AN INTRODUCTION TO AUTOMORPHIC REPRESENTATIONS

Abstract. These are notes from courses on automorphic representations given by Jayce R. Getz.

Contents

Introduction

The goal of this course is to introduce and study **automorphic representations**. Given a global field F and a reductive algebraic group G over F, then an automorphic representation of G is a $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$-module which is isomorphic to a subquotient of $L^2(G(F)\backslash G(\mathbb{A}_F))$. The first part of the course is dedicated to explicating the objects in this definition. The next goal is to state a rough version of the Langlands functoriality conjecture, motivated by the description of unramified admissible representations of reductive groups over nonarchimedian local fields. The discussion of unramified representations is complemented by a discussion of supercuspidal representations. Next we recall the notion of distinguished representations in global and local settings; this has emerged as an important concept, especially in relation to arithmetic-geometric applications of automorphic forms. The trace formula in simple settings is then desribed and proved. Finally, we end the course with a discussion of the relationship between automorphic representations and the cohomology of locally symmetric spaces.

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1. Background on adele rings

1.1. **Adeles.** The arithmetic objects of interest in this course are constructed using global fields. They can be defined axiomatically, but we take a more pedestrian approach. For more information consult chapter 5 of [RV99].

Definition 1.1. A **global field** F is a field which is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$ for some prime power $q = p^r$. Global fields over $\mathbb{Q}$ are called **number fields** while global fields over $\mathbb{F}_q(t)$ are called **function fields**.

To each global field F one can associate an adele ring $\mathbb{A}_F$. Before defining this ring, we recall the related notions of a valuation a place of a global field.

Definition 1.2. Let F be a global field. A (non-archimedian) **valuation** on F is a map

$$v : F \longrightarrow \mathbb{R} \cup \infty$$

such that for all $a, b \in F$

- $v(a) = \infty$ if and only if $a = 0$.
- $v(a) + v(b) = v(ab)$.
- $v(a + b) \geq \min(v(a), v(b))$.

These axioms are designed so that if one picks $0 < \alpha < 1$ then

$$\begin{aligned} |\cdot|_v : F &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \alpha^{v(x)} \end{aligned} \tag{1.1.1}$$

is a **non-archimedian absolute value** on F, in other words, it is a map to $\mathbb{R}_{\geq 0}$ satisfying the following axioms:

(1) $|a|_v = 0$ if and only if $a = 0$

(2) $|ab|_v = |a|_v|b|_v$

(3) $|a+b|_v \leq \max(|a|_v, |b|_v)$ (the non-archimedean triangle inequality).

A function $|\cdot| : F \longrightarrow \mathbb{R}_{\geq 0}$ that does not satisfy (3), but satisfies (1-2) and the following weakening of (3):

(3') $|a+b|_v \leq |a|_v + |b|_v$ (the usual triangle inequality)

is known as an **archimedean absolute value**. These absolute values all induce metrics on F, the metric induced by $|\cdot|_v$ is known as the **v-adic metric**. The completion of F with respect to this metric is denoted F_v.

Definition 1.3. A **place** of a global field F is an equivalence class of absolute values, where two absolute values are said to be equivalent if they induce the same topology on F. A place is (non)archimedian if it consists of (non)archimedian absolute values.

The places of a global field F fall into two categories: the finite and infinite places. In the function field case these are all nonarchimedean. If F is a number field then the finite places are in bijection with the prime ideals of its ring of integers $\mathcal{O}_F$; these are all non-archimedean. The infinite primes of a number field are in bijection with the embeddings $F \hookrightarrow \mathbb{C}$ up to complex conjugation; these are all archimedian.

The place v associated to a prime ϖ_v of $\mathcal{O}_F$ is the equivalence class of a absolute value attached to the valuation

$$v(x) := \max\{k \in \mathbb{Z} : x \in \varpi_v^k \mathcal{O}_F\}.$$

and the place v associated to an embedding $\iota : F \hookrightarrow \mathbb{C}$ is the equivalence class of the absolute value

$$|\iota x|^{[\iota(F):\mathbb{R}]}.$$

In the first case by convention we define $|x|_v := q^{-v(x)}$ where $q := |\mathcal{O}_F/\varpi|$, and in the second $|x|_\iota := |\iota x|^{[\iota(F):\mathbb{R}]}$.

If v is finite, then the **ring of integers** of F_v is

$$\mathcal{O}_{F_v} := \{x \in F_v : |x|_v \leq 1\};$$

it is a local ring with maximal ideal

$$\varpi_{F_v} = \{x \in F_v : |x|_v < 1\}$$

denotes the unique *maximal ideal* of $\mathcal{O}_{F_v}$. We will often write $\mathcal{O}_v$ and ϖ_v for $\mathcal{O}_{F_v}$ and ϖ_{F_v}, respectively.

Example 1.4. If $F = \mathbb{Q}$ and $p \in \mathbb{Z}$ is a finite prime, then completing $\mathbb{Q}$ at the p-adic absolute value gives the local field $\mathbb{Q}_p$. Its ring of integers is $\mathbb{Z}_p$ and the maximal ideal is $p\mathbb{Z}_p$. The residue field is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$.

Definition 1.5. Let F be a global field. The ring of **adeles** of F, denoted $\mathbb{A}_F$, is the restricted direct product of the completions F_v with respect to the rings of integers $\mathcal{O}_v$:

$$\mathbb{A}_F = \left\{ (x_v) \in \prod_v F_v : x_v \in \mathcal{O}_v \text{ for all but finitely many places } v \right\}.$$

The restricted product is usually denoted by a prime:

$$\mathbb{A}_F = \prod_v{}' F_v.$$

Note that the adeles are a subring of the full product $\prod_v F_v$. If S is a finite set of places of F then we write

$$\mathbb{A}_F^S = \prod_{v \notin S}{}' F_v, \quad F_S := \mathbb{A}_{F,S} = \prod_{v \in S} F_v$$

We endow $\mathbb{A}_F$ with the **restricted product topology**. This is defined by stipulating that open sets are sets of the form

$$U_S \times \prod_{v \notin S} \mathcal{O}_v$$

where S is a finite set of places of F including the infinite places and $U_S \subseteq F_S$ is an open set.

This is not the same as the topology induced on $\mathbb{A}_F$ by regarding it as a subset of the direct product $\prod_v F_v$. While $\prod_v F_v$ is not locally compact, for $\mathbb{A}_F$ one has the following:

Proposition 1.6. *The adeles $\mathbb{A}_F$ of a global field F are a locally compact hausdorff topological ring.*

Proof. We argue that $\mathbb{A}_F$ is locally compact and leave the other details to the reader. For any finite set of places S, the subset

$$\prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v$$

is an open subring of $\mathbb{A}_F$ for which the induced topology coincides with the product topology. The above subring is thus locally compact. Every $x \in \mathbb{A}_F$ is contained in some such subring, which shows that $\mathbb{A}_F$ is locally compact. □

There is a natural diagonal embedding $F \hookrightarrow \mathbb{A}_F$.

Lemma 1.7. *The subspace topology on F arising from the embedding $F \hookrightarrow \mathbb{A}_F$ is the discrete topology.*

Proof. Let $x \in F^\times$. For each finite place v of F let $n_v = v(x)$, so that $x \in \varpi_v^{n_v}$ but $x \notin \varpi_v^{n_v+1}$ for all v. Note that $n_v = 0$ for all but finitely many places. For each infinite place v let $U_v \subseteq F_v$ be the open ball of radius $\prod_{v<\infty} |x|_v^{-1}$ about x. Consider the open subset of $\mathbb{A}_F$ defined by

$$U = \prod_{v|\infty} U_v \times \prod_{v<\infty} \varpi_v^{n_v}.$$

Of course $x \in U$ by construction; suppose $y \in F$ is also contained in U. Recall that the product formula from algebraic number theory says that for any global field F and any $z \in F^\times$,

$$\prod_v |z|_v = 1.$$

Apply this to $x - y$; note that $|x-y|_v \le |x|_v$ for all finite places v. Thus

$$\prod_v |x-y|_v \le \prod_{v<\infty} |x|_v \times \prod_{v|\infty} |x-y|_v < 1$$

since $y \in U_v$. The product formula thus shows that we must have $x - y = 0$, and hence $F \cap U = \{x\}$. This shows that F obtains the discrete topology from $\mathbb{A}_F$. □

We often identify F with its image in $\mathbb{A}_F$.

Theorem 1.8 (Approximation)**.** *For every global field F, one has a decomposition*

$$\mathbb{A}_F = F_\infty + \prod_{v<\infty} \mathcal{O}_v + F.$$

Proof. See Theorem 5-8 of [RV99]. □

Claim 1.9. *For every global field F,*

$$\left(F_\infty + \prod_{v<\infty} \mathcal{O}_v\right) \cap F = \mathcal{O}_F.$$

Proof. One inclusion is obvious. For the other, if $x \in F$ satisfies $x \in \mathcal{O}_v$ for all finite places v then $x\mathcal{O}_F$ is a proper ideal of $\mathcal{O}_F$, and not just fractional, since

$$x\mathcal{O}_F = \prod_{v<\infty} \varpi_v^{v(x)}.$$

Thus $x \in \mathcal{O}_F$, which concludes the proof. □

Remark 1.10. The fact that $\mathbb{A}_F$ is a topological ring for the adelic topology relies on the fact that the local rings $\mathcal{O}_v$ are one dimensional. In higher dimensional settings, say for function fields of algebraic surfaces, one must be more creative when defining appropriate analogues of the adeles.

1.2. **Adelic points of affine schemes.** Let **Ring** denote the category of commutative rings with identity. If $R \in$ **Ring** then we obtain a functor

$$\begin{aligned} \mathrm{Spec}(R) : \mathbf{Ring} &\longrightarrow \mathbf{Set} \\ A &\longmapsto \mathrm{Hom}_{\mathbf{Ring}}(R, A). \end{aligned} \tag{1.2.1}$$

A **affine scheme** is can be defined to be a functor of this form, although alternate definitions are possible and often desirable. Thus the category of affine schemes is anti-equivalent to the category of rings. If S is a functor then we say it is **representable** by a ring R if $S = \mathrm{Spec}(R)$. In this case we write

$$\mathcal{O}(S) := R.$$

If $R \in$ **Ring**, then an R-scheme is a scheme S with a map $S \longrightarrow \mathrm{Spec}(R)$. A morphism $S_1 \longrightarrow S_2$ is a morphism commuting with the maps to R. An R-scheme $S = \mathrm{Spec}(A)$ is of finite type if it is finitely generated as an R-algebra.

We state the following theorem on topologizing the points of affine schemes points of schemes of finite type over a topological ring R.

Theorem 1.11. *Let R be a topological ring and let X be an affine scheme of finite type over R. Then there exists a unique way to topologize $X(R)$ such that:*

1. *the topology is functorial in X; that is if $X \to Y$ is a morphism of affine schemes of finite type over R, then the induced map on points $X(R) \to Y(R)$ is continuous;*
2. *the topology is compatible with fibre products; this means that if $X \to Z$ and $Y \to Z$ are morphisms of affine schemes, all of finite type over R, then the topology on $X \times_Z Y(R)$ is exactly the fibre product topology;*
3. *closed immersions of schemes $X \hookrightarrow Y$ correspond to topological embeddings $X(R) \hookrightarrow Y(R)$;*
4. *if $X = \mathrm{Spec}(R[T])$ then $X(R)$ is homeomorphic with R under the natural identification $X(R) \cong R$.*

Explicitly, if $A = \Gamma(X, \mathcal{O}_X)$ then $X(R) = \mathrm{Hom}_{R-\mathbf{alg}}(A, R)$ can be embedded in the product R^A. Give $X(R)$ the topology induced by the product topology on R^A.

If R is Hausdorff or locally compact, then so is $X(R)$.

Proof. See Conrad's note [Con] for the proof. The basic idea is to verify the statement in the case where $X = \mathbb{A}^k$ and then reduce to this case. □

2. Algebraic Groups

2.1. **Group schemes.** For a nice introduction to affine group schemes, consult Waterhouse's book [Wat79]. The notes [Mil12] handle more general situations. Fix a commutative ring k.

Definition 2.1. An **affine group scheme** over k is a functor

$$\boldsymbol{k}\textbf{-algebra} \longrightarrow \textbf{Group}$$

representable by a k-algebra. A **morphism** of affine group schemes $H \longrightarrow G$ is a natural transformation of functors from H to G.

In these notes we will only be interested in affine group schemes (as opposed to, say, elliptic curves), so we will often omit the word "affine."

Concretely, a natural transformation $H \to G$ is just a collection of group homomorphisms

$$H(R) \longrightarrow G(R)$$

for all k-algebras R such that if $R' \to R$ is a k-algebra homomorphism then the following diagram commutes:

$$\begin{array}{ccc} H(R') & \longrightarrow & G(R') \\ \downarrow & & \downarrow \\ H(R) & \longrightarrow & G(R) \end{array}$$

Example 2.2. The additive group $\mathbb{G}_a$ is the functor assigning to each k-algebra R its additive group, $\mathbb{G}_a(R) = (R, +)$. It is representable by the polynomial algebra $k[X]$:

$$\mathrm{Hom}_k\left(k[X], R\right) = R.$$

Example 2.3. The multiplicative group $\mathbb{G}_m$ is the functor assigning to each each k-algebra R its multiplicative group, $\mathbb{G}_m(R) = R^\times$. It is representable by $k[X,Y]/(XY-1)$.

Example 2.4. The general linear group GL_n for $n \geq 1$ is the functor taking a k-algebra R to the group of invertible matrices with coefficients in R. It is an affine group scheme represented by the k-algebra $k[X_{i,j} : 1 \leq i,j \leq n][Y]/(\det(X_{i,j}) \cdot Y - 1)$. Note that $\mathrm{GL}_1 = \mathbb{G}_m$.

Example 2.5. If one wishes to be coordinate free, then for any finite rank free k-module V one can define

$$\mathrm{GL}_V(R) := \{R\text{-module automorphisms } V \to V\}.$$

A choice of isomorphism $V \cong k^n$ induces an isomorphism $\mathrm{GL}_V \cong \mathrm{GL}_n$.

It is useful to have a isolate a few types of morphisms:

Definition 2.6. A morphism $H \to G$ is **injective** or an **embedding** if $\mathcal{O}(G) \to \mathcal{O}(H)$ is surjective.

More concretely, this means that $H = \mathrm{Spec}(R)$, where

$$R = \mathcal{O}(G)/I$$

for some ideal $I \leq \mathcal{O}(G)$.

Remark 2.7. If $H \to G$ is injective, then $H(R) \to G(R)$ is injective for all k-algebras R (this is an easy exercise). However, the converse is not true in general. It is true over a field [Mil12, Proposition 2.2].

We isolate a particularly important class of morphisms with the following definition:

Definition 2.8. A **representation** of an affine group G is a morphism $G \to \mathrm{GL}_V$. It is **faithful** if it is an injection

Definition 2.9. A group scheme G is said to be **linear** if it admits a faithful representation $G \to \mathrm{GL}_V$ for some V.

We will usually be concerned with linear algebraic groups. We shall see below in Theorem 2.13 that this is not much loss of generality if k is a field.

2.2. **Extension and restriction of scalars.** Let $k \to k'$ be a homomorphism of k-algebras. Given a k-algebra R, one obtains a k-algebra $R \otimes_k k'$. Moreover, given a k' algebra R', one can view it as a k-algebra in the tautological manner. This gives rise to a pair of functors

$$\otimes_k k' : k\textbf{-alg} \longrightarrow k'\textbf{-alg}$$
$$k'\textbf{-alg} \longrightarrow k\textbf{-alg}$$

known as **base change** and **restriction of scalars**, respectively.

Analogously, we have a **base change** functor

$$_{k'} : \mathbf{AffSch}_k \longrightarrow \mathbf{AffSch}_{k'}$$

given by $X_{k'}(R') = X(R')$.

Under certain circumstances we also have a **(Weil) restriction of scalars** functor

$$\mathrm{Res}_{k'/k} : \mathbf{AffSch}_{k'} \longrightarrow \mathbf{AffSch}_k$$

given by

$$\mathrm{Res}_{k'/k} X'(R) := X'(k' \otimes_k R).$$

For example, it is enough to assume that k'/k is finite and locally free [BLR90, Theorem 4, §7.6].

These constructions allow us to change the base ring k, and are quite useful. We note that the reason for care in the case of restriction of scalars is that it is not always the case that if X is an affine scheme then $\mathrm{Res}_{k'/k} X$ is again an affine scheme.

Example 2.10. The **Deligne torus** is

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_1 .$$

We have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$. Let V be a real vector space. To give a representation $\mathbb{S} \to \mathrm{GL}(V)$ is equivalent to giving a Hodge structure on V.

Example 2.11. Let d be a square free integer and $L = \mathbb{Q}(\sqrt{d})$. Taking the regular representation of L acting on L with basis $\{1, \sqrt{d}\}$ we see that $Res_{L/k}(\mathbb{G}_m)(\mathbb{Q}) \cong \left\{ \begin{pmatrix} a & db \\ b & a \end{pmatrix} | a^b - db^2 \neq 0 \right\}$.

For a good reference see for example [BLR90].

2.3. **Algebraic groups over a field.** We now assume that k is a field and let $k^{sep} \leq \bar{k}$ be a separable (resp. algebraic) closure of k. Much of the theory simplifies in this case.

One has the following definition:

Definition 2.12. An (affine) **algebraic group** over k is an affine group scheme of finite type over k.

Concretely, G is algebraic if G is represented by a quotient of $k[x_1, \dots, x_n]$ for some n.

One manner in which the theory simplifies in this case is exhibited by the following:

Theorem 2.13. *An algebraic group over k admits a faithful representation, and hence is linear.*

Proof. [Mil12, Theorem 9.1]. ☐

We now discuss the Jordan decomposition.

Definition 2.14. Let k be a perfect field. An element $x \in M_n(\overline{k})$ is said to be:

- **semi-simple** if there exists $g \in \mathrm{GL}_n(\overline{k})$ such that $g^{-1}xg$ is diagonal
- **nilpotent** if there exists $n \in \mathbf{N}$ such that $x^n = 0$
- **unipotent** if $(x - \mathrm{id})$ is nilpotent

For an arbitrary linear group we say that an element $g \in G$ is **semi-simple**, (resp **nilpotent**, resp **unipotent**) if $\phi(g)$ is so for some (any) faithful representation $\phi : G \longrightarrow \mathrm{GL}_n$.

Theorem 2.15. *(Jordan decomposition) Let G be an algebraic group over a perfect field k. Given $x \in G(\overline{k})$ there exist $x_s, x_u \in G(\overline{k})$ such that x_s is semi-simple, x_u is unipotent, $x = x_s x_u = x_u x_s$. Moreover, this decomposition is unique.*

Proof. [Mil12, Theorem 2.8]. ☐

We should point out that, in the theorem above, even though x may be a k point, neither x_s nor x_u need be.

At this point it is useful to introduce another condition on our algebraic groups, namely that of **smoothness**. Rather than take a digression to define this, we will use the following theorem to give an ad-hoc definition:

Theorem 2.16. *An algebraic group G over a field k is **smooth** if it is geometrically reduced, that is, if $\mathcal{O}(G) \otimes_k \bar{k}$ has no nilpotent elements.*

Proof. [Mil12, Proposition 8.3]. ☐

We will also require the notion of connectedness:

Definition 2.17. An affine scheme X is connected if the only idempotents in $\mathcal{O}(X)$ are 0 and 1. A group scheme is connected if its underlying affine scheme is connected.

If $k = \mathbb{Q}$, then G is connected if and only if $G(\mathbb{C})$ is connected as a topological space.

Definition 2.18. Let k be a perfect field and let G be a smooth algebraic group. The **unipotent radical** $R_u(G)$ of G is the maximal connected normal subgroup of G such that $G(\bar{k})$ consists of unipotent elements. The (**solvable**) **radical** is the maximal connected normal subgroup of $G_{\bar{k}}$ such that $G(\bar{k})$ is solvable.

We remark that since a unipotent subgroup is always solvable we always have $R_u(G) \subseteq R(G)$.

Remark 2.19. If k is not perfect then these definitions must be modified. See [Mil12] or [CGP10]. In fact, even if k is perfect, there are alternate, and perhaps better, definitions (see [Mil12]).

Definition 2.20. A smooth connected algebraic group G over a perfect field k is said to be **reductive** if $R_u(G) = \{\mathrm{id}\}$ and **semi-simple** if $R(G) = \{\mathrm{id}\}$.

Example 2.21.

- GL_n is reductive but not semi-simple since its center is normal.
- SL_n is semi-simple (which implies reductive)
- The group of upper triangular matrices in GL_n is not reductive (as it is solvable). We remark that unipotent groups are always upper-triangularizable (as groups) [Bor91, I.4.8].

Suppose that k is a perfect field and G is a reductive group over k. Then

$$G = Z_G G^{\mathrm{der}}$$

where $Z_G \leq G$ is its center and $G^{\mathrm{der}} \leq G$ is the **derived subgroup of** G. It is the algebraic subgroup such that

$$G^{\mathrm{der}}(\bar{k}) = \{xyx^{-1}y^{-1} : x, y \in G(\bar{k})\}$$

We note that since G is reductive, G^{der} is semisimple. One can alternately define G^{der} as the intersection of all normal subgroups $N \leq G$ such that G/N is commutative. We also note that

$$Z_G \cap G^{\mathrm{der}}$$

is the (finite) center of G^{der} [Mil12, XVII.5].

2.4. **Lie Algebras.** Now that we have defined reductive groups, we could ask for a classification of them, or more generally for a classification of morphisms $H \to G$ of reductive groups. The first step in this process is to linearize the problem using objects known as Lie algebras.

Definition 2.22. Let k be a ring. A **Lie algebra** (over k) is a k-module $\mathfrak{g}$ together with a pairing, called the **Lie bracket**

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

which satisfies the following:

(1) $[\cdot, \cdot]$ is bilinear.
(2) $[x, x] = 0$ for all $x \in \mathfrak{g}$.
(3) $[\cdot, \cdot]$ satisfies the Jacobi-identity, that is $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$

Morphisms of Lie algebras are simply k-module maps preserving $[\cdot, \cdot]$.

Remark 2.23. If k' is a k-algebra then $\mathfrak{g} \otimes_k k'$ inherits a Lie algebra structure in a natural manner.

Let $\mathbf{LAG}_k$ denote the category of linear algebraic groups over k and let $\mathbf{LieAlg}_k$ denote the category of Lie algebras over k.

There exists a functor

$$\mathrm{Lie} : \mathbf{LAG}_k \longrightarrow \mathbf{LieAlg}_k$$

defined by:

$$\mathrm{Lie}(G) = \ker(G(k[t]/t^2) \to G(k))$$

where the map $G(k[t]/t^2) \to G(k)$ is induced by the map $k[t]/t^2 \to k$ sending t to 0. We will define the bracket operation shortly. Usually one uses gothic German letters to denote Lie algebras, e.g.

$$\mathfrak{g} := \mathrm{Lie}(G).$$

Example 2.24. The kernel of the map $\mathrm{GL}_n(k[t]/t^2) \to \mathrm{GL}_n(k)$ is easily seen to be $\mathrm{id} + tA$ where $A \in M_n(k)$. Thus

$$\mathfrak{gl}_n = \mathrm{Lie}(\mathrm{GL}_n) \simeq M_n.$$

One can define the bracket in an ad-hoc manner as

$$[X, Y] := XY - YX.$$

We define the bracket in an ad-hoc manner for any linear algebraic group G by choosing a faithful representation

$$G \hookrightarrow \mathrm{GL}_n$$

and hence a map

$$\mathrm{Lie}(G) \longrightarrow \mathfrak{gl}_n$$

and defining the bracket on $\mathrm{Lie}(G)$ to be the restriction of the bracket on $\mathfrak{gl}_n$. This is of course an unsatisfactory definition as it is not intrinsic to $\mathfrak{g}$, but it will do for our purposes.

Example 2.25.

- The special linear group SL_n has lie algebra $\mathfrak{sl}_n = \{X \in M_n | Tr(X) = 0\}$.
- Let k be a field of characteristic not equal to 2 and let $\mathrm{SO}_{n,n}$ be the orthogonal group whose points in a k-algebra R are given by
$$\mathrm{SO}_{2n}(R) = \{g \in \mathrm{GL}_n(R) : g^t S g = S\}$$
where S is the symmetric matrix $\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ has Lie algebra
$$\mathfrak{so}_{2n} = \{X \in M_n | X^t S + SX = 0\}$$

2.5. **Tori.** Throughout this subsection we assume that k is a field.

Definition 2.26. An **algebraic torus** is a linear algebraic group T such that $T_{k^{sep}} \cong \mathbb{G}_m^n$ for some n. The integer n is called the **rank** of the torus.

Definition 2.27. A **character** of an algebraic group G is an element of $X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$. A **co-character** or one parameter subgroup is an element of $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$.

For k-algebras k' one usually abbreviates $X^*(G)_{k'} := X^*(G_{k'})$, etc.

One indication of the utility of the notion of characters is the following theorem:

Theorem 2.28. *The association*
$$T \longmapsto X^*(T)_{k^{sep}}$$
defines a contravariant equivalence of categories between the category of algebraic tori defined over k and finite dimensional $\mathbb{Z}$-torsion free $\mathbb{Z}[\mathrm{Gal}(k^{sep}/k)]$-modules.

Example 2.29.

- We define a special orthogonal group SO_2 by stipulating that for $\mathbb{Q}$-algebras R one has
$$\mathrm{SO}_2(R) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ and } a^2 + b^2 = 1 \right\}$$
Over any field containing a square root of -1 we can diagonalize this via:
$$\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} a - bi & 0 \\ 0 & a + bi \end{pmatrix}$$
In other words, $\mathrm{SO}_{2\mathbb{Q}(i)} \cong \mathbb{G}_m$.
- If L/k is any (separable) field extension then $\mathrm{Res}_{L/k}(\mathbb{G}_m)$ is an algebraic torus. Moreover one can show that:
$$X^*(\mathrm{Res}_{L/k}(\mathbb{G}_m))_L \simeq \bigoplus_{\tau} \mathbb{Z}_\tau$$
where the summation runs over the embeddings $\tau : L \to \bar{k}$ of L into an algebraic closure of k; this has a natural Galois action. In particular this example illustrates the connection between the "descent data" for etale algebras and that for the tori coming from their multiplicative group (see [BLR90], for example, for the definition of descent data).
- Let L/k be a separable extension and let
$$\mathrm{N}_{L/k} : \mathrm{Res}_{L/k}\mathbb{G}_m \longrightarrow \mathbb{G}_m$$
be the norm map; it is given on points by $x \mapsto \prod_{\tau \in \mathrm{Hom}_k(L, \bar{k})} \tau(x)$. Then the kernel of $\mathrm{N}_{L/k}$ is an algebraic torus. When $L = \mathbb{Q}(i)$ and $k = \mathbb{Q}$ this torus is isomorphic to the group SO_2 constructed above.

In the above examples we see that though an algebraic torus T satisfies $T_{k^{sep}} \cong \mathbb{G}_m^n$, it may not be the case that $T \cong \mathbb{G}_m^n$. The following definitions are therefore useful.

Definition 2.30. An algebraic torus T over a field k is said to be **split** if $T \simeq \mathbb{G}_m^n$ over k or if equivalently $X^*(T)_k \cong \mathbb{Z}^{\mathrm{rank}(T)}$. An algebraic torus T is said to be **anisotropic** if $X^*(T)_k = \{\mathrm{id}\}$.

Any torus T can be decomposed as $T = T^{\mathrm{ani}}T^{\mathrm{spl}}$ where $T^{\mathrm{spl}} \leq T$ is the maximal split subtorus, $T^{\mathrm{ani}} \leq T$ is the maximal anisotropic subtorus, and $T^{\mathrm{ani}} \cap T^{\mathrm{spl}}$ is finite.

2.6. **Maximal tori in reductive groups.** Let G be a reductive group over a perfect field k. It is a remarkable fact, known informally as Cartan-Weyl or highest weight theory, that the representation theory of G can be recovered by restricting representations to large abelian subgroups of G, namely maximal tori:

Definition 2.31. A torus $T \leq G$ is **maximal** if $T_{\bar{k}}$ is maximal among all tori of $G_{\bar{k}}$.

We will not discuss the highest weight theory, but the principle of studying representations of G by restricting various objects to maximal tori will play a role in what follows. We will also recall in the following subsection how one can recover $G_{\bar{k}}$ from a maximal torus in G along with certain auxilliary data (see Theorem 2.42).

Theorem 2.32. *Every reductive group over has a maximal torus* [Spr09, CH2 3.1.1]. *All maximal tori in $G(\overline{k})$ are conjugate under $G(\overline{k})$* [Bor91, IV.11.3].

In view of the second assertion of Theorem 2.32, the rank of a maximal torus of G is an invariant of G; it is known as the **rank** of G.

For the remainder of lecture G is a connected reductive group and $T \leq G$ is a maximal torus.

Definition 2.33. The **Weyl Group** of T in G is $W(G,T) := N_G(T)/Z_G(T)$, where $N_G(T)$ is the normalizer of T in G and $Z_G(T)$ is the centralizer of T in G.

Remark 2.34. The group $N_G(T), Z_G(T) \leq G$ are algebraic subgroups. The Weyl group is a finite group scheme in the sense that $W(G,T)(\bar{k}) = N_G(T)(\bar{k})/Z_G(T)(\bar{k})$ is finite.

Example 2.35.

- One maximal torus in GL_n is the torus of diagonal matrices. In this case we have that $W(G,T) \cong S_n$; it acts on the torus of diagonal matrices by by permuting the entries.
- If F/k is an etale k-algebra of rank n (for example a field extension of degree n) then choosing a basis for k we obtain an embedding
$$\mathrm{Res}_{F/k}\mathbb{G}_m \hookrightarrow \mathrm{GL}_n\,.$$
In this case $W(\mathrm{GL}_n, T)(k) \cong \mathrm{Aut}(F/k)$. Every maximal torus in GL_n arises in this manner for some F/k.

2.7. **Root data.** Let G be a reductive group over a perfect field k and let $T \leq G$ be a maximal torus. Our next goal is to associate to such a pair (G,T) a root datum $\Psi(G,T) = (X, V, \Phi, \Phi^\vee)$ that will characterize $G_{\bar{k}}$.

Let $\mathfrak{g}$ denote the Lie algebra of G. The natural action of G on $\mathfrak{g}$ is known as the *adjoint representation* of G on $\mathfrak{g}$:
$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$
For example, when $G = \mathrm{GL}_{n/k}$ this is the usual action of GL_n on M_n by conjugation.

If G contains a maximal torus that is split we say that G is **split**. Assume that G is split. Then $\mathrm{Ad}(T)$ consists of commuting semisimple elements, and therefore the action of T on $\mathfrak{g}$ is diagonalizable. For a character $\alpha \in X^*(T)$, let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k)\}. \tag{2.7.1}$$

Definition 2.36. The nonzero $\alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are called the *roots* of T in G. We let $\Phi(G,T)$ be the (finite) set of all such roots α, and call the corresponding $\mathfrak{g}_\alpha$ *root spaces.*

Theorem 2.37. *Let $T \subset G$ be as above, and let $\mathfrak{t} = \mathrm{Lie}(T)$. Then*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha.$$

Furthermore, each of the root spaces is one-dimensional [Spr09, Corollary 8.1.2].

Let V be the real vector space $\langle \Phi \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, where $\langle \Phi \rangle \subset X^*(T)(k)$ denote the ($\mathbb{Z}$-linear) span of $\Phi = \Phi(G,T)$. Then the pair (Φ, V) is a *root system*, according to the following:

Definition 2.38. Let V be a finite dimensional $\mathbb{R}$-vector space, and Φ a subset of V. We say that (Φ, V) is a *root system* if the following three conditions are satisfied:

(R1) Φ is finite, does not contain 0, and spans V;
(R2) For each $\alpha \in \Phi$ there exists a reflection s_α relative to α (i.e. an involution s_α of V with $s_\alpha(\alpha) = -\alpha$ and restricting to the identity on a subspace of V of codimension 1) such that $s_\alpha(\Phi) = \Phi$;
(R3) For every $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

A root system (Φ, V) is said to be of *rank* $\dim_{\mathbb{R}} V$, and to be *reduced* if for each $\alpha \in \Phi$, $\pm\alpha$ are the only multiples of α in Φ. The *Weyl group* of (Φ, V) is the subgroup of $\mathrm{GL}(V)$ generated by the reflections s_α:

$$W(\Phi, V) := \langle s_\alpha : \alpha \in \Phi \rangle \subseteq \mathrm{GL}(V).$$

Remark 2.39. If (Φ, V) is the root system associated with the split torus $T \leq G$ then (Φ, V) is reduced and

$$W(\Phi, V) \cong W(G,T)(k).$$

Let (Φ, V) be the root system associated with $T \subset G$. There exists a pairing

$$(\ ,\ ) : V \times V \to \mathbb{C}$$

for which the elements in the Weyl group become orthogonal transformations. Thus if $\alpha \in \Phi$ there exists a unique $\alpha^\vee \in X_*(T)$ such that

$$\langle -, \alpha^\vee \rangle := \alpha^\vee(-) = \frac{2(-,\alpha)}{(\alpha,\alpha)}$$

as maps $X^*(T) \to \mathbb{C}$. Let $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$, and $V^\vee := \langle \Phi^\vee \rangle \otimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 2.40. *The pair $(\Phi^\vee, V^\vee)$ is a root system.* □

A fundamental result to be stated below is that the quadruple $\Psi = (X^*(T), X_*(T), \Phi, \Phi^\vee)$ attached to $T \subset G$ contains enough information to characterize G, at least over $\overline{k}$.

Definition 2.41. A *root datum* is a quadruple $(X, Y, \Phi, \Phi^\vee)$ consisting of a pair of free abelian groups X, Y with a perfect pairing $\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}$, together with finite subsets $\Phi \subset X$, $\Phi^\vee \subset Y$ in 1-to-1 correspondence ($\Phi \ni \alpha \leftrightarrow \alpha^\vee \in \Phi^\vee$) such that

- $\langle \alpha, \alpha^\vee \rangle = 2$;
- If for each $\alpha \in \Phi$, we let $s_\alpha : X \to X$ be defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$, then $s_\alpha(\Phi) \subset \Phi$, and the group $\langle s_\alpha \mid \alpha \in \Phi \rangle$ generated by $\{s_\alpha\}$ is finite.

We say that a root datum is *reduced* if $\alpha \in \Phi$ only if $2\alpha \notin \Phi$.

An isomorphism of root data $(X, Y, \Phi, \Phi^\vee) \xrightarrow{\sim} (X', Y', \Phi', (\Phi')^\vee)$, is a group isomorphism $X \xrightarrow{\sim} X'$ inducing dual isomorphisms sending Φ to Φ' and $\Phi^\vee$ to $(\Phi')^\vee$, respectively.

Theorem 2.42 (Chevalley, Demazure)**.** *Assume $k = \overline{k}$. The map*

$$\left\{\begin{matrix} \textit{isomorphism classes of} \\ \textit{connected reductive groups} \\ \textit{over } k \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \textit{isomorphism classes of} \\ \textit{reduced root data} \end{matrix}\right\}$$

induced by $G \mapsto \Psi(G,T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$ is bijective.

If $(X, Y, \Phi, \Phi^\vee)$ is a root datum, then so is $(Y, X, \Phi^\vee, \Phi)$. The associated reductive algebraic group over $\mathbb{C}$ is denoted $\widehat{G}$ and is called the **complex dual of** G. We note that there is an isomorphism

$$W(G,T)_{\bar{k}} \xrightarrow{\sim} W(\widehat{G}, \widehat{T})(\mathbb{C}) \qquad (2.7.2)$$
$$s_\alpha \longmapsto s_{\alpha^\vee}.$$

Remark 2.43. Our original attributions for the above thereom were incorrect. Brian Conrad corrected us as follows: "Demazure introduced the notion of root datum so as to systematically keep track of a nontrivial central torus in the theory, but over an algebraically closed field all of the nontrivial content for the existence and isomorphism parts of the story is in the semisimple case, which is entirely due to Chevalley [Che58]. Demazure's contribution in [DG74, Expose XXII] was to solve the Existence and Isomorphism problems over $\mathbb{Z}$ (and so over any scheme). Actually, Chevalley did make constructions of everything over $\mathbb{Z}$, but without an intrinsic characterization of what he was doing (and without an Isomorphism Theorem) – this was the initial motivation for Demazure's work, to figure out the intrinsic significance of Chevalley's constructon over $\mathbb{Z}$."

One might ask if one could define in a natural way a morphism of root data, and thereby use root data to classify morphisms between reductive groups. If such a definition exists, we do not know it. However, it is the case that a great deal of information about morphisms between reductive groups can be deduced by considering root data. A systematic account of this for classical groups is given in Dynkin's work [Dyn52].

