.

❒

Corollary 4.11. *If $g\mathcal{A} \subset \mathcal{A}$ or $g\mathcal{A} \supset \mathcal{A}$, then $g\mathcal{A} = \mathcal{A}$.*

The above corollary allows us to define arbitrary apartments. Recall [Hum75, §21.3, Cor. A] that all maximal tori are conjugate in G.

Definition 4.12. If ${}^gT = gTg^{-1} \subset G$ is any maximal torus then, we define the ***apartment*** $\mathcal{A}({}^gT) = \mathcal{A}(G, {}^gT)$ associated with ${}^gT$ to be $g \cdot \mathcal{A}$.

Since any apartment is conjugate to the standard apartment $\mathcal{A}$, then, we may treat any apartment as the standard apartment. In order to make sense of the above definition, we need a proposition.

Proposition 4.13. *We have $g \cdot \mathcal{A} = h \cdot \mathcal{A}$ if and only if ${}^gT = {}^hT$. Equivalently, $\mathrm{stab}_G(\mathcal{A}) = N$.*

Proof.

We will prove the second (equivalent) statement. As remarked before, $N \subset \mathrm{stab}_G(\mathcal{A})$, so we need only show that if $g\mathcal{A} = \mathcal{A}$, then $g \in N = N_G(T(R))$. By Proposition 4.10, there exists an $n \in N$ such that $gx = nx$ for every $x \in \mathcal{A}$. Therefore gn^{-1} fixes $\mathcal{A}$, so by Lemma 4.5, gn^{-1} normalizes every G_x. Therefore gn^{-1} normalizes $\bigcap_{x\in\mathcal{A}} G_x = T(R)$, so $gn^{-1} \in N$ and $g \in N$.

❒

Thus we have more or less defined the apartment of ${}^gT$ to be the standard apartment of gGg^{-1}, and defined the parahorics corresponding to the apartment $g\cdot\mathcal{A}$ to be the conjugates of the standard parahorics G_x $(x \in \mathcal{A})$ under g. The building has allowed us to do this in such a way that it does matter which element g we choose to obtain ${}^gT$. One can define the apartment and associated parahorics for an arbitrary torus without the building, but it is difficult to do so in a consistent manner.

Note that if $[g, x] \in \mathcal{B}$ is any point, then $[g, x] \in g \cdot \mathcal{A} = \mathcal{A}({}^gT)$, so the union of all apartments is the whole building — in other words, the building is many apartments glued together. In fact, a stronger result is true:

Corollary 4.14 (to Bruhat Decomposition). *For any two elements of $\mathcal{B}(G)$, there is an apartment $\mathcal{A}'$ containing both of them.*

Proof.

Suppose that $x' = [g, x]$ and $y' = [h, y]$ are any two elements of $\mathcal{B}$. By Bruhat Decomposition (Theorem 3.36), we know that $g^{-1}h = p_x n p_y$ for $p_x \in G_x, p_y \in G_y$, and $n \in N$. Set $a = gp_x$, so $g^{-1}a = p_x \in G_x$ and $h^{-1}an = h^{-1}gp_x n = p_y \in G_y$. Thus $a^{-1}g \in N \cdot G_x$ and $a^{-1}h \in N \cdot G_y$, so by Lemma 4.9, $a^{-1}x', a^{-1}y' \in \mathcal{A}$ so $x', y' \in a \cdot \mathcal{A}$.

❒

Corollary 4.14 allows us to make the following important observation:

Proposition 4.15. *If $M \subset G$ is a generalized Levi subgroup, then $\mathcal{B}(M) \subset \mathcal{B}(G)$ canonically.*

Proof.

We know that the standard apartments $\mathcal{A}(M)$ and $\mathcal{A}(G)$ agree, which gives an inclusion $i : M \times \mathcal{A}(M) \hookrightarrow G \times \mathcal{A}(G)$.[11] We must show that this inclusion is compatible with the equivalence relations, i.e., if $g, h \in M$ and $x, y \in \mathcal{A}$, then $(g, x) \sim_M (h, y)$ in M if and only if $(g, x) \sim_G (h, y)$ in G. If $(g, x) \sim_M (h, y)$, then there is an $n \in N_M(T) \subset N_G(T)$ such that $nx = y$ and $g^{-1}hn \in M_x \subset G_x$; thus $(g, x) \sim_G (h, y)$. Therefore i descends to a well-defined map $\bar{i} : \mathcal{B}(M) \to \mathcal{B}(G)$. We claim that $\bar{i}$ is injective.

Let $x, y \in \mathcal{B}(M)$ be arbitrary, and suppose that $\bar{i}(x) = \bar{i}(y)$. By Corollary 4.14, we can choose an apartment $\mathcal{A}' \subset \mathcal{B}(M)$ containing x and y. Proposition 4.3 tells us that $\bar{i}|\mathcal{A}'$ is an injection, so we have $x = y$, as required.

❒

Having defined arbitrary apartments, we would also like to define arbitrary facets:

Definition 4.16. A subset $F \subset \mathcal{B}(G)$ is a ***facet*** if $F = g \cdot F'$ where $F' \subset \mathcal{A}$ is a facet and $g \in G$.

By Proposition 4.10, the facets contained in $\mathcal{A}$ are the same facets we defined before. It follows immediately from Proposition 3.32(i) that:

Proposition 4.17. *The action of G on the set of maximal facets in $\mathcal{B}(G)$ is transitive.*

We will also make use of the following fact:

Proposition 4.18. *If $x \in \mathcal{B}(G)$ is arbitrary, then there are only finitely many facets $F \subset \mathcal{B}(G)$ such that $x \in \overline{F}$. In other words, $\mathcal{B}(G)$ is locally finite.*

Proof.

By Proposition 4.15, we may assume that $G = G^c$. Let $\mathcal{F}(x)$ be the set of facets whose closure contains x. If y is a vertex contained in the closure of the facet containing x, then $\mathcal{F}(y) \supset \mathcal{F}(x)$, so it suffices to prove the proposition for vertices. Assume without loss of generality that x is a vertex in the standard apartment $\mathcal{A}$, and let $F \in \mathcal{F}(x)$. By definition, there is some facet $F' \subset \mathcal{A}$ and some $g \in G$ such that $gF' = F$. By Proposition 4.10, we can find an $n \in N$ such that $nF' \in \mathcal{F}(x)$, so as in the proof of

[11] Technically, this inclusion is non-canonical, but its image is canonical.

Proposition 3.32(ii), we may assume that g fixes x, i.e., $g \in G_x$. Therefore the orbit of F in $\mathcal{F}(x)$ corresponds to the quotient space $G_x/\operatorname{stab}_G(F)$, so it suffices to prove that the latter is finite, because $\mathcal{F}(x) \cap \mathcal{A}$ is finite. But $\operatorname{stab}_G(F) = G_F \subset G_x$, which is the inverse image in G_x of a parabolic subgroup P_F of G_x, so $G_x/G_F = \mathsf{G}_x/\mathsf{P}_F$, which is finite. In other words, the facets conjugate to F in $\mathcal{F}(x)$ are in one-to-one correspondence with the parabolic subgroups conjugate to P_F in G_x.

❒

The building comes equipped with a natural metric. In order to define this metric, we need a lemma:

Lemma 4.19. *Let $x, y \in \mathcal{B}(G)$, and let $\mathcal{A}_1$ be an apartment containing x and y. Any apartment $\mathcal{A}_2$ containing x and y also contains the line segment $\overline{xy} \subset \mathcal{A}_1$, and the length of this line in $\mathcal{A}_1$ (that is, the norm of the vector $x - y$) is the same as that in $\mathcal{A}_2$.*

Proof.

Assume without loss of generality that $\mathcal{A}_1 = \mathcal{A}$ is the standard apartment, and that $\mathcal{A}_2 = g \cdot \mathcal{A}$ for some $g \in G$. By Lemma 4.9, we have $g \in (N \cdot G_x) \cap (N \cdot G_y)$, which is $N \cdot (G_x \cap G_y)$ by Theorem 3.40. But $G_x \cap G_y = \bigcap_{z \in \overline{xy}} G_z$, so

$$g \in N \cdot \bigcap_{z \in \overline{xy}} G_z = \bigcap_{z \in \overline{xy}} (N \cdot G_z),$$

or equivalently, $\overline{xy} \subset \mathcal{A}_2$. The invariance of the length is immediate from Proposition 4.10.

❒

Lemma 4.19 allows us to make the following definition:

Definition 4.20. We define a metric d on $\mathcal{B}(G)$ as follows. Let $x, y \in \mathcal{B}(G)$, choose an apartment $\mathcal{A}'$ containing x and y, and let g be an element such that $g\mathcal{A}' = \mathcal{A}$. Define $d(x, y)$ to be the norm of the vector $gx - gy \in \mathcal{A}_s$, and give $\mathcal{B}(G)$ the metric topology.

By Proposition 4.10, $d(x, y)$ is well-defined and G-invariant. One can prove [BT72] that the triangle inequality is satisfied, so d indeed defines a metric. (This proof is harder and not vital to us, so we omit it.) Proposition 4.18 easily shows that:

Corollary 4.21. *For any positive $r \in \mathbf{R}$ and $x \in \mathcal{B}(G)$, the closed r-ball around x is compact, and therefore intersects only finitely many facets.*

Lastly, we prove a result that we will use later.

Proposition 4.22. *Suppose that $X \subset \mathcal{B}(G)$ is any minimal facet (i.e., $\pi_G(X)$ is one point). Then*

(i) If $y \in \mathcal{B}(G)$ and $G_y \cap G_X$ surjects onto G_X, then $y \in X$.

(ii) $N_G(G_X) = \operatorname{stab}_G(X) = \operatorname{stab}_N(X) \cdot G_X = C \cdot G_X$.

Proof.

Keep in mind that $G_y = G_z$ for any $y, z \in X$, and that $\operatorname{stab}_G(X)$ does not necessarily fix every $y \in X$.

(i) Suppose that $G_y \cap G_X$ surjects onto G_X. Choose an apartment $\mathcal{A}$ containing X and y. We claim that $y \in X$. If not, consider the line from any vertex x in X to y in $\mathcal{A}$. Since G_X is a maximal parahoric, we see that at any $z \notin X$ on this line, we have $G_X \cap G_z$ projecting onto a proper parabolic subgroup of G_X, so

$$G_y \cap G_X = \bigcap_{z \in \overline{xy}} G_z$$

will also project onto a proper parabolic of G_X, a contradiction. Therefore $y \in X$.

(ii) We will prove that

$$N_G(G_X) \subset \mathrm{stab}_G(X) \subset \mathrm{stab}_N(X) \cdot G_X \subset C \cdot G_X \subset N_G(G_X).$$

($N_G(G_X) \subset \mathrm{stab}_G(X)$): If $g \in N_G(G_X)$, then $G_X = gG_Xg^{-1} = G_{gX}$, so $G_X \cap G_{gX}$ surjects onto G_X, whence $gX = X$ and $g \in \mathrm{stab}_G(X)$.

($\mathrm{stab}_G(X) \subset \mathrm{stab}_N(X) \cdot G_X$): If $g \in \mathrm{stab}_G(X)$, then Proposition 4.10 tells us that there is an $n \in N$ such that $gy = ny$ for all $y \in X$, so $n \in \mathrm{stab}_N(X)$. But $g^{-1}n$ fixes every $y \in X$, so $g^{-1}n \in G_X$, i.e., $g \in \mathrm{stab}_N(X) \cdot G_X$.