Example 2.44. $G = \mathrm{GL}_n$. The group of diagonal matrices

$$T(R) := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mid t_i \in R^\times \right\}$$

is a maximal torus in G. The groups of characters and of cocharacters of T are both isomorphic to $\mathbb{Z}^n$ via

$$(k_1, \dots, k_n) \longmapsto \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_n^{k_n} \right)$$

and

$$(k_1, \dots, k_n) \longmapsto \left(t \mapsto \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n} \end{pmatrix} \right),$$

respectively. Note that with these identifications, the natural pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ corresponds to the standard "inner product" in $\mathbb{Z}^n$. The roots of G relative to T are the characters

$$e_{ij} : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}$$

for every pair of integers $(i,j) \in \{1, \dots, n\}^2$ with $i \neq j$, and the corresponding root spaces $\mathfrak{gl}_{n, e_{ij}}$ are the linear span of the $n \times n$ matrix with all entries zero except the (i,j)-th component. The coroot $e_{ij}^\vee$ associated with e_{ij} is the map sending t to the diagonal matrix with t in the ith entry and t^{-1} in the jth entry and 1 in all other entries.

Example 2.45. $G = \mathrm{Sp}_{2n}$. These are the split *symplectic groups*, whose R-valued points are given by

$$G(R) = \{g \in \mathrm{GL}_{2n}(R) \mid g^t J_{n,n} g = J_{n,n}\},$$

where $J_{n,n}$ is the block matrix $\left(\begin{smallmatrix} & J_n \\ -J_n & \end{smallmatrix}\right)$, with J_n being the $n \times n$ matrix $\left(\begin{smallmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{smallmatrix}\right)$. We discuss Sp_4. Its Lie algebra is

$$\mathfrak{sp}_4 = \{X \in \mathfrak{gl}_4 \mid X^t J_{2,2} + J_{2,2} X = 0\}.$$

The group of diagonal matrices

$$T(R) = \left\{ x(t_1, t_2) := \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \mid t_1, t_2 \in R^\times \right\}$$

is a maximal torus in $G = \mathrm{Sp}_4$. Consider the characters $e_1 : x(t_1, t_2) \mapsto t_1$ and $e_2 : x(t_1, t_2) \mapsto t_2$. Then the set of roots of G relative to T is

$$\Phi(G, T) = \{\pm(e_1 - e_2), \pm(e_1 + e_2), \pm 2(e_1 + e_2), \pm 2e_2\}.$$

The corresponding root spaces are easily computed. For example:

$$\begin{aligned} \mathfrak{sp}_{4,e_1-e_2} &= \{\left(\begin{smallmatrix} A & \\ & A' \end{smallmatrix}\right) \mid A = \left(\begin{smallmatrix} & v \\ & \end{smallmatrix}\right), A' = \left(\begin{smallmatrix} & -v \\ & \end{smallmatrix}\right)\} \\ \mathfrak{sp}_{4,2e_2} &= \{\left(\begin{smallmatrix} & B \\ & \end{smallmatrix}\right) \mid B = \left(\begin{smallmatrix} & b \\ & \end{smallmatrix}\right)\} \\ \mathfrak{sp}_{4,e_1+e_2} &= \{\left(\begin{smallmatrix} & B \\ & \end{smallmatrix}\right) \mid B = \left(\begin{smallmatrix} & \\ c & \end{smallmatrix}\right)\}. \end{aligned}$$

The coroots are given by $(ae_1 + be_2)^\vee(t) = \begin{pmatrix} t^a & & & \\ & t^b & & \\ & & t^{-b} & \\ & & & t^{-a} \end{pmatrix}$ We thus have

$$(e_1 - e_2)^\vee(t) = \begin{pmatrix} t & & & \\ & t^{-1} & & \\ & & t & \\ & & & t^{-1} \end{pmatrix}, \quad (2e_2)^\vee(t) = \begin{pmatrix} 1 & & & \\ & t^2 & & \\ & & t^{-2} & \\ & & & 1 \end{pmatrix},$$

etc.

Remark 2.46. There is a complete classification of all the possible reduced irreducible root systems. This is one of the main outcomes of the Weyl-Cartan theory. The exhaustive list is A_ℓ ($\ell \geq 1$), B_ℓ ($\ell \geq 1$), C_ℓ ($\ell \geq 3$) and D_ℓ ($\ell \geq 4$), corresponding to $\mathrm{SL}_{\ell+1}$, $\mathrm{SO}_{2\ell+1}$, $\mathrm{Sp}_{2\ell}$ and $\mathrm{SO}_{2\ell}$, respectively, and the exceptional E_6, E_7, E_8, F_4 and G_2.

2.8. **Borel subgroups.** We assume in this subsection that G is a reductive group over a perfect field k.

Definition 2.47. A closed subgroup $B \leq G$ is a *Borel subgroup* if $B_{\bar{k}} \leq G_{\bar{k}}$ is a maximal connected solvable subgroup. A closed subgroup $P \subset G$ is a *parabolic subgroup* if it contains a Borel subgroup.

Example 2.48. Conjugacy classes of parabolic subgroups of GL_n are parametrized by partitions of n.

Theorem 2.49. *A closed subgroup $P \subset G$ is parabolic if and only if the quotient G/P is representable by a projective scheme.*

Example 2.50. If $B \leq \mathrm{SL}_2$ is the subgroup of upper triangular matrices, then for k-algebras R one has an isomorphism

$$\begin{aligned} \mathrm{SL}_2 / B(R) &\longrightarrow \mathbb{P}^1(R) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\longmapsto [a : c] \end{aligned}$$

Parabolics always admit a **Levi decomposition**

$$P = MN$$

where N is the unipotent radical of P and $M \leq P$ is a reductive subgroup.

Remark 2.51. The group $G_{\bar{k}}$ trivially has Borel subgroups, but G need not have Borel subgroups. For example, if B is a division algebra over k and G is the algebraic group defined by

$$G(R) = (B \otimes_k R)^\times,$$

then G does not have a Borel subgroup.

Definition 2.52. A reductive group G is said to be *split* if there exists a maximal split torus $T \subset G$ (over k); it is said to be *quasi-split* if it contains a Borel subgroup.

Note that G is split only if it is quasi-split, but that the converse is not true, as evidenced by the following:

Example 2.53. Take $G = \mathrm{U}(1,1)$, that is

$$G(R) = \{g \in \mathrm{GL}_2(\mathbb{C} \otimes_{\mathbb{R}} R) \ : \ \overline{g}^t \left(\begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix} \right) g = \left(\begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix} \right)\}.$$

Then the subgroup of upper triangular matrices in G is a Borel subgroup of G. Thus G is quasi-split. It is not, however, split.

3. Automorphic representations

In this section we give the definitions of admissible and automorphic representations. At this point the definitions will undoubtably seems opaque and unmotivated to the uninitiated, but we will spend considerable time in the following sections elaborating on them. We recall that if R is a Hausdorf, locally compact topological ring and X an affine scheme of finite type over R then $X(R)$ is endowed with a canonical Hausdorf, locally compact topology by Theorem 1.11; it is the same as the toplogy obtained by choosing a closed immersion $X \to \mathbb{A}^k$ and giving $X(R) \hookrightarrow R^k$ the subspace topology.

3.1. **Haar measures.** If G is a locally compact group (for example $\mathrm{GL}_n(\mathbb{A}_F)$) then there exists a positive regular borel measure $d_L g$ on G that is left invariant under the action of G. :

$$\int_G f(xg) d_L g = \int_G f(g) d_L g \qquad \text{for all } x \in G.$$

Moreover, this measure is unique up to scalars. A **left Haar measure** is a choice of such a measure. There is also a right invariant positive borel measure $d_R g = d_L(g^{-1})$, again unique up to scalars. Such a measure is known as a **right Haar measure**.

Definition 3.1. A locally compact group G is **unimodular** if there is a (nonzero) constant C such that $d_R g = C d_L g$.

Example 3.2. All abelian groups, reductive groups and unipotent groups are uni-modular.

The points of Borel subgroups are, in general, not unimodular. For example, if $B \leq \mathrm{GL}_2$ is the Borel subgroup of upper triangular matrices then we can write

$$B(\mathbb{R}) = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} x^{\frac{1}{2}} & xy^{\frac{1}{2}} \\ 0 & y^{\frac{1}{2}} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R}) \right\}$$

With respect to this decomposition one can take $d_L g = \frac{dxdydu}{y^2|u|}$ and $d_R g = \frac{dxdydu}{y|u|}$.

For the remainder of this section we fix a Haar measure dg on $G(\mathbb{A}_F)$. Some of the constructions below depend on this choice, but only up to a scalar multiple.

3.2. **Non-archimedian Hecke algebras.** For the remainder of this section we let G be an affine group scheme of finite type over the ring of integers $\mathcal{O}_F$ of a global field F such that G_F is reductive.

Definition 3.3. Let S be a set of nonarchimedian places of F. A function f on $G(F_S)$ is **smooth** if it is locally constant. Similarly, if S contains all archimedian places of F, then a function on $G(\mathbb{A}_F^S)$ is **smooth** if it is locally constant.

With this definition in mind we can define as usual the space $C_c^\infty(G(F_S))$ of smooth, compactly supported functions on $G(F_S)$, etc. In particular, if F is a number field we define

$$\mathcal{H}^\infty := C_c^\infty(G(\mathbb{A}_F^\infty)$$

and if F is a function field we define

$$\mathcal{H} := C_c^\infty(G(\mathbb{A}_F)).$$

These are algebras under convolution of functions:

$$f * h(g) := \int_{G(\mathbb{A}_F)} f(x)h(x^{-1}g)dx. \tag{3.2.1}$$

In the number field case, $\mathcal{H}^\infty$ is known as the **non-archimedian Hecke algebra** or the **Hecke algebra away from infinity**. In the function field case, $\mathcal{H}$ is known as the **Hecke algebra**. If S is a set of nonarchimedian places of F then we let

$$\mathcal{H}_S := C_c^\infty(G(F_S)).$$

Let $\mathbb{1}_Y$ denote the characteristic function of a set Y. In the number field case one has

$$C_c^\infty(G(\mathbb{A}_F^\infty)) = \varinjlim_{S \not\supseteq \infty} C_c^\infty(G(F_S)) \otimes_{v \notin S \cup \infty} \mathbb{1}_{G(\mathcal{O}_v)}\,. \tag{3.2.2}$$

where the limit is over all finite sets of places of F not containing the infinite places, partially ordered by inclusion. In the function field case the analogous statement is true, though in this case there is no need to exclude the infinite places.

For the following definition, assume we are in the number field case:

Definition 3.4. A representation (π, V) of $\mathcal{H}^\infty$ is **admissible** if it is nondegenerate and for all compact open subgroups $K^\infty \leq G(\mathbb{A}_F^\infty)$ the space $V^{K^\infty} = \pi(\mathbb{1}_{K^\infty})V$ is finite dimensional.

Here an A-module M is **nondegenerate** if any element of M can be written as

$$a_1 m_1 + \cdots + a_k m_k$$

for $a_i \in A$ and $m_i \in M$. Of course, this would be trivial if $\mathcal{H}^\infty$ had an identity element, but it does not. It does however, admit approximations to the identity (see §7.3).

We make the analogous definition in the function field case, and also in the local case, i.e. where $\mathbb{A}_F^\infty$ is replaced by F_v for some non-archimedian place v of F.

3.3. **Archimedian Hecke algebras.** Assume for the moment that F is a number field. Then $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ is a real reductive Lie group (in other words, the real points of a reductive group over $\mathbb{R}$). We let

$$K_\infty \leq G(\mathbb{R} \otimes_{\mathbb{Q}} F)$$

be a maximal compact subgroup.

Example 3.5. If $G = \mathrm{GL}_2$ any maximal compact subgroup is conjugate to $K_\infty = \mathrm{O}_2(\mathbb{R})$.

Let

$$\mathcal{H}_\infty := \mathcal{H}(G(\mathbb{R} \otimes_{\mathbb{Q}} F), K_\infty) \tag{3.3.1}$$

be the convolution algebra of distributions of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ supported on K_∞. It is known as the **archimedean Hecke algebra** or the **Hecke algebra at infinity**.

Definition 3.6. A *fundamental idempotent* in $\mathcal{H}_\infty$ is an element of the form

$$\mathbb{1}_\sigma = \frac{1}{d(\sigma)\mathrm{meas}(K_\infty)} \chi_\sigma dK_\infty,$$

where $\sigma : K_\infty \to \mathrm{Aut}(V)$ is a representation of degree $d(\sigma) < \infty$, χ_σ is its character, and dK_∞ denotes a Haar measure giving unit volume to K_∞.

The convolution of $f \in C_c^\infty(G(\mathbb{R}))$ with a fundamental idempotent $\xi_\sigma \in \mathcal{H}_\infty$ is given by the formula

$$f * \xi_\sigma = \int_{K_\infty} f(\kappa)\xi_\sigma(\kappa) d(\sigma)^{-1} dK_\infty.$$

Definition 3.7. A continuous representation π of $G(F_\infty)$ on a Hilbert space V is **admissible** if for all irreducible representations σ of K the space $\pi(\mathbb{1}_\sigma)V$ is finite dimensional.

3.4. **Global Hecke algebras.** In the function field case, the global Hecke algebra is simply $\mathcal{H}$, defined as above. In the number field case, the global Hecke algebra is

$$\mathcal{H} := \mathcal{H}_\infty \otimes \mathcal{H}^\infty.$$

In the number field case, a representation (π, V) of $\mathcal{H}$ evidently decomposes as an exterior tensor product of representations (π_∞, V_∞) of $\mathcal{H}_\infty$ and $\mathcal{H}^\infty$. Such a representation (π, V) is called **admissible** if (π_∞, V_∞) and (π^∞, V^∞) are admissible.

In the number field case, let

$$A_G \leq Z_G(F_\infty) \tag{3.4.1}$$

denote the identity component of the real points of the greatest $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}}(Z_G)$. In the function field case[1], choose a single infinite place ∞_0 of F and let

$$A_G := Z_G(F_{\infty_0}).$$

Example 3.8. If $G = \mathrm{GL}_{2/\mathbb{Q}}$ then $A_G = \mathbb{R}_{>0}^\times I$, where I is the identity matrix.

Consider the space $L^2(G(F)A_G \backslash G(\mathbb{A}_F))$, where the Hermitian pairing is given by

$$(f_1, f_2) = \int_{G(F)A_G \backslash G(\mathbb{A}_F)} f_1(g)\overline{f_2(g)} dg.$$

Here the measure is induced by a Haar measure on $A_G \backslash G(\mathbb{A}_F)$; we will see later that $G(F)$ acts properly discontinuously on $A_G \backslash G(\mathbb{A}_F)$ and hence we obtain a measure on $G(F)A_G \backslash G(\mathbb{A}_F)$ by choosing a fundamental domain for the action of $G(F)$.

Remark 3.9. The reason for introducing A_G is that $G(F)A_G \backslash G(\mathbb{A}_F)$ has finite volume, whereas $G(F) \backslash G(\mathbb{A}_F)$ has finite volume if and only if the center of G is anisotropic.

The space $L^2(G(F)A_G \backslash G(\mathbb{A}_F))$ carries a natural action R of $\mathcal{H}$ by convolution:

$$\begin{aligned} R : \mathcal{H} \times L^2(G(F)A_G \backslash G(\mathbb{A}_F)) &\longrightarrow L^2(G(F)A_G \backslash G(\mathbb{A}_F)) \\ (f, \phi) &\longmapsto \left(g \mapsto \int_{G(\mathbb{A}_F)} \phi(gh) f(h) dh \right) \end{aligned}$$

[1] there are perhaps better ways to define A_G in the function field case

Here we regard ϕ as a function on $G(\mathbb{A}_F)$ trivial on A_G.

An important theorem of Harish-Chandra states that for any $\varphi \in L^2(G(F)A_G\backslash G(\mathbb{A}_F))$, the (dense) subspace of K_∞-finite vectors in $\overline{R(G(\mathbb{A}_F))\varphi}$ is admissible. Here the closure is taken with respect to the Hilbert space topology. We will return to this point, and the definition of K_∞-finite, in later sections.

In any case, we can finally formally state the following definition:

Definition 3.10. An *automorphic representation* of $G(\mathbb{A}_F)$ is an admissible representation of $\mathcal{H}$ which is isomorphic to a subquotient of $L^2(G(F)A_G\backslash G(\mathbb{A}_F))$.

Remark 3.11. One might ask why we restrict our attention to reductive groups G when defining automorphic representations. To explain this let us recall that a general algebraic group G over a number field F (not necessarily affine) has a unique linear algebraic subgroup $G^{aff} \trianglelefteq G$ such that $G^A := G/G^{aff}$ is an abelian variety by a famous theorem of Chevalley. Moreover, $G^{\mathbb{A}}$ is the semidirect product of its unipotent radical U and its maximal reductive quotient. Thus we might as well start by exploring what one might mean by an automorphic representation of a unipotent group and an abelian variety.

Consider an abelian variety A. In [Con] one finds a proof of a result of Weil stating that $A(\mathbb{A}_F)$ can still be defined as a topological space. Moreover $A(F) \leq A(\mathbb{A}_F)$ is still a discrete subgroup. However, it fails to be closed as soon as it is infinite [Con, Theorem 4.4] because A is proper over F. Thus the quotient $A(F)\backslash A(\mathbb{A}_F)$ is not Hausdorff, and dealing with such an object would probably be complicated.

Regarding unipotent groups, they can be thought of as a group obtained by extending the trivial group by copies of $\mathbb{G}_a$, the additive group. Let's consider the adelic quotient $\mathbb{G}_a(F)\backslash\mathbb{G}_a(\mathbb{A}_F)$. The irreducible representations of this group are all characters, and if we choose one nontrivial character ψ then via Pontryagin duality one sees that they are in bijection with F:

$$\begin{aligned} F &\xrightarrow{\sim} \widehat{F\backslash\mathbb{A}_F} \\ m &\longmapsto (x \longrightarrow \psi(mx)). \end{aligned}$$

More interesting representations can be obtained by considering, for example, the Heisenburg group, which plays a role in the theta correspondence, an important tool in automorphic representation theory that we unfortunately do not cover in these notes. It may be that there is no fundamental reason for concentrating on the reductive case rather than the affine case, although one might expect by comparison with the theory of actions of algebraic groups that certain things become harder if we try to work with general groups.[2]

4. Nonarchimedian Hecke algebras

In §3.2 we defined the nonarchimedian Hecke algebra F. In this section we discuss this algebra in more detail.

In this section we assume that F is a number field for notational simplicity; our discussion goes over with only notational changes in the function field case. In the number field case the nonarchimedian Hecke algebra was defined to be

$$\mathcal{H}^\infty := C_c^\infty(G(\mathbb{A}_F^\infty)),$$

where the subscript c indicates functions of compact support and the superscript ∞ indicates smooth functions. In this context, smooth means locally constant.

[2]M. Kim (personal communication, 10/2014) had the following interesting comment: " One justification for concentrating on the reductive case is that the groups arising as images of motivic Galois representations are conjectured to be reductive, but this is just for *pure* motives. Part of the difficulty with mixed motives could well be the lack of a uniform automorphic theory."

Lemma 4.1. *Any element $f \in C_c^\infty(G(\mathbb{A}_F^\infty))$ can be expressed as a finite linear combination of characteristic functions*

$$f = \sum_i c_i \, \mathbb{1}_{K^\infty a_i K^\infty}$$

for $K^\infty \subseteq G(\mathbb{A}_F^\infty)$ a compact open subgroup, $a_i \in G(\mathbb{A}_F)$ and $c_i \in \mathbb{C}$. □

Example 4.2. All compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}^\infty)$ are of the form

$$K_S \prod_{p \notin S} \mathrm{GL}_n(\mathbb{Z}_p)$$

for S a finite set of finite primes and K_S is a compact open subgroup of $\mathrm{GL}_n(\mathbb{Q}_S)$. The subgroup $\mathrm{GL}_n(\widehat{\mathbb{Z}}) = \prod_p \mathrm{GL}_n(\mathbb{Z}_p) \leq \mathrm{GL}_n(\mathbb{A}_\mathbb{Q}^\infty)$ is a maximal compact open subgroup, and all maximal compact open subgroups are conjugate to this maximal compact open subgroup [Ser06, Chapter IV, Appendix 1]. Examples of nonmaximal compact open subgroups are given by the kernel of the reduction map

$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/m)$$

for integers m.

If $a \in G(\mathbb{A}_F^\infty)$, then $a \in G(\widehat{\mathcal{O}}_F^S)$ for some finite set of place S. If $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup, then, upon enlarging S we can assume that $K^S = G(\widehat{\mathcal{O}}_F^S)$. For such a choice of S we have

$$\mathbb{1}_{K^\infty a K^\infty} = \mathbb{1}_{K_S a_S K_S} \otimes \mathbb{1}_{K^S}$$

for some finite set of finite places S. This reduces the study of nonarchimedian Hecke algebras to the study of the local Hecke algebras

$$C_c^\infty(G(F_v))$$

as v varies over nonarchimedian places v of F.

4.1. **Convolution.** As in (3.2.1) above, if $f, h \in C_c^\infty(G(F_v))$, then

$$f * h(g) := \int_{G(F_v)} f(x) h(x^{-1}g) dx.$$

Assume that v is nonarchimedian. In this case the convolution admits a purely combinatory definition. To state it, fix a compact open subgroup $K \leq G(F_v)$. Then for $\gamma \in G(F_v)$ we write

$$\mathbb{1}_\gamma := \mathbb{1}_{K\gamma K}\,.$$

We then have

$$1_\alpha * 1_\beta = \sum_{K\gamma K \in K \backslash G(F_v)/K} c_{\alpha,\beta,\gamma} 1_\gamma,$$

where the $c_{\alpha,\beta,\gamma}$ are defined as follows: put $K_\alpha = \alpha K \alpha^{-1} \cap K$, which is compact and open. It is thus of finite index in K. So we can write

$$K = \coprod_i x_i K_\alpha$$

for some finite number of $x_i \in K$. Similarly write

$$K = \coprod_j y_j \beta K_\beta$$

Then $c_{\alpha,\beta,\gamma}$ is the number of pairs (i, j) such that $\gamma x_i \alpha y_j \beta \in K$. For this see [Shi94, Chapter 3 §1].

4.2. **The spherical Hecke algebra.** Let G be a reductive group over F_v, where v is a nonarchimedian place of F. In general there can be several conjugacy classes of maimal compact open subgroups of $G(F_v)$. For example, in Sp_4 there are two conjugacy classes.

Definition 4.3. A reductive group G over F_v is **unramified** if it is quasi-split and splits over an unramified extension of F_v.

If G is unramified then there is a there exists a canonical conjugacy class of maximal compact open subgroups of $G(F_v)$: the class consisting of **hyperspecial** maximal compact subgroups. The hyperspecial maximal compact subgroup is of the form $\mathcal{G}(\mathcal{O}_{F_v})$ where $\mathcal{G}$ is a flat, linear group scheme such that $\mathcal{G}_{F_v} = G$ and $\mathcal{G}_{\mathcal{O}_v/\varpi_v}$ are both reductive and have the same "type." See [Tit79] and [Yu].

If G is unramified and $K \leq G(F_v)$ is a hyperspecial subgroup, then

$$C_c^\infty(G(F_v)//K)$$

is known as the **spherical Hecke algebra**.

Example 4.4. Let $G = \mathrm{GL}_2$. The spherical Hecke algebra in this case is

$$C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p) \mathbin{/\!/} \mathrm{GL}_2(\mathbb{Z}_p)).$$

Here the double slash means that these functions are invariant under the left and right actions of $\mathrm{GL}_2(\mathbb{Z}_p)$. Examples of functions in the spherical Hecke algebra are given by characteristic functions of compact open subgroups. Let

$$1_{(n,d)} = 1_{\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p^n & 0 \\ 0 & p^d \end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)}.$$

As n and d vary, these span the spherical Hecke algebra.

Example 4.5. Let $G = \mathrm{GL}_n$. The spherical Hecke algebra in this case is

$$C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p) \mathbin{/\!/} \mathrm{GL}_2(\mathbb{Z}_p))$$

and a basis is given by

$$\{1_{\mathrm{GL}_n(\mathbb{Z}_p)\lambda(p)\,\mathrm{GL}_n(\mathbb{Z}_p)}\}_{\lambda=(\lambda_1,\dots,\lambda_n),\lambda_1\geq\cdots\geq\lambda_n}$$

where

$$(\lambda_1, \dots, \lambda_n)(p) = \mathrm{diag}(p^{\lambda_1}, \dots, p^{\lambda_n}).$$

The Smith normal form for matrices over $\mathbb{Q}_p$, from the theory of elementary divisors, gives the decomposition

$$\mathrm{GL}_n(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Z}_p)T(\mathbb{Q}_p)\,\mathrm{GL}_n(\mathbb{Z}_p)$$

and it follows from this that the set above is a basis for the spherical Hecke algebra.

5. A bit of archimedian representation theory

Let (π, V) be a representation of $G(F_\infty)$ where V is assumed to be a Hilbert space. The example that one should keep in mind is the space

$$L^2(G(F)\backslash G(\mathbb{A}_F)).$$

The fact that this is not a space of smooth functions is often inconvenient. We now indicate how one can start with such a Hilbert space and end up with a space of smooth functions that essentially determines the original space and has the virtue that it can be described algebraically. The first step is to study smooth vectors in V, and the second is to combine this with the action of a maximal compact subgroup $K_\infty \leq G(F_\infty)$, thus passing from the analytic notion of an admissible representation to the algebraic notion of a $(\mathrm{Lie}(G(F_\infty)), K_\infty)$-module.

With the exception of §5.5 in this section G is a reductive group over an archimedean local field F (i.e. $\mathbb{R}$ or $\mathbb{C}$) and $K \leq G(F)$ is a maximal compact subgroup. Moreover

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{Res}_{F/\mathbb{R}} G)$$

is the Lie algebra of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G(\mathbb{R}) = G(F)$. Here the restriction of scalars is employed so that $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ if $F = \mathbb{C}$, if $F = \mathbb{R}$ then this construction just returns $\mathrm{Lie}(G_F)$. Finally π is a (continuous) representation of $G(F)$ on a Hilbert space V; we do not assume that π is unitary. Our basic reference for this section is [Bum97, Chapter 2].

5.1. **Smooth vectors.** In this subsection we define and discuss smooth vectors. Recall that there exists an exponential map

$$\exp\colon \mathfrak{g} \longrightarrow G(F).$$

Example 5.1. For GL_n, the Lie algebra $\mathfrak{gl}_n$ is the collection of $n \times n$ matrices. The exponential is simply the matrix exponential in this case.

Given $\phi \in V$ we write

$$\pi(X)\phi = \frac{d}{dt}\pi(\exp(tX))\phi|_{t=0}$$

whenever this makes sense. We will sometimes simply write $X\phi$ for $\pi(X)\phi$. We say that a vector $\phi \in V$ is C^1 if for all $X \in \mathfrak{g}$, the derivative $\pi(X)\phi$ is defined. We define C^k inductively by stipulating that $\phi \in V$ is C^k if ϕ is C^{k-1} and $X\phi$ is C^{k-1} for all $X \in \mathfrak{g}$. A vector $\phi \in V$ is C^∞ if it is C^k for all $k \geq 1$.

Definition 5.2. A vector $\phi \in V$ is said to be **smooth** if ϕ is C^∞. The subspace of smooth vectors is denoted by $V^\infty \leq V$.

Note that we can differentiate in V^∞:

Lemma 5.3. *The space V^∞ is invariant under $G(F)$.*

Proof. Let $g \in G(F)$ and let $X \in \mathfrak{g}$. Then

$$\begin{aligned} X(\pi(g)\phi) &= \lim_{t\to 0}\frac{1}{t}(\pi(\exp(tX)g)\phi - \pi(g)\phi) \\ &= \pi(g)\lim_{t\to 0}\frac{1}{t}(\pi(\exp(t\,\mathrm{Ad}(g^{-1})X)\phi - \phi) \end{aligned}$$

where $\mathrm{Ad}(g^{-1})X = g^{-1}Xg$. The limit exists if ϕ is C^1. This implies that $\pi(g)\phi$ is C^1. One shows that $\pi(g)\phi$ is C^k for all k if ϕ is so by induction. ☐

Lemma 5.4. *Let (π, V) be a Hilbert space representation of $G(F)$. Then the action of $\mathfrak{g}$ defined above is a Lie algebra representation.*

Proof. We will prove this assuming as known that $C^\infty(G(F))$ is a representation of the Lie algebra where the action is given by sending $X \in \mathfrak{g}$ to dX, that is, differentiation in the direction of X. We show how to reduce the lemma to this case.

Let $\phi_0 \in V$. We claim that

$$L\phi(g) := \langle \pi(g)\phi, \phi_0\rangle$$

defines a $\mathfrak{g}$-equivariant map

$$L\colon V^\infty \to C^\infty(G(F)).$$

To prove that L is an intertwining map it suffices to verify that

$$(dX \circ L)\phi(g) = ((L \circ X)\phi(g).$$

For this we compute:

$$\begin{aligned}\frac{d}{dt}(L\phi)(g\exp(tX))|_{t=0} &= \frac{d}{dt}\langle \pi(g)\pi(\exp(tX))\phi, \phi_0\rangle|_{t=0}\\ &= \langle \pi(g)X\phi, \phi_0\rangle\\ &= (L\circ X)\phi(g).\end{aligned}$$

Since we are assuming that ϕ is smooth, we see that this function is smooth as well.

In order to verify that V^∞ is a representation of $\mathfrak{g}$, we must check that

$$X(Y\phi) - Y(X\phi) = [X, Y]\phi$$

for all $X, Y \in \mathfrak{g}$ and all $\phi \in V^\infty$. By duality, it suffices to prove that

$$L(X(Y\phi)) - L(Y(X\phi)) = L([X, Y]\phi)$$

for all ϕ_0, but this is a consequence of the fact that $C^\infty(G(F))$ is a representation of $\mathfrak{g}$ as explained at the beginning of the proof.

□

Thus V^∞ affords a representation of the Lie algebra. Note that so far we don't even know if V^∞ is nonzero; it ought to be large in order for this notion to be useful. Fortunately, it is indeed large. To make this precise, if $f \in C_c^\infty(G(F))$ then define

$$\pi(f)\phi = \int_{G(F)} f(g)\pi(g)\phi dg.$$

Proposition 5.5.

(1) *If* $f \in C_c^\infty(G(F))$ *and* $\phi \in V$ *then* $\pi(f)\phi \in V^\infty$.

(2) *The space* V^∞ *is dense in* V.

Proof. Let

$$X * f(g) := (d/dt)f(\exp(-tXg)|_{t=0}.$$

Then

$$\begin{aligned}&\int_{G(F)} X * f(g)\pi(g)\phi dg\\ &= \frac{d}{dt}\int_{G(F)} f(\exp(-tX)g)\pi(g)\phi dg|_{t=0}\\ &= \frac{d}{dt}\int_{G(F)} f(g)\pi(\exp(tX)g)\phi dg|_{t=0}\\ &= \frac{d}{dt}\pi(\exp(tX))\pi(f)\phi|_{t=0}\\ &= X\pi(f)\phi.\end{aligned}$$

By induction we see that $\pi(f)\phi \in V^\infty$.

For the second claim let $\varepsilon > 0$. The map $G \times V \to V$ given by $(g, \phi) \mapsto \pi(g)\phi$ is continuous. This implies that there exists a neighbourhood $U \subseteq G(F)$ of the identity such that $|\pi(g)\phi - \phi| < \varepsilon$ for all $g \in U$. Take $f \in C_c^\infty(G(F))$ to be nonnegative with support in U and such that

$$\int_{G(F)} f(g)dg = 1.$$

Then

$$\begin{aligned}|\pi(f)\phi - \phi| &= \left|\int_{G(F)} f(g)(\pi(g)\phi - \phi)dg\right|\\ &\leq \int_{G(F)} f(g)\left|\pi(g)\phi - \phi\right| dg \leq \varepsilon\end{aligned}$$

which implies that $\pi(f)\phi$ is as close to ϕ as we wish. Hence V^∞ is dense. □

5.2. **Restriction to compact subgroups.** The representation theory of compact groups is much simpler than the representation theory of noncompact groups. For example, any irreducible representation of a compact group is unitarizable and finite-dimensional (see Theorem 5.9 below). A profitable strategy in the representation theory of general Lie groups is to analyze the restriction of a given representation to maximal compact subgroups. We discuss this in this section, starting by recalling the results from the representation theory of compact groups that we require.

Throughout $K \subseteq G(F)$ is a maximal compact subgroup.

Lemma 5.6. *Let (π, V) be a continuous representation of $G(F)$ on a Hilbert space V. There exists a Hermitian inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$ which gives the same topology as the given pairing on V on V but with respect to which $\pi|_K$ is unitary, i.e.*

$$\langle \pi(k)\phi_1, \pi(k)\phi_2 \rangle = \langle \phi_1, \phi_2 \rangle$$

for all $k \in K$ and $\phi_1, \phi_2 \in V$.

In particular, taking $G(F)$ to be compact we see that any representation of a compact Lie group is unitarizable.

Proof. Let $(\cdot, \cdot)$ denote the original Hilbert space pairing and $|\cdot|$ the original norm. Define

$$\langle \phi_1, \phi_2 \rangle = \int_K (\pi(k)\phi_1, \pi(k)\phi_2) dk$$

By construction it is K-invariant so we need only check the claim about the topology. For fixed $\phi \in V$, the map

$$\begin{aligned} K &\longrightarrow \mathbb{C} \\ k &\longmapsto (\pi(k)\phi, \pi(k)\phi) \end{aligned}$$

is continuous with compact domain, and hence has bounded image. Thus by the uniform boundedness principle the operator norm of $\pi(k)$ is bounded independently of k. In particular there is some $C > 0$ such that $|\pi(k)\phi| < C\,|\phi|$ for all $\phi \in V$. We can likewise find a similar bound for $\pi(k^{-1})$ and so there exists some $C > 0$ such that

$$C^{-1}\,|\phi| \leq |\pi(k)\phi| \leq C\,|\phi|$$

for all $\phi \in V$. From this we find that

$$|\phi|_{new}^2 := \int_K (\pi(k)\phi, \pi(k)\phi) dk$$

satisfies

$$C^{-2}\mathrm{meas}(K)\,|\phi| < |\phi|_{new}^2 < C^2\mathrm{meas}(K)\,|\phi|\,.$$

This implies the result. □

We now prepare for a proof of the Peter-Weyl theorem, which says that all of the representation theory of a compact Lie group K is contained in $L^2(K)$.

Definition 5.7. Let (π, V) be a continuous representation of a group G on a Hermitian vector space V. A **matrix coefficient of** π is a function of the form

$$\begin{aligned} m : G &\longrightarrow \mathbb{C} \\ g &\longmapsto (\pi(g)\phi_1, \phi_2) \end{aligned}$$

for some $\phi_1, \phi_2 \in V$.

Proposition 5.8. *Suppose K is a compact Lie group and (π_1, V_1) and (π_2, V_2) are two representations of K with π_2 unitary. If there exist matrix coefficients m_1, m_2 for π_1, π_2 respectively that are not orthogonal in $L^2(K)$ then there exists a non-trivial intertwining operator $L : V_1 \to V_2$.*

Proof. Write $(\cdot, \cdot)_i$ for the Hermitian pairing on V_i. Let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ be such that:

$$\int_K (\pi_1(k)x_1, y_1)_1 \overline{(\pi_2(k)x_2, y_2)_2} dk \neq 0$$

Let

$$L(\phi) = \int_K (\pi_1(k)\phi, y_1)_1 \pi_2(k^{-1})(y_2) dk.$$

We claim that L gives an intertwining map

$$L : V_1 \longrightarrow V_2.$$

Indeed,

$$\pi_2(g) \circ L(\phi) = \int_K (\pi_1(k)\phi, y_1)_1 \pi_2(gk^{-1})(y_2) dk$$

If we change variables $k \mapsto kg$ this becomes $L \circ \pi_1(g)\phi$ and so L is an intertwining operator. We now verify that it is nonzero. One has

$$\begin{aligned}
(L(x_1), x_2)_2 &= (\int_K (\pi_1(k)x_1, y_1)_1 \pi_2(k^{-1}) y_2 dk, x_2)_2 \\
&= \int_K (\pi_1(k)x_1, y_1)_1 (\pi_2(k^{-1}) y_2, x_2)_2 dk \\
&= \int_K (\pi_1(k)x_1, y_1)_1 \overline{(\pi_2(k)x_2, y_2)_2} dk \qquad (V_2 \text{ is unitary}) \\
&\neq 0.
\end{aligned}$$

Thus L is nonzero.

□

Theorem 5.9 (Peter-Weyl Theorem)**.** *Let $K \subset \mathrm{GL}_n(\mathbb{C})$ be a compact Lie group.*

(1) *The matrix coefficients of finite dimensional unitary representations of K are dense in $C(K)$ and $L^p(K)$ for all $1 \leq p \leq \infty$.*
(2) *Any irreducible unitary representation of K is finite dimensional.*
(3) *If (π, V) is a unitary representation of K, then V decomposes into a Hilbert space direct sum of irreducible unitary subrepresentations.*

Proof. We may regard K as a subset of $M_n(\mathbb{R})$ for some n. We shall further identify this with $\mathbb{R}^{n^2}$. We call a function on K polynomial if it is the restriction of a polynomial in $\mathbb{R}^{n^2}$ to K.

We claim that every polynomial function ϕ is a matrix coefficient for some finite dimensional representation. Indeed, let $r \in \mathbb{Z}_{>0}$ and let ρ_r be the representation of K on the polynomials of degree less than r via right multiplication (so $\rho_r(g)\phi(x) = \phi(xg)$). By Lemma 5.6 we can find a hermitian metric on the space of ρ_r making this representation unitary. By the Riesz representation theorem there exists some ϕ_0 in the space of ρ_r such that $\phi(1) = \langle \phi, \phi_0 \rangle$. We then have that $\phi(g) = \rho(g)\phi(1) = (\rho(g)\phi, \phi_0)$. This implies the claim. The first assertion of the theorem then follows from the Stone-Weierstrass theorem (density of polynomials in $L^p(K)$).

We now claim that if (π, V) is a non-zero unitary representation of K then V admits a non-zero finite-dimensional invariant subspace. Let $\psi(g) = (\pi(g)x, x)$ be a non-zero matrix coefficient. We can approximate ψ by polynomials. This implies there exists a polynomial not orthogonal to ψ in $L^2(K)$, say of degree r. Letting ρ_r be the representation in the previous paragraph we obtain a non-zero intertwining map $L : \rho_r \to V$ which implies we have a finite-dimensional invariant subrepresentation of V as claimed. Since (π, V) is unitary, this finite dimensional subspace admits a complement if it is not the whole space V, proving the section assertion of the theorem.

For assertion 3 we use Zorn's Lemma to construct a maximal subspace which is a direct sum of finite subrepresentations. It then follows from 2 that this is the whole space as the orthogonal complement has finite subrepresentations. □

5.3. **$(\mathfrak{g}, K)$-modules.** Let (π, V) be a representation of $G(F)$ on a Hilbert space and let $K \le G(F)$ be a maximal compact subgroup. We assume without loss of generality that $\pi|_K$ is unitary. For each equivalence class of irreducible representation σ of K we write:

$$V(\sigma) = \{v \in V : \langle \pi|_K v\rangle \cong \sigma\}$$

This is the **σ-isotypic** subspace. A vector in V (resp. a subspace) of V is said to have **K-type** σ if it is an element (resp. a subspace) of $V(\sigma)$. We note that V is admissible if and only if $\dim(V(\sigma)) < \infty$ for all σ.

Definition 5.10. The algebraic direct sum

$$V_{fin} := \oplus_{\sigma\in\widehat{K}} V(\sigma)$$

is the **space of K-finite vectors in V.**

In the definition we have used the useful notation $\widehat{K}$; it denotes the set of all equivalence classes of irreducible representations of K. Since all of these representations are unitarizable, the set $\widehat{K}$ is also refered to as the **unitary dual of K.**

Remark 5.11. We will later show in Proposition 5.15 that V_{fin} is dense in V, and hence V is the Hilbert space direct sum of the $V(\sigma)$.

We record the following proposition [Bum97, §2.4].

Proposition 5.12. *Let $\mathfrak{k} = \mathrm{Lie}(K)$. The following are equivalent:*

(1) *The vector $\phi \in V$ is K finite.*
(2) *The space $\langle \pi(k)\phi | k \in K\rangle$ is finite dimensional.*

If ϕ is smooth, then this is equivalent to

(3) *The space $\langle \pi(x)\phi | x \in \mathfrak{k}\rangle$ is finite dimensional.*

Thus K-finiteness can be detected using the Lie algebra of K.

Remark 5.13. We show in Proposition 5.15 that K-finite vectors are always smooth.

Definition 5.14. A **$(\mathfrak{g}, K)$-module** is a vector space V with a representation of π of $\mathfrak{g}$ and K which satisfy the following:

(1) The space V is a countable algebraic direct sum $V = \oplus_i V_i$ with each V_i a finite dimensional K-invariant vector space.
(2) For $X \in \mathfrak{k}$ and $\phi \in V$ we have:
$$\pi(X)\phi = X\phi = \frac{d}{dt}\exp(tX)\phi|_{t=0} = \lim_{t\to 0}\frac{1}{t}(\pi(\exp(tX)\phi) - \phi).$$
(3) For $k \in K$ and $X \in \mathfrak{g}$ we have $\pi(k)\pi(X)\pi(k^{-1})\phi = \pi(\mathrm{Ad}(k)X)\phi$.

We say that the $(\mathfrak{g}, K)$-module is **admissible** if we can choose the V_i to have have distinct K-types.

We prove in the following proposition that an admissible representation of $G(F)$ on a Hilbert space gives us a $(\mathfrak{g}, K)$-module:

Proposition 5.15. *Let (π, V) be an admissible Hilbert space representation of $G(F)$. Then:*

- *The K-finite vectors are smooth.*
- *The space of K-vectors $V_{\text{fin}} \leq V$ is dense and invariant under the action of $\mathfrak{g}$.*

Thus the space of K-finite vectors in an admissible Hilbert space representation of $G(F)$ is in a natural manner an admissible $(\mathfrak{g}, K)$-module.

Remark 5.16. The K-finite vectors are even real analytic in the sense that if $\phi \in V(\sigma)$ for some irreducible representation σ, then $g \mapsto \pi(g)\phi$ is a real analytic function of g [HC53].

We say that two Hilbert space representations are **infinitesimally equivalent** if their underlying $(\mathfrak{g}, K)$-modules are isomorphic. The notion of infinitesimal equivalence is strictly coarser than that of equivalence [Bum97, Exercise 2.6.1].

Proof. We assume without loss of generality that $\pi|_K$ is unitary. Write $V_0 := V^\infty \cap V^{fin}$.

We first prove that V_0 is dense in V. Let U be a neighborhood of 1 in $G(F)$ and let $\varepsilon > 0$. Suppose that f is a smooth function on $G(F)$ with support in KU such that

$$\int_{G(F)} f(g)dg = 1 \quad \text{and} \quad \int_{G(F)-U} |f(g)|dg < \varepsilon. \tag{5.3.1}$$

By making U and ε sufficiently small we can make $\pi(f)\phi$ as close as we like to ϕ for all ϕ in V (see the proof of Proposition 5.5).

It therefore suffices to show that for arbitrary U and $\varepsilon > 0$ we can choose an f satisfying (5.3.1) such that $\pi(f)\phi$ is K-finite. To construct such an f, let let $U_1 \subset G(F)$ and $W \subset K$ be neighborhoods of 1 such that $WU_1 \subset U$, and let f_1 be a nonnegative smooth function supported in U_1 such that $\int_{G(F)} f_1(g)dg = 1$. By the Peter-Weyl theorem, there exists a matrix coefficient f_0 of a finite dimensional representation of K such that $\int_K f_0(k)dk = 1$ and $\int_{K \smallsetminus W} |f_0(k)|dk < \varepsilon$. Let

$$f(g) := \int_K f_0(k) f_1(k^{-1}g)dk$$

Clearly, f has support contained in $KU_1 \subset KU$. Moreover, since $WU_1 \subset U$, if $k \in K$ is such that there exists $g \in G(F) \smallsetminus U$ with $f_1(k^{-1}g) \neq 0$ then $k \notin W$ (since otherwise would have $g = k(k^{-1}g) \in WU_1 \subset U$). Therefore:

$$\begin{aligned}
\int_{G(F)\smallsetminus U} |f(g)|dg &\leq \int_{G(F)\smallsetminus U} \int_K |f_0(k)| \cdot |f_1(k^{-1}g)|dkdg \\
&= \int_{G(F)-U} \int_{K\smallsetminus W} |f_0(k)| \cdot |f_1(k^{-1}g)|dkdg \\
&\leq \int_{K-W} |f_0(k)| \int_{G(F)} f_1(k^{-1}g)dgdk \\
&= \int_{K-W} f_0(k)dk < \varepsilon.
\end{aligned}$$

Thus f satisfies (5.3.1).

We now show that $\pi(f)\phi$ is K-finite. Let ρ be a finite dimensional unitary representation of which f_0 is a matrix coefficient. Thus $f_0(k) = \langle \rho(k)\xi, \zeta \rangle$ for some ξ, ζ in the space of ρ. Then if

$k_1 \in K$:

$$\begin{aligned} f(k_1^{-1}g) &= \int_K f_0(k) f_1(k^{-1}k_1^{-1}g)dk \\ &= \int_K \langle \rho(k)\xi, \zeta\rangle f_1(k^{-1}k_1^{-1}g)dk \\ &= \int_K \langle \rho(k_1^{-1})\rho(k)\xi, \zeta\rangle f_1(k^{-1}g)dk \\ &= \int_K \langle \rho(k)\xi, \rho(k_1)\zeta\rangle f_1(k^{-1}g)dk. \end{aligned}$$

Therefore, the linear span of the functions $f(k_1^{-1}g)$, is contained in the linear span of the functions $g \mapsto \int_K \langle \rho(k)\xi, \zeta\rangle f_1(k^{-1}g)dk$ for varying ξ, ζ in the space of ρ, and this space is finite dimensional. Thus the space spanned by the vectors $\pi(k_1)\pi(f)\phi = \int_{G(F)} f(g)\pi(k_1 g)\phi dg = \int_{G(F)} f(k_1^{-1}g)\pi(g)\phi dg$ as k_1 varies over K is finite dimensional, so $\pi(f)\phi \in V^{fin}$. Moreover, $\pi(f)\phi$ is smooth for any vector ϕ by Proposition 5.5. It follows that V_0 is dense in V^∞, which is dense in V.

We now prove that $V^{fin} \leq V^\infty$. First observe that V_0 is K-invariant since V^∞ is by Proposition 5.12. Let σ be an irreducible unitary representation of K. Then $V_0(\sigma) \leq V(\sigma)$. Since V_{fin} is an algebraic direct sum of the $V(\sigma)$ it suffices to show that $V_0(\sigma) = V(\sigma)$.

Since $V(\sigma)$ is finite-dimensional by admissibility, $V_0(\sigma)$ admits a well-defined orthogonal complement in $V(\sigma)$ (this is the only part of the proof where admissibility is used). If ϕ is in this orthogonal complement then ϕ is orthogonal to all of V_0, because it is orthogonal to $V(\tau)$ for every $\tau \neq \sigma$. Therefore $\phi = 0$, since V_0 is dense. This establishes that $V_0(\sigma) = V(\sigma)$, and hence $V_0 = V_{fin} \leq V^\infty$.

Finally, must show that V_{fin} is invariant under $\mathfrak{g}$. Let $\phi \in V_{fin}$, let R be the span of ϕ under K, and let

$$R_1 := \langle Y\phi \mid Y \in \mathfrak{g} \text{ and } \phi \in R\rangle,$$

which is clearly finite dimensional. We claim that R_1 is fixed by K.

Indeed, if $X \in \mathfrak{k} = \mathrm{Lie}(K)$, and $Y\phi \in R_1$, then $X(Y\phi) = [X,Y]\phi + Y(X\phi)$, which is an element in R_1. Therefore the elements of R_1 are K-finite by Proposition 5.12, and hence $Y\phi$ is K-finite for all $Y \in \mathfrak{g}$.

□

Remark 5.17. One motivation for introducing $(\mathfrak{g}, K)$-modules is that they can be classified, and, at least in special cases, the action of $\mathfrak{g}$ and K can be given explicitly. This gives important information on automorphic representations, our primary object of study. We refer the reader to §6.5 for a statement of the classification in the case of GL(2).

5.4. **The archimedean Hecke algebra.** Recall that $\mathcal{H}_\infty$ denotes the algebra of distributions on $G(F)$ with (compact) support contained in K. Let

$$U(\mathfrak{k}_\mathbb{C}) \text{ and } U(\mathfrak{g}_\mathbb{C})$$

denote the universal enveloping algebras of the complexifications $\mathfrak{k}_\mathbb{C} := \mathfrak{k} \otimes_\mathbb{R} \mathbb{C}$ (resp. $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$) of $\mathfrak{k}$ (resp. $\mathfrak{g}$). Here we regard the Lie algebra $\mathfrak{g}$ as a Lie algebra over the real numbers via the canonical identification $\mathfrak{g} = \mathrm{Lie}((\mathrm{Res}_{F/\mathbb{Q}}G)_\mathbb{R})$. Since $\mathfrak{k} \leq \mathfrak{g}$ is a subalgebra, it inherits the structure of a real Lie algebra.

Let $\mathcal{H}_K$ denote the convolution algebra of finite measures on K. One has an isomorphism

$$\mathcal{H} \cong \mathcal{H}_K \otimes_{U(\mathfrak{k}_\mathbb{C})} U(\mathfrak{g}_\mathbb{C})$$

The category of admissible (g, K)-modules and the category of admissible $\mathcal{H}_\infty$-modules are equivalent.

5.5. **An alternate definition.** Let G be a connected reductive group over a number field F.

Recall that if $\varphi \in L^2(G(F)A_G\backslash G(\mathbb{A}_F))$, then $\overline{R(G(\mathbb{A}_F))\varphi}$ is a $G(F_\infty)$-representation, and it is a result due to Harish-Chandra that the K-vectors in this space form an admissible $\mathcal{H}_\infty \times G(\mathbb{A}_F^\infty)$-module.

Therefore, the K_∞-finite vectors in $\overline{R(G(\mathbb{A}_F))\varphi}$ form an admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$-module. Thus an alternate formulation of the definition of an automorphic representation is the following:

Definition 5.18. An *automorphic representation* of $G(\mathbb{A}_F)$ is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module isomorphic to a subquotient of $L^2(G(F)A_G\backslash G(\mathbb{A}_F))$.

Here, of course, an admissible representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ is an exterior tensor product of an admissible representation of $(\mathfrak{g}, K_\infty)$ and $G(\mathbb{A}_F^\infty)$.

6. Automorphic forms

In this section we explain how automorphic representations relate to automorphic forms on locally symmetric spaces.

6.1. **Approximation.** We assume throughout this section that F is a number field. The appropriate analogues hold in the function field case, but they require different proofs.

Let G be a connected linear algebraic group (so we do not assume G to be reductive). Recall that $F^\times$ embeds into $\mathbb{A}_F^\times$ diagonally as a discrete subspace with non-compact quotient. In other words $GL_1(F)\backslash GL_1(\mathbb{A}_F)$ is non-compact, and actually it has infinite volume with respect to the Haar measure. An analogous phenomenon occurs for other groups, and this motivates the following definition:

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi : G(\mathbb{A}_F) \to \mathbb{R}_{>0}^\times).$$

Here $|x| := |x|_{\mathbb{A}_F} = \prod_v |x|_v$. Note that $G(F)$ is contained $G(\mathbb{A}_F)^1$ in virtue of the product formula. Moreover, $G(\mathbb{Q})$ is discrete in $G(\mathbb{A}_F)^1$ (this follows from the fact that G admits an embedding into affine space together with Lemma 1.7).

As above, let A_G be the identity component of the $\mathbb{R}$-points of the greatest $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}} Z_G$. Then

$$A_G G(\mathbb{A}_F)^1 = G(\mathbb{A}_F),$$

and the product is direct. We now discuss the topological features of the quotient

$$G(F)\backslash G(\mathbb{A}_F)^1.$$

For a proof of the following two theorems see [Bor63].

Theorem 6.1 (Borel)**.** *The group $G(\mathbb{A}_F)^1$ is unimodular. The quotient $G(F)\backslash G(\mathbb{A}_F)^1$ has finite volume with respect to the measure induced by a Haar measure on $G(\mathbb{A}_F)^1$, and $G(F)\backslash G(\mathbb{A}_F)^1$ is compact if and only if every unipotent element of $G(F)$ belongs to the radical of G.*

The following refinement of this result goes by the name of weak approximation:

Theorem 6.2 (Borel)**.** *The set $G(F)\backslash G(\mathbb{A}_F)/K^\infty$ is finite.*

Remark 6.3. In the special case $G = \mathrm{GL}_1$, $K^\infty := \widehat{\mathcal{O}}_F^\times$ the set above can be identified with the class group of F, and hence the theorem implies the finiteness of the class group.

Let

$$h := h(K^\infty) = |G(F)\backslash G(\mathbb{A}_F^\infty)/K^\infty|,$$

let $t_1, \dots, t_h$ denote a set of representatives for $G(F)\backslash G(\mathbb{A}_F)/K^\infty$, and let

$$\Gamma_i(K^\infty) := G(F) \cap t_i.A_G\backslash G(F_\infty)K^\infty.t_i^{-1}.$$

We then have a homeomorphism

$$\coprod_{i=1}^{h} \Gamma_i(K^\infty)A_G\backslash G(F_\infty) \longrightarrow G(F)A_G\backslash G(\mathbb{A}_F)/K^\infty \tag{6.1.1}$$

given on the ith factor by

$$\Gamma_i(K^\infty)g \longmapsto G(\mathbb{Q})(g,t_i)K^\infty.$$

Notice that the $\Gamma_i(K^\infty)$ are discrete subgroups of $G(F)$; they are moreover arithmetic in the following sense:

Definition 6.4. Let $G \leq \mathrm{GL}_n$ be a linear algebraic group. A subgroup $\Gamma \leq G(F)$ is **arithmetic** if it is commensurable with $\mathcal{G}(\mathcal{O}_F)$, where $\mathcal{G}$ is the schematic closure of G in $\mathrm{GL}_{n/\mathcal{O}_F}$.

Remark 6.5. One can show that the notion of arithmeticity does not depend on the choice of representation $G \leq \mathrm{GL}_n$.

Example 6.6. Consider $G = \mathrm{GL}_{2/\mathbb{Q}}$. Then $K^\infty = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is a maximal compact open subgroup, where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$. Moreover, if we denote by

$$\begin{aligned} K_0(N) &: = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : N|c\} \\ \Gamma_0(N) &: = K_0(N) \cap \mathrm{GL}_2(\mathbb{Z}) \end{aligned}$$

then

$$\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

Thus if ϕ is a complex function on the double coset space $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N)$, then it gives rise to a complex function on the quotient $\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})$, and viceversa.

If we let $K_\infty = \mathrm{SO}_2(\mathbb{R})$ then :

$$\Gamma_0(N)\backslash(\mathbb{C}-\mathbb{R}) \tilde{\to} \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N)K_\infty = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C}-\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

where on the left $\Gamma_0(N)$ acts via Möbius transformations.

6.2. **Classical automorphic forms.** In the previous subsection we discussed a homeomorphism

$$G(F)A_G\backslash G(\mathbb{A}_F)/K^\infty \xrightarrow{\sim} \coprod_i \Gamma_i(K^\infty)A_G\backslash G(F_\infty).$$

for certain (discrete) arithmetic subgroups $\Gamma \leq G(F)$. Clearly automorphic representations are related to functions on the left side. Automorphic forms, classically, were defined to be certain classes of functions on the right. We recall the definition in this section. In the next section we recast this definition adelically and show how automorphic forms give rise to automorphic representations.

Definition 6.7. A *norm* $\|\cdot\|$ on $G(F_\infty)$ is a function of the form

$$\|g\| = \mathrm{tr}(\sigma(g)^*\sigma(g))^{1/2}$$

where $\sigma\colon G(F_\infty) \to \mathrm{GL}(E)$ is a finite dimensional representation with finite kernel such that $\sigma|_{K_\infty}$ is unitary with respect to some Hermitian inner product. Here $*$ denotes the adjoint with respect to the given inner product.

Definition 6.8. A function $\phi\colon G(F_\infty) \to \mathbb{C}$ is said to be **slowly increasing** if there exists a norm $\|\cdot\|$, a constant C and a positive integer r such that

$$|f(x)| \leq C\|x\|^r$$

for all $x \in G(F_\infty)$.

The definition is independent of the choice of norm. Note somewhat paradoxically that a rapidly decreasing function is also slowly increasing.

Definition 6.9. Let $\Gamma \subseteq G(F)$ be an arithmetic subgroup. A function $\phi\colon G(F_\infty) \to \mathbb{C}$ is an **automorphic form** for Γ if

1. ϕ is smooth;
2. ϕ is slowly increasing;
3. $\phi(\gamma x) = \phi(x)$ for all $x \in G(F_\infty)$, $\gamma \in \Gamma$;
4. there exists an elementary idempotent $\xi \in \mathcal{H}_\infty$ such that $\xi\phi = \phi$. (This says ϕ has a particular K-type and is K-finite);
5. there exists an ideal $J \subseteq Z(\mathfrak{g}_\mathbb{C})$ of finite codimension such that $X\phi = 0$ for all $X \in J$.

Here $Z(\mathfrak{g}_\mathbb{C})$ is the center of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, where $\mathfrak{g} = \mathrm{Lie}((\mathrm{Res}_{F/\mathbb{Q}}G)_\mathbb{R})$.

We denote the space of automorphic forms, with notations as above, by

$$\mathcal{A}(\Gamma, \xi, J).$$

We will also put

$$\mathcal{A}(\Gamma, J) := \bigcup_{\xi} \mathcal{A}(\Gamma, \xi, J)$$

which is a $(\mathfrak{g}, K_\infty)$-modules.

Theorem 6.10. [HC68] *The space $\mathcal{A}(\Gamma, J)$ is an admissible $(\mathfrak{g}, K_\infty)$-module and hence $\mathcal{A}(\Gamma, \xi, J)$ is finite dimensional for each ξ* □

Remark 6.11. Automorphic forms in $\mathcal{A}(\Gamma, \xi)$ define sections of vector bundles over $\Gamma\backslash G(F_\infty)/K_\infty$ defined using the representation attached to ξ [?]. At least for certain ξ these sections can be viewed as differential forms with coefficients in vector bundles, and this explains why they are called "forms."

6.3. **Automorphic forms on adele groups.** The adelic definition of an automorphic form is analogous to the classical one:

Definition 6.12. An automorphic form on $G(\mathbb{A}_F)$ is a function $\phi\colon G(\mathbb{A}_F) \to \mathbb{C}$ such that

1. for all $y \in G(\mathbb{A}_F)$, $x \mapsto \phi(xy)$ is slowly increasing and smooth;
2. $\phi(\gamma x) = \phi(x)$ for all $\gamma \in G(F)$ and for all $x \in G(\mathbb{A}_F)$;
3. There exists a fundamental idempotent ξ such that $\xi\phi = \phi$;
4. there exists an ideal $J \subseteq Z(\mathfrak{g})$ such that $X\phi = 0$ for all $X \in J$;
5. $\phi(ag) = \phi(g)$ for all $a \in A_G$ and $g \in G(\mathbb{A}_F)$.

We let

$$\mathcal{A}(\xi, J)$$

be the space of functions satisfying the assumptions above.

If $\xi = \xi_\infty \otimes \xi_{K^\infty}$ then one has a bijection

$$\begin{aligned} \mathcal{A}(\xi_\infty \otimes \xi_{K^\infty}, J) &\xrightarrow{\sim} \oplus_{i=1} \mathcal{A}(\Gamma_i(K^\infty), \xi_\infty, J) \\ \phi &\longmapsto (x_i \mapsto \phi(x_i t_i)) \end{aligned} \tag{6.3.1}$$

with notation as in (6.1.1).

Definition 6.13. An automorphic form ϕ is said to be **cuspidal** if

$$\int_{N(\mathbb{A}_F)} \phi(ng)dn = 0$$

for all parabolic subgroups $P \subseteq G$ with Levi decomposition $P = MN$, and for all $g \in G(\mathbb{A}_F)$.

Let $\mathcal{A}^0(\xi, J) \subseteq \mathcal{A}(\xi, J)$ denote the subspace of cuspical automorphic forms. Analogously, the **cuspidal subspace**

$$L_0^2(G(F)A_G\backslash G(\mathbb{A}_F)) \subseteq L^2(G(F)A_G\backslash G(\mathbb{A}_F)).$$

is the space of functions ϕ such that $\int_{N(\mathbb{A}_F)} \phi(ng)dn = 0$ for almost every $g \in G(\mathbb{A}_F)$.

Remark 6.14. The space of smooth vectors in $L_0^2(G(F)A_G\backslash G(\mathbb{A}_F))$ subspace is preserved by $\mathfrak{g}$.

Definition 6.15. A *cuspidal automorphic representation* is an automorphic representation equivalent to a subrepresentation of $L_0^2(G(F)A_G\backslash G(\mathbb{A}_F))$.

Note that we do mean subrepresentation above, and not just subquotient as we had for automorphic representations.

One has

$$\bigcup_{\xi, J} \mathcal{A}^0(\xi, J) \subseteq L_0^2(G(F)A_G\backslash G(\mathbb{A}_F))$$

and this subspace is dense.

6.4. **From modular forms to automorphic forms.** We make the constructions above more explicit in a special case when $G = \mathrm{GL}_{2\mathbb{Q}}$. Let $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Z})$ be a congruence subgroup. For example, we could set Γ equal to

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : N|c\} . \tag{6.4.1}$$

Recall the following definition:

Definition 6.16. Let $k \in \mathbb{Z}_{>0}$ The space of **weight k modular forms** for Γ is the space $M_k(\Gamma)$ of functions $f\colon \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:

(1) $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \cap \mathrm{SL}_2(\mathbb{Z})$;
(2) f is holomorphic;
(3) f extends holomorphically to the cusps (see [Shi94] for details).

If f additionally vanishes at the cusps (see loc. cit.) we say that f is a **cusp form.** The space of weight k cusp forms is denoted $S_k(\Gamma_0(N))$.

Remark 6.17. If k is even, $M_k(\Gamma)$ can be identified with a subspace of the space of holomorphic differential forms $H^0(\Gamma\backslash\mathfrak{H}, \Omega^{\otimes k/2})$. Likewise, the space $S_k(\Gamma)$ can be identified with the space $H^0(\overline{\Gamma\backslash\mathfrak{H}}, \Omega^{\otimes k/2})$, where the bar denotes the Baily-Borel compactification.

We now relate the space $M_k(\Gamma_0(N))$ to a space of automorphic forms. The idea is to pull back an automorphic form on $\mathfrak{h}$ along the quotient map

$$\mathrm{GL}_2(\mathbb{R}) \longrightarrow \mathrm{GL}_2(\mathbb{R})/A_G O_2(\mathbb{R}) = \mathfrak{H}$$

here A_G is the collection of matrices $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ with $r > 0$ and $\mathfrak{H}$ is the complex upper half plane. We set

$$j(g, z) = \det(g)^{-1/2}(cz+d)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \mathrm{GL}_2(\mathbb{R})^+$. This is an example of an **automorphy factor** (see [**?**] for details).

Set

$$\phi_f(g) = j(g, i)^{-k} f(gi)\colon \ \mathrm{GL}_2(\mathbb{R})^+ \longrightarrow \mathbb{C}$$

where g acts on i by fractional linear transformations. This will give us an automorphic form, but to specify the type we require further notation.

Let $\xi_k \in \mathcal{H}_\infty$ be the elementary idempotent attached to the induction of the representation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{2\pi i k\theta} \tag{6.4.2}$$

of SO_2 to O_2. For the remainder of this section, set

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{gl}_2 \otimes_{\mathbb{R}} \mathbb{C}.$$

Lemma 6.18. *One has $Z(\mathfrak{g}_{\mathbb{C}}) = \langle \mathbb{C}\Delta, Z\rangle$ where $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and*

$$\Delta = (1/4)(H^2 + 2XY + 2YX)$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Proof. [Bum97, §2.5]. □

The element Δ is called the **Casimir element**.

Lemma 6.19. *For each integer $k \geq 1$ the $\mathbb{C}$-linear map $f \mapsto \varphi_f$ induces an isomorphism*

$$\begin{aligned} S_k(\Gamma) &\longrightarrow \mathcal{A}^0(\Gamma, \xi_k, \langle \Delta - \frac{1}{4}(k^2-1), Z\rangle) \\ f &\longmapsto \varphi_f. \end{aligned}$$

Remark 6.20. Let

$$K_0(N) := \widehat{\Gamma_0(N)} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : N|c \right\}$$

where the ˆ denote profinite completion. Composing the isomorphism of Lemma 6.19 with (6.3.1) we obtain an isomorphism

$$S_k(\Gamma) \longrightarrow \mathcal{A}^0(\xi_k \otimes \xi_{K_0(N)}, \langle \Delta - \frac{1}{4}(k^2-1)\rangle)$$

Proof. One computes that

(1) $\phi(\gamma g) = \phi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{R})^+$ and $\gamma \in \Gamma$;
(2) $\phi(g w_\theta) = e^{ik\theta}\phi(g)$ for $w_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$.

These immediately imply that ϕ is K_∞-finite, Γ-invariant and A_G-invariant. One also computes that $\Delta\phi = (1/4)(k^2-1)\phi$ for ϕ as above using the fact that ϕ is holomorphic. So ϕ is annihilated by

$$J = \langle \Delta - (1/4)(k^2-1)\,\mathrm{id}, Z\rangle.$$

□

6.5. **Digression:** (g, K_∞)**-modules.** In this subsection we discuss the classification of $(\mathfrak{g}, K_\infty)$-modules when

$$\mathfrak{g} := \mathfrak{gl}_2 \text{ and } K_\infty := O_2(\mathbb{R}).$$

All irreducible admissible $(\mathfrak{g}, K_\infty)$-modules are isomorphic to exactly one of the types described in the following three subsections.

Finite dimensional representations. Every finite-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ affords an irreducible admissible $(\mathfrak{g}, K_\infty)$-module

Discrete series and limits of discrete series. Suppose $k \geq 1$ and $\mu \in \mathbb{R}$. Then one has a $(\mathfrak{g}, K_\infty)$-module π_k whose space is

$$V_k = \bigoplus_{|\ell| \geq k, \quad \ell \equiv k \pmod 2} \mathbb{C} v_l$$

with the following action:

(1) $\pi_k(w_\theta) v_\ell = e^{i\ell\theta} v_l$ and $\pi_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\ell = v_{-\ell}$
(2) $X v_\ell = \frac{1}{2}((k+\ell) v_{\ell+2})$,
(3) $Y v_\ell = \frac{1}{2}((k-\ell) v_{\ell-2})$,
(4) $Y v_k = 0$, $X v_{-k} = 0$,
(5) $\Delta v_\ell = \frac{k(k-2)}{4} v_\ell$,
(6) $Z v_\ell = \mu v_\ell$.

Then π_k is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module. If $k \geq 2$ then these modules are known as **the discrete series of weight** k and when $k = 1$ they are known as the **limit of discrete series**. Any irreducible sub-$(\mathfrak{g}, K_\infty)$-module of $\mathcal{A}^0(\Gamma, \xi_k, \langle \Delta - \frac{1}{4}(k^2 - 1) \rangle)$ is equivalent to $D_k(0)$.

Principal series. Let $s_1, s_2 \in \mathbb{C}$ and let $\varepsilon \in \{0, 1\}$. Let

$$s := \frac{1}{2}(s_1 - s_2 + 1), \quad \lambda := s(1-s), \text{ and } \quad \mu = s_1 + s_2.$$

and suppose that there is no $k \equiv \varepsilon \pmod 2$ such that $\lambda = (k/2)(1 - k/2)$. Consider the space of functions

$$V = \left\{ f \in C^\infty(\mathrm{GL}_2(\mathbb{R})) \middle| \begin{array}{ll} f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g\right) & = y_1^{s_1+1/2} y_2^{s_2+1/2} f(g) \text{ and} \\ f\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g\right) & = (-1)^\varepsilon f(g) \end{array} \right\}.$$

This is certainly a smooth representation of $\mathrm{GL}_2(\mathbb{R})$, and hence affords a $(\mathfrak{g}, K_\infty)$-module. It is irreducible and admissible and is known as the **principal series**. One can compute that Δ acts by $-\lambda/4$ and Z acts by μ. The $SO_2(\mathbb{R})$-types are the characters corresponding to the integers congruent to $\varepsilon \pmod 2$.

7. Factorization

We have already studied automorphic representations (even defined them) as tensor products of representations of objects at the infinite places and at the finite places. It is useful to push this idea further. In fact, if π is an admissible representation of $\mathcal{H}_\infty \times \mathcal{H}^\infty$ one can write π as a restricted direct product:

$$\pi = \pi_\infty \otimes'_{v \nmid \infty} \pi_v$$

where the π_v are irreducible admissible representations of $G(F_v)$. We now explain what this means.

7.1. Restricted tensor products of modules.

We start by defining a restricted direct product of vector spaces. Let Ξ be a countable set, let $\Xi_0 \subset \Xi$ be a finite subset, let

$$\{W_v : v \in \Xi\}$$

be a family of $\mathbb{C}$-vector spaces and for each $v \in \Xi - \Xi_0$ let $\phi_{0v} \in W_v - 0$. For all sets

$$\Xi_0 \subseteq S \subseteq \Xi$$

of finite cardinality set $W_S := \prod_{v \in S} W_v$. If $S \subseteq S'$ there is a map

$$W_S \longrightarrow W_{S'} \tag{7.1.1}$$

defined by

$$\otimes_{v\in S} w_v \mapsto \otimes_{v\in S} w_v \otimes \otimes_{v\in S'-S}\phi_{0v}.$$

Consider the vector space

$$W := \otimes' W_v := \varinjlim_S W_S$$

where the transition maps are given by (7.1.1). This is is the **restricted tensor product** of the W_v with respect to the ϕ_{0v}. Thus W is the set of sequences

$$(w_v)_{v\in\Xi} \in \otimes_v W_v$$

such that $w_v = \phi_{0v}$ for all but finitely many $v \in \Xi$. We note that if we are given for each $v \in \Xi$ a $\mathbb{C}$-linear map

$$B_v : W_v \longrightarrow W_v$$

such that $B_v\phi_{0v} = \phi_{0v}$ for all but finitely many $v \in \Xi$ then this gives a map $B = \otimes_v B_v \colon W \to W$ defined by

$$B(\otimes w_v) = \otimes B_v w_v.$$

We now define a restricted directed product of algebras. Suppose we are given $\mathbb{C}$-algebras (not necessarily with unit) $\{A_v : v \in \Xi\}$ and idempotents $a_{0v} \in A_v$ for all $v \in \Xi - \Xi_0$. If $S \subseteq S'$ there is a map

$$A_S \longrightarrow A_{S'} \tag{7.1.2}$$

defined by

$$\otimes_{v\in S} a_v \mapsto \otimes_{v\in S} a_v \otimes \otimes_{v\in S'-S} a_{0v}.$$

Consider the algebra

$$A := \otimes' A_v := \varinjlim_S A_S$$

where the transition maps are given by (7.1.2). This is the **restricted tensor product** of the A_v with respect to the a_{0v}. Finally, if W_v is an A_v-module for all $v \in \Xi$ such that $a_{0v}\phi_{0v} = \phi_{0v}$ for almost all v, then $\otimes'_v W_v$ is an A-module.

Remark 7.1. The isomorphism class of W as an A-module in general depends on the choice of $\{\phi_{0v}\}$. However, if we replace the ϕ_{0v} by nonzero scalar multiples we obtain isomorphic A-modules.

Example 7.2. The ring $\mathbb{C}[X_1, X_2, \dots] = \otimes'_i \mathbb{C}[X_i]$ with a_{0i} the identity in $\mathbb{C}[X_i]$.

Example 7.3. One has

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \otimes' C_c^\infty(G(F_v))$$

with respect to the idemponts $e_{K_v} := \frac{1}{\mathrm{Vol}(K_v)}\mathbb{1}_{K_v}$ where K_v is a (choice of) hyperspecial subgroup. Implicit in this statement is the result that hyperspecial subgroups exist for almost all places (see [Tit79] for a precise statement).

7.2. **Flath's theorem.** Let Ξ_0 be a finite set of places of F including the infinite places. Enlarging Ξ_0 if necessary we assume that if $v \notin \Xi_0$ then G_{F_v} is unramified. If $v \notin \Xi_0$ we let $K_v \le G(F_v)$ be a choice of hyperspecial subgroup; it is unique up to conjugacy [Tit79].