($\mathrm{stab}_N(X) \cdot G_X \subset C \cdot G_X$): Let $n \in \mathrm{stab}_N(X)$, and suppose $\pi_G(X) = \{x_0\}$. Since n stabilizes $X = \{x_0\} \times (E_\Phi^*)^\perp$, n must act as a translation in $(E_\Phi^*)^\perp$. By definition, then, there is a $\lambda \in \Lambda_C$ such that n is translation by $c := \check{\lambda}(\varpi) \in C$. Thus $n^{-1}c$ fixes every point in X, i.e., $n^{-1}c \in G_X$. Therefore $n \in C \cdot G_X$.

($C \cdot G_X \subset N_G(G_X)$): Clear.

❐

4.2. Example: the building of $\mathrm{SL}_2(k)$

As an illustration, we analyze the full building of $G = \mathrm{SL}_2(k)$. It turns out that $\mathcal{B}$ is the infinite tree[12] whose vertices all have order $q + 1 = |\mathfrak{f}| + 1$ — that is, $\mathcal{B}$ is the homogeneous tree of degree $q+1$. The apartments are all of the lines (i.e., infinite reduced paths) in this graph. See Figure 4.1. We will use the notation in Example 2.28.

We start by defining the graph structure of $\mathcal{B}$.

Definition 4.23. Let the set of vertices be the 2-facets (i.e., the special vertices) and let the edges be the 0-facets, in the sense of Definitions 4.16 and 2.31. (Note that "vertices" and "special vertices" are the same thing.) If F is an edge, then the endpoints of F are $\overline{F} \setminus F$.

By Lemma 4.19, the graph structure is well-defined. Corollary 4.14 tells us that $\mathcal{B}$ is a connected graph.

Definition 4.24. If $x, y \in \mathcal{B}$ are vertices, then a ***(reduced) path*** from x to y is a finite sequence $F_1, \ldots, F_n$ of edges such that (i) $x \in \overline{F}_i \iff i = 1$ and $y \in \overline{F}_i \iff i = n$, (ii) each $\overline{F}_i \cap \overline{F}_{i+1}$ consists of only one vertex, and (iii) $\overline{F}_i \cap \overline{F}_j = \emptyset$ if $i \neq j \pm 1$. In other words, the path does not cross itself or double back. A ***line*** in $\mathcal{B}$ is an infinite sequence of edges $\ldots, F_{-1}, F_0, F_1, \ldots$ satisfying (ii) and (iii) above; i.e., a line is an infinite path.

[12]In fact, all Bruhat-Tits buildings are contractible.

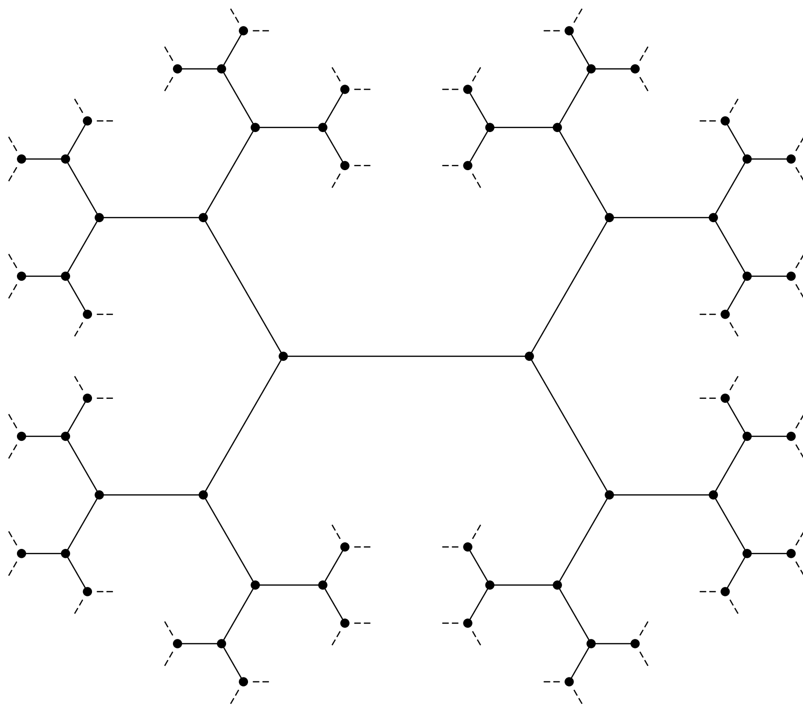

Figure 4.1: The building of $\mathrm{SL}_2(\mathbf{Q}_2)$.

Note that if x and y are contained in an apartment $\mathcal{A}$, then there is exactly one path in $\mathcal{A}$ connecting x and y (namely, the interval $[x, y] \subset \mathcal{A} \cong \mathbf{R}$). Our next goal is to show that $\mathcal{B}$ is a tree. We know from Lemma 4.19 that if x and y are vertices both contained in two apartments $\mathcal{A}$ and $\mathcal{A}'$, then the line segment $\overline{xy}$ in $\mathcal{A}$ is also contained in $\mathcal{A}'$, so that there is only one path contained in an apartment that connects x and y. It therefore suffices to show that any path between x and y is contained in some apartment.

Lemma 4.25. *If $\ldots, F_{-1}, F_0, F_1, \ldots$ is a line in $\mathcal{B}$, then there is an apartment $\mathcal{A}$ containing each F_i (so $\mathcal{A} = \bigcup_{i \in \mathbf{Z}} \overline{F}_i$). In particular, if $x, y \in \mathcal{B}$ are vertices and $F_1, \ldots, F_n$ is a path connecting x and y, then there is some apartment containing x, y, and each F_i.*

Proof.

We prove the second statement first. We will proceed by induction on n. The base case is clear. Assume that $F_1, \ldots, F_{n-1}$ are contained in the standard apartment $\mathcal{A}$, and assume for simplicity that $\overline{F}_{n-1} \cap \overline{F}_n = \{0\} \subset \mathcal{A}$, and that $\alpha(y) < 0$ for every $y \in F_i$ with $i < n$. Let F be the edge in $\mathcal{A}$ adjoining 0 that is not F_{n-1}. See Figure 4.2. It suffices to find a $g \in G$ such that $g \in G_{F_i}$ (so that $gF_i = F_i$) for every $i < n$, and such that $gF = F_n$, because then our path is contained in $g\mathcal{A}$.

Let $\Omega = \{x \in \mathcal{A} \;:\; \alpha(x) \leq 0\}$, so that $G_\Omega \subset G_{F_i}$ for $i < n$. (We will find $g \in G_\Omega$.) We see that

$$G_\Omega = \bigcap_{m \geq 0} \begin{bmatrix} R & \wp^m \\ \wp^{-m} & R \end{bmatrix} = \begin{bmatrix} R^\times & 0 \\ R & R^\times \end{bmatrix}.$$

From Proposition 4.17, we can find an $h \in G$ such that $hF = F_n$, and Proposition 4.10

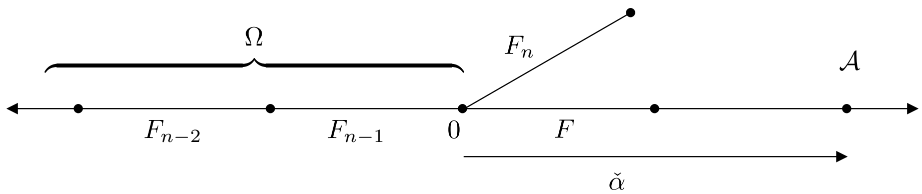

Figure 4.2: The setup in Lemma 4.25.

tells us that $h0 = 0$ because there is no element of n that sends $H_{\alpha-1}$ to $0 = H_{\alpha+0}$ (cf. Example 2.28). Lemma 4.5 tells us that $h \in G_0$, so we can write $h = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right]$. We will find a $g \in G_\Omega$ such that $h^{-1}g \in G_F$, i.e., $gF = hF = F_n$.

Note that $G_F = \left[\begin{smallmatrix} R & R \\ \wp & R \end{smallmatrix}\right]$. If $a \in \wp$, then $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] h = \left[\begin{smallmatrix} c & d \\ -a & -b \end{smallmatrix}\right] \in G_F$, so $h \in N \cdot G_F$ and $hF \subset \mathcal{A}$, so $hF = F$ or $hF = F_{n-1}$ (since h fixes 0). In the first case, we are done, and we are assuming the second case cannot happen. Therefore we may assume that $a \notin \wp$. Let y be an element of R such that $ya = c \bmod \wp$, and let $g = \left[\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right] \in G_\Omega$. Then we have

$$h^{-1}g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} d - by & -b \\ -c + ay & a \end{bmatrix} \in G_F$$

so we are done.

Now for the first statement. Let $\ldots, F_{-1}, F_0, F_1, \ldots$ be any path, and number the maximal facets in the standard apartment $\mathcal{A}$ in order as $\ldots, F'_{-1}, F'_0, F'_1, \ldots$. Let

$$\Omega_n = \bigcup_{-n \le i \le n} \overline{F}_i \qquad \text{and} \qquad \Omega'_n = \bigcup_{-n \le i \le n} \overline{F}'_i.$$

By the above, we know that for each n we can find a h_n with $h_n F'_i = F_i$ for each $-n \le i \le n$ (we may have to renumber the F'_i in order to do this, because not all vertices are conjugate). The set of $h \in G$ with this property is thus $K_n := h_n G_{\Omega'_n}$. We have that $K_n \subset K_m$ when $n \ge m$, so $K_0 \supset K_1 \supset K_2 \supset \cdots$ is an infinite decreasing sequence of nonempty closed sets contained in K_0. If the intersection $K = \bigcap_{n=0}^\infty K_n$ were empty, then $\{K_0 \setminus K_n\}_{n=1}^\infty$ would form an open cover of K_0. Since $K_0 = h_0 G_{F'_0}$ is compact, this cover would have a finite subcover, which would imply that some $K_n = \emptyset$, a contradiction. Therefore $K \neq \emptyset$, so if $h \in K$, then $hF'_i = F_i$ for all $i \in \mathbf{Z}$. Therefore our line is contained in $h \cdot \mathcal{A}$.

❐

Now we know that $\mathcal{B}$ is a tree whose lines are the apartments. It remains to find the order of the vertices.

Lemma 4.26. *The order of any vertex in $\mathcal{B}$ is $|\mathfrak{f}| + 1 = q + 1$.*

Proof.

Let $x \in \mathcal{B}$ be a vertex, and assume for simplicity that x is the origin in the standard apartment $\mathcal{A}$. Let F be the edge adjoining x such that $\alpha(y) > 0$ for $y \in F$, as in the proof of Lemma 4.25. Note that $G_x = \left[\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right]$ and $G_F = \left[\begin{smallmatrix} R & R \\ \wp & R \end{smallmatrix}\right] \subset G_x$. The edges adjoining x correspond to the cosets G_x/G_F, as follows: if gF adjoins x, then as in the proof of Lemma 4.25, $gx = x$, so $g \in G_x$; also, $gF = hF$ iff $g^{-1}hF = F$ iff $g^{-1}h \in G_F$.