Definition 7.4. A $C_c^\infty(G(\mathbb{A}_F^\infty))$-module W is **factorizable** if we can write

$$W \cong \otimes'_v W_v \tag{7.2.1}$$

where the restricted direct product is with respect to elements $\phi_{0v} \in W_v^{K_v}$, $\dim(W_v^{K_v}) = 1$, and the isomorphism (7.2.1) intertwines the action of $C_c^\infty(G(\mathbb{A}_F^\infty))$ with the action of $\otimes'_v C_c^\infty(G(F_v))$, the restricted direct product being with respect to the idempotents e_{K_v}.

In view of the assumption that $\dim(W_v^{K_v}) = 1$ the module $\otimes'_v W_v$ only depends on the choice of the ϕ_{0v} up to isomorphism. Note that in this case W is admissible and irreducible if and only if the W_v are for all v.

Theorem 7.5 (Flath). *Every admissible irreducible representation W of $C_c^\infty(G(\mathbf{A}_F^\infty))$ is factorizable.*

We will prove Flath's theorem below. The most interesting part is proving that $\dim(W_v^{K_v}) = 1$ for almost all v; this is accomplished via Gelfand's lemma. We postpone discussion of Gelfand's lemma until §8 below.

7.3. **Proof of Flath's theorem.** For the remainder of this section we let G be a locally compact totally disconnected group. The basic example of interest to us is the case where G is the points of a reductive group in a local nonarchimedian field.

Definition 7.6. A representation of G on a complex vector space V is **smooth** if the stabilizer of any vector in V is open in G.

Equivalently, V is smooth if and only if $V = \bigcup_{K \leq G} V^K$ where the union is over all compact open subgroups $K \leq G$.

Remark 7.7. In this definition we do not assume that the representation V is continuous, or for that matter even give a topology on V. In fact, V is smooth if and only if the representation is continuous if we give $G(F)$ its usual topology and V the discrete topology.

The Hecke algebra

$$C_c^\infty(G)$$

is the convolution algebra of compactly supported smooth functions on G; this is consistent with our earlier definition in the case where G is the points of a reductive group in a local nonarchimedian field.

Recall the notion of a non-degenerate $C_c^\infty(G)$-module from after Definition 3.4. For the following lemma, see [Car79]:

Lemma 7.8. *There is an equivalence of categories between non-degenerate $C_c^\infty(G)$-modules and smooth representations of G.* □

Using this equivalence we prove the following irreducibility criterion:

Proposition 7.9. *A smooth G-module V is irreducible if and only if V^K is an irreducible $C_c^\infty(G//K)$-module for all compact open subgroups $K \leq G$.*

To prove the proposition we recall some properties of $C_c^\infty(G)$. As mentioned after (3.4), the condition of non-degeneracy has content because $C_c^\infty(G)$ has no identity element. However, it has approximate identities in the following sense. For each compact open subgroup $K \leq G$ let

$$e_K := \frac{1}{\mathrm{meas}(K)} \mathbb{1}_K \,.$$

Then e_K acts as the identity on

$$C_c^\infty(G//K) = e_K C_c^\infty(G) e_K$$

and on $V^K := e_K V$. This observation is in fact the key to proving Lemma 7.8.

Proof of Proposition 7.9. Suppose $W = W_1 \oplus W_2$ as $C_c^\infty(G)$-modules. Then

$$W^K = W_1^K \oplus W_2^K$$

as $C_c^\infty(G//K)$-modules for some compact open $K \le G$ by smoothness. Conversely, if $W^K = W_1^K \oplus W_2^K$ for some compact open $K \le G$ then

$$(C_c^\infty(G)W_1)^K = e_K C_c^\infty(G)W_1 = e_K C_c^\infty(G) e_K e_K W_1 = C_c^\infty(G//K)W_1^K \neq W$$

hence $C_c^\infty(G)W_1 \neq W$. □

An **admissible** representation of $C_c^\infty(G)$ is a nondegenerate $C_c^\infty(G)$-module V such that $e_K V$ is finite dimensional for all compact open subgroups $K \le G$ (this is consistent with the definition of §3.4. In view of Lemma 7.8 there ought to be a way to phrase this condition in the context of the associated smooth G-module. This is accomplished by the following definition:

Definition 7.10. A representation V of G is **admissible** if it is smooth and V^K is finite dimensional for every compact open subgroup $K \le G$.

It is immediate that a representation V of G is admissible if and only if its associated $C_c^\infty(G)$-module is admissible.

We are now in a position to prove a weak version of Theorem 7.5:

Theorem 7.11. *Let G_1, G_2 be locally compact totally disconnected groups and let $G = G_1 \times G_2$.*

1. *If V_i is an admissible irreducible representation of G_i for $1 \le i \le 2$ then $V_1 \otimes V_2$ is an admissible irreducible representation of G.*
2. *If V is an admissible irreducible representation of G then there exists admissible irreducible representations V_i of G_i for $1 \le i \le 2$ such that $V \cong V_1 \otimes V_2$. Moreover the isomorphism classes of the V_i are uniquely determined by V.*

Proof. We first prove (1). By the irreducibility criterion Proposition 7.9 for every compact open subgroup $K_1 \times K_2 \le G_1 \times G_2$ the representation $V_1^{K_1} \otimes V_2^{K_2}$ of $C_c^\infty(G_1//K_1) \times C_c^\infty(G_2//K_2)$ is irreduble. But

1. $C_c^\infty(G_1 \times G_2) = C_c^\infty(G_1) \times C_c^\infty(G_2)$,
2. $C_c^\infty(G_1 \times G_2 /\!/ K_1 \times K_2) = C_c^\infty(G_1 /\!/ K_1) \times C_c^\infty(G_2 /\!/ K_2)$, and
3. $(V_1 \otimes V_2)^{K_1 \times K_2} = V_1^{K_1} \otimes V_2^{K_2}$,

so this implies that $V_1 \otimes V_2$ is admissible and irreducible.

Conversely, let W be an admissible G-module. Choose $K = K_1 \times K_2$ such that $W^K \neq 0$ (this is possible by smoothness). Then since W^K is finite dimensional there exists finite-dimensional $C_c^\infty(G_i//K_i)$-modules $W_i(K_i)$ and an isomorphism of $C_c^\infty(G//K)$ modules $W^K \to W_1(K_1) \otimes W_2(K_2)$. Varying K, we obtain a decomposition

$$W \cong W_1 \otimes W_2$$

as $C_c^\infty(G) \cong C_c^\infty(G_1 \times G_2)$-modules, where

$$W_1 := \varinjlim_{K_1} W_1(K_1) \text{ and } W_2 := \varinjlim_{K_2} W_2(K_2).$$

□

Proof of Theorem 7.5. For each finite set of finite places S containing the (finite) set of places v where G_{F_v} is ramified let $K^S := \prod_v K_v \le G(\mathbb{A}_F^S)$ be a fixed compact open subgroup with K_v hyperspecial for every v.

Choose an isomorphism

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \otimes_v' C_c^\infty(G(F_v)) \tag{7.3.1}$$

where the restricted direct product is constructed using the idempotents e_{K_v} for hyperspecial $K_v \leq G(F_v)$. Use (7.3.1) to identify these two algebras for the remainder of the proof. We then have a well-defined subalgebra

$$A_S := C_c^\infty(G(F_S^\infty)) \otimes e_{K^S} \leq C_c^\infty(G(\mathbb{A}_F^\infty)),$$

where $e_{K^S} = \otimes_{v \notin S} e_{K_v}$ is $\frac{1}{\mathrm{meas}(K^S)} \mathbb{1}_{K^S}$ (see Example 7.3). By corollary 8.9 below and Theorem 7.11 above, as a representation of A_S we have an isomorphism

$$W^{K^S} \cong \otimes_{v \in S} W_v \otimes W^S$$

where W^S is a one-dimensional $\mathbb{C}$-vector space on which e_{K^S} acts trivially. Hence, by admissibility,

$$W = \bigcup_S W^{K_S} \cong \varinjlim_S \otimes_{v \in S} W_v \otimes W^S. \tag{7.3.2}$$

with respect to the obvious transition maps (compare §7.1). On the other hand, (7.3.1) induces an identification

$$C_c^\infty(G(\mathbb{A}_F^\infty)) = \bigcup_S A_S = \varinjlim_S A_S \tag{7.3.3}$$

where the direct limit is taken with respect to the obvious transition maps (again, compare §7.1), and it is clear that (7.3.2) is equivariant with respect to (7.3.3). ☐

8. Gelfand pairs

In the proof of Theorem 7.5 above we used without proof the fact that if G is a connected reductive group over a nonarchimedian local field F, V is an irreducible admissible representation of $G(F)$ and $K \leq G(F)$ is a hyperspecial subgroup then $\dim(V^K) = 1$. This important result is proven via an application of Gelfand's lemma (see Corollary 8.9), which is a lemma used to establish the existence of so-called Gelfand pairs. Since this notion is incredibly useful in representation theory, we devote this section to it. Throughout G is a totally disconnected locally compact group.

Definition 8.1. If V is a representation of G, then the **contragredient representation** $V^\vee$ is the representation afforded by all linear forms $V \to \mathbb{C}$ that are fixed by some open subgroup in G.

Note that the contragredient of an admissible representation is admissible.

Definition 8.2. Suppose $H \subset G$ is a closed subgroup. The pair (G, H) is a **Gelfand pair** if for all irreducible admissible representations V of G we have that $\dim \mathrm{Hom}_H(V, \mathbb{C}) \dim \mathrm{Hom}_H(V^\vee, \mathbb{C}) \leq 1$.

Studying representations of G by studying their restrictions to subgroups H is often a profitable strategy in representation theory. If (G, H) is Gelfand, this is especially true. One can think of the multiplicity one property as an analogue of Schur's lemma in a sense we now explain. Since we have not proven Schur's lemma in the context of admissible representations, we provide a proof now:

Lemma 8.3 (Schur's lemma, extended to smooth irreducibles by Jacquet)**.** *Let (π, V) be a smooth irreducible representation of G. Assume that the topology of G has a countable basis. Then any endomorphism of V commuting with π is necessarily scalar.*

Remark 8.4. The hypothesis on G is valid if G is the F-points of a reductive group over a local nonarchimedian field F.

Proof. Let $T : V \to V$ be a nonzero intertwining map. Assume for the sake of a contradiction that $T \neq \lambda \,\mathrm{Id}$ for all $\lambda \in \mathbb{C}$. Then for all $\lambda \in \mathbb{C}$, the map $T - \lambda \,\mathrm{Id} : V \to V$ is a nonzero intertwining map since it is nonzero and π is irreducible. Let $R_\lambda = (T - \lambda \,\mathrm{Id})^{-1}$. Then we claim that the R_λ are linearly independent over $\mathbb{C}$ as λ varies. Indeed, suppose $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are distinct and let $a_1, \dots, a_n \in \mathbb{C}^\times$. The linear combination $\sum a_i R_{\lambda_i}$ decomposes as a product

$$\sum_i a_i R_{\lambda_i} = \left(\prod_i R_{\lambda_i} \right) P(T),$$

where $P(X) = \sum_i a_i \prod_{j \neq i} (X - \lambda_j)$. Factor $P(X) = a \prod_i (X - m_i)$ for $a \in \mathbb{C}^\times$ and $m_i \in \mathbb{C}$. Then since $P(T)$ is invertible each $T - m_i \,\mathrm{Id}$ is invertible. Thus $\sum a_i R_{\lambda_i}$ is invertible, and the R_{λ_i} are thus linearly independent. This proves the claim, which implies that the endomorphisms of π form a $\mathbb{C}$-vector space of uncountable dimension.

Now let $\phi \in V - \{0\}$, so that V is generated by ϕ. Thus V is generated by $\{\pi(g)\phi : g \in G\}$ over $\mathbb{C}$. By smoothness, ϕ is stabilized by a compact open subgroup stabilizer $K \leq G$. Since G/K is countable by assumption, it follows that V has countable dimension.

On the other hand there is an injective map from the $\mathbb{C}$-algebra of intertwining maps $T : V \to V$ to V given by $T \mapsto T(\phi)$. Thus the $\mathbb{C}$-algebra of intertwining maps $T : V \to V$ has countable dimension. This contradiction implies the proposition. ☐

As an easy corollary, we have that if the topology of G has a countable basis and π is an irreducible admissible representation of G, then there is a quasi-character $\omega : Z_G \longrightarrow \mathbb{C}^\times$ such that $\pi(z)$ acts via multiplication by $\omega_\pi(z)$ for $z \in Z_G$. The character ω_π is called the **central character of** π. As another application, we have the following lemma:

Lemma 8.5. *Regard G as a subgroup of $G \times G$ via the diagonal embedding. Then $(G \times G, G)$ is a Gelfand pair if and only if for all irreducible admissible representations (π, V) of G any intertwining map $V \to V$ is a scalar.*

This explains our earlier claim that the Gelfand property can be thought of as an analogue of Schur's lemma. It is in fact a generalization of this property.

In order to give the proof, we observe that for any representation V of G the space

$$\mathrm{End}(V) := \mathrm{Hom}(V, V)$$

is naturally a $G \times G$-module, where one copy of G acts via precomposition and the other via postcomposition. Let

$$\mathrm{End}_{sm}(V) \leq \mathrm{End}(V)$$

denote the subspace consisting of **smooth endomorphisms**, that is, endomorphisms that are left and right invariant by a compact open subgroup $K \leq G$.

Proof. The space of $\mathrm{End}_{sm}(V)$ isomorphic as a representation of $G \times G$ to $V \otimes V^\vee$, and the subspace of intertwining maps $V \to V$ is naturally isomorphic to

$$(V \otimes V^\vee)^G,$$

where here we regard G as embedded diagonally in $G \times G$. ☐

One important characterization of Gelfand pairs in a special case is the following:

Proposition 8.6. *If $K \subset G$ is compact and open then (G, K) is a Gelfand pair if and only if $C_c^\infty(G//K)$ is commutative.*

For an application of this proposition, see §9.1 below.

Proof. If $C_c^\infty(G//K)$ is commutative then $\dim(V^K) = 1$ for all irreducible admissible representations V by Proposition 7.9 so (G, K) is a Gelfand pair. Conversely, note that we have an equivalence of categories

$$\begin{aligned}\{\text{reps } V \text{ of } G \text{ generated by } V^K\} \longrightarrow & \{C_c^\infty(G//K)\text{-modules}\}\\ V \longmapsto & V^K\end{aligned}$$

which sends irreducible representations to irreducible representations. Moreover, one can check that any representation of G that is generated by V^K is in fact admissible since we are in a nonarchimedian setting. If (G, K) is Gelfand, then $\dim V^K = 1$ for all irreducible admissible representations V, and hence every irreducible representation of $C_c^\infty(G//K)$ is one-dimensional. It follows that $C_c^\infty(G//K)$ is commutative. □

Gelfand's lemma gives a method for proving that a given pair is Gelfand. To state it we use the following definition:

Definition 8.7. The space of distributions $D'(G)$ on G is the linear dual of $C_c^\infty(G)$.

Note that in contrast to the archimedian case we place no conditions of continuity on the linear functions. We have that $G \times G$ acts on $D'(G)$ via its left and right actions on functions of $C_c^\infty(G)$.

Lemma 8.8 (Gelfand's Lemma)**.** *Assume the topology of G has a countable basis and let $H \leq G$ be a closed subgroup. If there is an involution ι of G which stabilizes H and acts trivially on $D'(G)^{H\times H}$ then (G, H) is a Gelfand pair.*

Proof. Assume that the topology of G has a countable basis and let V be an irreducible representation of G, and let $\ell : V \to \mathbb{C}$ and $m : V^\vee \to \mathbb{C}$ be nonzero H-invariant linear forms. Define linear maps

$$F_\ell : C_c^\infty(G) \longrightarrow V^\vee$$
$$F_m : C_c^\infty(G) \longrightarrow V$$

by

$$F_\ell(f)(v) = \int_G f(g)\ell(gv)dg$$
$$F_m(f)(v^\vee) = \int_G f(g)m(gv^\vee)dg$$

respectively. Since $V^\vee$ and V are irreducible, by Schur's lemma [Car79, §1.4(c)] these maps are determined up to scaling by their kernels. We consider the composite map:

$$B : C_c^\infty(G) \times C_c^\infty(G) \xrightarrow{F_m \times F_\ell} V \times V^\vee \xrightarrow{\langle\,,\,\rangle} \mathbb{C},$$

where $\langle\ ,\ \rangle$ is the tautological (G-invariant) pairing. Note that $B(f_1, f_2) = m(F_\ell(f_1 * f_2))$. Extending linearly B defines a distribution on $G \times G$ right invariant under $H \times H$ and left invariant under G (embedded diagonally). For $f \in C_c^\infty(G)$, define $\tilde{f} := f(\iota(g^{-1}))$. Since $f \mapsto m(F_\ell(f))$ is bi-H-invariant, it is fixed by ι. Thus $m(F_\ell(f)) = m(F_\ell(\tilde{f}))$. We now take $f = f_1 * f_2$, and we have $\tilde{f} = \tilde{f}_2 * \tilde{f}_1$, since ι is an involution. Thus we see that $B(f_1, f_2) = B(\tilde{f}_2, \tilde{f}_1)$, so the left kernel of B determines the right kernel of B. Hence m determines $\ker(F_\ell)$, and therefore determines ℓ up to scaling. But since m was arbitrary, we must have $\dim_H \mathrm{Hom}(V, \mathbb{C}) \leq 1$. A symmetric argument implies that $\dim_H \mathrm{Hom}(V^\vee, \mathbb{C}) \leq 1$. □

We close this section with the application of Gelfand's lemma that motivated our discussion in the first place:

Corollary 8.9. *If F is a nonarchimedean local field, G a connected reductive F-group, and $K \leq G(F)$ a hyperspecial subgroup, then $(G(F), K)$ is a Gelfand pair. More generally, if G is a connected reductive group over a global field F unramified outside of S and $K^S := \prod_{v \notin S} \leq G(\mathbb{A}_F^S)$ is a subgroup with $K_v \leq G(F_v)$ hyperspecial for all v, then $(G(\mathbb{A}_F^S), K^S)$ is a Gelfand pair.*

Proof. In [Car79, Theorem 4.1] one finds a sketch of a proof that $C_c^\infty(G(F)//K)$ is commutative, which implies the first statement by Proposition 8.6.

For the adelic statement we note that

$$C_c^\infty(G(\mathbb{A}_F^S)//K^S) \cong \otimes'_{v \notin S} C_c^\infty(G(F_v)//K_v)$$

so the algebra $C_c^\infty(G(\mathbb{A}_F^S)//K^S)$ is commutative, hence the pair $(G(\mathbb{A}_F^S), K^S)$ is Gelfand.

In the special case of $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O}_F)$, recall that any double coset KgK is equal to KaK with a a diagonal matrix by the structure theorem for finitely generated modules over a principal ideal domain. Thus we can show that (G, K) is Gelfand using Gelfand's lemma. $\square$

9. Unramified representations

By Flath's theorem, if π is an automorphic representation of $G(\mathbb{A}_F)$, then

$$\pi \cong \otimes'_v \pi_v$$

where for almost every v the representation π_v is unramified in the sense that it contains a (unique) vector fixed under a hyperspecial subgroup $K \leq G(F_v)$. In this section we discuss the classification of unramified representations. It turns out that they can be explicitly parametrized in terms of conjugacy classes in the dual group of G (see Theorem 9.3). This fundamental fact will later be used in §10 to formally state the Langlands functoriality conjecture.

In this section we let G be a connected reductive group over a nonarchimedean local field F.

9.1. Unramified representations.

Our purpose here is to study unramified representations. Recall that G is **unramified** if G is quasi-split and split over an unramified extension of F.

Definition 9.1. An admissible irreducible representation (π, V) of $G(F)$ is called **unramified** or **spherical** if G is unramified and $V^K \neq 0$ for some (any) hyperspecial subgroup $K \leq G(F)$.

Let $K \leq G(F)$ be a hyperspecial subgroup. Then the subalgebra

$$C_c^\infty(G(F)//K) \leq C_c^\infty(G(F))$$

is known as the **spherical Hecke algebra** of $G(F)$ (with respect to K). Let $f \in C_c^\infty(G(F)//K)$ and let π be unramified. Then $\pi(f)$ acts via a scalar on V^K and hence on all of V. It is sensible to denote the scalar by $\mathrm{tr}(\pi(f))$. The map

$$\begin{aligned} C_c^\infty(G(F)//K) &\longrightarrow \mathbb{C} \\ f &\longmapsto \mathrm{tr}(\pi(f)) \end{aligned}$$

is called the **Hecke character** of π.

Proposition 9.2. *An unramified representation π is determined up to isomorphism by its Hecke character.*

Proof. Recall that a smooth representation of $G(F)$ generated by V^K is determined by the action of $C_c^\infty(G(F)//K)$ on V^K (compare the proof of Proposition 8.6). $\square$

9.2. **The Satake isomorphism.** If we don't know anything about $C_c^\infty(G(F)//K)$, then we could hardly count this result as useful. However, it turns out that $C_c^\infty(G(F)//K)$ has a simple description:

Theorem 9.3 (Satake)**.** *Assume that G is split. There is an isomorphism of algebras*

$$\mathcal{S} : C_c^\infty(G(F)//K) \longrightarrow \mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})(\mathbb{C})}$$

where $\widehat{G}$ is the complex connected reductive algebraic group with root datum dual to that of G and $\widehat{T} \le \widehat{G}$ is a maximal torus. □

Remark 9.4. A similar, but more complicated statement is true for any unramified group G, see [Car79, §4.2].

To have a feeling for this result, let us consider the special case of GL_n following [Tam63]. The Hecke algebra $C_c^\infty(\mathrm{GL}_n(F)//\mathrm{GL}_n(\mathcal{O}_F))$, as a $\mathbb{C}$-module, has a basis given by

$$\mathbb{1}_\lambda := \mathrm{GL}_n(\mathcal{O}_F)\begin{pmatrix} \varpi_1^{\lambda_1} & & \\ & \ddots & \\ & & \varpi_n^{\lambda_n}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_F)$$

with $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_i \ge \lambda_{i+1}$ for all $1 \le i \le n-1$ (this follows from the theory of elementary divisors). As a $\mathbb{C}$-algebra it is generated by $\mathbb{1}_{\lambda(r)}$ with $\lambda(r) = (1, \dots, 1, 0, \dots, 0)$ (r ones and $n-r$ zeros) for $1 \le r \le n$ and $\lambda = (-1, \dots, -1)$.

On the generating set above the Satake isomorphism is given by

$$\mathcal{S}(\mathbb{1}_{\lambda(r)}) = q^{r(n-r)/2}\mathrm{tr}\,(\wedge^r \mathbb{C}^n) \tag{9.2.1}$$

where $\mathbb{C}^n$ is the tautological representation of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{tr}\,(\wedge^r\mathbb{C}^n)$ is the trace of a diagonal matrix in $\mathrm{GL}_n(\mathbb{C})$ acting on the given representation.

We now record the following consequence of Proposition 9.2 and Theorem 9.3:

Corollary 9.5. *Assume that G is split. There is a bijection*

$$\left\{\begin{matrix}\textit{semi-simple}\\ \textit{conjugacy classes}\\ \textit{in } \widehat{G}(\mathbb{C})\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{isomorphism classes}\\ \textit{of irreducible unramified}\\ \textit{representations of } G(F)\end{matrix}\right\}.$$

Proof. We have

$$\mathrm{Hom}_{\mathbb{C}\text{-algebras}}(\mathbb{C}[\widehat{T}]^{W(\widehat{G},\widehat{T})(\mathbb{C})}, \mathbb{C}) = \widehat{T}(\mathbb{C})/W(\widehat{G},\widehat{T})(\mathbb{C})$$

On the other hand every semisimple conjugacy class in $\widehat{G}(\mathbb{C})$ intersects $\widehat{T}(\mathbb{C})$ and two elements of $\widehat{T}(\mathbb{C})$ are conjugate in G if and only if they are in the same $W(\widehat{G},\widehat{T})(\mathbb{C})$-orbit. □

This result is our first example of a functorial correspondence in the sense of Langlands. We will return to functorial correspondences in §10 below. By way of terminology the semisimple conjugacy class attached to an isomorphism class of unramified representations is called its **Langlands class**. The eigenvalues of a representative of the conjugacy class are called its **Satake parameters**.

9.3. **Principal series.** We now explain how to explicitly realize unramified representations. Let $P \subset G$ is a parabolic subgroup, $M \subset P$ is its Levi subgroup, and N its unipotent radical, so that we have $P = MN$.

Letting

$$\mathfrak{n} := \mathrm{Lie}(N)$$

we obtain a representation

$$\mathrm{Ad}_{\mathfrak{n}} : M(F) \longrightarrow \mathrm{GL}(\mathfrak{n})$$

by restricting the adjoint action of $G(F)$ on $\mathrm{Lie}(G)$. The **modular character**

$$\delta := \delta_P : M(F) \to \mathbb{C} \tag{9.3.1}$$

is defined by

$$\delta(m) := |\det(\mathrm{Ad}(m))|.$$

Let (σ, V) be a smooth irreducible representation of $M(F)$. Using the modular character we define the **induced representation** $I(\sigma) := \mathrm{Ind}_P^G(\sigma)$ to be the (smooth) representation of $G(F)$ on the space of functions

$$\left\{ f : G(F) \longrightarrow V : \begin{array}{c} f(mng) = \delta(m)^{1/2}\sigma(m)f(g) \text{ for all } (m,n,g) \in M \times N \times G(F), \\ \text{and there exists a compact open } K \le G \\ \text{such that } f(gk) = f(g) \text{ for all } g \in G(F) \end{array} \right\}$$

Here G acts via right translation:

$$I(\sigma)(g)f(x) = f(xg).$$

This is an example of an **induced representation**. The factor of $\delta^{1/2}$ is present so that if σ is unitarizable then $I(\sigma)$ is also unitarizable (see Lemma 9.6). We note that this procedure yields a functor

$$\mathrm{Ind}_P^G : \mathrm{SmRep}M(F) \rightsquigarrow \mathrm{SmRep}G(F),$$

from smooth representations of $M(F)$ to smooth representations of $G(F)$.

Lemma 9.6.

(1) *If σ is admissible then $I(\sigma)$ is admissible.*
(2) *One has $I(\sigma)^\vee \cong I(\sigma^\vee)$*
(3) *If σ is unitary then $I(\sigma)$ is unitarizable.*

Proof. There is a compact open subgroup $K \le G(F)$ such that

$$G(F) = KP(F)$$

by the **Iwasawa decomposition** [Tit79, §3.3.2]. Thus $G(F)/P(F)$ is compact and (1) follows.

Let V denote the space of $I(\sigma)$, W the space of $I(\delta_P^{1/2})$ Define a map

$$\begin{aligned} \Pi : C_c^\infty(G(F)) &\longrightarrow W \\ f &\longmapsto \int_{M(F)}\int_{N(F)} f(mg)dmdn. \end{aligned}$$

It is surjective and intertwines the natural action of $G(F)$ on $C_c^\infty(G(F))$ by right translation with the action of $I(\delta)$. Consider assertion (2). For $\phi_1 \in V, \phi_2 \in V^\vee$ choose f such that $\Pi(f) = \phi_1\phi_2 \in W$. One can check that the pairing

$$\langle \phi_1, \phi_2 \rangle := \int_{G(F)} f(g)dg \tag{9.3.2}$$

is well-defined. It is trivially $G(F)$-invariant and nondegenerate, and this realizes $I(\sigma^\vee)$ as the contragredient of $I(\sigma)$. This proves assertion (3). The proof of assertion (3) is similar; one assumes that $\phi_1, \phi_2 \in V$ and replaces $\phi_1\phi_2$ with $\phi_1\phi_2^\vee$, where the $\phi_2^\vee$ denotes the dual vector with respect to the given Hermitian pairing. ☐

We now isolate a particular case of the construction above. Let $A \le G$ be a maximal split torus, let $M = C_G(A)$ be its centralizer, and let $P \le G$ be a minimal parabolic containing M. There is a map

$$\mathrm{ord}_M : M(F) \to X_*(M)$$

defined by

$$\langle \mathrm{ord}_M(m), \chi\rangle = |\chi(m)|.$$

where $|\cdot|$ is the norm on F. Denote by $\Lambda(M) \subset X_*(M)$ the image of ord_M and let $M(F)^o$ be the group fitting into the exact sequence:

$$1 \to M(F)^o \to M(F) \to \Lambda(M) \to 1. \tag{9.3.3}$$

By definition, $m \in M(F)^o$ if and only if $\lambda(m) \in \mathcal{O}_F^\times$ for all $\lambda \in X^*(M)$. Thus $M(F)^0$ is open.

Definition 9.7. A quasi-character $\chi : M(F) \to \mathbb{C}$ is said to be *unramified* if $\chi|_{M(F)^o}$ is trivial.

Definition 9.8 (Unramified principal series)**.** Assume that G is unramified and that χ is an unramified character of M. Then the representation

$$I(\chi) := \mathrm{Ind}_P^G(\chi)$$

is an **unramified principal series representation**.

Remark 9.9. Of course, we have not checked that $I(\chi)$ is unramified, or even irreducible. In fact, it is sometimes reducible. When $I(\chi)$ is irreducible it is indeed the case that $I(\chi)$ is unramified.

Let

$$W = N(A)(F)/M(F).$$

It acts on $X_*(M)$ and leaves $X_*(A) \subset X_*(M)$ and $\Lambda(M)$ invariant. For $w \in W$, define the character χ^w by $\chi^w(m) := \chi(x_w^{-1} m x_w)$, where x_w represents w in $V(A)$. We say that χ is **regular** if $\chi^w = \chi$ only when $w = 1$. One has the following fundamental theorem:

Theorem 9.10. *The following hold:*

1. *The representations $I(\chi)$ and $I(\chi^w)$ are isomorphic for all $w \in W$, and these account for all the possible isomorphisms among the $I(\chi)$.*
2. *Every unramified representation is isomorphic to a unique subquotient of a unique $I(\chi)$.*

Consider now the following concrete instance of the above construction. Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be characters (not-necessarily unitary). Then $I(\chi_1, \chi_2)$ is the space of locally constant functions

$$f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$$

with $f(\left(\begin{smallmatrix} a_1 & * \\ & a_2\end{smallmatrix}\right) g) = \chi_1(a_1)\chi_2(a_2)\frac{|a_1|^{1/2}}{|a_2|^{1/2}} f(g)$. Note that χ_1, χ_2 are unramified if and only if they restrict trivially to $\mathbb{Z}_p^\times$, in which case they are completely determined by the value at a uniformizer ϖ, thus by the complex number s_i such that $\chi_i(\varpi) = q^{s_i}$ $(i = 1, 2)$. By part (1) of Theorem 9.10, the parameters (s_1, s_2) and (s_2, s_1) correspond to the same representation, so by part (2) of Theorem 9.10, we obtain a bijection

$$\left\{\begin{matrix}\text{semi-simple}\\ \text{conjugacy classes}\\ \text{in } \widehat{\mathrm{GL}}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{isomorphism classes}\\ \text{of irreducible unramified}\\ \text{representations of } \mathrm{GL}_2(\mathbb{Q}_p)\end{matrix}\right\}.$$

We are led to ask if this is the same bijection as that given by Corollary 9.5. This is indeed the case. We will not prove it is the same bijection, but we will describe how to set up the bijection using principal series instead of Hecke characters.

Assume for simplicity that G is split. We wish to construct another bijection between the two sets in Corollary 9.5.

Recall the exact sequence (9.3.3), and consider the complex torus

$$\widehat{T} := \mathrm{Spec}(\mathbb{C}[\Lambda(T)]).$$

Thus $\widehat{T}(\mathbb{C}) = \mathrm{Hom}(\Lambda(T), \mathbb{C})$. Let X^0 denote the set of unramified characters of $M(F)$. Thus any element of X^0 is a character of $M(F)/M(F)^0 \cong \Lambda(M)$, and hence defines a point in $\widehat{T}(\mathbb{C})$. If two characters are conjugate under W, then the two points are conjugate under $W(\widehat{G}, \widehat{T})(\mathbb{C})$, where $\widehat{G}$ is the dual group of G (see §2.7). Using Theorem 9.10 we therefore obtain a bijection

$$\left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of irreducible unramified} \\ \text{representations of } G(F) \end{matrix} \right\} \longrightarrow \widehat{T}(\mathbb{C})/W(\widehat{G}, \widehat{T})(\mathbb{C}) = \left\{ \begin{matrix} \text{semi-simple} \\ \text{conjugacy classes} \\ \text{in } \widehat{G}(\mathbb{C}) \end{matrix} \right\}. \tag{9.3.4}$$

sending χ to the corresponding point of $\widehat{T}(\mathbb{C})$. To check that this is the same bijection as that of Corollary 9.5, one must compute the effect of an element of the spherical Hecke algebra on an unramified representation, but we will not go into this here.

10. Statement of the Langlands conjectures and functoriality

Let G be a connected reductive group which is unramified over a nonarchimedian local field F_v. In Corollary 9.5 we set up a bijection

$$\{\text{semisimple conjugacy class in } \widehat{G}(\mathbb{C})\} \longleftrightarrow \{\text{isom. classes of unramified reps of } G(F_v)\}.$$

In this section we describe the (mostly conjectural) generalization of this, first to ramified representations and second to global fields. These generalizations are what is known as Langlands functoriality.

10.1. **The Weil group.** Let F be a global or local field, let $\bar{F}$ be an algebraic closure of F and let

$$\mathrm{Gal}_F := \mathrm{Gal}(\bar{F}/F)$$

be the absolute Galois group of F. It is endowed with the profinite topology.

A continuous homomorphism $\mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{C})$ necessarily has finite image. On the other hand, there are many continuous homomorphisms $\mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ with infinite image.

Example 10.1. If E is an elliptic curve over $\mathbb{Q}$ without CM, then the Tate module of E gives give a representation with image containing $\mathrm{SL}_2(\mathbb{Z}_\ell)$ for almost every ℓ [Ser72].

Example 10.2. A more elementary example is given by the cyclotomic character, it is the character

$$\chi_\ell : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathbb{Z}_\ell^\times$$

defined as follows. If $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$ and ζ_n is a primitive ℓ^nth root of unity then $\sigma(\zeta_n) = \zeta_n^{a_{\sigma,n}}$ for some $a_{\sigma,n} \in (\mathbb{Z}/\ell^n)^\times$. We then define

$$\chi_\ell(\sigma) = \varprojlim_n a_{\sigma,n}.$$

One can check that if $p \neq \ell$ then $\chi_\ell(\mathrm{Frob}_p) = p$.

It would be nice to view all of these examples as complex-valued representations of a single group. Weil and later Deligne introduced refinements of the Galois group to do just this. The definitions we now give are taken from [Tat79]

Definition 10.3. Let F be a local or global field. Then a *Weil group* for F is a tuple $(W_F, \phi, \{r_E\})$ where W_F is a topological group,

$$\phi\colon W_F \to \mathrm{Gal}(\overline{F}/F)$$

is a continuous homomorphism with dense image, and, for each finite extension E/F,

$$W_E := \phi^{-1}(\mathrm{Gal}_E),$$

and

$$r_E : C_E \longrightarrow W_E^{\mathrm{ab}} \tag{10.1.1}$$

is an isomorphism, where

$$C_E := \begin{cases} E^\times \text{ if } F \text{ is local} \\ E^\times \backslash \mathbb{A}_E^\times \text{ if } F \text{ is global.} \end{cases}$$

These data are required to satisfy the following assumptions:

(1) For each finite extension E/F the composite

$$C_E \xrightarrow{r_E} W_E^{ab} \xrightarrow{\text{induced by } \phi} \mathrm{Gal}_E^{ab}$$

is the reciprocity map of class field theory.

(2) Let $w \in W_F$ and $\sigma = \phi(w) \in \mathrm{Gal}_F$. For each E the following diagram is commutative:

$$\begin{tikzcd} C_E \arrow[r, "r_E"] \arrow[d, "\sigma"'] & W_E^{ab} \arrow[d] \\ C_{E^\sigma} \arrow[r, "r_{E^\sigma}"'] & W_{E^\sigma}^{ab} \end{tikzcd}$$

commutes.

(3) For $E' \subseteq E$

$$\begin{tikzcd} C_{E'} \arrow[r, "r_E"] \arrow[d, "\text{inclusion}"'] & W_{E'}^{ab} \arrow[d, "\text{transfer}"] \\ C_E \arrow[r, "r_{E'}"'] & W_E^{ab} \end{tikzcd}$$

commutes.

(4) Finally, the map

$$W_F \to \varprojlim W_{E/F}$$

is an isomorphism, where $W_{E/F} = W_F/\overline{W_E^{ab}}$, the bar denoting closure.

If the Weil group exists, it is unique up to isomorphism.