Therefore it suffices to find coset representatives of G_x/G_F.[13]

We claim that $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right]$ for $t \in R/\wp$ are such coset representatives. Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_x$ be arbitrary. If $a \in \wp$, then $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} c & d \\ -a & -b \end{smallmatrix}\right] \in G_F$. If $a \notin \wp$, then let $t \in R$ be an element such that $ta = c \bmod \wp$, so that $\left[\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ c-ta & d-tb \end{smallmatrix}\right] \in G_F$. An easy calculation shows that none of our coset representatives is equivalent mod G_F, so we are done.

❒

In summary, we have:

Theorem 4.27. *The building $\mathcal{B}(\mathrm{SL}_2(k))$ is the infinite tree whose vertices all have order $q + 1 = |\mathsf{f}| + 1$. The apartments are the lines in this graph, which are therefore in one-to-one correspondence with the conjugates of T.*

5. An application of the building to representation theory

The Bruhat-Tits building offers us a more detailed understanding of the structure theory of our groups, and gives us a convenient way to visualize that structure. This in turn enables us to think in new ways about anything relating to our groups; in particular, about p-adic representation theory. Moy and Prasad found one of the first important applications of the theory of the building to the representation theory of p-adic Chevalley groups. They defined a notion of the depth of an irreducible admissible complex representation of G, and were able to classify all depth-zero irreducible super-cuspidal representations of G via Bruhat-Tits theory. In this chapter, we present this classification, along with the full example of SL_2 in Section 5.3.

For an alternate approach, see Morris [Mor99], who has proved the same result using Hecke algebras.

5.1. The depth of a representation

We have been stating that our goal is to find super-cuspidal representations of G by inducing from compact open subgroups. Moy and Prasad's theory of unrefined minimal K-types is one attempt at determining from which representations we should be inducing. In summary, the depth of a representation of G is how "deep" we must look into a filtration of the subgroups G_x before we find nontrivial fixed vectors. Most of the material presented in this section is for context; we will only be interested in depth-zero representations.

First we must define the Moy-Prasad filtration of the parahoric subgroups.

Definition 5.1. In the notation of Definition 3.1, for any nonnegative real number r and any $x \in \mathcal{A}$, we define

$$G_{x,r} = \langle T(1 + \wp^{\lceil r \rceil}),\ \mathfrak{X}_\alpha(\wp^{-\lfloor \alpha(x) - r \rfloor})\ :\ \alpha \in \Phi \rangle \subset G_x.$$

$$G_{x,r^+} = \langle T(1 + \wp^{\lfloor r \rfloor + 1}),\ \mathfrak{X}_\alpha(\wp^{1 - \lceil \alpha(x) - r \rceil})\ :\ \alpha \in \Phi \rangle \subset G_{x,r}.$$

[13] Note that $\mathsf{G}_x/\mathsf{G}_F \cong \mathsf{G}/\mathsf{B}$, where B is the image of G_F in G, and is thus a Borel subgroup. Thus Lemma 4.26 really follows from the fact that G/B is isomorphic as an algebraic variety to the projective space $\mathbf{P}^1(\mathsf{f})$. (In general, a parabolic subgroup of an algebraic group G of the type that we are considering is a subgroup P such that the quotient G/P is projective.)

Example 5.2. Consider $G = \mathrm{SL}_2(k)$, with root space $\mathfrak{X}_\alpha = \left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]$ (cf. Example 2.28). Some of the filtration subgroups $G_{x,r}$ are given in Figure 5.1. Note that in general, the group $G_{x,r}$ does not depend only on the facet containing x; rather, the group changes when one crosses a line at which $r \pm \alpha(x)$ is an integer. Also note that $G_{x,r^+} = G_{x,r+\epsilon}$, where ϵ is very small. As for the groups $G_{x,r}$ with (x,r) on the lines: at the points $(x,r), (y,s)$, and (z,t) in the Figure, we have

$$G_{x,r} = \begin{bmatrix} 1+\wp & \wp^2 \\ R & 1+\wp \end{bmatrix} \qquad G_{y,s} = \begin{bmatrix} 1+\wp & \wp \\ \wp^2 & 1+\wp \end{bmatrix} \qquad G_{z,t} = \begin{bmatrix} 1+\wp^2 & \wp^2 \\ \wp^2 & 1+\wp^2 \end{bmatrix}.$$

Note that if (x,r) is on a line, then $G_{x,r} = G_{x,r-\epsilon}$, where ϵ is very small.

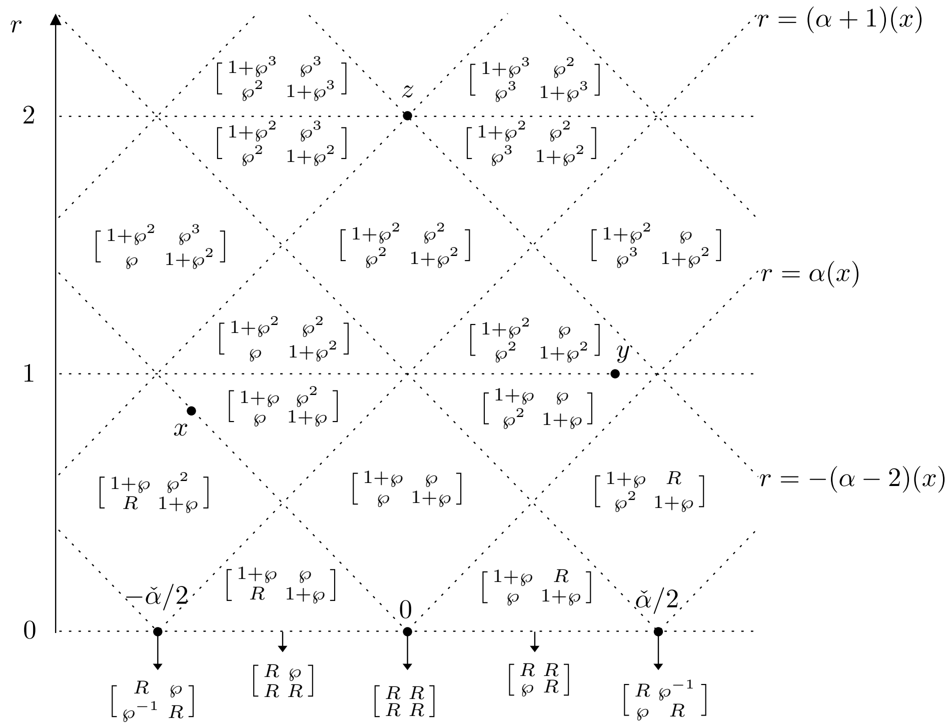

Figure 5.1: Some Moy-Prasad filtration subgroups for $\mathrm{SL}_2(k)$. The affine apartment for $\mathrm{SL}_2(k)$ is the bottom line in the figure.

Remark 5.3. There are several things to note about Definition 5.1:

- $G_{x,r} \subset G_{x,s}$ when $r > s$.
- $G_{x,0} = G_x$ and $G_{x,0^+} = G_x^+$.
- For any $g \in G$, one can put $G_{g\cdot x,r} = gG_{x,r}g^{-1}$ and $G_{g\cdot x,r^+} = gG_{x,r^+}g^{-1}$, thus defining these objects for all $x \in \mathcal{B}(G)$.
- One can show that G_{x,r^+} is always normal in $G_{x,r}$, and that the quotient is always finite, and is abelian if $r > 0$.
- For any $x \in \mathcal{B}$ and $r \geq 0$, there is an $\epsilon \geq 0$ such that $G_{x,r^+} = G_{x,r+\epsilon}$.

- It is relatively easy to see that for any $x \in \mathcal{B}(G)$, the $G_{x,r}$ form a neighborhood basis of the identity in G consisting of compact open subgroups.
- The obvious analog of Proposition 3.10(ii) holds for any $G_{x,r}$; Propositions 3.10(i) and 3.12(i) hold upon replacing G_x^+ with $G_{x,r}$ if $r > 0$.

The subgroups $G_{x,r}$ are called the ***Moy-Prasad filtration subgroups***. The minimal r such that a representation has nontrivial G_{x,r^+}-fixed vectors turns out to be an important property of the representation:

Theorem 5.4 ([MP94, Theorem 5.2]**).** *If (π, V) is any smooth representation of G, then there is a nonnegative rational number $r = \varrho(\pi)$ with the property that r is the minimal number such that $V^{G_{x,r^+}}$ is nonzero for some $x \in \mathcal{B}(G)$.*

With the above theorem in mind, we can define:

Definition 5.5. The ***depth*** of an admissible irreducible representation π of G is the number $r = \varrho(\pi)$.

Therefore, a depth-zero irreducible super-cuspidal representation is an irreducible super-cuspidal representation (π, V) of G such that $V^{G_x^+} \neq \{0\}$ for some $x \in \mathcal{B}(G)$ (see Definition 1.1).

Definition 5.6. A ***depth-zero minimal*** K***-type*** is a pair (G_x, τ), where $x \in \mathcal{B}(G)$ and τ is a cuspidal representation (cf. Definition 1.2) of the finite group G_x, inflated to G_x.

It turns out that any depth-zero irreducible super-cuspidal representation (π, V) with G_x^+-fixed vectors contains a depth-zero minimal K-type on restriction to $V^{G_x^+}$. This a somewhat surprising relation between the notion of a super-cuspidal representation of G and a cuspidal representation of G. The relation goes the other way too: we will show that all depth-zero irreducible super-cuspidal representations (π, V) of G can be obtained by inducing a depth-zero minimal K-type from a vertex x.

5.2. The main result

We will classify all depth-zero irreducible super-cuspidal representations (π, V) of G in two steps. First we will show that if $V^{G_x^+}$ is nonzero, then x is a vertex, i.e., G_x is a maximal parahoric. Second, we will show that if (G_x, τ) is a depth-zero minimal K-type and x is a vertex, then the irreducible super-cuspidal representations which contain τ on restriction to G_x (which are thus necessarily depth-zero since the space of G_x^+-fixed vectors is nonzero) are exactly those representations obtained by induction from an irreducible representation of $N_G(G_x)$ that contains τ on restriction to G_x.

The first step is as follows. Let G_x be a *nonmaximal* parahoric subgroup of G (i.e., x is not a vertex), and choose an apartment $\mathcal{A} = \mathcal{A}(G, T)$ containing x. Let $M(x)$ be the generalized Levi subgroup associated with x and T, and let $M \subset G$ be a proper Levi subgroup containing $M(x)$ (we can find one of these by Lemma B.5). Let P be a (proper) parabolic subgroup with Levi decomposition $P = MU$, and assume that P is standard; let $\overline{P} = M\overline{U}$ be the opposite parabolic with respect to M.

During the proof of the following theorem, we will keep this specific example in mind:

suppose $G = \mathrm{SL}_3(k)$, with the notation as in Example 2.14. Suppose that x is as in Figure 5.2, so that $M = M(x) \cong \mathrm{GL}_2(k)$, as in Example 2.29. Thus we have

$$M = \begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \quad P = \begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix} \quad U = \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} \quad \overline{P} = \begin{bmatrix} * & * & \\ * & * & \\ * & * & * \end{bmatrix} \quad \overline{U} = \begin{bmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{bmatrix}.$$

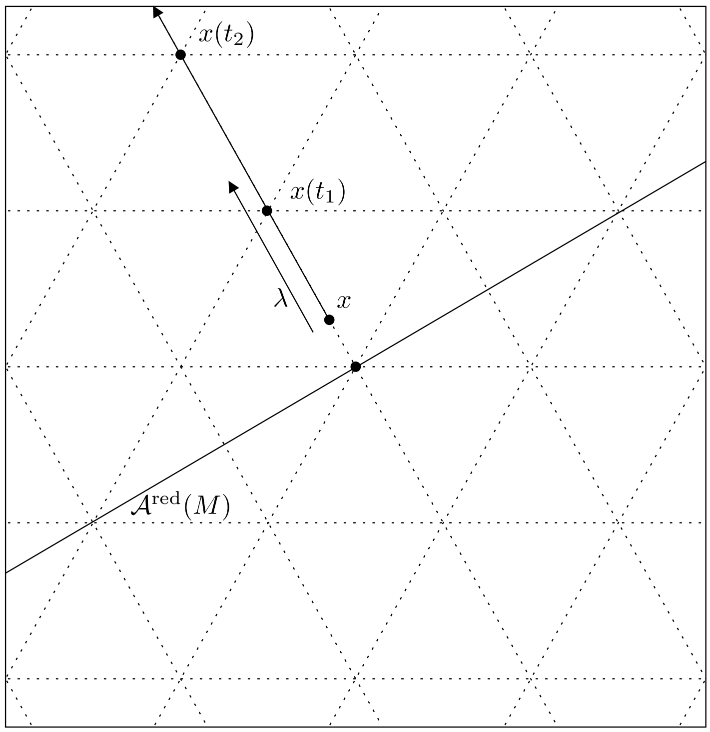

Figure 5.2: The setup in the proof of Theorem 5.7.

Theorem 5.7 ([MP96, Prop. 6.7]**).** *Let (π, V) be an irreducible admissible representation of G. Then the natural projection map*

$$J : V^{G_x^+} \to V_U^{M_x^+}$$

onto the Jacquet module is an isomorphism.

Proof.

Since G_x^+ has an Iwahori factorization with respect to $P = MU$ and since $M_x^+ = M \cap G_x^+$, Jacquet's Lemma [Cas, Theorem 3.3.3] tells us that J is a well-defined surjective map.

If $v \in V^{G_x^+}$ is nonzero, then the goal is to prove that $v \notin V(U)$. By [BZ76, Lemma 2.33], $v \in V(U)$ if and only if $\int_K \pi(x)v\, dx = 0$ for some compact open subgroup $K \subset U$. We observe that if $K_1 \subset K_2$ are both compact open subgroups of U and $\int_{K_1} \pi(x)v\, dx = 0$ then

$$\int_{K_2} \pi(x)v\, dx = \sum_{\overline{g} \in K_2/K_1} \int_{gK_1} \pi(x)v\, dx = \sum_{\overline{g} \in K_2/K_1} \pi(g) \int_{K_1} \pi(x)v\, dx = 0$$

where the (finite) sum runs over some set of coset representatives for K_2/K_1. Therefore if $K_1 \subset K_2 \subset \cdots$ are compact open subgroups of U such that $\bigcup_n K_n = U$, it suffices to show that $\int_{K_n} \pi(x)v\,dx \neq 0$ for all n.

We will find such K_n using the building. Intuitively, what we will do is the following: if we move away from x on a path $x(t)$ in a direction orthogonal to the reduced apartment of M, then the U part of $G_{x(t)}$ will grow larger — that is, the compact open subgroup $U \cap G_{x(t)}$ will eventually contain every element of U. Then we will only have to show that

$$\int_{U\cap G_{x(t)}} \pi(x)v\,dx \neq 0 \tag{5.1}$$

for all t. We will prove that the integral (5.1) is never zero by contradiction: we will suppose that there were some smallest t for which it is zero, then we will use a fact about the representation theory of finite groups of Lie type to show that, in fact, (5.1) is nonzero for slightly larger t.

Our first task is to define the path $x(t)$. In our example of $\mathrm{SL}_3(k)$, choose a $\lambda \in \mathcal{A}(\mathrm{SL}_3(k))$ orthogonal to $\mathcal{A}^{\mathrm{red}}(M)$, such that $\beta(\lambda) > 0$, as indicated in Figure 5.2. Let $x(t) = x + t\lambda$. Since $\lambda \perp \alpha$, the only generators of $G_{x(t)}$ that will change with t are those corresponding to the roots $\pm\beta, \pm(\alpha+\beta)$, so each $G_{x(t)} \cap M = M_x$. At the point x, we have

$$G_x = \begin{bmatrix} R & R & R \\ R & R & R \\ \wp & \wp & R \end{bmatrix} \qquad M_x = \begin{bmatrix} R & R & \\ R & R & \\ & & R \end{bmatrix}.$$

At times t_1 and t_2 in the figure, we have

$$G_{x(t_1)} = \begin{bmatrix} R & R & \wp^{-1} \\ R & R & \wp^{-1} \\ \wp & \wp & R \end{bmatrix} \qquad G_{x(t_2)} = \begin{bmatrix} R & R & \wp^{-2} \\ R & R & \wp^{-2} \\ \wp^2 & \wp^2 & R \end{bmatrix}$$

so that

$$G_{x(t_1)} \cap U = \begin{bmatrix} 1 & & \wp^{-1} \\ & 1 & \wp^{-1} \\ & & 1 \end{bmatrix} \qquad G_{x(t_2)} \cap U = \begin{bmatrix} 1 & & \wp^{-2} \\ & 1 & \wp^{-2} \\ & & 1 \end{bmatrix}.$$

In general, we will have

$$G_{x(t)} \cap U = \begin{bmatrix} 1 & & \wp^{-m} \\ & 1 & \wp^{-m} \\ & & 1 \end{bmatrix}$$

with $m \to \infty$ as $t \to \infty$, so that $\bigcup_{t\to\infty} G_{x(t)} \cap U = U$.

It is not hard to find such a path $x(t)$ in the general case. Let Φ_M be the root system of M; we know that $\check{\Phi}_M$ does not span $\mathcal{A}_s^{\mathrm{red}}$ by Lemma B.5. Fix a $\lambda \in \mathcal{A}_s^{\mathrm{red}}$ such that $\langle \alpha, \lambda \rangle = 0$ for every $\alpha \in \Phi_M$ — i.e., λ is orthogonal to $\mathcal{A}_s^{\mathrm{red}}(M) \subsetneq \mathcal{A}_s^{\mathrm{red}}$ — and such that $\langle \alpha, \lambda \rangle > 0$ for every $\alpha \in \Phi^+ \setminus \Phi_M$. Note that $\Phi^+ \setminus \Phi_M$ consist of the roots whose root spaces generate U. Consider the ray $x(t) = x + t\lambda$ for $t \geq 0$, starting at x and traveling in the λ direction in $\mathcal{A}$. It is important to note that for any nonnegative $t \in \mathbf{R}$, the set $\Phi_{x(t)}$ will contain Φ_x because $\langle \alpha, x(t) \rangle$ does not change for $\alpha \in \Phi_x$. Since $\langle \alpha, x(t) \rangle$ changes linearly with t for $\alpha \notin \Phi_x$, we will have $\Phi_{x(t)} = \Phi_x$ for all t except for a discrete set $t = t_1, t_2, \ldots$ with $t_i \to \infty$. These are the t at which $x(t)$ crosses a hyperplane. Confer Figure 5.2.

Set $U_{x(t)} = U \cap G_{x(t)}$ and $\overline{U}_{x(t)} = \overline{U} \cap G_{x(t)}$. Proposition 3.10(ii) tells us that if $s \geq t$, then

$$U_{x(s)} \supset U_{x(t)} \qquad \text{and} \qquad \overline{U}_{x(s)} \subset \overline{U}_{x(t)}.$$

Since every element of U has a unique expression of the form (3.2), it is easy to see that, as in the case of $\mathrm{SL}_3(k)$, U is the union of the nested compact open subgroups

$U_{x(t)}$. Note that our Haar measure on U can thus be taken to be bi-invariant (cf. [BZ76, Prop. 1.19(b)]).

Checking root systems and using Proposition 3.10(ii), we find that G_x^+ has an Iwahori decomposition (Proposition 3.12(i)) into

$$G_x^+ = U_{x(0)} M_x^+ \overline{U}_{x(0)}.$$

Since v is fixed under G_x^+, then, we have $\int_{U_{x(0)}} \pi(x)v\,dx \neq 0$. Now suppose that there were some $t > 0$ such that

$$\int_{U_{x(t)}} \pi(x)v\,dx = 0. \tag{5.2}$$

One can check that $U_{x(t)}$ does not change on the regions $0 \leq t < t_1$ and $t_i \leq t < t_{i+1}$ for $i \geq 1$. Therefore there is a minimal t — call it s — such that (5.2) holds, and this s is one of the distinguished times t_i.

Let $y = x(s)$, $U_y = U_{x(s)}$, and $\overline{U}_y = \overline{U}_{x(s)}$. Let $G_{y^-} = G_{x(t)}$ for t slightly less than s (so that the facet containing $x(t)$ contains y in its closure), let $G_{y^+} = G_{x(t)}$ for t slightly greater than s, and define $U_{y^-}, U_{y^+}, \overline{U}_{y^-}$, and $\overline{U}_{y^+}$ similarly. In our example, if $s = t_1$, then

$$G_y = \begin{bmatrix} R & R & \wp^{-1} \\ R & R & \wp^{-1} \\ \wp & \wp & R \end{bmatrix} \qquad G_y^+ = \begin{bmatrix} 1+\wp & \wp & R \\ \wp & 1+\wp & R \\ \wp^2 & \wp^2 & 1+\wp \end{bmatrix}$$

$$G_{y^-} = \begin{bmatrix} R & R & R \\ R & R & R \\ \wp & \wp & R \end{bmatrix} \qquad G_{y^-}^+ = \begin{bmatrix} 1+\wp & \wp & R \\ \wp & 1+\wp & R \\ \wp & \wp & 1+\wp \end{bmatrix}$$

$$G_{y^+} = \begin{bmatrix} R & R & \wp^{-1} \\ R & R & \wp^{-1} \\ \wp^2 & \wp^2 & R \end{bmatrix} \qquad G_{y^+}^+ = \begin{bmatrix} 1+\wp & \wp & \wp^{-1} \\ \wp & 1+\wp & \wp^{-1} \\ \wp^2 & \wp^2 & 1+\wp \end{bmatrix}.$$

As the example makes clear, we see that as before, we have Iwahori decompositions

$$G_{y^-}^+ = U_{y^-} M_x^+ \overline{U}_{y^-} \qquad \text{and} \qquad G_{y^+}^+ = U_{y^+} M_x^+ \overline{U}_{y^+}.$$