Example 10.4. Let F be a local field and for all finite extensions E/F let k_E be the residue field of E and q_E the cardinality of k_E. Put $\overline{k} = \bigcup_E k_E$. Then in this case W_F is the dense subgroup of Gal_F generated by the $\sigma \in \mathrm{Gal}_F$ such that on $\overline{k}$, σ acts as $x \mapsto x^{q_E^n}$ for some $n \in \mathbb{Z}$. Then $r_E(a)$ acts as $x \mapsto x^{|a|}$ on $\overline{k}$.

Example 10.5. In the global function field case, the Weil group is defined as in the previous example, but one replaces the residue field above with the constant field.

Example 10.6. For $F = \mathbb{C}$, then $W_F = \mathbb{C}^\times$, ϕ is the trivial map and $r_F = \mathrm{id}$.

Example 10.7. For $F = \mathbb{R}$, then $W_F = \overline{F}^\times \cup j\overline{F}^\times$ where $j^2 = -1$ and $jcj^{-1} = \overline{c}$. Here ϕ takes $\overline{F}^\times$ to 1 and $j\overline{F}^\times$ to the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

For number fields one doesn't have a nice intrinsic description like in the above examples. Almost by definition of the Weil group, one obtains the following correspondence:

Theorem 10.8 (The Langlands correspondence for GL_1)**.** *There is a bijection between*

$$\{\textit{isomorphism classes of irreducible automorphic representations of } \mathrm{GL}_1(\mathbb{A}_F)\}$$

and

$$\{\textit{continuous representations } \chi : W_F \to \mathrm{GL}_1(\mathbb{C})\}$$

Proof. An irreducible automorphic representation of $\mathrm{GL}_1(\mathbb{A}_F)$ can be identified with a character of $F^\times\backslash\mathbb{A}_F^\times$, which is isomorphic to W_F^{ab} by definition of the Weil group. □

Example 10.9. Under this correspondence the cyclotomic character χ_ℓ corresponds to the character $|\cdot|$ (the inverse of the idelic norm), provided that r_E is normalized so that uniformizers of prime ideals are sent to the inverse of the corresponding Frobenius element.

10.2. **The Weil-Deligne group.** For better or worse the Weil group is still not big enough; we will make this precise later. The correct enlargement of W_F when F is a number field should be the as yet hypothetical Langlands group. On the other hand, the correct enlargement of W_F when F is a nonarchimedian local field is known, so in this subsection we restrict to this case.

When F is a local field the **Weil-Deligne group** is

$$W_F' := W_F \times \mathrm{SL}_2(\mathbb{C}). \tag{10.2.1}$$

We now prepare to define a representation of W_F'. Let k be the residue field of F, $\bar{k}$ a choice of algebraic closure, and $\mathrm{Gal}_k := \mathrm{Gal}(\bar{k}/k)$. The action of Gal_F preserves the ring of integers and the (unique) prime ideal of the ring of integers of any finite extension field E/F contained in $\bar{F}$. There is thus an exact sequence

$$1 \longrightarrow I_F \longrightarrow \mathrm{Gal}_F \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

where I_F is the **inertia subgroup**, which can be defined as the kernel of the map $\mathrm{Gal}_F \to \mathrm{Gal}_k$.

The Weil group W_F is a subgroup of $\mathrm{Gal}(\bar{F}/F)$ (compare Example 10.4) and contains I_F. We thus have an exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathrm{Gal}(\bar{k}/k).$$

Remark 10.10. The last map is not surjective. Its image is isomorphic to $\mathbb{Z}$, whereas $\mathrm{Gal}(\bar{k}/k)$ is isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.

Again using Example 10.4 we see that

$$W_F \cong I_F \ltimes \langle \mathrm{Fr} \rangle$$

where Fr is a Frobenius element.

Let G be a reductive group over $\mathbb{C}$.

Definition 10.11. A **representation of W_F' into** $G(\mathbb{C})$ is a homomorphism

$$\varphi : W_F' \longrightarrow G(\mathbb{C})$$

such that φ is trivial on an open subgroup of I_F, $\varphi(\mathrm{Fr})$ is semisimple, and $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is induced by a morphism of algebraic groups $\mathrm{SL}_2 \to G$.

By a representation of W_F' we mean a representation of W_F' into $\mathrm{GL}_n(\mathbb{C})$ for some integer n.

Remark 10.12. There are various equivalent definitions of a representation of the Weil-Deligne group in the literature, see [GR10].

10.3. **Local Langlands for the general linear group.** Let F be a nonarchimedian local field. The local Langlands conjecture for $\mathrm{GL}_n(F)$, now a theorem, can be stated vaguely as follows:

Theorem 10.13 (Local Langlands correspondence)**.** *There is a bijection, satisfying certain desirata, between representations of W'_F into $\mathrm{GL}_n(\mathbb{C})$ and irreducible admissible representations of $\mathrm{GL}_n(F)$.*

When $n = 1$ this is essentially local class field theory. The theorem was proved by Harris-Taylor first [HT01], then a simplified proof was found by Henniart [Hen00]. One can find a precise statement of the theorem in either of these references. The bijections are required to preserve L and ε factors

Remark 10.14. In the archimedian case there is a bijection between representations of W_F into $\mathrm{GL}_n(\mathbb{C})$ and irreducible admissible representations of $\mathrm{GL}_n(F)$; there is no need for a Weil-Deligne group in this case.

10.4. **The Langlands dual.** In Corollary 9.5 we saw that we need to introduce the complex dual $\widehat{G}(\mathbb{C})$ of a reductive group G in order to classify representations of $G(F)$ To obtain analogues of Theorem 10.13 for other groups, it is necessary to develop the notion of the Langlands dual group, which is an extension of $\widehat{G}(\mathbb{C})$ by a Galois group. In Theorem 10.13 this wasn't necessary because GL_n is split and $\widehat{\mathrm{GL}}_n = \mathrm{GL}_n(\mathbb{C})$.

For the moment, let G be a connected reductive group over a global or local field F and let $T \leq G$ a maximal torus. To these data we associated in §2.7 a root datum $\Psi(G, T) = (X^*(T), X_*(T), \Phi, \Phi^\vee)$. If G is split, then we set

$$^LG := \mathrm{Gal}(\bar{F}/F) \times \widehat{G}(\mathbb{C})$$

where $\widehat{G}$ is the complex dual group defined as in §2.7.

In the nonsplit case the definition of the Langlands dual is somewhat complicated. However, it uses some notions that appear in a variety of contexts, so we will give the full definition. For the moment, let G be a connected reductive group over $\mathbb{C}$.

Definition 10.15. A set of **positive roots** $\Phi^+ \subset \Phi$ is a set of roots such that

(1) For all elements $\alpha \in \Phi$ exactly one of α and $-\alpha$ is an element of Φ.
(2) For $\alpha, \beta \in \Phi^+$ the sum $\alpha + \beta \in \Phi^+$.

A **simple root** in Φ is a root in Φ^+ that cannot be written as a sum of two elements of Φ^+.

The set of roots dual to Φ^+ in $\Phi^\vee$ forms a set of positive roots in $\Phi^\vee$. We denote by $\Delta \subseteq \Phi$ the set of simple roots with respect to some set of positive roots. The set of dual roots $\Delta^\vee \subseteq \Phi^\vee$ is a maximal set of simple roots with respect to the dual set of positive roots. The sets Δ, $\Delta^\vee$ span Φ and $\Phi^\vee$, respectively, as $\mathbb{Z}$-modules and thus determine Φ and $\Phi^\vee$. A tuple

$$(X, Y, \Delta, \Delta^\vee)$$

is called a **based root datum** if $(X, Y, \Phi, \Phi^\vee)$ is a root datum, $\Delta \subseteq \Phi$ is a maximal set of simple roots (with respect to some set of positive roots) and $\Delta^\vee$ is the dual set. There is an obvious notion of isomorphism of root data; it is simply a pair of linear isomorphisms on the first two factors preserving the pairing and the sets of simple roots.

We let $\Psi_0(G, B, T) := (X^*(T), X_*(T), \Delta, \Delta^\vee)$ be a choice of based root datum, and $\Psi(G, T)$ the root datum it defines. The reason for the B in this notation is the following lemma:

Lemma 10.16. *The choice of a set of simple roots $\Delta \subseteq \Phi$ is equivalent to the choice of a Borel subgroup $B \leq G$ containing T.*

Before giving the proof we recall that for each $\alpha \in \Phi$ there exists a unique homomorphism

$$\exp_\alpha \colon \mathfrak{g}_\alpha \to G(\mathbb{C})$$

such that $t\exp_\alpha(x)t^{-1} = \exp(\alpha(t)x)$ and $\mathrm{Lie}(\exp_\alpha)$ is the natural inclusion $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}$.

For example if $G = \mathrm{GL}_n$ and $\alpha = e_{ij}$ in the notation of (2.44) then

$$\exp_\alpha(x) = \sum_{n\geq 0} \frac{e_{ij}^\vee(x)^n}{n!}$$

We denote by

$$U_\alpha := \mathrm{Im}(\exp_\alpha).$$

Proof. Given a choice of a set of simple roots Δ the group

$$B = \langle T, \{U_\alpha\}_{\alpha\in\Delta}\rangle \tag{10.4.1}$$

is a Borel subgroup, and conversely given a a Borel there exists a unique maximal set of simple roots Δ such that such that (10.4.1) is true. □

Definition 10.17. A **pinning** of G is a tuple

$$(B, T, \{u_\alpha\}_{\alpha\in\Delta})$$

where $T \leq B \leq G$ is a maximal torus and a Borel subgroup, respectively, Δ is the set of simple roots with respect to this Borel, and $u_\alpha \in U_\alpha - 1$ for all α.

We let $\mathrm{Aut}((B, T, \{u_\alpha\}_{\alpha\in\Delta})$ be the group of automorphisms of G that preserve B and T and the set u_α. Then it is a fact that restriction defines an isomorphism

$$\mathrm{Aut}((B, T, \{u_\alpha\}_{\alpha\in\Delta})) \cong \mathrm{Aut}(\Psi_0(G, B, T))$$

where we use Δ to define $\Psi_0(G, B, T)$.

One has a split exact sequence

$$1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\Psi_0(G, B, T)) \longrightarrow 1. \tag{10.4.2}$$

Splittings of this sequence are in bijection with pinnings.

We now assume that G is a connected reductive group over a local or global field F. To simplify matters we assume that G is quasi-split, and so has a Borel subgroup $B \leq G$ and a maximal torus $T \leq B \leq G$. We obtain a based root datum

$$\Psi_0(G, B, T) := \Psi_0(G_\mathbb{C}, B_\mathbb{C}, T_\mathbb{C})$$

and since B and T are defined over F there is an action of $\mathrm{Gal}(\bar{F}/F)$ on $\Psi_0(G, B, T) := (X^*(T), X_*(T), \Delta, \Delta^\vee)$.

We note that $(X_*(T), X^*(T), \Delta^\vee, \Delta)$ is the based root datum of some tuple $\widehat{G} \geq \widehat{B} \geq \widehat{T}$, that is, it is equal to $\Psi_0(\widehat{G}, \widehat{B}, \widehat{T})$. We therefore obtain an action of $\mathrm{Gal}(\bar{F}/F)$ on $\Psi_0(\widehat{G}, \widehat{B}, \widehat{T})$.

Via a choice of pinning of $\widehat{G}$ we obtain a section of the map

$$\mathrm{Aut}(\widehat{G}) \to \mathrm{Aut}(\Psi_0(\widehat{G}, \widehat{B}, \widehat{T}))$$

and hence a map $\mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Aut}(\widehat{G})$. We define the **Langland dual group of** G to be the semidirect product

$$^L G := \widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(\bar{F}/F)$$

with respect to this action. More colloquially, any group of this form is said to be an **L-group**. A morphism of L-groups

$$^L H \longrightarrow {}^L G$$

is simply a homomorphism commuting with the projections to $\mathrm{Gal}(\bar{F}/F)$ such that its restriction to the neutral components is induced by a map of algebraic groups $\widehat{H} \longrightarrow \widehat{G}$.

10.5. L**-parameters.** Let G be a reductive group over a local field F.

Definition 10.18. An L**-parameter** is a representation of W_F' into ${}^LG$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$. Two L-parameters are equivalent if they are conjugate by an element of $\widehat{G}(\mathbb{C})$.

We note that given an L-parameter $\varphi : W_F' \to {}^LH$ and an L-map

$$ {}^LH \longrightarrow {}^LG $$

we obtain an L-parameter $\varphi : W_F' \to {}^LG$.

10.6. **The local Langlands correspondence and functoriality.** There is a conjectural partition of the set of equivalence classes of irreducible representations of $G(F)$ into disjoint sets called L-packets. In the case $G = \mathrm{GL}_n$ the L-packets are singletons, that is, an L-packet is just the equivalence class of a given irreducible admissible representation.

The analogue of the Langlands correspondence for general groups over local fields is the following vaguely stated conjectural statement:

Conjecture 10.19 (Local Langlands correspondence)**.** *There is a bijection between L-packets and equivalence classes of L-parameters satisfying various desiderata.*

In the last subsection we saw that an L-map induces a map

$$ \{L\text{-parameters } W_F' \to {}^LH\} \longrightarrow \{L\text{-parameters } W_F' \to {}^LG\}. $$

Local Langlands functoriality is the statement that there is a corresponding map of L-packets:

Conjecture 10.20 (Local Langlands functoriality)**.** *Given an L-map ${}^LH \to {}^LG$ there is a corresponding transfer of L-packets compatible with the local Langlands correspondence.*

Given these two conjectures, together with the fact that the local Langlands correspondence is a theorem for GL_n, one can often construct an ad-hoc definition of L-packets. More specifically, if we have an L-map

$$ {}^LH \longrightarrow {}^L\,\mathrm{GL}_n $$

that is injective, and we find a natural way to associate to equivalence classes of admissible representations of $H(F)$ equivalence classes of admissible representations of $\mathrm{GL}_n(F)$. One can then define an L-packet on H to be the set of equivalence classes of admissible representations of $H(F)$ that map to a given admissible representation of $\mathrm{GL}_n(F)$ (remember, L-packets on GL_n are defined to be equivalence classes of irreducible admissible representations). This is the approach taken in forthcoming work of Arthur and Mok.

10.7. **Global Langlands functoriality.** Let F be a global field. In the global case, it is not known what group (if any) parametrizes L-packets of (irreducible) automorphic representations, in analogy with how L-parameters conjecturally parametrize L-packets of admissible representations in the local setting as in Conjecture 10.19. However, the local functorial correspondence of Conjecture 10.20 has a natural analogue. Namely, one conjectures the existence of a partition of equivalence classes of admissible automorphic representations of $G(\mathbb{A}_F)$ into disjoint sets called L-packets such that the following conjecture holds:

Conjecture 10.21. *Given an L-map*

$$ R : {}^LH \longrightarrow {}^LG $$

there is a corresponding transfer of L-packets of automorphic representations that is compatible with the local correspondence at all places.

We leave it to the reader to formulate the appropriate notion of compatibility here.

Remark 10.22. For some applications, both local and global, it may not be the case that L-packets provide the most natural manner of organizing automorphic representations. For nontempered representations (a term we have not defined) Arthur packets or A-packets are more appropriate. For real groups, it also seems that the notion of a Vogan packet is often useful. We will not discuss these notions here; indeed, even our treatment of L-packets is rather vague.

As a special case, assume that H and G are split at a nonarchimedian place v. In this case it is expected that if $\pi = \otimes_v \pi_v$ is an element of an L-packet of automorphic representations of H and π_v is unramified, then every element of the L-packet is unramified, and hence the corresponding local L-packet at v is nothing but the representation trivial on I_{F_v} and $\mathrm{SL}_2(\mathbb{C})$ sending Fr to the Langlands class $A(\pi_v) \in \widehat{H}(\mathbb{C})$ of π_v. Then any transfer of the L-packet of π should also be unramified at v and have Langlands class $R(A(\pi_v))$.

10.8. ***L*-functions.** It has been said that the reason for studying automorphic representations in the first place was because of their relation to the theory of L-functions. Unfortunately we will not be giving adequate space to this subject, in part because there are already useful surveys of the theory available [**?**]. W

At least for GL_n, there are two ways of associating an L-function to an automorphic representation. One essentially goes back to Hecke, and was generalized to GL_n by Jacquet, Shalika and Piatetskii-Shapiro (see loc. cit.). There is another way to associate L-functions to automorphic representations, essentially using the local Langlands correspondence. Part of the content of the local Langlands conjectures is that these two constructions give the same answer.

To start, we suppose v is a place of F and that we are given an L-parameter $\varphi : W'_{F_v} \to {}^LG$ and a representation

$$r : {}^LG_{F_v} \to \mathrm{GL}(V)$$

we associate an L-function as follows:

$$L(s, r \circ \varphi) := \det\left(1 - \mathrm{Fr}_v q_v^{-s} | V^{I_{F_v}}\right)^{-1} \tag{10.8.1}$$

We also define an ε factor. If $\psi : F_v \to \mathbb{C}^\times$ is an additive character, $dx^\times = |x|^{-1}dx$ is a Haar measure on $F_v^\times$, and $r \circ \varphi$ is a quasi-character χ, i.e. a homomorphism

$$\chi : W_F \longrightarrow \mathrm{GL}_1(\mathbb{C}),$$

then we define $\varepsilon(s, \chi, \psi, dx)$ to be the nonzero complex number such that

$$\frac{\int_{F_v^\times} \widehat{f}(x)\chi^{-1}(x)|x|dx^\times}{L(s-1, \chi^{-1})} = \varepsilon(s, \chi, \psi, dx)\frac{\int_{F_v^\times} f(x)\chi(x)dx^\times}{L(s, \chi)} \tag{10.8.2}$$

where $\widehat{f}(y) := \int_{F_v} f(x)\psi(xy)dx$ is the Fourier transform of f. Langlands and Deligne defined in general a ε-factor $\varepsilon(s, \varphi \circ r, \psi, dx)$ that is invariant under induction.

If an L-packet of an admissible representation π_v of $G(F_v)$ corresponds to an L-parameter $\varphi : W'_F \to {}^LG_{F_v}$ we define

$$L(s, \pi_v, r) := L(s, \varphi \circ r) \quad \text{ and } \quad \varepsilon(s, \pi_v, r, \psi, dx) = \varepsilon(s, \varphi \circ r, \psi, dx).$$

Remark 10.23. Note that admissible representations in the same L-packet necessarily have the same L and ε-functions. This is the reason for the terminology "L-packet." Sometimes one says that elements in the same L-packet are L-indistinguishable.

If r was the localization of an L-map $r : {}^L G \to \mathrm{GL}_n(\mathbb{C})$ then we set

$$L(s,\pi,r) := \prod_v L(s,\pi_v,r) \quad \text{and} \quad \varepsilon(s,\pi,r) := \prod_v \varepsilon(s,\pi_v,r,\psi_v,dx_v)$$

where $dx = \prod_v dx_v$, $\psi := \prod_v \psi_v$, $\int_{F\backslash \mathbb{A}_F} dx = 1$ and $\psi|_F = 1$. As the notation indicates, the global ε-factor does not depend on the choice of ψ_v and dx_v. The basic conjectures regarding these L-functions are as follows:

Conjecture 10.24. *The L-function $L(s,\pi,r)$ is meromorphic as a function of s, is bounded in vertical strips, and satisfies the functional equation $L(s,\pi,r) = \varepsilon(s,\pi,r)L(1-s,\pi^\vee,r^\vee)$.*

When r is the tensor product map from ${}^L\,\mathrm{GL}_n \times \mathrm{GL}_m$ to $\mathrm{GL}_{mn}(\mathbb{C})$ then this theorem is known (see [Cog07] for a survey). There are now other cases known due to work on converse theory; some are surveyed in loc. cit. , some fall under the general heading of the Langlands-Shahidi method (see [CKM04] for a survey). If $r : {}^L\,\mathrm{GL}_m \times \mathrm{GL}_n \to \mathrm{GL}_{mn}(\mathbb{C})$ is the tensor product representation then we set

$$L(s,\pi_1 \times \pi_2) := L(s,\pi_1 \times \pi_2, r) \quad \text{and} \quad \varepsilon(s,\pi_1\times\pi_2) := \varepsilon(s,\pi_1\times\pi_2,r).$$

These are known as **Rankin-Selberg** L-functions.

Example 10.25. Assume that $G = \mathrm{GL}_n$ and that π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. For almost every place v of F the local factor π_v is unramified (see Theorem 7.5). Let

$$A(\pi_v) := \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

be its Langlands class (see Corollary 9.5). In this case one has

$$L(s,\varphi_v) = \prod_i (1 - a_i q_\nu^{-s})^{-1}$$

In the case where $G = \mathrm{GL}_n$, the local parameters of a cuspidal automorphic representation at the unramified places determine the representation [JS81b] [JS81a]:

Theorem 10.26 (Jacquet and Shalika)**.** *If π_1, π_2 are cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ such that $\pi_{1v} \simeq \pi_{2v}$ for all but finitely many v then $\pi_1 \simeq \pi_2$.*

This theorem is known as **strong multiplicity one**, although it does not imply that a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ occurs with multiplicity one in the cuspidal subspace of $L^2(\mathrm{GL}_n(F)\backslash^1\,\mathrm{GL}_n(\mathbb{A}_F))$, though this latter statement is also true.

For a beautiful generalization of this work we refer the reader to forthcoming work of Dinakar Ramakrishnan on automorphic analogues of the Chebatarev density theorem.

Remark 10.27. Theorem 10.26 is false for essentially every group that is not a general linear group.

10.9. **Nonarchimedian representation theory.** For the moment let F be a global field. In Theorem 7.5 we showed that an admissible representation of $G(\mathbb{A}_F)$ factors as a restricted direct product:

$$\pi \cong \otimes_v' \pi_v.$$

This provides us with a link between automorphic representations and admissible representations of $G(F)$ as v varies; automorphic representations are simply products of admissible representations

Remark 10.28. Not every product of admissible representations gives rise to an automorphic representation (compare Theorem 10.26).

It therefore makes sense to study admissible representations place by place. The archimedian case is better understood from the viewpoint of, e.g. functoriality. We will concentrate on the nonarchimedian case. Thus for the remainder of this section we let F be a local nonarchimedian field and G a connected reductive F-group.

11. The philosophy of cusp forms

Let $P \leq G$ be a parabolic subgroup and let $MN = P$ be its Levi decomposition. We have shown in §9.3 how to construct a functor

$$\mathrm{Ind}_M^G := \mathrm{Ind}_P^G : \mathrm{SmRep}(M(F)) \longrightarrow \mathrm{SmRep}(G(F))$$

from the category of smooth representations of $M(F)$ to the category of smooth representations of $G(F)$ (recall that smooth representations are those representations such that every vector is fixed by an open subgroup). The functor maps irreducibles to irreducibles and unitary representations to unitary representations by Lemma 9.6.

One is immediately led to ask what subset of the set of irreducible representations of $G(F)$ are induced. The answer is part of what Harish-Chandra calls the **philosophy of cusp forms**. To state it vaguely, we recall the following definition:

Definition 11.1. A **supercuspidal** (resp. **quasicuspidal**) representation of $G(F)$ is an admissible (resp. smooth) representation such that all matrix coefficients of the representation are compactly supported modulo $Z_G(F)$.

The philosophy of cusp forms states that the only irreducible admissible representations of $G(F)$ that are not induced are the so-called supercuspidal representations, and every other representation is induced from a supercuspidal representation of the Levi-subgroup of some parabolic $P \leq G$.

Remark 11.2. One should think of supercuspidal representations as being analogous to cuspidal representations in the global setting; the corresponding statement in that setting was proven by Langlands in his important and notoriously difficult work [Lan76]. In passing we note that Langlands has stated that he came to his conjectures on functoriality while working on loc. cit.

We will develop a few basic functors useful for working with representations of reductive groups over nonarchimedian fields and state a result that makes the philosophy of cusp forms mentioned above precise in Corollary 11.9 below. We will also discuss the notion of a trace of an admissible representation, and in addition develop some of the basic properties of supercuspidal representations, including the fact that they admit coefficients (see Proposition **??**), a fact that we will later use in our treatment of simple trace formulae.

The basic reference for this subject seems to be an unpublished paper of Casselman available at http://www.math.ubc.ca/ cass/research/pdf/p-adic-book.pdf. The paper [Car79] contains sketches of proofs and refers to this manuscript for the details.

11.1. **Jacquet functors.** There are other ways of defining a supercuspidal representation; in order to describe them we need to describe left-adjoints to the functors Ind_M^G. We define a **restriction functor** or **Jacquet functor**

$$\mathrm{Res}_P^G : \mathrm{SmRep}(G(F)) \longrightarrow \mathrm{SmRep}(G(F)) \tag{11.1.1}$$

by assigning to a representation (π, V)

$$(\pi|_P \otimes \delta_P^{1/2}, V/V(N))$$

where

$$V(N) := \langle \pi(n)\phi - \phi : \phi \in V, n \in N(F) \rangle$$

The following is easy to verify:

Proposition 11.3. *Induction and restriction are exact and transitive.* □

The following is a version of Frobenius reciprocity:

Proposition 11.4. *Restriction is left adjoint to induction; in other words*

$$\mathrm{Hom}_{G(F)}(V, \mathrm{Ind}_P^G(W)) = \mathrm{Hom}_{M(F)}(\mathrm{Res}_P^G(V), W).$$

Proof. There is a $G(F)$-equivariant map

$$\begin{aligned}\Lambda : \mathrm{Ind}_P^G(W) &\longrightarrow W\\ f &\longmapsto f(1)\end{aligned}$$

where $1 \in G(F)$ is the identity; it is clearly surjective. Thus we have a $G(F)$-equivariant map

$$\Lambda \circ (\cdot) : \mathrm{Hom}_{G(F)}(V, \mathrm{Ind}_P^G(W)) \longrightarrow \mathrm{Hom}_{M(F)}(\mathrm{Res}_P^G(V), W)$$

given by composition with Λ. Here we are using the fact that N acts trivially on $\mathrm{Ind}_P^G(W)$ and thus any $G(F)$-equivariant map $V \to \mathrm{Ind}_P^G(W)$ factors through $\mathrm{Res}_P^G(V)$. To construct the inverse, suppose we are given an $M(F)$-intertwining map $f : \mathrm{Res}_P^G(V) \to W$. We define $\Phi : V \to \mathrm{Ind}_P^G(W)$ to be

$$\Phi_\phi(g) := f(g \cdot \phi).$$

□

One would hope that the functor Res_P^G takes admissible representations to admissible representations, and this is indeed the case:

Theorem 11.5 (Jacquet)**.** *The restriction functor* Res_P^G *takes admissible representations to admissible representations.*

Jacquet also proved the following elegant characterization of supercuspidal representations using these functors:

Theorem 11.6 (Jacquet)**.** *A smooth representation π of $G(F)$ is quasi-cuspidal if and only if $Res_P^G(\pi) = 0$ for all parabolic subgroups $P \leq G$.*

Remark 11.7. It is because of these theorems that the restriction maps are often called Jacquet functors.

We will not prove this result because the reductive group structure theory required is somewhat notationally intricate. However we will prove some corollaries.

We have the following proposition, whose proof we defer for a moment:

Proposition 11.8. *A quasi-cuspidal irreducible representation is supercuspidal.*

Jacquet's results above allow us to deduce the following concrete manifestation of the philosophy of cusp forms:

Corollary 11.9. *If π is a smooth irreducible representation of $G(F)$ then*

(1) *There exists a parabolic subgroup $P = MN \leq G$, a supercuspidal representation σ of $M(F)$ and an embedding $\pi \hookrightarrow Ind_P^G(\sigma)$.*
(2) *The representation π is admissible.*

Proof. The first assertion implies the second, as induction preserves admissibility by Lemma 9.6. For the first, we proceed by induction on the dimension of G (as an F-algebraic group, say). If G has dimension 1 then it is a torus, and so the result is trivial.

Assume that for all proper parabolic subgroups P there is no embedding $\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma)$ where σ is a smooth irreducible representation of $M(F)$. This is equivalent to the statement that there is no nonzero $G(F)$-map $\pi \to \mathrm{Ind}_P^G(\sigma)$ by the irreducibility of π. Applying Frobenius reciprocity (Proposition 11.4) we see that

$$\mathrm{Hom}_{M(F)}(\mathrm{Res}_P^G(\pi), \sigma) = 0$$

for all parabolic subgroups $P = MN$ and all smooth representations σ of $M(F)$, which implies π is supercuspidal by Theorem 11.6.

Now assume that there is a proper parabolic subgroup P such that $\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma)$. By the exactness of restriction we obtain an injection

$$\mathrm{Res}_P^G(\pi) \hookrightarrow \sigma,$$

so we can apply our inductive hypothesis. □

We now prove Proposition 11.8.

Proof of Proposition 11.8. Fix $\phi \in V - \{0\}$. For all compact open subgroups $K \leq G(F)$ one has

$$V^K = \pi(e_K)V = \langle \pi(e_K)\pi(g)v : g \in G(F) \rangle.$$

We must show this space is finite dimensional. Assume to the contrary that there exists $(g_n)_{n \in \mathbb{Z}_{>0}} \subseteq G(F)$, all inequivalent modulo $Z_G(F)$, such that $\pi(e_K)\pi(g_n)\phi$ are linearly independent. Let $W \subseteq V^K$ be an arbitrary $\mathbb{C}$-vector space such that

$$V^K = W \oplus \langle \pi(e_k)\pi(g_n)v : n \in \mathbb{Z}_{>0} \rangle.$$

As $V = V^K \oplus \ker \pi(e_K)$, we can define $\phi^\vee \in \mathrm{Hom}(V, \mathbb{C})$ such that

$$\langle \phi^\vee, \pi(e_K)\pi(h_n)v \rangle = n$$

for all n and $\phi^\vee|_{W \oplus \ker(e_K)} = 0$. Thus $\phi^\vee$ is fixed by K and hence an element of $V^\vee$. On the other hand, by construction the support of the matrix coefficient $\langle \widetilde{v}, \pi(g)v \rangle$ is not compact modulo the center. This implies the claim. □

11.2. **Traces, characters, coefficients.** Let π be an admissible representation of $G(F)$. Then for all $f \in C_c^\infty(G(F))$ one has an operator

$$\pi(f) : V \longrightarrow V.$$

There is a compact open subgroup $K \leq G(F)$ such that $f \in C_c^\infty(G(F)//K)$, and hence $\pi(f)$ induces an operator

$$\pi(f) : W \longrightarrow W$$

for any finite dimensional subspace $V^K \leq W \leq V$. We define the **trace of** $\pi(f)$ by

$$\mathrm{tr}(\pi(f)) := \mathrm{tr}(\pi(f)|_W) \tag{11.2.1}$$

for any such W. This defines a distribution

$$\theta_\pi := \mathrm{tr}(\pi) : C_c^\infty(G(F)) \to \mathbb{C}$$

called the **character** of π. Of course this depends on a choice of Haar measure. Let $G^{reg} \leq G$ denote the subscheme consisting of regular semsimple elements; this is the subscheme such that

$$G^{reg}(R) := \{\gamma \in G(R) : C_\gamma^\circ \leq G \text{ is a maximal torus}\}.$$

The following is a fundamental and deep result (see [HC99]):

Theorem 11.10 (Harish-Chandra)**.** *The distribution θ_π is represented by a locally constant function with support in $G^{reg}(F)$.* □

In other words there is a locally constant function θ_π on $G^{reg}(F)$ such that

$$\operatorname{tr}\pi(f) = \int_{G(F)} \theta_\pi(g)f(g)dg$$

for all $f \in C_c^\infty(G(F))$. This result tells us that we can almost regard θ_π as a function.

In most settings, there is a version of linear independence of characters, and this is no exception:

Proposition 11.11 (Linear independence of characters)**.** *If $\pi_1, \dots, \pi_n$ is a finite set of admissible irreducible representations, such that $\pi_i \cong \pi_j$ implies $i = j$, then the distributions θ_{π_i} are linearly independent.*

Proof. As usual, we use compact open subgroups to reduce to a finite dimensional situation. Fix a compact open subgroup $K \leq G(F)$ such that $V_i^K \neq 0$ for all i. This implies that $\{V_i^K\}$ is a finite family of finite dimensional $\mathbb{C}$-vector spaces with an action of $C_c^\infty(G(F)//K)$. They are all simple, that is irreducible, for this action. Moreover, they are pairwise nonisomorphic as Hecke-modules. Let A be the image of $C_c^\infty(G(F)//K)$ in $\prod_i \operatorname{End}_{\mathbb{C}}(V_i^K)$. Then A is a finite-dimensional $\mathbb{C}$-algebra and the V_i^K are a finite family of finite-dimensional nonisomorphic simple A-modules. Hence the traces $\theta_{\pi_i}|_{C_c^\infty(G(F)//K)}$ are linearly independent (this follows from a standard argument using the Jacobson density theorem). ☐

Thus traces can be used to distinguish between a finite set of representations. In particular, if $\{\pi_1, \dots, \pi_n\}$ is a finite set of pairwise nonisomorphic irreducible representations then linear independence of characters implies that we can find an $f \in C_c^\infty(G(F))$ such that

$$\operatorname{tr}(\pi_i(f)) = 0 \text{ if and only if } i \neq 1,$$

for example.

One can ask for more. Let π be an admissible irreducible representation. A **coefficient** of π is a smooth function $f_\pi \in C_c^\infty(G(F))$ such that $\operatorname{tr}\pi(f_\pi) \neq 0$ and $\operatorname{tr}\pi_1(f_\pi) = 0$ for $\pi_1 \not\cong \pi$. If $Z_G(F)$ is noncompact, we can weaken the last condition to $\pi_1 \not\cong \pi \otimes \chi$ for some character $\chi\colon G(F) \to \mathbb{C}^\times$. Thus if a coefficient for π exists, we can use it to isolate π among **any** set of irreducible admissible representations, finite or not. For general π, it is not necessarily true that such f exist. In the archimedian setting, it is good to think of this in terms of the Heizenberg uncertainty principle, namely that the Fourier transform of a compactly support smooth function cannot again be compactly supported, so we cannot isolate a particular phase using a compactly supported function. However, in special cases this principle fails, in particular in the archimedian case for so-called discrete series representations, and in the nonarchimedian case for supercuspidal representations, as the following proposition shows:

Proposition 11.12. *Assume that $Z_G(F)$ is compact, and let π be a cuspidal representation of $G(F)$. Then for all $f \in C_c^\infty(G(F))$ there exists a unique $f_\pi \in C_c^\infty(G(F))$ such that*

$$\pi(f_\pi) = \pi(f) \text{ and}$$
$$\pi_1(f_\pi) = 0 \text{ if } \pi_1 \neq \pi.$$

Note that, as an immediate consequence, we see that coefficients exist for supercuspidals.

We require some preparation before giving the proof. Given a smooth representation (π, V) of $G(F)$, we have a smooth representation σ of $G(F) \times G(F)$ on

$$\operatorname{End}_{sm}(V) \leq \operatorname{End}(V),$$

the space of smooth endomorphisms (see below Lemma 8.3). It is given explicitly by

$$\sigma(g_1, g_2)\phi = \pi(g_2^{-1}) \circ \phi \circ \pi(g_1).$$

There is an isomorphism of $G(F) \times G(F)$-representations

$$\alpha : V^\vee \otimes V \longrightarrow (\mathrm{End}_{sm}(\pi), \sigma)$$

given by

$$\alpha(\phi^\vee \otimes \phi)(\phi_1) = \langle \phi^\vee, \phi_1 \rangle \phi,$$

Assume that π is admssible. Then $\mathrm{End}_{sm}(V)$ is again admissible, since for any compact subgroups $K_1, K_2 \subset G(F)$, we can find a compact $K \subset G(F)$ with $K \times K \subset K_1 \times K_2$, and therefore

$$\mathrm{End}_{sm}(\pi)^{K_1 \times K_2} \cong \mathrm{End}_{sm}(V_\pi^{K_1 \times K_2}) \cong V_\pi^{K_1} \otimes (V_\pi^\vee)^{K_2} \subset V_\pi^K \otimes (V_\pi^\vee)^K$$

is of finite dimension.

One also has a map

$$\beta : (\mathrm{End}_{sm}(\pi), \sigma) \longrightarrow (C^\infty(G(F)), \rho)$$

given by

$$\beta(\phi)(g) := \mathrm{tr}(\pi(g) \circ \phi),$$

where ρ acts via $\rho(g_1, g_2)(f)(h) = f(g_1^{-1} h g_2)$.

Proof of Proposition 11.12. Since $Z_G(F)$ is compact by assumption and π is supercuspidal, we have $\beta(\mathrm{End}_{sm}(\pi)) \leq C_c^\infty(G(F))$. Since there are $\phi \in V$ and $\phi^\vee \in V^\vee$ such that

$$\langle \phi^\vee, \phi \rangle = \langle \phi^\vee, \pi(1)\phi \rangle \neq 0,$$

we have that $\alpha \circ \beta$ is not identically zero, hence β is not identically zero. It follows from the irreducibility of $\pi^\vee \times \pi$ that β is an embedding.