Also, by Proposition 3.22, the image of G_{y^-} in $\mathsf{G}_y = G_y/G_y^+$ is a parabolic subgroup $\overline{\mathsf{P}}_y$, and the image of G_{y^+} in G_y is the opposite parabolic P_y. In addition, $G_{y^-}^+$ is the inverse image in G_y of the unipotent radical $\overline{\mathsf{U}}_y$ of $\overline{\mathsf{P}}_y$, and $G_{y^+}^+$ is the inverse image on G_y of the inipotent radical U_y of P_y. In our example,

$$\overline{\mathsf{P}}_y = G_{y^-}/G_y^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \\ \mathsf{f} & \mathsf{f} & \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \end{bmatrix} \qquad \overline{\mathsf{U}}_y = G_{y^-}^+/G_y^+ = \begin{bmatrix} 1 & & \\ & 1 & \\ \mathsf{f} & \mathsf{f} & 1 \end{bmatrix}$$

$$\mathsf{P}_y = G_{y^+}/G_y^+ = \begin{bmatrix} \mathsf{f} & \mathsf{f} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \\ & & \mathsf{f} \end{bmatrix} \qquad \mathsf{U}_y = G_{y^+}^+/G_y^+ = \begin{bmatrix} 1 & & \mathsf{f} \\ & 1 & \mathsf{f} \\ & & 1 \end{bmatrix}.$$

Set

$$w := \int_{G_{y^-}^+} \pi(x)v\,dx.$$

Since $\overline{U}_{y^-} \subset \overline{U}_{x(0)} \subset G_x^+$, we have that v is fixed by $M_x^+ \overline{U}_{y^-}$, so

$$w = \int_{U_{y^-} M_x^+ \overline{U}_{y^-}} \pi(x)v\,dx = \operatorname{meas}(M_x^+ \overline{U}_{y^-}) \int_{U_{y^-}} \pi(x)v\,dx \neq 0$$

since s is the minimal t for which the integral (5.2) is zero. Let (τ, W) be the natural (finite-dimensional) representation of G_y obtained by restricting π to $V^{G_y^+}$ and passing to the quotient $\mathsf{G}_y = G_y/G_y^+$. Since w is fixed by $G_{y^-}^+$, we have that $w \in W^{\overline{\mathsf{U}}_y}$. There is a result about the representation theory of finite groups of Lie type, proved in [HL94] (but cf. [MP96, Prop. 6.1(2)]), which says that integration of $\tau(\mathsf{x})v$ over U_y defines an isomorphism of $W^{\overline{\mathsf{U}}_y}$ with W^{U_y}. Therefore, since $w \neq 0$, we have

$$
\begin{aligned}
0 &\neq \int_{\mathsf{U}_y} \tau(\mathsf{x})w\, d\mathsf{x} \\
&= (\text{const}) \cdot \int_{G_{y^+}^+} \pi(x)w\, dx && \text{(since } G_{y^+}^+ \text{ projects onto } \mathsf{U}_y\text{)} \\
&= (\text{const}) \cdot \int_{U_{y^+} M_x^+ \overline{U}_{y^+}} \pi(x)w\, dx \\
&= (\text{const}) \cdot \int_{U_{y^+}} \pi(x)w\, dx && \text{(since } G_{y^-}^+ \supset M_x^+\overline{U}_{y^+} \text{ fixes } w\text{)} \\
&= (\text{const}) \cdot \int_{U_{y^+}} \int_{U_{y^-}} \pi(xy)v\, dy\, dx \\
&= (\text{const}) \cdot \int_{U_{y^-}} \int_{U_{y^+}} \pi(xy)v\, dx\, dy && \text{(by Fubini's theorem)} \\
&= (\text{const}) \cdot \int_{U_{y^+}} \pi(x)v\, dx && \text{(since } U_{y^-} \subset U_{y^+} \text{ and our Haar measure is bi-invariant).}
\end{aligned}
$$

This contradicts our assumption that $\int_{U_{x(s)}} \pi(x)v\, dx = 0$. Therefore

$$\int_{U_{x(t)}} \pi(x)v\, dx \neq 0$$

for all $t \geq 0$, so $v \notin V(U)$ and the theorem is proved.

❒

Corollary 5.8. *Let (π, V) be a depth-zero irreducible super-cuspidal representation of G.*

(i) $V^{G_x^+}$ can only be non-zero when x is a vertex. In other words, the depth-zero minimal K-types contained in V come from maximal parahorics.

(ii) Any irreducible subrepresentation of the natural finite-dimensional representation (τ, W) of G_x obtained by restricting π to $V^{G_x^+}$ is cuspidal.

Proof.

(i) First note that since (π, V) is irreducible and smooth then by [Jac75], (π, V) is admissible. Suppose $x \in \mathcal{B}(G)$ with G_x nonmaximal. If $V^{G_x^+} \neq \{0\}$, then by Theorem 5.7, $V_U^{M_x^+} \neq \{0\}$ for a nonmaximal parabolic P with Levi decomposition MU. But $V_U = \{0\}$ when V is super-cuspidal, so this cannot happen; therefore G_x is a maximal parahoric.

(ii) Let P be a proper parabolic subgroup of G_x with Levi decomposition MU, and let $v \in W$. It suffices to show that $v \in W(\mathsf{U})$, or equivalently,

$$\sum_{\mathsf{u} \in \mathsf{U}} \tau(\mathsf{u})v = 0 \tag{5.3}$$

(this result is analogous to [BZ76, Lemma 2.33]; see [Car85, p.293]). Assume without loss of generality that x is in the standard apartment $\mathcal{A}$ of G. Let $y_0 \in \mathcal{A}_s(\mathsf{G}_x)$ be a vector such that $\mathsf{P} = \mathsf{P}_{y_0}$; note that $y_0 \neq 0$ (cf. Proposition 2.13). By the results in Section 3.2, the vector y_0 corresponds to some point $y \in \mathcal{A}$ near x. The point y is not a vertex, so by part (i), $V^{G_y^+} = \{0\}$. But G_y^+ is the inverse image in G_x of U, and the sum (5.3) is U-invariant, so (5.3) must be zero.

❐

Now we can move on to the second step of our classification, i.e., we will actually induce any depth-zero irreducible super-cuspidal representation of G from a cuspidal representation of a maximal parahoric. This will require doing quite a bit of representation theory first.

Notation 5.9. In the rest of this section, we fix a depth-zero minimal K-type (G_x, σ), where x is a vertex. Let $F_x = N_G(G_x)$, and let $\mathcal{E}(\sigma)$ be the irreducible representations of F_x (up to equivalence) which contain σ on restriction to G_x.

We also remind the reader about our notation for induction and compact induction (cf. Page 4):

Notation 5.10. We define ind and Ind to be the compact induction and induction functors, respectively.

Remark 5.11. One can partition $\mathcal{E}(\sigma)$ into finitely many equivalence classes by setting $\tau_1 \sim \tau_2$ if there is an unramified character χ of G (cf. [Cas, §1.6]) such that $\tau_1 = \tau_2 \otimes \chi$. Note that $\mathrm{ind}_{F_x}^G(\tau \otimes \chi) = (\mathrm{ind}_{F_x}^G \tau) \otimes \chi$.

Proposition 4.22(ii) implies the following:

Corollary 5.12. *If $x \in \mathcal{B}(G)$ is a vertex, then $F_x = C \cdot G_x$ is compact mod C, and therefore it is compact mod its center.*

There are several things to note about the elements of $\mathcal{E}(\sigma)$.

Lemma 5.13. *Let $\tau \in \mathcal{E}(\sigma)$.*

(i) The elements of $\mathcal{E}(\sigma)$ are exactly the irreducible components of $\mathrm{Ind}_{G_x}^{F_x} \sigma$.

(ii) τ is finite-dimensional.

(iii) τ is trivial on G_x^+. The natural representation τ of G_x satisfies the property

$$\sum_{\mathsf{u} \in \mathsf{U}} \tau(\mathsf{u}) = 0$$

when U is the unipotent radical of any proper parabolic subgroup of G_x.

Proof.

The first statement follows from Frobenius reciprocity and the fact that G_x is compact. Let $\tau \in \mathcal{E}(\sigma)$, and let V_τ be a representation space for τ. Since F_x is compact modulo its center and τ is irreducible and smooth, we have $V_\tau = \langle g \cdot v \ : \ g \in F_x \rangle$ is finite-dimensional for any $v \in V_\tau$.

Let (π, V) be the induced representation $\mathrm{Ind}_{G_x}^{F_x} \sigma$. It suffices to prove the third statement for π. Let X be the (minimal) facet containing x. We have that if $g \in F_x = \mathrm{stab}_G(X)$, then $gG_X^+g^{-1} = G_X^+$. Therefore if $f \in V$, $g \in F_x$, and $h \in G_x^+$, then

$$(h \cdot f)(g) = f(gh) = \sigma(ghg^{-1}) \cdot f(g) = f(g)$$

as required. Therefore $(\mathrm{res}_{G_x} \pi, V)$ is naturally a representation of G_x. If $\mathsf{U} \subset \mathsf{G}_x$ is the unipotent radical of any proper parabolic subgroup of G_x, then since σ is cuspidal, we have [Car85, p.293]

$$\sum_{\mathsf{u} \in \mathsf{U}} \sigma(\mathsf{u}) = 0.$$

If $g \in F_x$, then Proposition 4.22(ii) says that $g = c \cdot h$ for some $c \in C$ and $h \in G_x$. Therefore the automorphism $\mathsf{x} \mapsto g\mathsf{x}g^{-1} : \mathsf{G}_x \xrightarrow{\sim} \mathsf{G}_x$ is an inner automorphism, so $g\mathsf{U}g^{-1}$ is the unipotent radical of a proper parabolic subgroup of G_x as well. (Alternately, if U is the unipotent radical of the parabolic subgroup corresponding to some $y_0 \in \mathcal{A}_s(\mathsf{G}_x)$, then $g\mathsf{U}g^{-1}$ is the unipotent radical of the parabolic subgroup corresponding to $g \cdot y_0$.) Thus if $f \in V$, we have

$$\sum_{\mathsf{u} \in \mathsf{U}} (\mathsf{u} \cdot f)(g) = \sum_{\mathsf{u} \in \mathsf{U}} \sigma(g\mathsf{u}g^{-1}) \cdot f(g) = \sum_{\mathsf{u} \in g\mathsf{U}g^{-1}} \sigma(\mathsf{u}) \cdot f(g) = 0.$$

❒

An immediate consequence of Lemma 5.13 is the following:

Lemma 5.14. *Let $\tau \in \mathcal{E}(\sigma)$ have representation space V_τ, and let $H \subset G_x$ be a subgroup whose image in G_x contains the unipotent radical of some proper parabolic subgroup. Then $V_\tau^H = \{0\}$.*

Proof.

Suppose that $\mathsf{U} \subset \mathsf{G}_x$ is a unipotent radical of a proper parabolic subgroup contained in the image of H in G_x, and let $v \in V_\tau^H$. Then

$$|\mathsf{U}| \cdot v = \sum_{\mathsf{u} \in \mathsf{U}} \tau(\mathsf{u}) \cdot v = 0$$

by Lemma 5.13. Therefore $v = 0$.

❒

Corollary 5.15. *Let the notation be as in Lemma 5.14, let $y \in \mathcal{B}(G)$, and let $r \geq 0$. There is some constant $c > 0$ independent of y and r such that if $V_\tau^{G_x \cap G_{y,r}} \neq \{0\}$, then $d(x, y) \leq c \cdot r$. In particular, if $V_\tau^{G_x \cap G_y^+} \neq \{0\}$, then $y = x$.*

Proof.

First we need to establish a geometric fact. Consider the unit sphere $S^{\ell-1} \subset \mathcal{A}_s^{\mathrm{red}}$. For a subset $\Theta \subset \Delta$ and any $\epsilon > 0$, let

$$H_{\Theta,\epsilon} = \{x \in \mathcal{A}_s^{\mathrm{red}} \ : \ \alpha(x) < \epsilon \text{ for all } \alpha \in \Theta.\}$$

and let $H_\Theta = H_{\Theta,0} = \mathcal{A}_s^{\mathrm{red}} \cap \left(\bigcap_{\alpha \in \Theta} H_\alpha\right)$. If $\Theta \subset \Delta$ with $|\Theta| = \ell - 1$, then $H_\Theta \cap S^{\ell-1}$ consists of two points. Therefore if $\{\alpha\} = \Delta \setminus \Theta$, then there is an $\epsilon > 0$ such $H_{\Theta,\epsilon} \cap S^{\ell-1}$

does not intersect $H_{\{\alpha\},\epsilon} \cap S^{\ell-1}$. Since Δ is finite, we can choose ϵ such that the above is true for any $\alpha \in \Delta$.

Continuing with the proof, let $y \in \mathcal{B}(G)$ be arbitrary, and choose an apartment $\mathcal{A}'$ containing x and y. Consider the vector $v = y - x \in \mathcal{A}_s$; let w be the unit vector in the direction of v. Choose a Weyl chamber $D \subset \mathcal{A}_s$ such that $y \in x + \overline{D}$, and let Δ be the set of simple roots corresponding to D, so that $\alpha(v) \geq 0$ for all $\alpha \in \Delta$. By the previous paragraph, there is some $\epsilon > 0$ and some $\alpha \in \Delta$ such that $\alpha(w) \geq \epsilon$. Let $\Theta = \Delta \setminus \{\alpha\}$, and let $P = P_\Theta$ as in Section 2.1; write U for the unipotent radical of P. The group U is generated by root spaces $\mathfrak{X}_\beta$ such that $\beta(w) \geq \epsilon$, i.e., $\beta(y - x) \geq \epsilon \cdot d(x, y)$. But by Remark 5.3, we know that

$$G_{y,r} \cap U = \prod_{\beta \in \Phi^+ \setminus \Theta} \mathfrak{X}_\beta(\wp^{-\lfloor \beta(y) - r \rfloor}) \supset \prod_{\beta \in \Phi^+ \setminus \Theta} \mathfrak{X}_\beta(\wp^{-\lfloor \beta(x) + \epsilon \cdot d(x,y) - r \rfloor})$$

so if $d(x, y) > r/\epsilon$, then $G_{y,r} \cap G_x \cap U = G_x \cap U$. But $G_x \cap U$ projects onto the unipotent radical of the parabolic in G_x corresponding to P, so by Lemma 5.14, we have $V^{G_x \cap G_{y,r}} = \{0\}$.

❐

Using the above, we may prove some key facts about $\operatorname{ind}_{F_x}^G \tau$ for $\tau \in \mathcal{E}(\sigma)$.

Lemma 5.16. *Let $\tau \in \mathcal{E}(\sigma)$.*

(i) $\operatorname{ind}_{F_x}^G \tau$ is an admissible and semisimple G-module.

(ii) If $\operatorname{Hom}_{F_x}(\tau, \operatorname{res}_{F_x} \operatorname{ind}_{F_x}^G \tau)$ is one-dimensional, then $\operatorname{ind}_{F_x}^G \tau$ is irreducible.

Proof.

Let (π, V) be the induced representation $\operatorname{ind}_{F_x}^G \tau$, and let V_τ be a representation space for τ.

(i) In order to show admissibility it suffices to show that $V^{G_{x,r}}$ is finite-dimensional for any $r > 0$ (cf. Remark 5.3). One has

$$(\operatorname{ind}_{F_x}^G \tau)^{G_{x,r}} = \bigoplus_{\sigma \in F_x \backslash G / G_{x,r}} V_\tau^{F_x \cap g_\sigma G_{x,r} g_\sigma^{-1}} = \bigoplus_{\sigma \in F_x \backslash G / G_{x,r}} V_\tau^{G_x \cap G_{g_\sigma \cdot x, r}}$$

where g_σ is any representative of the double coset $\sigma \in F_x \backslash G / G_{x,r}$ (cf. [BZ76, Lemma 2.24]). However, by Corollaries 5.15 and 4.21, only a finite number of the above summands is nonzero, so since V_τ is finite-dimensional, $V^{G_{x,r}}$ must be finite-dimensional as well.

Since τ is irreducible, Schur's Lemma tells us that the center Z of F_x acts by a character on τ, and therefore on V. As in [Cas, Lemma 5.2.5], then, one can find a real-valued character χ of G such that Z acts by a unitary character on $\chi \otimes \pi$. Thus since taking the tensor product with a character does not affect semisimplicity, we may assume without loss of generality that τ, and therefore π, is unitary. But any admissible unitary representation is semisimple (cf. [Cas, Prop. 2.1.14]), so we are done.

(ii) We continue to assume that π is unitary. Suppose that $W \subset V$ is a proper G-subrepresentation, so $\operatorname{Hom}_G(W, V) = \operatorname{Hom}_G(W, \operatorname{ind}_{F_x}^G \tau) \neq \{0\}$. Frobenius reciprocity thus gives $\operatorname{Hom}_{F_x}(\operatorname{res}_{F_x} W, \tau) \neq 0$, so since V is semisimple, W is too, so τ embeds into $\operatorname{res}_{F_x} W$. But semisimplicity also gives $V = W \oplus W^\perp$ as G-submodules, so $\operatorname{Hom}_{F_x}(\tau, \operatorname{res}_{F_x} V)$ is at least two-dimensional.

❐

Now we can prove the main theorem.

Theorem 5.17 ([MP96, Prop. 6.6]**).** *Given $\tau \in \mathcal{E}(\sigma)$, the representation $\operatorname{ind}_{F_x}^G \tau$ is an irreducible super-cuspidal representation of G. Moreover, any irreducible representation of G which contains σ on restriction to G_x is isomorphic to $\operatorname{ind}_{F_x}^G \tau$ for some $\tau \in \mathcal{E}(\sigma)$.*

Proof.

Let (π, V) be the induced representation $\operatorname{ind}_{F_x}^G \tau$. If π is irreducible, then by [Mau64], it is super-cuspidal, so by Lemma 5.16, it suffices to show that $\operatorname{Hom}_{F_x}(\tau, \operatorname{res}_{F_x} \pi) = \mathbf{C}$. Let $\mathcal{H} = \mathcal{H}(G/\!/F_x, \tau)$ be the Hecke algebra, i.e., the space of all $\varphi : G \to \operatorname{End}(V_\tau)$ such that (a) φ has compact support mod F_x, and (b) $\varphi(hgh') = \tau(h)\varphi(g)\tau(h')$ for all $h, h' \in F_x$ and $g \in G$. It is easy to show that the map $\mathcal{H}(G/\!/F_x, \tau) \xrightarrow{\sim} \operatorname{Hom}_{F_x}(\tau, \operatorname{res}_{F_x} \pi)$ taking $\varphi \mapsto \psi_\varphi$ given by $\psi_\varphi(v)(g) = \varphi(g)v$ is a natural isomorphism, so it suffices to show that $\mathcal{H}(G/\!/F_x, \tau) = \mathbf{C}$. Let $\mathcal{H}(F_x g F_x)$ be the subspace of $\mathcal{H}$ consisting of functions with support in the double coset $F_x g F_x$, so that $\mathcal{H}$ is the direct sum of all $\mathcal{H}(F_x g F_x)$, as g ranges through representatives of the double cosets $F_x \backslash G / F_x$. We will show that $\mathcal{H}(F_x g F_x) = \{0\}$ unless $F_x g F_x = F_x$; note that $\mathcal{H}(F_x)$ is one-dimensional because it consists of scalar multiples of the map $h \mapsto \tau(h)$.

Let $F_x g F_x$ be an arbitrary double coset, and let $\varphi \in \mathcal{H}(F_x g F_x)$ be nonzero, so $\varphi(g) \neq 0$. If ghg^{-1} is any element of $G_x \cap g G_x^+ g^{-1} = G_x \cap G_{gx}^+$, then we have

$$\tau(ghg^{-1})\varphi(g) = \varphi(gh) = \varphi(g)\tau(h) = \varphi(g)$$

because τ is trivial on G_x^+. Therefore $\varphi(g) \in V^{G_x \cap G_{gx}^+}$, so by Corollary 5.15, we must have $gx = x$, i.e., $g \in F_x$. Thus $\mathcal{H}$ is one-dimensional and π is irreducible and super-cuspidal.

As for the other direction, suppose that (π, V) is an irreducible representation of G which contains σ on restriction to G_x. As in the proof of Lemma 5.16, we can assume that C acts by a unitary character on V, so integration over $C \backslash F_x$ defines a F_x-invariant Hermetian form on V. Therefore we may assume that $\operatorname{res}_{F_x} \pi$ is unitary, so since π is an irreducible G-representation, π is admissible, so $\operatorname{res}_{F_x} \pi$ is semisimple. Since $\operatorname{Hom}_{G_x}(\sigma, \operatorname{res}_{G_x} \pi) \neq \{0\}$, it is easy to see that there is some irreducible subrepresentation τ of $\operatorname{res}_{F_x} \pi$ containing σ, so, by semisimplicity, there is some $\tau \in \mathcal{E}(\sigma)$ such that $\operatorname{Hom}_{F_x}(\operatorname{res}_{F_x} \pi, \tau) \neq \{0\}$. By Frobenius reciprocity for the compact induction functor ind [BZ76, 2.29], since F_x and G are both unimodular, we have $\operatorname{Hom}_G(\pi, \operatorname{ind}_{F_x}^G \tau) \neq \{0\}$, so since both are irreducible, they must be equivalent.

❐

Corollary 5.18 (to Theorems 5.17 and 5.7). *If $\tau \in \mathcal{E}(\sigma)$, then the induced representation $\operatorname{ind}_{F_x}^G \tau$ is an irreducible super-cuspidal representation of G (necessarily of depth zero), and any depth-zero irreducible super-cuspidal representation of G arises from some vertex $x \in \mathcal{B}$ in this way.*

5.3. Example: depth-zero super-cuspidals of $\mathrm{SL}_2(k)$

At this point it is illuminating explicitly to write down the depth-zero irreducible super-cuspidal representations of $G = \mathrm{SL}_2(k)$. This is not terribly difficult because $C = \{1\}$.

Throughout this example, keep in mind the results in Section 4.2; in particular, recall that in $\mathcal{B}(\mathrm{SL}_2(k))$, all vertices are special.

Notation 5.19. If (π, V) is a representation of any group G and $g \in G$ is any element, then we define the representation $({}^g\pi, V)$ by ${}^g\pi(h)v = \pi(ghg^{-1})v$.

The map $v \mapsto \pi(g)v$ defines an equivalence $\pi \cong {}^g\pi$.

Notation 5.20. If σ is a cuspidal representation of $\mathsf{G} = \mathrm{SL}_2(\mathfrak{f})$ and $x \in \mathcal{B}(G)$ is a vertex, then we put $(\pi_{\sigma,x}, V_{\sigma,x}) = \mathrm{ind}_{G_x}^G \sigma$.

By Proposition 4.22(ii), we have $G_x = N_G(G_x)$ since $C = \{1\}$, so the results in Section 5.2 tell us that each $\pi_{\sigma,x}$ is an irreducible super-cuspidal representation of $\mathrm{SL}_2(k)$ (of depth zero), and that all such representations occur as one of the $\pi_{\sigma,x}$.