Since $V^\vee \otimes V$ is an irreducible representation of $G(F) \times G(F)$, β is an embedding. Consider

$$\begin{aligned} \beta' : (C_c^\infty(G(F)), \rho) &\longrightarrow (\mathrm{End}(\pi)^\infty, \sigma) \\ f &\longmapsto \pi(f). \end{aligned} \tag{11.2.2}$$

Then $\beta' \circ \beta$ is an endomorphism of the irreducible representation $\mathrm{End}_{sm}(\pi)$ of $G(F) \times G(F)$. Hence $\beta' \circ \beta$ is scalar by Schur's lemma, say $\beta' \circ \beta = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{C}$. We will now show that $\lambda \neq 0$ and that we can take

$$f_\pi := \lambda^{-1} \beta \circ \beta'(f).$$

First,

$$\pi(\beta \circ \beta'(f)) = \beta' \circ \beta \circ \beta'(f) = \lambda \beta'(f) = \lambda \pi(f).$$

To show that $\lambda \neq 0$, note that we can find $f \in C_c^\infty(G(F))$ such that $\pi(f) \neq 0$ (take, for example, f to be the characteristic function of a sufficiently small compact open subgroup). Thus $\beta'(f) = \pi(f) \neq 0$, and since β is an embedding, we deduce that $\beta \circ \beta'(f) \neq 0$.

Second, let (π_1, V_1) be a smooth irreducible representation of $G(F)$, and let $\phi_1 \in V_1$ be a non-zero vector. Let

$$\begin{aligned} \gamma_1 : (C_c^\infty(G(F)), \sigma|_{1 \times G(F)}) &\longrightarrow (V_1, \pi_1) \\ f &\longmapsto \pi_1(f)\phi_1. \end{aligned} \tag{11.2.3}$$

As a representation of $G(F)$ one has that $\mathrm{End}_{sm}(\pi)|_{1 \times G(F)}$ isomorphic to a direct sum of copies of π, and thus the same is true of $\gamma_1(\beta(\mathrm{End}_{sm}(\pi), \sigma|_{1 \times G(F)})$. Thus $\gamma_1(\beta(\mathrm{End}_{sm}(\pi))) = 0$ unless $\pi_1 = \pi$ (since whenever the former is nonzero we obtain an intertwining operator between π_1 and π). It follows that $\pi_1(f_\pi) = 0$ if $\pi_1 \not\cong \pi$. This completes the proof of the proposition. $\square$

12. Simple trace formulae and relative trace formulae

In this section, let G be a connected reductive group over a number field F and let $H \leq G$ be a connected reductive subgroup. In the previous section we encountered traces of admissible representations of $G(F_v)$ for nonarchimedian representations. We saw in Proposition 11.11 that the trace of a representation determines the representation, just as in the familiar case of representations of finite groups.

In this section we explore a powerful tool for studying traces in the global adelic setting, namely the trace formula. We will simultaneously study the relative trace formula, which has emerged as an important generalization of the usual trace formula.

12.1. **Distinction.** We require some notation. Let

$$T \leq G, P = MN \geq T$$

denote a maximal F-split torus in G and and a choice of minimal parabolic containing it. We set $?_H := ? \cap H$. The choice of parabolic P (resp. P_H) is equivalent to a set of simple roots Δ (resp. Δ_H) in the roots of T in G (resp. T_H in H). In the case where G is quasi-split, this correspondence is recalled in §10.4, for the general case a nice brief survey is given in [Mur05, §7].

Set

$$A^G := \mathrm{Res}_{F/\mathbb{Q}} T(\mathbb{R})^0 / A_G$$

where the 0 denotes the connected component in the real topology. For any positive real number r we set

$$A_r^G := \{t \in A^G : t^\alpha > r \text{ for all } \alpha \in \Delta\}$$

Definition 12.1. A function

$$\phi : G(F)A_G \backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

is **rapidly decreasing** if it is smooth and for all compact subsets $\Omega \subset G(F)A_G\backslash G(\mathbb{A}_F)$ and $r \in \mathbb{R}_{>0}$ there is a constant C such that one has

$$|\phi(tx)| \leq Ct^{\alpha p}$$

for all $t \in A_r^G$ and $\alpha \in \Delta$.

For ease of notation, let

$$L^2 := L^2(G(F)A_G\backslash G(\mathbb{A}_F)) \quad \text{and} \quad L_0^2 := L_0^2(G(F)A_G\backslash G(\mathbb{A}_F))$$

respectively. Let $\chi : H(\mathbb{A}_F) \to \mathbb{C}$ be a quasi-character trivial on $(A_G \cap H(\mathbb{A}_F)).H(F)$. The following proposition appears in [AGR93].

Proposition 12.2. *Then for all rapidly decreasing (smooth) functions $\phi \in L^2$, the period integral*

$$P_\chi(\phi) = \int_{(A_G\cap H(\mathbb{A}_F)).H(F)\backslash H(\mathbb{A}_F)} \phi(g)\chi(g)\mathrm{d}g$$

is absolutely convergent.

We will prove this in a moment.

Let (π, V) be a cuspidal automorphic representation of $A_G\backslash G(\mathbb{A}_F)$ and let

$$L^{2,cusp}(\pi)$$

be the π-isotypic subspace. We recall the following basic theorem (see **??**):

Theorem 12.3. *If $\phi \in L^{2,cusp}$ and $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ then one has an estimate*

$$|R(f)\phi(tx)| \leq C_f t^{\alpha p}||\phi||$$

for all $t \in A_r^G$ and $\alpha \in \Delta$, where the constant C_f is independent of ϕ. In particular, $R(f)\phi$ is rapidly decreasing.

□

As a corollary, we obtain the following:

Corollary 12.4. *In particular, if ϕ is an automorphic form (that is, K_∞-finite and $Z(\mathfrak{g}_{\mathbb{C}})$-finite) then ϕ is rapidly decreasing.* □

In view of this theorem, the following definition is reasonable.

Definition 12.5. A cuspidal automorphic representation π of $A_G\backslash G(\mathbb{A}_F)$ is said to be (H,χ)-*distinguished* if $P_\chi(\phi) \neq 0$ for some smooth $\phi \in L^{2,cusp}(\pi)$. When (H,χ) is understood, or irrelevant, we simply say that it is *distinguished*.

Example 12.6.

(1) Consider the diagonal embedding $\Delta : H \to H \times H$. We ask ourselves which representations π of $H \times H(\mathbb{A}_F)$ are H-distinguished. Any such representation π can be decomposed as $\pi_1 \times \pi_2$ with $(\pi_1, V_1), (\pi_2, V_2)$ representations of $H(\mathbb{A}_F)$. As a map

$$V_1 \times V_2 \longrightarrow \mathbb{C}(\phi_1, \phi_2) \longmapsto P(\phi_1 \otimes \phi_2)$$

the period integral is invariant under $\Delta H(\mathbb{A}_F)$. Thus $\pi = \pi_1 \times \pi_2$ is distinguished if and only if $\pi_1 \cong \pi_2^\vee$. That is, the representations π of the form $\pi \times \pi^\vee$ for π a representation of $H(\mathbb{A}_F)$ are the only distinguished representations of ΔH.

(2) Take $G = \mathrm{GL}_{2/\mathbb{Q}}$ and $H = \mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_1$ for a quadratic extension $K/\mathbb{Q}$. This data corresponds to an embedding $K \hookrightarrow \mathrm{GL}_2(\mathbb{Q})$. Then the notion of distinction is related to Heegner points. We note that $A_G \cap H(\mathbb{A}_F).H(F)\backslash H(\mathbb{A})/K_H$ is a *finite* number of points for every $K_H \subset H(\mathbb{A}_F)$ compact subgroup.

(3) For $K/\mathbb{Q}$ a real quadratic extension take $G = \mathrm{Res}_{E/\mathbb{Q}}$ and $H = \mathrm{GL}_{2/\mathbb{Q}}$. Studying distinction in this case amounts to studying modular curves in Hilbert modular surfaces. It was the study of this case that led Harder, Langlands, and Rapoport to define the notion of distinction in [HLR86].

Proof of Proposition 12.2. By reduction theory, there is an $0 < r < 1$ such that

$$H(\mathbb{A}_F) = A_G \cap H(F_\infty)H(F)A_r^H \omega K_H$$

where ω is relatively compact subgroup of $N_H(\mathbb{A}_F)M_H(\mathbb{A}_F)$ and $K_H \leq H(\mathbb{A}_F)$ is a maximal compact subgroup (combine [Bor07, §5.2] and strong approximation).

Thus, since χ is unitary, the integral $P_\chi(\phi)$ converges absolutely provided that

$$\int_{A_r^H N_H^o M_H} |\phi(anm)| da dn dm$$

converges. But this follows easily from the definition of a rapidly decreasing function. □

12.2. **Studying traces and distinction.** We now explain a fundamental idea first applied to the study of automorphic forms by Selberg. Let

$$L^2 := L^2(A_G G(F)\backslash G(\mathbb{A}_F)).$$

For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ consider the integral operator

$$\begin{aligned} R(f) : L^2 &\longrightarrow L^2 \\ \phi &\mapsto \int_{A_G \backslash G(\mathbb{A}_F)} f(x)\phi(gx)\mathrm{d}x. \end{aligned}$$

Just manipulating formally for the moment, we see that

$$\begin{aligned} R(f)\phi(x) &= \int_{A_G \backslash G(\mathbb{A}_F)} f(y)R(y)\phi(x)\mathrm{d}y \\ &= \int_{A_G \backslash G(\mathbb{A}_F)} f(x)\phi(xy)\mathrm{d}y \\ &= \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}y)\phi(y)\mathrm{d}y \\ &= \int_{A_G G(F) \backslash G(\mathbb{A}_F)} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\phi(y)\mathrm{d}y. \end{aligned}$$

In other words, $R(f)$ is an integral operator with kernel

$$K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

This is the **geometric expansion** of the kernel. Let

$$\Omega_1 \times \Omega_2 \subset A_G \backslash G(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)$$

be compact subsets. If

$$(x, y) \in \Omega_1 \times \Omega_2$$

then the only nonzero summands in $K_f(x, y)$ correspond to γ satisfying $\gamma \in \Omega_1 \mathrm{Supp}(f)\Omega_2$. Thus $K_f(x, y)$ is smooth as a function of x and y separately, and using this observation the formal manipulations above can be justified.

There is also a **spectral expansion** of the kernel, which we explain after recalling some basic definitions regarding integral operators.

Definition 12.7. Let V be a Hilbert space. An operator $A : V \to V$ is **Hilbert-Schmidt** if $A(V)$ has a countable basis $(\phi_i)_{i=1}^\infty$ consisting of eigenvectors for A, say $A(\phi_i) = \lambda_i \phi_i$, such that

$$\sum_{i=1}^\infty |\lambda_i|^2$$

is finite. The operator $A : V \to V$ is **of trace class** if it is Hilbert-Schmidt and in the notation above

$$\sum_{i=0}^\infty |\lambda_i| < \infty$$

If A is Hilbert-Schmidt we let

$$|A| = |A|_{HS} = \sum_{i=1}^\infty |\lambda_i|^2$$

this is known as the **Hilbert-Schmidt norm** of A. If A is trace class we define its **trace** to be

$$\mathrm{tr}(A) := \sum_{i=1}^\infty |\lambda_i|.$$

These notions are independent of the choice of basis.

If G^{der} is not anisotropic, then $R(f)$ is in general not of trace class. However, if we denote by

$$R^{cusp}(f) := L^{2,cusp} \longrightarrow L^{2,cusp}$$

its restriction to the cuspidal subspace, this is of trace class [Don82]. In fact even the restriction of $R(f)$ to the discrete spectrum is of trace class [**?**]

Thus we can write

$$\mathrm{tr}(R_0(f)) = \sum_\pi m(\pi)\mathrm{tr}(\pi(f));$$

where the sum is over equivalence classes of cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are trivial on A_G (or, more briefly, cuspidal automorphic representations of $A_G\backslash G(\mathbb{A}_F)$) and $m(\pi)$ is the (finite) **multiplicity** of π. Thus $R_0(f)$ has kernel

$$K_f^{cusp}(x,y) := \sum_\pi K_{\pi(f)}(x,y),$$

where

$$K_{\pi(f)}(x,y) = K^{\mathrm{cusp}}_{\pi(f)}(x,y) = \sum_{\phi\in B(\pi)} \pi(f)\phi(x)\overline{\phi(y)}$$

for $B(\pi)$ an orthonormal basis of $L^2_0(\pi)$, the π-isotypical subspace of L^2_0.

Remark 12.8. Note that $\pi(f)B(\pi)$ need not be finite dimensional in general, but it is if f is K_∞-finite by admissibility.

As defined, $K_{\pi(f)}(x,y)$ is only a function in the L^2 sense; it may not make sense to evaluate it at any point. But integrating $K_{\pi(f)}(x,y)$ along $\Delta : A_G\backslash G(\mathbb{A}_F) \to A_G\backslash G(\mathbb{A}_F) \times A_G\backslash G(\mathbb{A}_F)$ is always well-defined, as

$$\sum_\pi \int_{A_G G(F)\backslash G(\mathbb{A}_F)} K_{\pi(f)}(x,y)\mathrm{d}x = \sum_\pi m(\pi)\mathrm{tr}(\pi(f)),$$

because $(\phi_1,\phi_2) \mapsto \int_{A_G G(F)\backslash G(\mathbb{A}_F)} \phi_1(g)\overline{\phi_2(g)}dg$ is the pairing defining the metric on L^2.

Selberg's idea was to use the two different expressions for $K_f(x,y)$ to give a geometric expansion of

$$\mathrm{tr}(R^{cusp}(f)) = \int_{A_G\backslash G(\mathbb{A}_F)} K_f^{cusp}(x,x)dx$$

in the case $G = \mathrm{SL}_{2/\mathbb{Q}}$ and use it to give estimates on the number of automorphic forms satisfying certain properties. Motivated by the work of Langlands on Eisenstein series [Lan76], Arthur has spent most of his career making this idea work for arbitrary G despite the formidable complications due to the fact that since $G(F)A_G\backslash G(\mathbb{A}_F)$ is not compact it is only $K_f^{cusp}(x,y)$, and not $K_f(x,y)$, that is integrable along the diagonal. In our treatment we will always make additional restrictive assumptions that make an appeal to Arthur's work unnecessary. Of course, to remove the assumptions requires Arthur's work. Moreover, since we will treat relative trace formula simultaneously, there is removing these assumptions in general would require more than what is currently known.

12.3. **The trace formula for compact quotient.** We now give an example of a trace formula in the simplest possible case, that is, when G^{der} is anisotropic. For F-algebras R and $\gamma \in G(R)$ let

$$G_\gamma(R) := \{g \in G(R) : g\gamma g^{-1} = \gamma\}$$

be the centralizer of γ. One has an absolutely convergent **orbital integral**

$$O_\gamma(f) = \int_{G_\gamma^\circ(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)\frac{dg}{dt} \tag{12.3.1}$$

and a volume term

$$\tau(G_\gamma^\circ) := \mathrm{meas}_{dt}(A_G G_\gamma^\circ(F)\backslash G_\gamma^\circ(\mathbb{A}_F)) \tag{12.3.2}$$

When dt is chosen appropriately this is a **Tamagawa number**, which can be regarded as a generalized class number.

One has the following theorem:

Theorem 12.9. *If G^{der} is anisotropic*

$$\sum_\pi m_\pi \mathrm{tr}(\pi(f)) = \sum_\gamma \tau(G_\gamma) O_\gamma(f)$$

where the sum on the left is over isomorphism classes of automorphic representations of $A_G\backslash G(\mathbb{A}_F)$ and the sum on the right is over $G(F)$-conjugacy classes of $\gamma \in G(F)$.

Proof. The quotient $A_G G(F)\backslash G(\mathbb{A}_F)$ is compact. General results in functional analysis imply that $R(f)$ is therefore a trace-class operator and

$$\mathrm{tr}(R(f)) = \sum_\pi m_\pi \mathrm{tr}(\pi(f)) = \int_{A_G G(F)\backslash G(\mathbb{A}_F)} K_f(x,x)dx.$$

On the other hand, manipulating integrals formally for the moment,

$$\begin{aligned}
\int_{G(F)A_G\backslash G(\mathbb{A}_F)} K_f(x,x)dx &= \int_{G(F)A_G\backslash G(\mathbb{A}_F)} \sum_{\gamma\in G(F)} f(x^{-1}\gamma x)dx \\
&= \int_{G(F)A_G\backslash G(\mathbb{A}_F)} \sum_{\gamma/\sim} \sum_{\delta \in G_\gamma^\circ(F)\backslash G(F)} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&= \sum_{\gamma/\sim} \int_{A_G G_\gamma^\circ(F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)dx \\
&= \sum_{\gamma/\sim} \int_{G_\gamma^\circ(F)\backslash G_\gamma^\circ(\mathbb{A}_F)} \int_{G_\gamma^\circ(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}t^{-1}\gamma t x)dt\frac{dx}{dt} \\
&= \sum_{\gamma/\sim} \mathrm{meas}_{dt}(G_\gamma^\circ(F)\backslash G_\gamma^\circ(\mathbb{A}_F)) \int_{G_\gamma^\circ(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)\frac{dg}{dt}.
\end{aligned}$$

where the $\sim$ denotes the equivalence relation of $G(F)$-conjugacy. These formal manipulations can be justified using the fact that $A_G G(F)\backslash G(\mathbb{A}_F)$ is compact. □

12.4. **Relative traces.** In this subsection we introduce a generalization of the notion of a trace that is due to Jacquet.

Let π be a cuspidal automorphic representation of $A_G\backslash G(\mathbb{A}_F)$. Let $H \leq G \times G$ be a connected reductive subgroup and let $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ be a character trivial on $H(F)A_G \times A_G \cap H(F_\infty)$. If f is K_∞-finite let

$$\begin{aligned}
\mathrm{rtr}(\pi(f)) := \mathrm{rtr}_\chi(\pi(f)) :&= \int_{H(F)A_G\times A_G\cap H(F_\infty)\backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_\ell,h_r)\chi(h_\ell,h_r)dh_\ell h_r \\
&= \sum_{\phi\in\mathcal{B}(\pi)} \int_{H(F)A_G\times A_G\cap H(F_\infty)\backslash H(\mathbb{A}_F)} \frac{R(f)\phi(h_\ell)\bar{\phi}(h_r)dh_\ell h_r}{||\phi||} \\
&= \sum_{\phi\in\mathcal{B}(\pi)} \frac{\mathcal{P}_\chi(R(f)\phi\times\bar{\phi})}{||\phi||^2}
\end{aligned} \tag{12.4.1}$$

Any function ϕ contributing to this sum is necessarily K_∞-finite and hence smooth (even real analytic, since it is also $Z(\mathfrak{g}_\mathbb{C})$-finite). Using Theorem 12.3 we can also show that it is rapidly decreasing. Moreover, since f is K_∞-finite the sum over ϕ has only finitely many nonzero summands. We conclude that the integral defining $\mathrm{rtr}(\pi(f))$ is absolutely convergent. We refer to $\mathrm{rtr}(\pi(f))$ as a **relative trace**.

12.5. **A simple relative trace formula.** Motivated by the computation in §12.3 we will now give a relative trace formula that is valid under restrictive assumptions on $f \in C_c^\infty(G(\mathbb{A}_F))$ even if the quotient $G(F)A_G\backslash G(\mathbb{A}_F)$ is noncompact. It is a slight generalization of the main theorem of [Hah09] which in turn follows the tradition of the simple trace formula of Deligne and Kazhdan. Let $H \leq G \times G$ be a reductive subgroup and let $\chi : A_G \times A_G \cap H(F_\infty)\backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$ be a unitary character trivial on $H(F)$.

We will first state the main theorem and then define the new terminology in the coming sections.

Theorem 12.10. *Let v_1, v_2 be places of F (not necessarily distinct). Let $f = f_{v_1v_2} \otimes f^{v_1v_2} \in C_c^\infty(G(\mathbb{A}_F))$ be a K_∞-finite function such that $R(f)$ has cuspidal image and*

(1) *f_{v_1} is supported on relatively elliptic elements.*
(2) *f_{v_2} is supported on relatively connected semisimple elements.*

Then:

$$\sum_\gamma \tau(H_\gamma) RO_\gamma(f) = \sum_\pi \mathrm{rtr}(\pi(f))$$

where the sum on γ is over relatively connected semisimple relevant classes and the sum on π is over isomorphism classes of cuspidal automorphic representations of $A_G\backslash G(\mathbb{A}_F)$.

Here we say that a cuspidal automorphic representation of $G(\mathbb{A}_F)$ is a cuspidal automorphic representation of $A_G\backslash G(\mathbb{A}_F)$ if its central character is trivial on A_G.

12.6. **Functions with cuspidal image.** Lindenstrauss and Venkatesh have defined a large class of functions with purely cuspidal image that essentially have no kernel when restricted to the cuspidal spectrum [LV07]. Before their work, there was a standard example of such functions that was used to good effect in studying local factors of automorphic representations. We recall it now.

Let v be a finite place of F.

Definition 12.11. A function $f_v \in C_c^\infty(G(F_v))$ is said to be *F-supercuspidal* if $\int_{N(F_v)} f(gnh)dn = 0$ for all proper parabolic subgroups $P = MN$ of G defined over F and all $g, h \in G(F_v)$.

Lemma 12.12. *If $f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ is F-supercuspidal at some place v then $R(f)$ has cuspidal image.*

Proof. For $\varphi \in L^2$, $R(f)\varphi$ is smooth and hence can be integrated over any compact subset. Let $P = MN$ be a proper F-rational parabolic. For all $x \in A_G \backslash G(\mathbb{A}_F)$ we have

$$\begin{aligned}
\int_{N(F)\backslash N(\mathbb{A}_F)} R(f)\varphi(nx)dn &= \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{A_G\backslash G(\mathbb{A}_F)} f(g)\varphi(nxg)dgdn \\
&= \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{A_G\backslash G(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dgdn \\
&= \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{A_G N(F)\backslash G(\mathbb{A}_F)} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta g)\varphi(g)dgdn \\
&= \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{A_G N(F)\backslash G(\mathbb{A}_F)} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}g)\varphi(g)dgdn \\
&= \int_{A_G N(F)\backslash G(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{y \in N(F)} f(x^{-1}n^{-1}g)\varphi(g)dndg \\
&= \int_{A_G N(F)\backslash G(\mathbb{A}_F)} \int_{N(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dndg \\
&= 0.
\end{aligned}$$

$\square$

Essentially all examples of supercuspidal functions are obtained using the following lemma:

Lemma 12.13. *Assume that $Z_G(F_v)$ is compact for some v and that (π_v, V) is a supercuspidal representation of $G(F_v)$. If f_v is a matrix coefficient of π_v then f_v is F-supercuspidal.*

Remark 12.14. The assumption that $Z_G(F_v)$ is compact is not essential; compare the discussion of truncated matrix coefficients in [HL04].

Proof. Let $P = MN$ be a proper parabolic subgroup of G. If $\int_{N(F_v)} f(gnh)dn \neq 0$ for some $g, h \in G(F_v)$ then upon realizing V as a subspace of $C_c^\infty(G(F_v))$ as in the proof of Proposition 11.12 we obtain a nonzero map

$$\begin{aligned}
V_{\pi_v} &\longrightarrow V_{\pi_v}/V_{\pi_v}(N) \\
\varphi_v &\longmapsto \left(g \mapsto \int_{N(F_v)} f(gnh)dn\right)
\end{aligned}$$

contradicting the supercuspidality of π_v.

$\square$

12.7. **Orbits and stabilizers.** For this subsection only let F be an arbitrary field of characteristic zero and let G/F be a reductive group. For $H \subset G \times G$ we have a natural action of H on G via:

$$(h_\ell, h_r) \cdot g := h_\ell g h_r^{-1}. \tag{12.7.1}$$

We assume for simplicity that H is connected and reductive.

If R is an F-algebra and $\gamma \in G(R)$ we let H_γ be the stabilizer of γ. It is a linear algebraic group over R [MFK94]. Moreover denote by $O(\gamma)$ the orbit of γ under this action. Finally let

$$\Delta : G \longrightarrow G \times G$$

denote the diagonal embedding.

Definition 12.15. An element $\gamma \in G(R)$ is said to be:

- **relatively semisimple** if $O(\gamma)$ is closed (this implies H_γ is reductive [BR85]).

- **relatively elliptic** if H_γ is anisotropic modulo $Z_H \cap \Delta(Z_G)$.
- **relatively regular** if $O(\gamma)$ has maximal dimension among all $\gamma' \in G(F)$).
- **relatively connected semisimple** if γ is semisimple and H_γ is connected.

Definition 12.16. Let R be an F-algebra. A **relative class**, or simply a **class**, is an element of

$$\Gamma_r(R) = H(R)\backslash G(R).$$

A **geometric relative class**, or simply a **geometric class**, is an element of

$$\Gamma_r^{geom}(R) = Im(G(R) \to (H\backslash G)(R)).$$

As an important special case, consider the situation when $G = H \times H$ and H is viewed as the diagonal subgroup of G. We refer to this as the **group case**. In this case $O(\gamma)$ is the conjugacy class of γ and G_γ is the centralizer of γ. In this case all of the notions above reduce to the usual notions of semisimplicity, and regularity as explained in the classic paper [Ste65]. In this case we usually omit the adjective "relatively" from the definitions above.

Example 12.17. Assume that we are in the group case with $H = \mathrm{GL}_n$. In this case if γ is regular semisimple, then it is elliptic if and only if $\mathbb{Q}[\gamma]/\mathbb{Q}$ is a division algebra. In this case the map

$$\Gamma_r(F) \longrightarrow \Gamma_r(\bar{F})$$

is injective. For most H this is false, and this is what leads to the theory of endoscopy.

12.8. **Relative Orbital Integrals.** We now revert back to the global setting. Let H and G be connected reductive F-groups with $H \leq G \times G$ and let $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ be a quasi-character trivial on $H(F)A_G \times A_G \cap H(F_\infty)$.

Definition 12.18. A relatively semisimple element $\gamma_v \in G(F_v)$ is **relevant** if χ_v is trivial on $H_\gamma^0(F_v)$. An element $\gamma \in G(\mathbb{A}_F)$ is **relevant** if γ_v is relevant for all v.

The point of this definition is that irrelevant elements will not end up contributing to the trace formula. We note that if χ is trivial then all relatively semisimple elements are relevant.

Definition 12.19. Let v be a place of F. For $f_v \in C_c^\infty(F_v)$ and γ_v relevant we define the *relative orbital integral*:

$$RO_{\gamma_v}^\chi(f_v) = \int_{H_\gamma^0(F_v)\backslash H(F_v)} \chi_v((h_\ell, h_r)) f_v(h_\ell^{-1}\gamma_v h_R)\frac{d(h_\ell, h_r)}{dt_\gamma}$$

We remark that $d(h_\ell, h_r)$ is Haar measure on $H(F_v)$ and dt_γ is Haar measure on H_γ so the resulting measure is a Radon measure (inner regular and locally finite).

Proposition 12.20. *If γ is relevant relatively semisimple then the integral $RO_\gamma(f)$ is absolutely convergent.*

Proof. Since the measure $d(h_\ell, h_r)/t_\gamma$ is a Radon measure on $H_\gamma^0(F_v)\backslash H(F_v)$, to show the integral is well-defined and absolutely convergent it is enough to construct a pull-back map

$$C_c^\infty(G(F_v)) \longrightarrow C_c^\infty(H_\gamma^0(F_v)\backslash H(F_v))$$

attached to the natural map

$$H_\gamma^0(F_v)\backslash H(F_v) \longrightarrow G(F_v).$$

We claim that the image of this map is closed. The orbit of γ in the sense of the algebraic group action of H on G is closed in the Zariski topology by assumption, and it follows that $H(F_v) \cdot \gamma$ is also closed in the v-adic topology [RR96, §2, below B.]. On the other hand,

$$H_\gamma(F_v)\backslash H(F_v) \longrightarrow H(F_v) \cdot \gamma$$

is not only a bijection but also a homeomorphism [Ser06, §II.5]. The existence of the pull-back follows.

□

12.9. **Relative orbital integrals are 1 at almost all places.** Let

$$f = \prod_v f_v \in C_c^\infty(G(\mathbb{A}_F)).$$

We have shown that $RO_\gamma^\chi(f_v)$ is well-defined for each v. This, together with the following proposition, shows that the global relative orbital integral $RO_\gamma^\chi(f)$ is well-defined:

Proposition 12.21. *Suppose that $\gamma \in G(F)$ is relevant and relatively connected semisimple. Then for almost every v,*

$$\mathrm{RO}_\gamma^\chi(\mathbf{1}_{K_v}) = \int_{H_\gamma(F_v)\cap K_v\times K_v\backslash H(F_v)\cap K_v\times K_v} d(h_\ell, h_r)/dt_\gamma = 1,$$

where $K_v = G(\mathcal{O}_{F_v})$ is hyperspecial and the Haar measures gives volume 1 to the interesction with $K_v \times K_v$.

Remarks.
(1) The set of v for which $\mathrm{RO}_\gamma^\chi(\mathbf{1}_{K_v}) = 1$ depends on γ.
(2) The author does not know how to prove the analogue of this proposition when H_γ is not connected. In the group case one uses the theory of z-extensions, but the analogue in this setting is not known, or at least is not in the literature.

We require the following two important results:

Theorem 12.22 (Lang)**.** *If G is a connected linear algebraic group over a finite field $\mathbb{F}$ then $H^1(\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}), G)$ is a singleton.*

(see [Lan56]).

The following is a general analogue of Hensel's lemma.

Theorem 12.23. *Let v be a nonarchimedean place of F and $\mathbb{F}_v$ the residue field of $\mathcal{O}_{F_v}$. If $X/\mathcal{O}_{F_v}$ is a smooth scheme of finite type, then $X(\mathcal{O}_{F_v}) \to X(\mathbb{F}_v)$ is surjective.*

(see [BLR90, §2.3]).

Corollary 12.24. *If X is a homogeneous space for a connected group scheme G over $\mathcal{O}_{F_v}$ then $X(\mathcal{O}_{F_v})$ is nonempty.*

Proof. By Theorem 12.23 it suffices to show that $X(\mathbb{F}_v)$ is nonempty. Since $X_{\mathbb{F}_v}$ is a homogeneous space for $G_{\mathbb{F}_v}$, it is nonempty if and only if the corresponding class in the pointed set $H^1(\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}), G)$ is the trivial class. Thus we conclude by Theorem 12.22. □

Proof of Proposition 12.21. Let S be a finite set of nonarchimedean places of F. Let $G \to \mathrm{GL}_n$ be a faithful representation for some n and let $\mathcal{G}$ be the schematic closure of G in $\mathrm{GL}_{n\mathcal{O}_F^S}$. Define $\mathcal{H}$ similarly. Upon enlarging S if necessary we can and do assume that $\mathcal{H} \le \mathcal{G}$ are smooth and $\mathcal{H}(\mathcal{O}_{F_v})$, $\mathcal{G}(\mathcal{O}_{F_v})$ are hyperspecial for all $v \notin S$ (see [Tit79]). Let $\gamma \in \mathcal{G}(\mathcal{O}_F^S)$ be a relatively connected semisimple element. We let $\mathcal{H}_\gamma$ be the schematic closure of H_γ in $\mathcal{H}$. Upon enlarging S if necessary we can assume that $\mathcal{H}_\gamma$ is smooth and is the stabilizer of γ in the scheme-theoretic sense. We can even assume that

$$\mathcal{H}(\mathcal{O}_{F_v}) = H(F) \cap \mathcal{G} \times \mathcal{G}(\mathcal{O}_{F_v}) \le H(F_v)$$
$$\mathcal{H}_\gamma(\mathcal{O}_{F_v}) = H_\gamma(F_v) \cap \mathcal{H}(\mathcal{O}_{F_v}) \le H_\gamma(F_v)$$

are hyperspecial subgroups [Tit79].

Let $\mathcal{O}(\gamma)/\mathcal{O}_F^S$ be the set-theoretic image of the morphism

$$\mathcal{H} \times \gamma \longrightarrow \mathcal{G}$$

given by the action of $\mathcal{H}$ on γ. It is a constructible subset of (the underlying topological space of) $\mathcal{G}$. The set

$$Z := \{s \in \mathrm{Spec}(\mathcal{O}_F^S) : \mathcal{O}(\gamma)_s \text{ is closed } \}$$

is a constructible subset of $\mathrm{Spec}(\mathcal{O}_F^S)$ [GW10, §E.1]. In other words it is a finite union of subsets of $\mathrm{Spec}(\mathcal{O}_F^S)$ of the form $U \cap V$ where U is open and V is closed. Since $\mathcal{O}(\gamma)_F$ is closed by assumption Z contains the generic point $\eta \in \mathrm{Spec}(\mathcal{O}_F^S)$, there is an open set U and a closed set V in $\mathrm{Spec}(\mathcal{O}_F^S)$ such that

$$\eta \in U \cap V.$$

On the other hand, the only closed set containing η is the whole of $\mathrm{Spec}(\mathcal{O}_F^S)$. We conclude that Z contains an open set. Thus, upon enlarging S if necessary, we can assume that $\mathcal{O}(\gamma)$ is closed and hence equal to its schematic closure.

Since $\mathcal{H}_F$ is dense in $\mathcal{H}$ the orbit $\mathcal{O}(\gamma)$ can also be described as the Zariski closure of $O(\gamma)$ in $\mathcal{G}$. Since $\mathcal{O}(\gamma)_F$ is homogeneous under the action of H, upon enlarging S again if necessary we can assume that $\mathcal{O}(\gamma)$ is smooth over $\mathcal{O}_F^S$ and that the map $\mathcal{H} \to \mathcal{O}(\gamma)$ is also smooth.

Thus for $v \notin S$

$$\mathcal{O}(\gamma)(\mathcal{O}_{F_v}) = O(\gamma)(F) \cap \mathcal{G}(\mathcal{O}_{F_v}).$$

It therefore suffices to show that if $\gamma' \in \mathcal{O}(\gamma)(\mathcal{O}_{F_v})$ then it is in the $\mathcal{H}(\mathcal{O}_{F_v})$-orbit of γ. Let $\gamma' \in \mathcal{O}(\gamma)(\mathcal{O}_{F_v})$ and consider the $\mathcal{O}_{F_v}$-scheme whose points in an $\mathcal{O}_{F_v}$-algebra R are

$$X(R) := \{(h_\ell, h_r) \in \mathcal{H}(R) : h_\ell^{-1}\gamma h_r = \gamma'\}.$$

We claim that there is some $(h_\ell, h_r) \in X(\mathcal{O}_{F_v}^{ur})$, where $\mathcal{O}_{F_v}^{ur}$ is the ring of integers of the maximal unramified extension of F_v. Indeed, X may be identified with the fiber of the smooth map $\mathcal{H} \longrightarrow \mathcal{O}(\gamma)$ over γ', and hence X is a smooth subscheme of $\mathcal{H}$. The map

$$X(\widehat{\mathcal{O}}_{F_v}) \longrightarrow X(\bar{\mathbb{F}}_v)$$

is therefore surjective by Theorem 12.23. On the other hand, $X(\bar{\mathbb{F}}_v)$ is nonempty since the map $\mathcal{H} \longrightarrow \mathcal{O}(\gamma)$ is surjective, and we conclude that $X(\mathcal{O}_{F_v}^{ur})$ is nonempty as claimed.

Thus we have a map

$$\begin{aligned} \mathcal{H}_{\gamma \mathcal{O}_{F_v}^{ur}}(R) &\longrightarrow X_{\mathcal{O}_{F_v}^{ur}}(R) \\ t &\longmapsto (th_\ell, th_r) \end{aligned}$$

which realizes X as a trivial $\mathcal{H}_{\gamma \mathcal{O}_{F_v}^{ur}}$-torsor. Since $\mathcal{X}$ was originally defined over $\mathcal{O}_{F_v}$ we conclude that X is a $\mathcal{H}_\gamma$-torsor and apply Corollary 12.24 to deduce the proposition. □

12.10. **The geometric side.** In view of Proposition 12.20 and Proposition 12.21 for $f \in C_c^\infty(G(\mathbb{A}_F))$ and relevant relatively connected semisimple $\gamma \in G(F)$ the global relative orbital integral

$$RO_\gamma^\chi(f) := \int_{H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) f(h_\ell^{-1}\gamma h_r) \frac{d(h_\ell, h_r)}{dt_\gamma} \tag{12.10.1}$$

is absolutely convergent. Let

$$\begin{aligned} A_{G,H} &:= A_H \cap A_G \times A_G \\ A &:= A_H \cap \Delta(A_G) \end{aligned} \tag{12.10.2}$$

where $\Delta : G \to G \times G$ is the diagonal embedding. Fix Haar measures da_G on A_G, $d(a_\ell, a_r)$ on $A_{G,H}$ and da on A.