Remark 5.21. Let σ be a cuspidal representation of G with representation space V_σ. Let $v_0 \in V_\sigma$ be nonzero, and let $f \in V_{\sigma,x}$ be the function given by

$$f(h) = \begin{cases} 0 & \text{if } h \notin G_x \\ \tau(h)v_0 & \text{if } h \in G_x. \end{cases}$$

Then there is an embedding of V_σ into $V_{\sigma,x}$ given by $v_0 \mapsto f$.

The parameters σ, x are in fact the correct ones to use, as the following proposition shows.

Proposition 5.22. *The representations $\pi_{\sigma,x}$ and $\pi_{\sigma',x'}$ are equivalent if and only if x is conjugate to x' under G and $\sigma \cong \sigma'$ as representations of* G.

Remark 5.23. This proposition has a generalization to any generalized Levi G of a simply-connected p-adic Chevalley group.

Proof of Proposition 5.22.

($\Longleftarrow$) Find a $g \in G$ such that $gx = x'$. Then $\pi_{\sigma,x} \cong {}^g\pi_{\sigma,x} \cong \pi_{\sigma,x'}$ (cf. Remark 4.8). Since $\sigma \cong \sigma'$, the functorial nature of ind tells us that

$$\pi_{\sigma,x'} = \mathrm{ind}_{G_{x'}}^G \sigma \cong \mathrm{ind}_{G_{x'}}^G \sigma' = \pi_{\sigma',x'}.$$

($\Longrightarrow$) Suppose that $\pi_{\sigma,x} \cong \pi_{\sigma',x'}$. Let V_σ (resp. $V_{\sigma'}$) be a representation space for σ (resp. σ'). First we will show that x and x' must be conjugate under G. Let $y \in \mathcal{B}(G)$ be any point, and let $g_\sigma \in G$ be any representative for the coset $\sigma \in G_x \backslash G / G_y^+$. One can show that

$$(\mathrm{ind}_{G_x}^G \sigma)^{G_y^+} \cong \bigoplus_{\sigma \in G_x \backslash G / G_y^+} V_\sigma^{G_x \cap g_\sigma G_y^+ g_\sigma^{-1}} = \bigoplus_{\sigma \in G_x \backslash G / G_y^+} V_\sigma^{G_x \cap G_{g_\sigma y}^+} \tag{5.4}$$

under the map $f \mapsto (f(g_\sigma))_\sigma$ (cf. [BZ76, Lemma 2.24]). If $gx \neq x'$ for all $g \in G$, then by Corollary 5.15, we have $V_{\sigma,x}^{G_{x'}^+} = \{0\}$. But there is an embedding of $V_{\sigma'} \hookrightarrow \mathrm{res}_{G_{x'}} V_{\sigma',x'}$, whose image is contained in $V_{\sigma',x'}^{G_{x'}^+}$ because $V_{\sigma'} = V_{\sigma'}^{G_{x'}^+}$, which contradicts the assumption that $\pi_{\sigma,x} \cong \pi_{\sigma',x'}$. Therefore x and x' are conjugate under G, so we may assume that $x = x'$.

Let $i_x : V_\sigma \hookrightarrow \mathrm{res}_{G_x} V_{\sigma,x}$ be the embedding given by Remark 5.21. Equation (5.4) and Corollary 5.15 show that the image of i_x is in fact all of $V_{\sigma,x}^{G_x^+}$, so i_x restricts to an isomorphism of $\mathrm{res}_{G_x} V_{\sigma,x}^{G_x^+}$ with V_σ; similarly, $\mathrm{res}_{G_x} V_{\sigma',x}^{G_x^+}$ is isomorphic to $V_{\sigma'}$. Therefore the isomorphism of $V_{\sigma,x}$ with $V_{\sigma',x}$ restricts to an isomorphism of V_σ with $V_{\sigma'}$, completing the proof.

❒

Example 2.28 and Proposition 4.10 imply that there are exactly two orbits of (special) vertices in $\mathcal{B}(\mathrm{SL}_2(k))$, so all that is left to do is to find the cuspidal characters of $\mathrm{SL}_2(\mathfrak{f})$. Assume from now on that the characteristic of $\mathfrak{f}$ is odd. The character table of $\mathrm{SL}_2(\mathfrak{f})$ is given in [Car85, p.155]; using [Car85, Prop. 9.1.1], one finds that an irreducible character χ is cuspidal if and only if $\sum_{a\in\mathfrak{f}} \chi\left(\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]\right) = 0$. With this in mind, one calculates that the only cuspidal characters are as follows:

- For every complex character ω of the cyclic group C_{q+1} of order $q+1$ such that $\omega^2 \neq 1$, there is a cuspidal character $R(\omega)$ (called the Deligne-Lusztig virtual character, denoted $-R_{\mathbf{T}_s}^{\mathbf{G}}(\omega)$ in [Car85]). We have $R(\omega) = R(\omega')$ if and only if $\omega' = \omega^{\pm 1}$. Therefore there are $\frac{q-1}{2}$ such characters.
- Associated to the nontrivial character ω_0 of C_{q+1} such that $\omega_0^2 = 1$ are two cuspidal characters $\chi^{\pm 1}$.

Therefore we have proved that:

Theorem 5.24. *If q is odd, then there are exactly $q+1$ depth-zero irreducible supercuspidal representations of $\mathrm{SL}_2(k)$.*

A. Appendix: Non-simply-connectedness and $\mathrm{SO}_4(k)$

It is unfortunately not the case that the Chevalley group generated by a set of Chevalley data (Lie algebra, representation, etc.) is always the algebraic group associated with those data. For instance, if we exponentiate the standard representation of $\mathfrak{so}_4(\mathbf{C})$, we would like to obtain all of $\mathrm{SO}_4(k)$ (that is, all elements that preserve a quadratic form); unfortunately, in general we only obtain a subgroup of $\mathrm{SO}_4(k)$, as we will demonstrate in this appendix.[14]

The Dynkin diagram of SO_4 is of type D_2, which agrees with $A_1 \times A_1$; i.e., there are two orthogonal simple roots α and β, and the root system is $\{\pm\alpha, \pm\beta\}$. Therefore the Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ has a two-dimensional Cartan subalgebra $\mathfrak{h}$, spanned by elements Z_α and Z_β, and four root spaces $\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}$, spanned by $X_{\pm\alpha}$ and $X_{\pm\beta}$. The spaces $\langle X_{\pm\alpha}, Z_\alpha\rangle$ and $\langle X_{\pm\beta}, Z_\beta\rangle$ are two orthogonal standard copies of $\mathfrak{sl}_2$. Therefore the weight lattice is $\Lambda_W = \mathbf{Z}(\alpha/2) \oplus \mathbf{Z}(\beta/2)$, and the root lattice Λ_R is of course $\mathbf{Z}\alpha \oplus \mathbf{Z}\beta$, which is index four in the weight lattice.

We will realize $\mathfrak{g}_{\mathbf{C}}$ in matrix form as follows. We will take our quadratic form to be $M = \left[\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right]$, where I_2 is the identity matrix. Then, as in [FH91], the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is given

[14] However, if k is algebraically closed or we generate the simply-connected form, then the Chevalley group will agree with the algebraic group — recall that a simply-connected algebraic group of the type that we are considering is generated by its unipotent elements.

by block matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where B and C are skew-symmetric and $A = -{}^tD$. We may take

$$Z_\alpha = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \quad X_\alpha = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & -1 & 0 \end{bmatrix} \quad X_{-\alpha} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}$$
$$Z_\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \quad X_\beta = \begin{bmatrix} 0 & & & 1 \\ & 0 & -1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad X_{-\beta} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & -1 & 0 & \\ 1 & & & 0 \end{bmatrix}.$$

One checks that these span $\mathfrak{g}_{\mathbf{C}}$, and that they satisfy the appropriate relations — in other words, this is a Chevalley basis of $\mathfrak{g}$. The standard four-dimensional representation $V^{\mathbf{C}}$ is the one that generates SO_4. The weight spaces are spanned by the coordinate vectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$; a calculation shows that the weights are $\pm\alpha \pm \beta$, so the representation lattice is index two in the weight lattice — in other words, SO_4 is not simply connected.

Using [Ste68, Corollary 2], one can see that a lattice V invariant under the set (1.4) is $\mathbf{Z}\mathbf{e}_1 \oplus \cdots \oplus \mathbf{Z}\mathbf{e}_4$, so $V^k = k^4$, with $X_{\pm\alpha}, X_{\pm\beta}$ acting as the matrices above. Therefore we calculate

$$x_\alpha(t) = \exp(tX_\alpha) = \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & \\ & & -t & 1 \end{bmatrix} \qquad x_{-\alpha}(t) = \exp(tX_{-\alpha}) = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ & & 1 & -t \\ & & & 1 \end{bmatrix}$$
$$x_\beta(t) = \exp(tX_\beta) = \begin{bmatrix} 1 & & & t \\ & 1 & -t & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad x_\beta(t) = \exp(tX_{-\beta}) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -t & 1 & \\ t & & & 1 \end{bmatrix}$$
$$w_\alpha(t) = \begin{bmatrix} 0 & t & & \\ -t^{-1} & 0 & & \\ & & 0 & t^{-1} \\ & & -t & 0 \end{bmatrix} \qquad h_\alpha(t) = \begin{bmatrix} t & & & \\ & t^{-1} & & \\ & & t^{-1} & \\ & & & t \end{bmatrix}$$
$$w_\beta(s) = \begin{bmatrix} 0 & & & s \\ & 0 & -s & \\ & s^{-1} & 0 & \\ -s^{-1} & & & 0 \end{bmatrix} \qquad h_\beta(s) = \begin{bmatrix} s & & & \\ & s & & \\ & & s^{-1} & \\ & & & s^{-1} \end{bmatrix}$$

so the Cartan subgroup T is given by $\{\mathrm{diag}(st, st^{-1}, s^{-1}t^{-1}, s^{-1}t) \; : \; s, t \in k^\times\}$ First note that when $(s,t) = \pm(1,1)$, then $h_\alpha(s)h_\beta(t) = 1$, so elements of T cannot be uniquely expressed as products of $h_\alpha(s)$ and $h_\beta(t)$.

We know that the centralizer in G of T is T itself (cf. Proposition B.3). Thus in order to show that $G \neq \mathrm{SO}_4(k)$, it suffices to find a diagonal element of $\mathrm{SO}_4(k)$ which is not contained in T. Assume that there exists some $a \in k^\times \setminus (k^\times)^2$. Then it is easy to see that $\mathrm{diag}(a, 1, a^{-1}, 1)$ preserves our form M but is not contained in T. Therefore $G \neq \mathrm{SO}_4(k)$ (unless k is algebraically closed, for instance).

Remark A.1. Although the Chevalley group SO_4 is not simply-connected, one can realize $\mathrm{SO}_4(k)$ as a generalized Levi subgroup of G_2. Therefore one can use the results in this paper to analyze $\mathrm{SO}_4(k)$ after all.

B. Appendix: Chevalley group miscellanea

This thesis is primarily concerned with the affine structure of our groups, but there are a few slightly nonstandard pieces of spherical structure of G that we should include for completeness. Therefore, in this appendix, G will be be any Chevalley group over an arbitrary field k.