We note that $A \leq H_\gamma(F_\infty)$ for all $\gamma \in G(F)$, and

$$\tau(H_\gamma^\circ) := \mathrm{meas}_{dt_\gamma}(H_\gamma^\circ(F)A\backslash H_\gamma^\circ(\mathbb{A}_F)) \tag{12.10.3}$$

is finite if γ is relatively elliptic.

We fix Haar measures on A and define

$$\begin{aligned} f^1(x) &:= \int_{A_G/(A_{G,H}/A)} f(ax)d\cdot a \\ f^0(x) &:= \int_{A_G} f(ax)da_G \end{aligned} \tag{12.10.4}$$

Here we are abusing notation and viewing $A_{G,H}/A$ as a subgroup of A_G via the map

$$\begin{aligned} A_{G,H}/A &\longrightarrow A \\ (a_\ell, a_r) &\longmapsto a_\ell^{-1}a_r \end{aligned}$$

Recall our assumption that $H \leq G \times G$ is connected. The following theorem is roughly half of Theorem 12.10:

Theorem 12.25. *Assume that there exist places v_1 and v_2 of F such that f_{v_1} is supported on relatively elliptic elements and f_{v_2} is supported on relatively connected semisimple elements. Then*

$$\sum_{[\gamma]\in\Gamma_\gamma(F)} \tau(H_\gamma)\,\mathrm{RO}_\gamma^\chi(f^1) = \int_{H(F)A_{H,G}\backslash H(\mathbb{A}_F)} \chi(h_\ell,h_r)K_{f^0}(h_\ell,h_r)d(h_\ell,h_r).$$

Moreover, the sum on the left is finite and the integral on the right is absolutely convergent.

In the theorem we use the notation $[\gamma]$ for the class of γ; we will continue to use this convention. To prove the theorem it is convenient to first prove the following proposition:

Proposition 12.26. *Let $C \subseteq G(\mathbb{A}_F)$ be a compact subset such that C_v is hyperspecial for almost all v. Then there exist only finitely many $[\gamma] \in \Gamma_r(F)$ with γ relatively connected semi-simple and $H(\mathbb{A}_F)\cdot\gamma \cap C \neq \emptyset$.*

We will first assume this proposition and prove the previous theorem:

Proof of Theorem 12.25. Suppose that $\gamma \in G(F)$ is relevant and relatively connected semi-simple. Then

$$\left|\mathrm{RO}_\gamma^\chi(f)\right| < \infty$$

and $|\tau(H_\gamma)| < \infty$ as observed earlier in this subsection. It is easy to see that this implies that $|RO_\gamma^\chi(f^1)| < \infty$ as well. Let C be the closure of the support of f, so $C \subseteq G(\mathbb{A}_F)$. Then by Proposition 12.26 the sum

$$\sum_{H(F)\gamma} \left|\tau(H_\gamma)\,\mathrm{RO}_\gamma^\chi(f^1)\right|$$

is finite, hence convergent. This implies that

(12.10.5)

$$\begin{aligned} \sum_{[\gamma]} \tau(H_\gamma)\,\mathrm{RO}_\gamma^\chi(f) &= \sum_{[\gamma]\in\Gamma_r(F)} \tau(H_\gamma) \int_{(A_{G,H}/A)H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \chi(h_\ell,h_r)f^1(h_\ell^{-1}\gamma h_r)d(h_\ell,h_r)/dt_\gamma \\ &= \sum_{[\gamma]} \tau(H_\gamma) \int_{A_{G,H}H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F))} \chi(h_\ell,h_r)f^0(h_\ell^{-1}\gamma h_r)d(h_\ell,h_r)/dt_\gamma \end{aligned}$$

where the sums are over relevant relatively connected semisimple classes in $\Gamma_r(F)$ that are also relatively elliptic. Notice that

$$\int_{H_\gamma(F)A_{G,H}\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) f^0(h_\ell^{-1}\gamma h_r) d(h_\ell, h_r)/dt_\gamma = 0$$

if γ is not relevant, because in this case

$$\int_{AH_\gamma(F)\backslash H_\gamma(\mathbb{A}_F)} \chi(h_\ell, h_r) d(h_\ell, h_r)_\gamma = 0.$$

Thus (12.10.5) is equal to

$$\begin{aligned}
&\sum_{[\gamma]\in\Gamma_r(F)} \int_{H_\gamma(F)A_{G,H}\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) f^0(h_\ell^{-1}\gamma h_r) d(h_\ell, h_r) \\
&= \int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r^{-1}) \sum_{\gamma\in G(F)} f^0(h_\ell^{-1}\gamma h_r) d(h_\ell, h_r) \\
&= \int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r) K_{f^0}(h_\ell, h_r) d(h_\ell h_r).
\end{aligned}$$

□

Now we must prove the proposition:

Proof of Proposition 12.26. Let

$$B : G \longrightarrow X$$

where X is the categorical quotient of G by H with respect to the action (12.7.1). Note that if $\gamma, \gamma' \in G(F)$ are relatively semisimple then $B(\gamma) = B(\gamma')$ if and only if γ and γ are in the same geometric relative class. Let $C \subset G(\mathbb{A}_F)$ be a compact such that C_v is hyperspecial for almost all v. Then $B(C) \cap X(F)$ is a finite set, because $X(F) \subseteq X(\mathbb{A}_F)$ is discrete. Thus there are only finitely many relatively semisimple classes in $[\gamma] \in \Gamma_r^{geo}(F)$ such that $H(\bar{F}_v)\gamma_v \cap C_v \neq \emptyset$ for all v.

Assume that γ is relatively connected semisimple and that $H(\bar{F}_v)\gamma_v \cap C_v \neq \emptyset$ for all v. There exists a finite set of places S such that if $v \notin S$, then $\gamma_v \in C_v$ and if γ'_v is in the geometric class of γ_v and its class intersects C_v, then $H(F_v) \cdot \gamma_v = H(F_v) \cdot \gamma'_v$, that is, γ_v and γ'_v define the same class in $\Gamma_r(F_v)$. This can be deduced from the proof of Proposition 12.21.

On the other hand, Galois cohomology implies that there are only finitely many semisimple elements of $\Gamma_r(F_v)$ in a given geometric class, that is, elements of $\Gamma_r(\overline{F}_v)$.

In sum, there exists finitely many $[\gamma'] \in \Gamma_r(\mathbb{A}_F)$ such that $H(\mathbb{A}_F)\gamma' \cap C \neq \emptyset$ and $\gamma_v \in H(F_v)\gamma'_v$ for all v.

We have a natural map

$$\Gamma_r(F) \longrightarrow \Gamma_r(\mathbb{A}_F).$$

The fiber of this map over a relatively connected semisimple element injects into a group

$$\mathfrak{E}(H_\gamma, H, \mathbb{A}_F/F)$$

by [Lab99, Proposition 1.8.4] which is finite by [Lab99, Lemma 1.8.5]. □

12.11. **The spectral side.** As above let $f \in C_c^\infty(G(\mathbb{A}_F))$ and let

$$f^0(x) := \int_{A_G} f(xa)da.$$

As above, let

$$L^{2,cusp} := L^{2,cusp}(A_G G(F)\backslash G(\mathbb{A}_F)).$$

For any $f \in C_c^\infty(G(\mathbb{A}_F))$ as above we let

$$R^{cusp}(f^0) : L^{2,cusp} \longrightarrow L^{2,cusp}$$

denote the operator given by convolution by f^0. We write

$$R^{cusp}(f^0) = \sum_\pi m(\pi)\pi(f^0)$$

where the sum is over equivalence classes of cuspidal automorphic representations of $A_G\backslash G(\mathbb{A}_F)$ and $m(\pi)$ is the multiplicity of π in $L^{2,cusp}$. Thus, if we let $L^{2,cusp}(\pi)$ denote the π-isotypic subspace of $L^{2,cusp}$, the restriction $\pi(f^0)$ of $R^{cusp}(f^0)$ to $L^{2,cusp}(\pi)$ is

$$\pi(f^0) : L^{2,cusp}(\pi) \longrightarrow L^{2,cusp}(\pi).$$

We let $\mathcal{B}(\pi)$ be an orthonormal basis of $L^{2,cusp}(\pi)$ with respect to the pairing

$$\langle \phi_1, \phi_2 \rangle = \int_{G(F)A_G\backslash G(\mathbb{A}_F)} \psi_1(g)\overline{\phi_2(g)}dg.$$

Then $m(\pi)\pi(f^0)$ has kernel

$$K_{\pi(f^0)}(x,y) = \sum_{\phi \in \mathcal{B}(\pi)} (\pi(f^0)\phi)(x)\phi(y).$$

A priori, this expression only converges in L^2, but Arthur proves as a very special case of [Art83, Lemma 4.5 and 4.8] that there is a unique square-integrable function smooth in x and y separately that represents $K_{\pi(f^0)}$; henceforth we let $K_{\pi(f^0)}$ be this function.

In the special case where f is K^∞-finite we defined the relative trace $rtr(\pi(f^0))$ in (12.4.1) above. In general we define

$$rtr(\pi(f^0)) := rtr_{(H,\chi)}(\pi(f^0)) := \int_{H(F)A_{G,H}\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_R)K_{\pi(f^1)}(h_\ell, h_r)d(h_\ell, h_r). \tag{12.11.1}$$

We will check in the course of the proof of Proposition 12.28 that this is absolutely convergent.

Remark 12.27. Notice that if $rtr(\pi(f^0)) \neq 0$ then $\pi \times \pi^\vee$ is (H,χ)-distinguished.

We now prove the following proposition:

Proposition 12.28. *Let $f \in C_c^\infty(G(\mathbb{A}_F))$, and assume that $R(f^0)$ has image in L^2_0. Then*

$$\int_{H(F)A_{H,G}\backslash H(\mathbb{A}_F)} \chi(h_\ell, h_r)K_{f^0}(h_\ell, h_r)d(h_\ell, h_r) = \sum_\pi rtr(\pi(f^0)).$$

Moreover, the integral on the left and the sum on the right are absolutely convergent.

Proof. By assumption, $R(f^0)$ has image in the cuspidal spectrum, and hence the operator $R(f^0)$ is of trace class by a result of Donnelly [Don82]. We therefore have the convergent L^2-expansion

$$K_{f^0}(x,y) := \sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)} \frac{R(f^0)\phi(x)\bar{\phi}(y)}{||\phi||^2} = \sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)} \frac{(R(f^0)\phi)(x)\bar{\phi}(y)}{||\phi||^2} \tag{12.11.2}$$

By the Dixmier-Malliavin lemma [DM78] we can write

$$f^0 = f_1 * f_2 * f_3$$

for $f_1, f_2, f_3 \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$. Letting

$$f^\vee(g) := f(g^{-1})$$

we note that

$$\sum_{\phi\in\mathcal{B}(\pi)} \frac{(R(f)\phi)(x)\bar{\phi}(y)}{||\phi||^2}$$
$$\sum_{\phi\in\mathcal{B}(\pi)} \frac{\phi(x)(R(f^\vee)\bar{\phi})(y)}{||\phi||^2}$$

because they both represent the same kernel.

Thus the kernel (12.11.2) becomes

$$\begin{aligned} &\sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)} \frac{(R(f_1*f_2*f_3)\phi)(x)\bar{\phi}(y)}{||\phi||^2} \\ &= \sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)} \frac{(R(f_2*f_3)\phi)(x)(R(f_1^\vee)\bar{\phi})(y)}{||\phi||^2} \\ &= R(f_3)\times R(f_1^\vee)\left(\sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)}\left(\frac{(R(f_3)\phi)(x)\bar{\phi}(y)}{||\phi||^2}\right)\right) \end{aligned} \tag{12.11.3}$$

In the notation before (12.2), Theorem 12.3 implies that for all compact subsets $\Omega \subset (G(F)A_G\backslash G(\mathbb{A}_F))^2$ and $r \in \mathbb{R}_{>0}$ one has

$$\left| R(f_3)\times R(f_1^\vee)\left(\sum_{\pi}\sum_{\phi\in\mathcal{B}(\pi)}\left(\frac{(R(f_3)\phi)(t_2x)\bar{\phi}(t_1y)}{||\phi||^2}\right)\right)\right| \ll (t_1,t_2)^{\alpha p}\sum_{\pi}\mathrm{tr}(\pi(f_3*f_3^*)) \tag{12.11.4}$$

for all $t \in A_r^{G\times G}$ and $\alpha \in \Delta$. Note that $\mathrm{tr}(\pi(f_3*f_3^*)) \geq 0$ and $\sum_\pi \mathrm{tr}(\pi(f_3*f_3^*)) < \infty$ since the operator $R(f_3*f_3^*)$ is of trace class by the result of Donnelly cited above. This implies the proposition.

□

12.12. **Some specializations.**

Products of reductive subgroups. Assume that $H = H_1 \times H_2 \leq G \times G$ and $\chi = 1$. Then the spectral side of the simple relative trace formula can be used to study automorphic representations distinguished by H_1 and H_2.

The simple trace formula. Let

$$C_\gamma(R) := \{g \in G(R) : g\gamma g^{-1} = \gamma\}$$

be the centralizer of γ.

Assume that $H = \mathrm{diag}(G)$. Then

$$rtr(\pi(f)) = m(\pi)tr(\pi(f))$$

and $RO_\gamma(f) = O_\gamma(f)$ where for each place v of F

$$O_\gamma(f) = \int_{C_\gamma^\circ(F_v)\backslash G(F_v)} f(g^{-1}\gamma g)d\dot{g}.$$

The simple twisted trace formula. Let σ be an automorphism of G. Let

$$H(R) := \{(g, {}^\tau g) \in G \times G(R) : g \in G(R)\}.$$

In this case

$$rtr(\pi(f)) = m(\pi)tr(\pi(f \circ \tau))$$

and

$$RO_\gamma(f) = TO_\gamma(f) := \int_{(C_\gamma^\sigma)^\circ(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g^{-\sigma}\gamma g)dg$$

where

$$C_\gamma^\sigma(R) := \{g \in G(R) : g^{-\sigma}\gamma g = \gamma\}.$$

Remark 12.29. The theory of twisted endoscopy deals with the comparison of twisted trace formulae and trace formulae and, at present, is the primary tool that has been used to establish instances of Langlands functoriality. The last few sections of [Art05] contain a survey.

13. Applications of the simple relative trace formula and related issues in distinction

In this section we discuss applications of the simple relative trace formula to problems in the theory of distinguished representations. We restrict ourselves to existence results, which are the easiest thing to prove using these formulae.

13.1. **Applications of the simple trace formula.** The least refined application of the relative trace formula (which is still nontrivial) is the following proposition:

Proposition 13.1. *Let G be a (connected) reductive group with $Z_G(F_{v_i})$ compact for $v_1, \dots, v_n$, and let $\rho_{v_1}, \dots, \rho_{v_n}$ be a collection of supercuspidal representations. Then there exists a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ such that $\rho_{v_i} \cong \pi_{v_i}$ for all i.*

We will prove Proposition 13.1 as a special case of **??** below, at least in the case where the centralizer $C_{\gamma,G}$ is connected for all semisimple $\gamma \in G(F)$, though this assumption is unnecessary.

13.2. **Globalizations of distinguished representations.** Proposition **??** implies in particular that cuspidal automorphic representations always exist. If we fix a reductive group H and a quasi-character $\chi : H(F)A_{G,H}\backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$ we can also ask if there exists cuspidal automorphic representations that are (H, χ)-distinguished. That this is a more subtle point is illustrated by the following theorem [AGR93]

Theorem 13.2 (Ash, Ginzburg and Rallis)**.** *If (G, H) is one of the following pairs of groups over $\mathbb{Q}$ then no H-distinguished cuspidal automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ exist.*

(1) $(\mathrm{GL}_{n+k}, \mathrm{SL}_n \times \mathrm{SL}_k)$ *for* $n \neq k$.
(2) $(\mathrm{GL}_{2n}$ *or* $\mathrm{GL}_{2n+1}, \mathrm{Sp}_{2n})$.

(3) $(\mathrm{SO}(n,n), \mathrm{SL}_n)$ *for n odd.*
(4) $(\mathrm{Sp}_{2(n+k)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2k})$.
(5) $(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2(n-l)})$ *with $4l < n$.*
(6) $(O(Q), O(Q_l))$ *where $Q = Q_l \oplus Q_l^{\perp}$ with $2dim(Q_L^{\perp})$ less than the Witt index (i.e. dimension of the maximal isotropic subspace) of Q.*

Remark 13.3. In the setting of the theorem, there may still be distinquished automorphic representations, just not cuspidal ones.

13.3. **The local analogue of distinction.** Interestingly, despite Theorem 13.2, one can still obtain a conditional analogue of Proposition 13.1. To state it, we recall the local version of the notion of distinction. Thus let v be a nonarchimedian place of F, let $H_{F_v} \leq G_{F_v}$ be an algebraic subgroup, and let $\chi_v : H(F_v) \longrightarrow \mathbb{C}^{\times}$ be a quasi-character.

Definition 13.4. An admissible representation (π_v, V) of $G(F_v)$ is $(H(F_v), \chi_v)$**-distinguished** if there is a linear functional $\lambda : V \to \mathbb{C}^{\times}$ such that

$$\lambda(h \cdot \phi) = \chi_v^{-1}(h)\lambda(\phi)$$

for all $h \in H(F_v)$.

Assume now that $G = \mathrm{GL}_n$ and H is reductive a reductive F-subgroup of GL_n. Assume moreover that $\chi : H(\mathbb{A}_F) \longrightarrow \mathbb{C}^{\times}$ is a (unitary) character trivial on $H(F)$.

Theorem 13.5. *Let G be a (connected) reductive group with $Z_G(F_{v_i})$ compact for $v_1, \ldots, v_n$, and let $\rho_{v_1}, \ldots, \rho_{v_n}$ be a collection of supercuspidal representations.*

Assume that π_v is supercuspidal and $(H(F_v), \chi_v)$-distinguished. Then there is an (H, χ)-distinguished cuspidal automorphic representation σ of $G(\mathbb{A}_F)$ such that $\sigma_v \cong \pi_v$.

Proof. See [HM02]. □

14. More on distinction

Let $H \leq G$ be a connected reductive subgroup of a connected reductive group G over a number field F. We would like to make some comments on the problem of understanding automorphic representations of $G(\mathbb{A}_F)$ distinguished by (H, χ). Currently even a conjectural understaning of this problem seems out of reach. However, given recent (submitted) work of Sakellardis and Venkatesh special classes of examples seem to be more tractable. We recall the following definition:

Definition 14.1. An algebraic subgroup $H \leq G$ is said to be **spherical** if $\mathrm{Hom}_H(V, V_{triv})$ is at most one dimensional for all algebraic finite irreducible $\rho\colon G \to V$. Here V_{triv} denotes the trivial representations.

Notice that this says nothing a-priori about the multiplicity of $H(\mathbb{A}_F)$-invariant functionals on automorphic representations of $A_G\backslash G(\mathbb{A}_F)$, although it seems to be the case that when H is spherical these multiplicities can be controlled.

14.1. **Symmetric subgroups.** We will actually restrict our attention still further. Let $\sigma : G \to G$ be an automorphism of order 2 and let

$$G^{\sigma}(R) := \{g \in G(R) : g^{\sigma} = g\}$$

be the subgroup fixed by σ. Finally let $H = (G^{\sigma})^{\circ}$. In this case we refer to G/H as a symmetric space and H as a symmetric subgroup. We have the following theorem of Vust [Vus90]:

Theorem 14.2 (Vust)**.** *If G/H is a symmetric space then it is a spherical variety.*

We now give a list of examples of involutions σ and G^{σ}.

Example 14.3.

- We can take σ to be conjugation by an element of order two. For example by $\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$ gives $\mathrm{GL}_{m+n}^{\sigma} = \mathrm{GL}_m \times \mathrm{GL}_n$ where I_k is the k by k identity matrix.
- $\sigma(g) := g^{-t}$ gives the orthogonal group (this represents the unique outer isomorphism of GL_n over the algebraic closure)).
- Composition of conjugation by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ with $g \mapsto g^{-t}$ gives Sp_{2n}.
- Let M/F be a quadratic extension and let $\bar{\cdot}$ be the nontrivial Galois element then have:
$$U_{m,n}(R) = \left\{ g \in \mathrm{GL}_{m+n}(M \otimes R) | \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} \overline{g}^{-t} \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} = g \right\}.$$

There is a great deal of helpful geometry available in this special case that allows us to make our earlier definitions of relative classes concrete. The set of relative classes $\Gamma_r(R)$ with respect to the subgroup
$$H \times H \leq G \times G$$
is
$$H(R) \backslash G(R)/H(R).$$
and the set of geometric relative classes becomes
$$\Gamma_r^{geo}(R) = \mathrm{Im}(G(R) \longrightarrow H\backslash G/H(R))$$
To study these classes it is useful to introduce the **moment map**
$$B_\sigma : G \longrightarrow G$$
given on points by $g \mapsto gg^{-\sigma}$. We denote by Q the scheme-theoretic image of B_σ. Thus we have an isomorphism
$$B_\sigma : H\backslash G/G^\sigma \longrightarrow H\backslash Q \tag{14.1.1}$$
and the natural map $H\backslash G/H \to H\backslash G/G^\sigma$ is étale. Here H acts on Q via conjugation since
$$B_\sigma(h_1 g h_2) = h_1 B_\sigma(g) h_1^{-1}$$
for $h_i \in G^\sigma(R)$ and $g \in G(R)$. It is not hard to check that projection to the first factor of $H \times H$ induces an isomorphism
$$H_\gamma \longrightarrow C_{\gamma\gamma^{-\sigma},H} \tag{14.1.2}$$
where the latter group is the intersection of the centralizer in G of $\gamma\gamma^{-\sigma}$ with H.

We have the following proposition:

Proposition 14.4. *An element $\gamma \in G(F)$ is relatively semisimple if and only if $B_\sigma(\gamma)$ is semisimple. An element $\gamma \in G(F)$ is relatively ? if $B_\sigma(\gamma)$ is ?, where ? $\in$ {ellipticsemisimple, semisimple}.*

Proof. The first statement is proven in [Ric82], and the secon statement is clear from (14.1.1) and (14.1.2). ∎

We have seen that the notion of a maximal torus in a reductive group is absolutely crucial. In the case of symmetric varieties, the following definition constitutes a substitute:

Definition 14.5. A torus $T \subseteq G$ is said to be **σ-split** if for all F-algebras R and all $t \in T(R)$ one has $t^{-1} = t^\sigma$.

Remark 14.6. Beware that in [Ric82] a σ-split torus is called a "σ-anisotropic torus."

It is not hard to see that if T is any σ-split torus then $T \leq Q$. Indeed, every element of t is in the image of the isogeny $t \mapsto t^2 = tt^{-\sigma}$. Moreover, σ-split tori exist. If T is any σ-stable torus then

$$T = T_\sigma T^\sigma$$

where $T_\sigma \cap T^\sigma$ is a finite group scheme consisting of elements of order 2. Note that $T_\sigma \cap T^\sigma$ is a finite group scheme consisting of elements of order 2. We have a Weyl group attahced to this torus

$$W(T_\sigma, H) = N_H(T_\sigma)/C_H(T_\sigma),$$

sometimes called the **little Weyl group**. We have the following theorem [Ric82]:

Theorem 14.7 (Richardson)**.** *Let $T_\sigma \subseteq G$ be a maximal σ-split torus. The inclusion $T_\sigma \hookrightarrow Q$ induces an isomorphism*

$$T_\sigma/W(T_\sigma, H) \tilde{\to} H\backslash Q.$$

Remark 14.8. Take $G = H \times H$ and $\sigma\colon G \to G$ to be the map $(h_1, h_2) \mapsto (h_2, h_1)$. In this special case the theorem above reduces to the Chevalley restriction theorem.

In the same paper [Ric82], Richardson proves the following result: given $q \in Q(F)$, it admits a Jordan decomposition $q = q_s q_n$ with $q_s, q_n \in Q(\overline{F})$ with q_s and q_n commuting, q_s is semisimple and q_n is unipotent. The orbit Hq_s is the unique closed orbit in the Zariski closure of Hq and Hq is closed if and only if $q = q_s$. Thus we have a fairly explicit description of relatively semisimple elements in the case of symmetric varieties.

14.2. **Cases when one can characterize distinction.** Using the exact sequence (10.4.2), one can classify the involutory automorphisms of a given reductive algebraic group, and hence the symmetric subgroups. In the case where the group G is GL_{nF} the symmetric subgroups that can be obtained are all isomorphic over an algebraic closure of F to one of the following:

- $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ where $n_1 + n_2 = n$,
- O_n, and
- Sp_n.

If the group G is $\mathrm{Res}_{M/F}\mathrm{GL}_n$ for a quadratic extension M/F one can also obtain inner forms of the unitary group and GL_n. Due to a great deal of work spearheaded by Jacquet one now has a fairly complete understanding of each of these cases with the exception of O_n; this will be recalled in this subsection.

Remark 14.9. If the group G is not a general linear group or restriction of scalars of such a group, much less is known. However, the author and Wambach have outlined a program to relate distinction of automorphic representations of classical groups to distinction of automorphic representations of general linear groups. The paper will appear in AJM.

Jacquet originally developed the relative trace formula to prove the following theorem [Jac04], [Jac05]:

Theorem 14.10 (Jacquet)**.** *Let M/F be a quadratic extension, let $G = \mathrm{Res}_{M/F}\,\mathrm{GL}_n$ and let H be a quasi-split unitary group. Then a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ is distinguished by H if and only if $\pi^\sigma \cong \pi$, where $\langle\sigma\rangle = \mathrm{Gal}(M/F)$.*

Remark 14.11. The condition that $\pi \cong {}^\sigma\pi$ is equivalent to the statement that Note that the condition in the theorem above is equivalent to π being a lift from $\mathrm{GL}_n(\mathbb{A}_F)$, by work of Arthur and Clozel [AC89].

Similarly, in [FZ95] one finds the following theorem:

Theorem 14.12. *If M/F is a quadratic extension, then an automorphic respresentation of* $\mathrm{Res}_{M/F}\,\mathrm{GL}_n(\mathbb{A}_F)$ *is distinguished by* $\mathrm{GL}_n(\mathbb{A}_F)$ *if and only if the Asai L-function has a pole at $s = 1$.*

Remark 14.13. It should follow from recent work of Mok following work of Arthur that automorphic representations $\mathrm{Res}_{M/F}\,\mathrm{GL}_n(\mathbb{A}_F)$ whose Asai L-function has a pole at $s = 1$ are precisely those that are (stable or unstable) lifts from a unitary group.

For another example, take $H = \mathrm{GL}_n \times \mathrm{GL}_n \subseteq \mathrm{GL}_{2n}$ to be the the fixed points of $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. We have a character

$$\chi_{s_0} \colon H(\mathbb{A}_F) \to \mathbb{C}^\times$$

given by the formula

$$\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \mapsto |\det q_1/\det q_2|^{s_0-1/2}\,\chi(\det q_1/\det q_2)\eta(\det q_2),$$

where χ and η are characters of $\mathbb{A}_F$. In this setting Jacquet and Friedberg [FJ93] prove the following:

Theorem 14.14 (Friedberg-Jacquet)**.** *A cuspidal automorphic representation π on $G(\mathbb{A}_F)$ is (H, μ_s)-distinguished if and only if $L(s, \Lambda^2\pi \otimes \eta)$ has a pole at $s = 1$ and $L(s_0, \pi \otimes \chi) \neq 0$.*

We have actually already dealt with the case $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \leq \mathrm{GL}_{n_1+n_2}$ where $n_1 \neq n_2$, see Theorem 13.2.

Ash and Ginzburg use Theorem 14.14 to construct p-adic L-functions under a technical hypothesis, see [AG94].

Remark 14.15. If $\eta = 1$ and π has trivial central character, the condition that $L(s, \Lambda^2\pi)$ has a pole at $s = 1$ is equivalent to the statement that π is a lift of an automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A}_F)$ [CKPSS04], [GRS01].

Here is another case when we can characterize distinction [JR92].

Theorem 14.16 (Jacquet-Rallis)**.** *There are no cuspidal automorphic representations on* GL_{2n} *which are distinguished by* Sp_{2n}.

It is not clear to the author what is known in the case where $G^\sigma = \mathrm{O}_n$ and $G = \mathrm{GL}_n$. In the case $n = 2$ one has a complete result in [Jac86]. In general those representations of $\mathrm{GL}_n(\mathbb{A}_F)$ distinguished by O_n should be functorial lifts from a metaplectic group.

14.3. **The relationship between distinguished representations and functorial lifts.** In each of the cases discussed in the previous subsection, the cuspidal representations of a group G distinguished by a symmetric subgroup $H = (G^\sigma)^\circ$, if they existed, were those that were functorial lifts from another group G' (or metaplectic group) perhaps satisfying some addition desiderata. To the author's knowledge there is no exact conjectural recipe for the description of the other group or the additional desiderata, although it may be contained in forthcoming work of Sakellaridis and Venkatesh. However, in [JLR93] one finds a rough heuristic for what G' should be. We close our discussion of distinction with an explanation of this heuristic.

Assume for simplicity that G is semisimple and simply connected and let $T_\sigma \leq Q$ denote a maximal σ-split torus. Let Φ_0 denote the reduced root system attached to the set of roots of $T_{\sigma\mathbb{C}}$ in $G_{\mathbb{C}}$. Then

$$(X^*(T_{\sigma\mathbb{C}}), \Phi_0)$$

is a reduced root system and hence is the reduced root system of a semisimple group over $\bar{F}$. Jacquet conjectures that the group G' should be a form of this group, although what one means by form should be interpetted somewhat broadly.

15. The cohomology of locally symmetric spaces

The topic in this last section is the automorphic description of the cohomology of arithmetic locally symmetric spaces. In the following subsection we discuss the geometry of locally symmetric spaces. We then discuss local systems on these spaces in §15.2. One reference for this material, and its generalization to various compactifications of locally symmetric spaces is [Get12]. The special case of Shimura varieties is mentioned in §15.4.

We then briefly introduce $(\mathfrak{g}, K_\infty)$-cohomology in §**??** and show how it and automorphic representations can be used to give a complete description of the cohomology of locally symmetric spaces with coefficients in local systems as a module under Hecke-correspondences in §**??**. We close by describing some applications of this theory.

In this section we set

$$\mathbb{A} := \mathbb{A}_{\mathbb{Q}}$$

to ease notation. Unless otherwise specified G is an affine group scheme flat of finite type over $\mathbb{Z}$ such that $G_{\mathbb{Q}}$ is connected and reductive (in other words we fix a model of G over $\mathbb{Z}$).

Moreover we let $K \leq G(\mathbb{A}^\infty)$ is a compact-open subgroup, $K_\infty \leq G(\mathbb{R})$ is a compact subgroup containing the maximal connected compact subgroup K_∞^0 (in the real topology) and

$$A_G \leq G(\mathbb{R})$$

is the identity component in the real topology of the maximal $\mathbb{Q}$-split torus in the center of G. Finally we set

$$X := A_G G(\mathbb{R})/K_\infty.$$

15.1. **Locally symmetric spaces.** With the notation from above, any connected component of X is a symmetric space; one can essentially take this to be the definition of a symmetric space, although it is not the most natural definition. We let

$$\mathrm{Sh}^K := \mathrm{Sh}(G,X)^K := G(\mathbb{Q})\backslash X \times G(\mathbb{A}^\infty)/K \tag{15.1.1}$$

and refer to it as a Shimura manifold. As explained in (6.1.1) this is a finite union of locally symmetric spaces. Indeed, if we take a set of representatives $(g_i)_{i\in I}$ for the finite set $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/K$; then

$$\begin{aligned} \coprod_{i\in I} \Gamma_i\backslash X &\xrightarrow{\sim} \mathrm{Sh}^K \\ \Gamma_i x &\longmapsto G(\mathbb{Q})(x, g_i)K \end{aligned} \tag{15.1.2}$$

where $\Gamma_i = g_i^{-1}Kg_i \cap G(\mathbb{Q})$.

It is useful to have in mind the following definitions:

Definition 15.1. A subgroup $\Gamma \leq G(\mathbb{Q})$ is **arithmetic** if it is commensurable with $G(\mathbb{Z})$, and **congruence** if it is of the form $G(\mathbb{Q}) \cap K$ for some compact open subgroup $K \leq G(\mathbb{A}^\infty)$.

Remark 15.2. The famous congruence subgroup problem asks whether every arithmetic subgroup is congruence. The answer is in general no, but the groups G for which an arithmetic subgroup can be noncongruence are quite limited.

A useful assumption on a subgroup $\Gamma \leq G(\mathbb{Q})$ is that it is torsion-free. Unfortunately, the property of being torsion free is not preserved under certain contstructions, such as intersecting with a parabolic subgroup and mapping to the Levi quotient. A more robust condition is that of being neat.

An element $g \in \mathrm{GL}_n(\mathbb{Q})$ is *neat* if the subgroup of $\bar{\mathbb{Q}}^\times$ generated by its eigenvalues is torsion-free. An arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ is **neat** if given any (equivalently, one faithful) representation

$\rho : G \to \mathrm{GL}_n$, $\rho(g)$ is neat for all $g \in \Gamma$. Clearly, if Γ is neat, then all its subgroups and homomorphic images are neat.

Lemma 15.3. *Any congruence subgroup $\Gamma \le G(\mathbb{Q})$ contains a congruence subgroup of finite index that is neat.*

Proof. When $G = \mathrm{GL}_n$, we can take the neat subgroup to be of the form

$$\{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I \mod N\}$$

for a sufficiently divisible integer N. The general case follows from this case. □

Definition 15.4. A compact open subgroup $K \le G(\mathbb{A}^\infty)$ is **neat** if

$$G(\mathbb{Q}) \cap g^{-1}Kg$$

is neat for all $g \in G(\mathbb{A}^\infty)$.

In fact, one only has to check this condition for all $g \in G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/K$, which is a finite set. An elaboration of the proof of Lemma 15.3 implies the following:

Lemma 15.5. *If $K \le G(\mathbb{A}^\infty)$, then K contains a neat subgroup of finite index.* □

Our motivation for introducing this notion is the following:

Lemma 15.6. *If K is* neat*, the* Sh^K *is a smooth manifold.*

In fact the conclusion of the lemma is valid under the weaker assumption that $g^{-1}Kg \cap G(\mathbb{Q})$ is torsion-free for all $g \in G(\mathbb{A}^\infty)$. The proof is a consequence of basic results on discrete group actions on manifolds.

In §3.4 we defined Hecke operators attached to elements of $C_c^\infty(G(\mathbb{A}))$. We recall that these functions are finite linear combinations of characteristic functions of double cosets KgK for compact open subgroups $K \le G(\mathbb{A}^\infty$ and $g \in G(\mathbb{A}^\infty)$. We now explain how these operators can be realized geometrically as correspondences. Let $K, K' \le G(\mathbb{A}^\infty)$ be compact open subgroups, and $g \in G(\mathbb{A}^\infty)$ be such that $K' \subset gKg^{-1}$. The we have a map

$$\begin{aligned} T_g : \mathrm{Sh}^{K'} &\longrightarrow \mathrm{Sh}^K \\ G(\mathbb{Q})(x, hK') &\longmapsto G(\mathbb{Q})(x, hgK) \end{aligned} \tag{15.1.3}$$

On the other hand, if $\Gamma, \Gamma' \subset G(\mathbb{Q})$ are arithmetic subgroups, and $\gamma \in G(\mathbb{Q})$ is such that $\Gamma' \subset \gamma\Gamma\gamma^{-1}$, we also have

$$T_\gamma : \Gamma'\backslash X \longrightarrow \Gamma\backslash X$$

given by $\Gamma' x \mapsto \Gamma\gamma^{-1}x$; these are a finite étale maps.