Let $B \subset G$ be a Borel subgroup containing T with unipotent radical U and opposite unipotent radical $\overline{U}$. For $w \in W = N/T$, let U_w be the product of the root subgroups for roots $\alpha \in \Phi^+$ such that $w(\alpha) \in \Phi^-$. From [Car89, Cor. 8.4.4], we have:

Theorem B.1. *For every $g \in G$, there exist unique $u \in U, n \in N$, and $u_w \in U_w$, such that $w = nT$ and $g = unu_w$.*

Corollary B.2. $N \cap (U\overline{U}) = N \cap (\overline{U}U) = \{1\}$

Proof.

We have $N \cap (U\overline{U}) = (N \cap (\overline{U}U))^{-1}$, so it suffices to show that $N \cap (U\overline{U}) = \{1\}$. Suppose that $n = u\overline{u}$ where $n \in N$ and $u \in U, \overline{u} \in \overline{U}$. Since all positive root systems are conjugate under W, we can find an $n_1 \in N$ such that $u_w := n_1^{-1}\overline{u}n_1 \in U$, so $nn_1 = un_1u_w$. Since $n_1u_wn_1^{-1} \in \overline{U}$, both sides of this equation are now written in canonical form, so we can use Theorem B.1 to conclude that $u_w = u = 1$ and therefore $\overline{u} = n = 1$. ❐

Another fact which we use several times is the following:

Proposition B.3. *Suppose that k is an infinite field of characteristic zero and that G is simply-connected. Then the torus T is its own centralizer in G.*

Proof.

Let Z be the centralizer in G of T. Since N is the normalizer of T, we have $Z \subset N$. Suppose that there were some $n \in Z \setminus T$. Then n projects onto a nontrivial element w of the Weyl group, so there is some root $\alpha \in \Delta$ such that $w\alpha \neq \alpha$. Therefore by [Ste68, Lemma 20(b)], $nh_\alpha(1)n^{-1} = h_{w\alpha}(\pm 1)$, which is not equal to $h_\alpha(1)$ by simply-connectedness. This is a contradiction. ❐

We also need a useful fact about root systems. Recall that Λ_R is the root lattice, contained in the Euclidean space E. A sort of "Graham-Schmidt" procedure works on this lattice:

Proposition B.4. *Let $x_1, \ldots, x_n \in \Lambda_R$ be linearly independent. There is an orthogonal set $y_1, \ldots, y_n \in \Lambda_R$ such that $\operatorname{span}(x_1, \ldots, x_i) = \operatorname{span}(y_1, \ldots, y_i)$ for each $1 \le i \le n$.*

Proof.

We will prove the proposition by induction. The base case $n = 1$ is trivially true. Since Φ contains roots of at most two lengths a and b such that $a^2/b^2 \in \mathbf{Z}$, we may assume that the inner product (x, y) is an integer for every $x, y \in \Lambda_R$ (because $\langle x, y\rangle = 2(x, y)/(y, y) \in \mathbf{Z}$ for $x, y \in \Phi$).

Suppose that we have some $y_1, \ldots, y_{n-1}$ which satisfy the inductive hypotheses. We claim that there are $m \in \mathbf{Z} \setminus \{0\}$ and $a_1, \ldots, a_{n-1} \in \mathbf{Z}$ such that

$$(mx_n - a_1y_1 - \cdots - a_{n-1}y_{n-1}, y_i) = m(x_n, y_i) - a_i(y_i, y_i) = 0$$

for each $1 \le i \le n-1$; then we may set $y_n = mx_n - a_1y_1 - \cdots - a_{n-1}y_{n-1} \in \Lambda_R$. It is certainly possible to find such m and a_i because each $(x_n, y_i) \in \mathbf{Z}$ and $(y_i, y_i) \in \mathbf{Z}$. ❐

A related fact, used to prove Theorem 5.7, is as follows.

Lemma B.5. *If $x \in \mathcal{A}$ is not a vertex, then $M(x)$ is contained in a proper (not generalized) Levi subgroup M of G, whose co-root system Φ_M does not span $\mathcal{A}_s^{\mathrm{red}}$.*

Proof.

Note that x is not a vertex iff Φ_x does not span E_Φ, which is true iff $\check{\Phi}_x$ (the set of co-roots of the roots in Φ_x) does not span $\mathcal{A}_s^{\mathrm{red}}$. Let $E_x^* \subsetneq \mathcal{A}_s^{\mathrm{red}}$ be the span of $\check{\Phi}_x$, and choose a nonzero $y \in (E_x^*)^\perp \cap \mathcal{A}_s^{\mathrm{red}}$. Proposition 2.13 tells us that

$$M = \langle T,\ \mathfrak{X}_\alpha \ :\ \alpha \in \Phi,\ \alpha(y) = 0 \rangle$$

is a Levi subgroup of G containing $M(x)$, which is proper because $y \neq 0$. The root system Φ_M of M is $\{\alpha \in \Phi \ :\ \alpha(y) = 0\}$; since M is a proper Levi subgroup, there is a choice of simple roots Δ and a proper subset $\Theta \subset \Delta$ such that Θ is also a system of simple roots for Φ_M; in particular, $\check{\Phi}_M$ does not span $\mathcal{A}_s^{\mathrm{red}}$ either.

❒

Remark B.6. Our proof of Theorem 5.7 (which is not substantively different from [MP96, Prop. 6.7]) holds for any M as in Lemma B.5, which is a slightly stronger result than the one stated in [MP96]. However, Moy and Prasad offer a more elegant way to construct such an M in [MP96, §6.3].

C. Index of Notation

The following notational conventions are in place:

- If X is an object associated with a generalized Levi subgroup G of G^c, then X^c is the corresponding object associated with the full Chevalley group G^c, and X is the corresponding object defined over the residue field, when this makes sense.
- If $X \subset \Lambda_R$ is a subset of the root lattice, then $\check{X} \subset \check{\Lambda}_R$ is the corresponding subset of the co-root lattice.
- If X is a vector space, then X^* is its linear dual.

List of symbols, in quasi-alphabetical order:

$(\cdot,\cdot)$	The inner product on E and E^*.	8
$\langle\cdot,\cdot\rangle$	$= 2(\gamma,\delta)/(\delta,\delta)$ is always an integer for $\gamma,\delta \in \Lambda_R$.	8
$\langle\cdot,\cdot\rangle$	The canonical pairing of E with E^*.	8
$\sim$	The equivalence relation on $G \times \mathcal{A}$ used to make the building.	42
$[g,x]$	The equivalence class of $(g,x) \in G \times \mathcal{A}$ under $\sim$.	42
$\mathcal{A}_s(\mathsf{G})$	The spherical apartment of G.	16
$\mathcal{A}_s^{\mathrm{red}}(\mathsf{G})$	The reduced spherical apartment of G.	16
$\mathcal{A}(G)$	The standard affine apartment of G.	20
$\mathcal{A}({}^gT)$	The apartment in $\mathcal{B}(G)$ of the torus gTg^{-1}.	43
$\mathcal{A}^{\mathrm{red}}(G)$	The reduced affine apartment of G.	21
$\mathcal{B}(G)$	The Bruhat-Tits building of G.	42
C	The split center of G.	14
d	The metric on $\mathcal{B}(G)$.	45
Δ	A set of simple roots for Φ.	12
E	The Euclidean space containing the abstract root system Φ^c.	8
E_Φ	$\subset E$ is the subspace of E spanned by the root system Φ.	12
E_Φ^*	The dual of E_Φ, which can be identified with the subspace of E^* spanned by $\check{\Phi}$.	12
$\mathcal{E}(\sigma)$	The irreducible representations of F_x which contain σ on restriction to G_x.	56
f	$= R/\wp$ is the residue field of k.	15
F_x	$= N_G(G_x)$.	56
$\mathfrak{g}$	A fixed Lie algebra defined over the integers, coming from a complex semisimple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ and a Chevalley basis.	8
$\mathfrak{g}_\gamma$	$= \mathbf{Z} \cdot X_\gamma$ is the (integer) root space in $\mathfrak{g}$ defined by the root γ.	8
G	The generalized Levi subgroup of G^c associated with Φ.	12
G_x	The parahoric subgroup of G associated to the point $x \in \mathcal{A}(G)$.	28
G_x^+	The pro-unipotent radical of G_x.	28
G_F, G_F^+	The parahoric subgroup and its pro-unipotent radical corresponding to a facet $F \subset \mathcal{A}$.	28

G_Ω	The “parahoric” subgroup of G associated to a region $\Omega \subset \mathcal{A}$.	32
$G_{x,r}$	The Moy-Prasad filtration subgroups of G_x.	50
G_{x,r^+}	$= \bigcup_{s>r} G_{x,s}$.	50
G_x	The quotient group G_x/G_x^+.	33
$\mathfrak{h}$	The (integer) Cartan subalgebra of $\mathfrak{g}$.	8
$h_\gamma(t)$	$= w_\gamma(t)w_\gamma(1)^{-1}$ for $t \in k^\times$.	11
H_γ	The hyperplane in E^* fixed by r_γ.	15
$H_{\alpha+n}$	The hyperplane in $\mathcal{A}$ killed by the functional $\alpha + n$.	20
$\mathcal{H}_n$	The union of all n-facets.	26
Ind	The induction functor.	56
ind	The compact induction functor.	56
k	A finite extension of a p-adic field $\mathbf{Q}_p$.	15
ℓ	$= \lvert\Delta\rvert$.	12
L	$= \lvert\Phi\rvert$.	12
Λ_C	$= \Lambda_R^c \cap E_\Phi^\perp$.	14
Λ_R	The root lattice, which is the lattice generated by Φ.	12
$M(x)$	The generalized Levi subgroup of G associated to a point $x \in \mathcal{A}$.	26
N	The normalizer in G of T if $\lvert k\rvert > 3$.	14
N_x	$= N \cap G_x$ is the inverse image in G_x of N_x.	36
N_x	The normalizer in G_x of T if $\lvert\mathsf{f}\rvert > 3$.	36
ω	A nontrivial discrete valuation on k.	15
$\wp$	The unique prime ideal in R.	15
P_x	The parabolic subgroup of G associated with $x \in \mathcal{A}_s(\mathsf{G})$.	17
ϖ	A fixed uniformizing element in k.	15
π_{G}	The orthogonal projection map $\mathcal{A}_s(\mathsf{G}) \to \mathcal{A}_s^{\mathrm{red}}(\mathsf{G})$.	16
π_G	The orthogonal projection map $\mathcal{A}(G) \to \mathcal{A}^{\mathrm{red}}(G)$.	21
Φ	$\subset E$ is the root system of our generalized Levi subgroup G.	12
$\Phi^\pm$	Some choice of positive and negative roots in Φ.	12
Φ_x	The closed sub-root system corresponding to a point $x \in \mathcal{A}$.	26
Ψ	The affine root system associated to the root system Φ.	28
q	$= \lvert\mathsf{f}\rvert$.	15
R	The ring of integers in k.	15
r_γ	The reflection of E or E^* over the hyperplane orthogonal to the root γ or the co-root $\check{\gamma}$.	15
$r_{\alpha+n}$	The reflection over the hyperplane $H_{\alpha+n}$.	20
$\mathfrak{R}(G)$	The category of smooth representations of G.	3
$\varrho(\pi)$	The depth of the representation π.	51
T	The Cartan subgroup of G^c, which is *fixed*.	11

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