The geometric realization of the characteristic function of the double coset KgK is the correspondence

$$\begin{tikzcd} & \mathrm{Sh}^{K\cap gKg^{-1}} \arrow[dl, "T_I"'] \arrow[dr, "T_g"] & \\ \mathrm{Sh}^K & & \mathrm{Sh}^K \end{tikzcd} \tag{15.1.4}$$

It acts on functions via pullback along T_I followed by pushforward along T_g. In other words, if we set

$$T(g) := T_{g*} \circ T_I^* \tag{15.1.5}$$

then for any function $\varphi : \mathrm{Sh}^K \longrightarrow \mathbb{C}$,

$$T(g)\varphi = R(\mathbb{1}_{KgK})\varphi. \tag{15.1.6}$$

15.2. **Local systems.** A well-known classical fact is that if Γ is a congruence subgroup of $\mathrm{GL}_{2\mathbb{Q}}$ contained in the subgroup of matrices with positive determinant and $\mathfrak{H}$ is the upper half-plane then the cohomology group

$$H^1(\Gamma\backslash\mathfrak{H}, \mathbb{C})$$

can be decomposed as a direct sum of three summands, one isomorphic to the vector space $S_2(\Gamma)$ of §6.4, one isomorphic to the space

$$\{f(\bar{z}) : f \in S_2(\Gamma)\}$$

of antiholomorphic forms, and one isomorphic to a certain space of Eisenstein series. In order to give a geometric interpretation of modular forms of weight bigger than 2, it is necessary to introduce local systems. We now recall their construction in the case at hand.

Until further notice $\Gamma \leq G(\mathbb{Q})$ is a neat arithmetic subgroup. Let V be a left $G(\mathbb{Q})$-module equipped with the discrete topology, and form the quotient

$$\mathcal{V} := \mathcal{V}^{\Gamma} = \Gamma\backslash(V \times X) \tag{15.2.1}$$

by the diagonal action of Γ on the product. We say that the diagram given by the natural projection

$$\mathcal{V} := \mathcal{V}^{\Gamma} \longrightarrow \Gamma\backslash X \tag{15.2.2}$$

is a **local system**. For example, if V is a representation of Γ over $\mathbb{C}$, then this is the total space of a locally constant sheaf of $\mathbb{C}$-vector spaces. For each open set $U \subseteq \Gamma\backslash X$ we let

$$\mathcal{V}|_U := \{s : U \longrightarrow \mathcal{V}\} \tag{15.2.3}$$

be the abelian group of sections of the map (15.2.2). Then the functor

$$\begin{aligned} \{U \subseteq \Gamma\backslash X\} &\longrightarrow \mathrm{Ab} \\ U &\longmapsto \mathcal{V}|_U \end{aligned} \tag{15.2.4}$$

from the category of open sets of $\Gamma\backslash X$ with morphisms given by inclusions to the category of abelian groups is a sheaf. It is called the **sheaf of sections of $\mathcal{V}$**. It's sheaf cohomology is denoted

$$H^\bullet(\Gamma\backslash X, \mathcal{V}) := H^\bullet(\Gamma\backslash X, \mathcal{V}^\Gamma) \tag{15.2.5}$$

We now turn to the adelic setting. As above, let V be a representation of G. For $K \subset G(\mathbb{A}^\infty)$ compact open, define

$$\mathcal{V} := \mathcal{V}^K := G(\mathbb{Q})\backslash V \times X \times G(\mathbb{A}^\infty)/K,$$

where for $\gamma \in G(\mathbb{Q})$, $\gamma(v, x, gK) = (\gamma.v, \gamma.x, \gamma gK)$. We thus have a natural map $\mathcal{V}^K \longrightarrow \mathrm{Sh}^K$. This is again a local system, and we define the associated sheaf of sections as above.

The relationship between the two constructions given above can be described as follows. Fix $g \in G(\mathbb{A}^\infty)$, and let $\Gamma = gKg^{-1} \cap G(\mathbb{Q})$. We then have an embedding

$$\begin{aligned} \iota : \Gamma\backslash X &\longrightarrow \mathrm{Sh}^K \\ \Gamma x &\longmapsto G(\mathbb{Q})(x, g)K \end{aligned} \tag{15.2.6}$$

and on the corresponding local systems, if V is a representation of $G(\mathbb{Q})$, then $\iota^*\mathcal{V}^K \cong \mathcal{V}^\Gamma$.

For $g \in G(\mathbb{A}^\infty)$, and K, K' with $K' \subset gKg^{-1}$, we have an isomorphism of sheaves

$$\begin{aligned} \theta : \mathcal{V}^{K'} \xrightarrow{\sim} T_g^*\mathcal{V}^K &= G(\mathbb{Q})\backslash V \times X \times G(\mathbb{A})/K \times_{\mathrm{Sh}^K} \mathrm{Sh}^{K'} \\ (v, (x, hK)) &\longmapsto (v, (xhgK, xhK)). \end{aligned} \tag{15.2.7}$$

This isomorphism is called a **lift of the correspondence**. The reason for this is that the isomorphism allows us to define a diagram

$$\begin{tikzcd} & \mathcal{V}^{K\cap gKg^{-1}} \arrow[dl, "\pi_1"'] \arrow[dr, "T_g\circ\theta"] & \\ \mathcal{V}^K & & \mathcal{V}^K \end{tikzcd} \tag{15.2.8}$$

and a map of sheaves $T(g) := T_{g*} \circ \theta \circ \pi_1^*$. This induces maps

$$T(g) : H^\bullet(\mathrm{Sh}^K, \mathcal{V}) \longrightarrow H^\bullet(\mathrm{Sh}^K, \mathcal{V}).$$

15.3. **A classical example.** We return to the example at the beginning of the previous subsection. Let $G = \mathrm{GL}_{2,\mathbb{Q}}$,

$$\begin{aligned} K_0(c) &: = \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : g \equiv \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \pmod{c}\} \\ \Gamma_0(c) &: = G(\mathbb{Q}) \cap K_0(c) \end{aligned}$$

For $k > 2$ the Shimura isomorphism yields an injection

$$\omega : S_k(\Gamma_0(c)) \xrightarrow{\omega\oplus\bar{\omega}} H^\bullet(\mathrm{Sh}^K, \mathcal{V}) \tag{15.3.1}$$

Here

$$\begin{aligned} \omega(f) &:= f(z)(-X + zY)^{k-2}dz) \\ \bar{\omega}(f) &:= f(\bar{z})(-X + \bar{z}Y)^{k-2}d\bar{z}. \end{aligned}$$

where $\mathcal{V}$ is the sheaf attached to the dual of the $(k-2)$st symmetric power of the standard representation $\mathbb{C}^2$, which we can regard as homogeneous polynomials of degree $k-2$ in two variables. The cokernel can be explained in terms of Eisenstein series. As we have explained, the right hand side is a module under Hecke correspondences. There are also "classical Hecke operators" indexed by

$$\gamma \in \{g \in M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q}) : g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{c}\}$$

defined by

$$\begin{aligned} T(\gamma) : S_k(\Gamma_0(c)) &\longrightarrow S_k(\Gamma_0(c)) \\ f &\longmapsto \sum_{\gamma_i\Gamma_0(c)\in\Gamma_0(c)\gamma\Gamma_0(c)} f|_k\gamma_i \end{aligned} \tag{15.3.2}$$

where

$$f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) := \det(\gamma)^{k/2}(cz+d)^{-k}f(z)$$

Then the operator $T(\gamma)$ on the left of (15.3.1) is intertwined with the operator $T((\gamma^\infty)^{-1}\det(\gamma_\infty))$ on the right. The reason for the inverse is that the Eichler-Shimura isomorphism transforms invariance properties of a function under the left action of $\Gamma_0(c)$ to invariance properties of a function under the right action of $K_0(c)$. For more details see [**?**].

15.4. **Shimura data.** Recall the definition of *Deligne torus*

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_\mathrm{m}). \tag{15.4.1}$$

Note that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. The following definition is due to Deligne.

Definition 15.7. Let G be a connected reductive group and let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_\mathbb{R}$ The pair (G, X) is a **Shimura datum** if

(SV1) For $h \in X$, the characters $z/\bar{z}$, 1, and $\bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_\mathbb{C}$ defined by h (where $G^{\mathrm{ad}} := G/Z_G$ is the adjoint group);

(SV2) $\mathrm{ad}(h(\sqrt{-1}))$ is a Cartan involution of G^{ad};
(SV3) G^{ad} has no $\mathbb{Q}$-factors on which the projection of h is trivial.

The condition (SV2) is equivalent to the statement that

$$G^{\mathrm{ad},(\theta)}(\mathbb{R}) := \{g \in G^{\mathrm{ad}}(\mathbb{C}) \ : \ g = \theta(\bar{g})\}$$

is compact. Conditions (SV1) and (SV2) together imply that X is a Hermitian symmetric space for G.

Since Sh^K is a complex manifold in this case, a natural question is whether $\mathrm{Sh}(G,X)^K$ can be realized as the complex points of some variety. This is indeed the case by a basic theorem of Baily and Borel which states that if K is neat then $\mathrm{Sh}(G,X)^K$ can be given the structure of the complex points of smooth quasi-projective scheme over $\mathbb{C}$ in a canonical manner. One can say even more.

For each $x \in X$, we have a cocharacter

$$u_x(z) := h_{x_{\mathbb{C}}}(z,1),$$

where $x_{\mathbb{C}}$ denotes be base change of x to $\mathbb{C}$. This certainly defines an element of

$$G\backslash \mathrm{Hom}(\mathbb{G}_m \longrightarrow G)(\mathbb{C})$$

where the implied action is via conjugation. Let $E(G,X) \subset \mathbb{C}$ be the field of definition of u_x. It is a number field, independent of the choice of $x \in X$. It is called the **reflex field** of (G,X).

Theorem 15.8. *For each neat $K \leq G(\mathbb{A}^\infty)$ there exists a smooth quasi-projective variety $M(G,X)^K$ defined over the reflex field E of (G,X) such that*

$$\mathrm{Sh}(G,X)^K = M(G,X)^K(\mathbb{C}).$$

and all the correspondence $T(g)$ are defined over $E(G,X)$. Furthermore, there is a canonical such model, characterized by the Galois action on certain special points.

Definition 15.9. The *Shimura variety* attached to (G,X) is the projective limit

$$M := M(G,X) := \varprojlim_K M(G,X)^K$$

of the canonical models of $\mathrm{Sh}^K = \mathrm{Sh}(G,X)^K$.

Definition 15.10. A morphism of Shimura data, $(G,X) \longrightarrow (G',X')$, is a morphism of algebraic groups $G \to G'$ (over $\mathbb{Q}$) sending X to X'.

Definition 15.11. A morphism of Shimura varieties, $\mathrm{Sh}(G,X)^K \longrightarrow \mathrm{Sh}(G',X')^K$, is an inverse system of regular maps compatible with the action of Hecke correspondences.

Theorem 15.12. *A morphism of Shimura data induce a morphism of Shimura varieties over the compositum $E(G,X)E(G',X')$. Moreover, it is a closed immersion if $G \to G'$ is injective.*

We end with some examples of Shimura data

(1) $G = \mathrm{GL}_{2,\mathbb{Q}}$; $h(a+b\sqrt{-1}) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$.
(2) $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$; for example, for $F/\mathbb{Q}$ totally real, $h = \prod_\sigma \left(\begin{smallmatrix} a^\sigma & b^\sigma \\ -b^\sigma & a^\sigma \end{smallmatrix}\right)$. The associated Shimura varieties are known as Hilbert modular varieties.
(3) $G = \mathrm{GSp}_{2n}$, where

$$\mathrm{GSp}_{2n}(R) = \{g \in \mathrm{GL}_{2n}(R) \ : \ g^t J g = c(g) J \text{ for some } c(g) \in R^\times\}$$

for $J = \left(\begin{smallmatrix} & & 1 \\ & ⁄ & \\ 1 & & \end{smallmatrix}\right)$; $h(a+b\sqrt{-1}) = \left(\begin{smallmatrix} aI & -bJ \\ bJ & aI \end{smallmatrix}\right)$. The associated Shimura varieties are known as Siegel modular varieties.

15.5. $(\mathfrak{g}, K_\infty)$**-cohomology.** As above let V be a finite-dimensional representation of G. A fundamental tool for describing $H^\bullet(\mathrm{Sh}^K, \mathcal{V})$ as a Hecke module is $(\mathfrak{g}, K_\infty)$-cohomology. We explain its definition in this section. The basic reference is [BW00a].

Now, let $\mathfrak{g} \geq \mathfrak{k}$ be the complexifications of the Lie algebras of $G_{\mathbb{R}}$ and K_∞ respectively. Let $\mathcal{A}$ be a $\mathfrak{g}$-module (not necessarily of finite dimension). Let

$$C^\bullet(\mathfrak{g}, \mathcal{A}) = \mathrm{Hom}(\Lambda^\bullet \mathfrak{g}, \mathcal{A}) = \wedge^\bullet \mathfrak{g}^\vee \otimes_{\mathbb{C}} \mathcal{A}.$$

We define a differential on the complex by:

$$(df)(x_0, \dots, x_g) = \sum (-1)^i x \cdot f(x_0, \dots, \hat{x}_i, \dots, x_g) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_g)$$

We then let $H^\bullet(\mathfrak{g}, \mathcal{A})$ denote the cohomology of this complex:

$$H^\bullet(\mathfrak{g}, \mathcal{A}) = H^\bullet(C^\bullet(\mathfrak{g}, \mathcal{A}), d).$$

Consider

$$C^\bullet(\mathfrak{g}, \mathfrak{k}, \mathcal{A}) = \{f \in C(g, V) | \iota_x f = \theta_x f = 0 \ \forall x \in \mathfrak{k}\}$$

where ι_x and θ_x are the interior product and derivative respectively. In other words

$$(\iota_x f)(x_1, \dots, x_{g-1}) = f(x, x_1, \dots, x_{g-1}) \tag{15.5.1}$$

and

$$(\theta_x f)(x_1, \dots, x_g) = \sum_i f(x_1, \dots, [x, x_i], \dots, x_g) + x f(x_1, \dots, x_g). \tag{15.5.2}$$

This implies that

$$C^\bullet(\mathfrak{g}, \mathfrak{k}, \mathcal{A}) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}).$$

Notice that one has $\theta_x = d \circ \iota_x + \iota_x \circ d$ for $x \in \mathfrak{g}$ and thus this new complex is d stable. Consequently we can take the cohomology of this complex. This cohomology is called the $(\mathfrak{g}, \mathfrak{k})$-cohomology of $\mathcal{A}$; it is denoted by $H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A})$.

Because of the fact that $\mathrm{Lie}\, K_\infty = \mathrm{Lie}\, K_\infty^+$ this cohomology is insensitive to the connected components of K_∞. It is therefore desirable to refine it.

Assume now that $\mathcal{A}$ is a $(\mathfrak{g}, K_\infty)$-module. The group K_∞ acts in a natural manner on $C(\mathfrak{g}, \mathfrak{k}, \mathcal{A})$ via the action of K_∞ on $\mathcal{A}$ and the adjoint action on $\mathfrak{g}$. For this action define

$$C^\bullet(\mathfrak{g}, K_\infty, \mathcal{A}) = C^\bullet(\mathfrak{g}, \mathfrak{k}, \mathcal{A})^{K_\infty}.$$

Denote by $H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A})$ the cohomology of this complex.

Definition 15.13. The $(\mathfrak{g}, K_\infty)$**-cohomology** of the $(\mathfrak{g}, K_\infty)$-module $\mathcal{A}$ is $H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A})$.

15.6. **The relationship between** $(\mathfrak{g}, K_\infty)$**-cohomology and the cohomology of Shimura manifolds.** Let V be a finite dimensional representation of G. We now relate $(\mathfrak{g}, K_\infty)$-cohomology and the cohomology of Sh^K with coefficients in $\mathcal{V}$

Fix a basis ω^i, $1 \leq i \leq m := \dim_{\mathbb{R}}(X)$ of left invariant 1-forms on $A_G \backslash G(\mathbb{R})/K_\infty$. For

$$I = \{i_1, \dots, i_q\} \subset \{1, \dots, m\}$$

with $i_j < i_n$ for $j < n$ set

$$\omega^I := \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}.$$

Then any differential q-form η on $\Gamma A_G \backslash G(\mathbb{R})$ can be written as

$$\eta = \sum_I f_I \omega^I$$

with $f_I \in C^\infty(\Gamma A_G \backslash G(\mathbb{R}))$. Let

$$\mathcal{A} := C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K^\infty).$$

Let

$$A^q(\mathrm{Sh}^K, \mathcal{V})$$

denote the space of differential q-forms on Sh^K with coefficients in $\mathcal{V}$. Any element of $A^q(\mathrm{Sh}^K, \mathcal{V})$ can be written as

$$\eta = \sum_I f_I \omega^I$$

with $f_i \in \mathcal{A} \otimes V$. This yields an identification

$$A^\bullet(\mathrm{Sh}^K, \mathcal{V}) = C^\bullet(\mathfrak{g}, K_\infty; \mathcal{A}) \tag{15.6.1}$$

commuting with the differentials, which in turn yields an isomorphism

$$H^\bullet(\mathrm{Sh}^K, \mathcal{V}) \cong H^\bullet(\mathfrak{g}, K_\infty, \mathcal{A})$$

This gives an explicit link between cohomology and automorphic representations. We remark that this map is Hecke equivaraiant in the sense that for $g \in G(\mathbb{A}^\infty)$ the action of the correspondence $T(g)$ is intertwined with the action of $\mathbf{1}_{KgK}$.

This whole construction motivates the following definition:

Definition 15.14. A vector $\phi \in C^\infty(G(\mathbb{Q})A_G\backslash G(\mathbb{A}))^K$ is **cohomological** if there exists a representation V of G, $v \in V$ and ω^I on $G(\mathbb{R})$ such that

$$\phi\omega^I \otimes v \in C^\bullet(\mathfrak{g}, \mathfrak{k}_\infty; C^\infty(G(\mathbb{Q})A_G\backslash G(\mathbb{A})) \otimes V)^K$$

defines a non-zero class in $H^\bullet(\mathfrak{g}, K_\infty, C^\infty(G(\mathbb{Q})A_G\backslash G(\mathbb{A})))^K$.

If we wish to specify the representation V we could speak of a V**-cohomological** vector instead.

Now $C^\infty(G(\mathbb{Q})A_G\backslash G(\mathbb{A}))$ is naturally a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}^\infty)$-module. Let us decompose it under the action of $G(\mathbb{A}^\infty)$.

Let $\{\pi\}$ be a set of representatives for equivalence classes of cuspidal automorphic representations of $A_G\backslash G(\mathbb{A})$. Letting

$$L^{2,cusp} := L^{2,cusp}(G(\mathbb{Q})A_G\backslash G(\mathbb{A}))$$

as usual we have

$$L^{2,cusp} := \bigoplus_\pi \pi^{\oplus m(\pi)}$$

where $m(\pi)$ is the multiplity of π in $L^{2,cusp}$. Hence

$$\begin{aligned} H^\bullet(\mathrm{Sh}^K, \mathcal{V}) \supset H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{V}) &= H^\bullet(\mathfrak{g}, K_\infty; (\mathcal{A} \cap L^{2,cusp}) \otimes V))^K \\ &= \bigoplus_\pi \left(H^\bullet(\mathfrak{g}, K_\infty; \pi \otimes V))^K\right)^{\oplus m(\pi)} \end{aligned} \tag{15.6.2}$$

The group $H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{V})$ is known as the **cuspidal cohomology**. Its complement is described in terms of so-called Eisenstein cohomology; for some information about the decomposition of the whole of the cohomology see [BLS96]. It follows that, as a module under $C_c^\infty(G(\mathbb{A}^\infty) /\!/ K)$, one has

$$H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{V}) = \oplus_\pi \left(H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty \otimes V) \otimes (\pi^\infty)^K\right)^{m(\pi)}$$

We refer to

$$H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{V})(\pi^\infty) := \oplus_{\pi' : \pi'_\infty \cong \pi_\infty} \left(H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty \otimes V) \otimes (\pi'^\infty)^K\right)^{m(\pi)} \tag{15.6.3}$$

as the π-isotypic component of the cohomology.

Definition 15.15. A cuspidal automorphic representation π of $A_G\backslash G(\mathbb{A})$ is **cohomological** if there is a representation V of G such that $H^{\bullet}(\mathfrak{g}, K_{\infty}, \pi_{\infty} \otimes V) \neq 0$. A vector ϕ in the space of π is **cohomological** if there exists an embedding $\pi \to L^{2,cusp}$ such that the image of ϕ is cohomological.

Thus if there is a vector in the space of π that is cohomological, then π itself is cohomological.

15.7. **The relation to distinction.** Usually in the literature one finds references to cohomological representations but no references to cohomological vectors. Despite this, the notion of a cohomological vector is of great importance.

Suppose that $H \leq G$ are connected reductive $\mathbb{Q}$-groups and that their corresponding symmetric spaces are chosen so that $X_H \hookrightarrow X$. Then there is an embedding

$$\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}^{\infty})} \hookrightarrow \mathrm{Sh}(G, X)^K.$$

Assume that one can define a cohomology class attached to this subset (if $\mathrm{Sh}(G, X)^K$ is compact this is always possible). Assume that π is a cohomological representation. One would like an answer to the following question:

> If π is H-distinguished, then is there a cohomological vector ϕ in the space of π such that $\mathcal{P}_H(\phi) \neq 0$?

For many reasons, including investigation of the Tate conjecture for Shimura varieties, one would like to know an answer to this question. Indeed, a positive answer to this question allows us to construct an explicit nonzero cycle in the π^{∞}-isotypic component of $H^{\bullet}_{cusp}(\mathrm{Sh}^K, \mathcal{V})$.

In general, the answer to the question is no; for example, there could be no cycles of the appropriate dimension. However, in special cases it can be answered in the affirmative.

Remark 15.16. Suppose $H \times H = G$. If we take $\Delta(H)$ to be the diagonal copy of H, then a cohomological representation of $G(\mathbb{A})$ that is distinguished by $\Delta(H)$ has a cohomological vector ϕ in its space such that $\mathcal{P}_H(\phi) \neq 0$. The proof of this statement is essentially an application of Schur's lemma.

15.8. **More on $(\mathfrak{g}, K_{\infty})$-cohomology.** We have seen that the question of whether or not a given automorphic representation contributes to the cohomology of a Shimura manifold with coefficients in a local system is completely determined by the $(\mathfrak{g}, K_{\infty})$-cohomology of its factor at infinity. One thing that makes this remark so useful is the fact that $(\mathfrak{g}, K_{\infty})$-cohomology is a very pleasant object with which to work. In this subsection we list some properties of these groups; the canonical reference is [BW00b].

Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as Lie algebras and $K_{\infty} = K_1 \times K_2$, $V = V_1 \otimes V_2$ (exterior tensor product). Then one has a Künneth forumla:

Theorem 15.17 (Künneth formula)**.** *One has natural isomorphisms*

$$H^k(\mathfrak{g}, K_{\infty}, V) = \bigoplus_{p+q=k} H^p(\mathfrak{g}_1, K_1, V_1) \oplus H^q(\mathfrak{g}_2, K_2, V_2).$$

One also has a version of Poincare duality:

Theorem 15.18 (Poincare duality)**.** *If $m = \dim X$, then letting $\mathfrak{a} := \mathrm{Lie}(A_G) \otimes_{\mathbb{R}} \mathbb{C}$ one has*

$$H^q(\mathfrak{a}\backslash\mathfrak{g}, K_{\infty}; \pi \otimes V) \cong H^{m-q}(\mathfrak{a}\backslash\mathfrak{g}, K_{\infty}; \pi_{\infty}^{\vee} \otimes V^{\vee})^{\vee}.$$

Thus, in particular, to compute the $(\mathfrak{g}, K_{\infty})$-cohomology of a $(\mathfrak{g}, K_{\infty})$-module it suffices to understand the case where $\mathfrak{g}$ is simple over $\mathbb{R}$ and when $q \leq \dim X/2$.

15.9. **The Vogan-Zuckerman classification.** Vogan and Zuckerman have provided a classification of all $(\mathfrak{g}, K_\infty)$-modules with nonzero cohomology. We briefly recall this result. For simplicity assume that $\mathfrak{g}$ is simple and let K_∞ is connected. We let

$$\mathfrak{k}_0 := \mathrm{Lie}(K_\infty).$$

Since K_∞ is compact, the adjoint operator $\mathrm{Ad}(x)$ acting on $\mathfrak{g}$ is diagonalizable for all $x \in \mathfrak{k}_0$, with real eigenvalues, and complex conjugation switches positive and negative eigenvalues. Let $\mathfrak{q}$ denote the sum of the non-negative eigenspaces, let $\mathfrak{u}$ be the sum of the positive eigenspaces, and let $\mathfrak{l}$ be the sum of the zero eigenspaces. Then $\mathfrak{q} \subseteq \mathfrak{g}$ is a parabolic subalgebra and

$$\mathfrak{q} = \mathfrak{l} + u.$$

There is an involutory automorphism θ of $\mathfrak{g}$, called the **Cartan involution**, such that $\mathfrak{k}$ is the fixed points of θ. It will preserve a parabolic subalgebra as above. It is convenient to call such a $\mathfrak{q}$ simply a θ-stable parabolic subalgebra, though not all parabolic subalgebras stable under θ arise in this way.

Suppose that V is a representation of $\mathfrak{g}$ such that the highest weight with respect to a θ-stable Borel subalgebra is fixed by θ. Attached to this data Vogan and Zuckerman define a $(\mathfrak{g}, K_\infty)$-module $A_\mathfrak{q}(V)$ (in the notation of [BW00b, VI.5]), and explicitly compute its $(\mathfrak{g}, K_\infty)$-cohomology. They then prove the following result:

Theorem 15.19 (Vogan-Zuckerman)**.** *If π_∞ is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module and*

$$H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty \otimes V^\vee) \neq 0,$$

then $\pi_\infty \cong A_\mathfrak{q}(V)$.

(see [BW00b] and [VZ84]).

15.10. **Cohomology in low degree.** For motivation, we state the Lefschetz theorem from algebraic geometry. Let X be a smooth projective n-dimensional variety over $\mathbb{C}$, and let $Y \subseteq X$ denotes a hyperplane section such that $X - Y$ is smooth (this will hold generically). One has the following result:

Theorem 15.20 (Lefschetz)**.** *The induced map on cohomology*

$$H^k(X, \mathbb{Q}) \longrightarrow H^k(Y, \mathbb{Q})$$

is an isomorphism for $k \leq n-1$ and injective for $k = n-1$.

The moral of this theorem is that in low degree, all of the cohomology of a variety comes from a subvariety of lower dimension in a precise sense. Bergeron and Clozel have proven similar results for locally symmetric spaces, even which are not of hermitian type. We refer the reader to [BC05] and more recent work of Bergeron.

In this section we will recall a much more elementary result that state that in low degrees the cohomology of locally symmetric varieties is very simple. We refer to [BW00b] for the proofs. Let $\mathfrak{P} < \mathfrak{g}$ denote the -1 eigenspace of the Cartan involution θ.

Proposition 15.21. *If V is an irreducible finite-rank representation of G then*

$$H^\bullet(\mathfrak{g}, K_\infty, V) = \left\{ \begin{array}{ll} 0 & \text{if } V \text{ is nontrivial,} \\ (\wedge^\bullet \mathfrak{P})^{K_\infty} & \text{if } V \text{ is trivial.} \end{array} \right\}.$$

Now suppose that G is semisimple. Then we have injective maps

$$j \colon H^q(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^q(\mathrm{Sh}^K, \mathbb{C})$$

These classes represent somewhat trivial parts of the cohomology. In [Bor74] one finds a definition of an integer $m(\mathfrak{g})$ related to the curvature of X and a proof of the following theorem:

Theorem 15.22 (Matsushima)**.** *The map j above is bijective for $k \leq m(\mathfrak{g})$. Moreover if $q \leq m(\mathfrak{g})$ then every harmonic form in $H^q(\mathrm{Sh}^K)$ is invariant under $G(\mathbb{R})$.*

Remark 15.23. Moreover, the assumption that the coefficients are $\mathbb{C}$ is also not necessary at the expense of complicating the statement of the theorem, and in fact nontrivial coefficients tend to have more vanishing cohomology groups in low degrees (see [BW00b, §II.10, Theorem]).

15.11. **Galois representations.** Let $\mathrm{Sh}(G, X)$ be a Shimura variety. Since it is defined over a number field, its étale cohomology provides a vital link between automorphic representations and Galois representations. This link has been exploited notably by R. Taylor, his collaborators, and students, to prove spectacular results attaching Galois representations to important classes of automorphic representations and conversely establishing the modularity of Galois representations. In fact the field is evolving so quickly that we will not attempt to provide references, but instead invite the reader to search them out for him or herself.

References

[AC89] James Arthur and Laurent Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989. 75

[AG94] Avner Ash and David Ginzburg. p-adic L-functions for $\mathrm{GL}(2n)$. *Invent. Math.*, 116(1-3):27–73, 1994. 76

[AGR93] Avner Ash, David Ginzburg, and Steven Rallis. Vanishing periods of cusp forms over modular symbols. *Math. Ann.*, 296(4):709–723, 1993. 58, 72

[Art83] James Arthur. The trace formula for reductive groups. In *Conference on automorphic theory (Dijon, 1981)*, volume 15 of *Publ. Math. Univ. Paris VII*, pages 1–41. Univ. Paris VII, Paris, 1983. 70

[Art05] James Arthur. An introduction to the trace formula. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 1–263. Amer. Math. Soc., Providence, RI, 2005. 72

[BC05] Nicolas Bergeron and Laurent Clozel. Spectre automorphe des variétés hyperboliques et applications topologiques. *Astérisque*, (303):xx+218, 2005. 85

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. 8, 11, 66

[BLS96] A. Borel, J.-P. Labesse, and J. Schwermer. On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields. *Compositio Math.*, 102(1):1–40, 1996. 83

[Bor63] Armand Borel. Some finiteness properties of adele groups over number fields. *Inst. Hautes Études Sci. Publ. Math.*, (16):5–30, 1963. 29

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974. 85

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. 9, 12

[Bor07] Armand Borel. Automorphic forms on reductive groups. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 7–39. Amer. Math. Soc., Providence, RI, 2007. 59

[BR85] Peter Bardsley and R. W. Richardson. Étale slices for algebraic transformation groups in characteristic p. *Proc. London Math. Soc. (3)*, 51(2):295–317, 1985. 64

[Bre94] Ralph J. Bremigan. Quotients for algebraic group actions over non-algebraically closed fields. *J. Reine Angew. Math.*, 453:21–47, 1994.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. 22, 26, 27, 33

[BW00a] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000. 82

[BW00b] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000. 84, 85, 86

[Car79] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979. 36, 40, 41, 42, 53

[CGP10] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 17 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2010. 9

[Che58] Claude Chevalley. Séminaire C. Chevalley, 1956–1958. Classification des groupes de Lie algébriques. pages ii+166 + ii+122 pp. (mimerographed), 1958. 14

[CKM04] James W. Cogdell, Henry H. Kim, and M. Ram Murty. *Lectures on automorphic L-functions*, volume 20 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2004. 52

[CKPSS04] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi. Functoriality for the classical groups. *Publ. Math. Inst. Hautes Études Sci.*, (99):163–233, 2004. 76

[Cog07] James W. Cogdell. L-functions and converse theorems for GL_n. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 97–177. Amer. Math. Soc., Providence, RI, 2007. 52

[Con] Brian Conrad. Weil and grothendieck approaches to adelic points. Personal website. 6, 19

[DG74] C. Demazure and P. Grothendieck. *Schémas en groupes*, volume 151–153 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 1974. 14

[DM78] Jacques Dixmier and Paul Malliavin. Factorisations de fonctions et de vecteurs indéfiniment différentiables. *Bull. Sci. Math. (2)*, 102(4):307–330, 1978. 71

[Don82] Harold Donnelly. On the cuspidal spectrum for finite volume symmetric spaces. *J. Differential Geom.*, 17(2):239–253, 1982. 61, 71

[Dyn52] E. B. Dynkin. Maximal subgroups of the classical groups. *Trudy Moskov. Mat. Obšč.*, 1:39–166, 1952. 14

[FJ93] Solomon Friedberg and Hervé Jacquet. Linear periods. *J. Reine Angew. Math.*, 443:91–139, 1993. 76

[FZ95] Yuval Z. Flicker and Dmitrii Zinoviev. On poles of twisted tensor L-functions. *Proc. Japan Acad. Ser. A Math. Sci.*, 71(6):114–116, 1995. 75

[Get12] Mark Getz, Jayce R. and Goresky. *Hilbert modular forms with coefficients in intersection homology and quadratic base change*, volume 298 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2012. 77

[GR10] Benedict H. Gross and Mark Reeder. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.*, 154(3):431–508, 2010. 47

[GRS01] David Ginzburg, Stephen Rallis, and David Soudry. Generic automorphic forms on $\mathrm{SO}(2n+1)$: functorial lift to $\mathrm{GL}(2n)$, endoscopy, and base change. *Internat. Math. Res. Notices*, (14):729–764, 2001. 76

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises. 67

[Hah09] Heekyoung Hahn. A simple twisted relative trace formula. *Int. Math. Res. Not. IMRN*, (21):3957–3978, 2009. 63

[HC53] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953. 27

[HC68] Harish-Chandra. *Automorphic forms on semisimple Lie groups*. Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62. Springer-Verlag, Berlin, 1968. 31

[HC99] Harish-Chandra. *Admissible invariant distributions on reductive p-adic groups*, volume 16 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999. Preface and notes by Stephen DeBacker and Paul J. Sally, Jr. 55

[Hen00] Guy Henniart. Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique. *Invent. Math.*, 139(2):439–455, 2000. 48

[HL04] Michael Harris and Jean-Pierre Labesse. Conditional base change for unitary groups. *Asian J. Math.*, 8(4):653–683, 2004. 64

[HLR86] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. *J. Reine Angew. Math.*, 366:53–120, 1986. 59

[HM02] Jeffrey Hakim and Fiona Murnaghan. Globalization of distinguished supercuspidal representations of $\mathrm{GL}(n)$. *Canad. Math. Bull.*, 45(2):220–230, 2002. 73

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich. 48

[Jac86] Hervé Jacquet. Sur un résultat de Waldspurger. *Ann. Sci. École Norm. Sup. (4)*, 19(2):185–229, 1986. 76

[Jac04] Hervé Jacquet. Kloosterman identities over a quadratic extension. *Ann. of Math. (2)*, 160(2):755–779, 2004. 75

[Jac05] Hervé Jacquet. Kloosterman identities over a quadratic extension. II. *Ann. Sci. École Norm. Sup. (4)*, 38(4):609–669, 2005. 75

[JLR93] Hervé Jacquet, King F. Lai, and Stephen Rallis. A trace formula for symmetric spaces. *Duke Math. J.*, 70(2):305–372, 1993. 76

[JR92] Hervé Jacquet and Stephen Rallis. Symplectic periods. *J. Reine Angew. Math.*, 423:175–197, 1992. 76

[JR96] Hervé Jacquet and Stephen Rallis. Uniqueness of linear periods. *Compositio Math.*, 102(1):65–123, 1996.

[JS81a] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.*, 103(4):777–815, 1981. 52

[JS81b] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981. 52

[Lab99] Jean-Pierre Labesse. Cohomologie, stabilisation et changement de base. *Astérisque*, (257):vi+161, 1999. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen. 69

[Lan56] Serge Lang. Algebraic groups over finite fields. *Amer. J. Math.*, 78:555–563, 1956. 66

[Lan76] Robert P. Langlands. *On the functional equations satisfied by Eisenstein series*. Lecture Notes in Mathematics, Vol. 544. Springer-Verlag, Berlin, 1976. 53, 61

[LV07] Elon Lindenstrauss and Akshay Venkatesh. Existence and Weyl's law for spherical cusp forms. *Geom. Funct. Anal.*, 17(1):220–251, 2007. 63

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. 64

[Mil12] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/. 7, 9, 10

[Mur05] Fiona Murnaghan. Linear algebraic groups. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 379–391. Amer. Math. Soc., Providence, RI, 2005. 58

[Ric82] R. W. Richardson. Orbits, invariants, and representations associated to involutions of reductive groups. *Invent. Math.*, 66(2):287–312, 1982. 74, 75

[RR96] Cary Rader and Steve Rallis. Spherical characters on $\mathfrak{p}$-adic symmetric spaces. *Amer. J. Math.*, 118(1):91–178, 1996. 65

[RV99] Dinakar Ramakrishnan and Robert J. Valenza. *Fourier analysis on number fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. 3, 5

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972. 45

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. 20, 66

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1. 20, 32

[Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second edition, 2009. 12, 13

[Ste65] Robert Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965. 65

[Tam63] Tsuneo Tamagawa. On the ζ-functions of a division algebra. *Ann. of Math. (2)*, 77:387–405, 1963. 42

[Tat79] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979. 45

[Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979. 21, 35, 43, 66, 67

[Vus90] Thierry Vust. Plongements d'espaces symétriques algébriques: une classification. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(2):165–195, 1990. 73

[VZ84] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984. 85

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. 7

[Yu] J.K. Yu. Smooth models associated to concave functions in bruhat-tits theory. (Preprint). 21