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RIGIDLY BREAKING POTENTIAL FLOWS AND A COUNTABLE ALEXANDROV THEOREM FOR POLYTOPES

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We study all the ways that a given convex body in d dimensions can break into countably many pieces that move away from each other rigidly at constant velocity, with no rotation or shearing. The initial velocity field is locally constant a.e., but may be continuous and/or fail to be integrable. For any choice of mass-velocity pairs for the pieces, such a motion can be generated by the gradient of a convex potential that is affine on each piece. We classify such potentials in terms of a countable version of a theorem of Alexandrov for convex polytopes, and prove a stability theorem. For bounded velocities, there is a bijection between the mass-velocity data and optimal transport flows (Wasserstein geodesics) that are locally incompressible.

Given any rigidly breaking velocity field that is the gradient of a continuous potential, the convexity of the potential is established under any of several conditions, such as the velocity field being continuous, the potential being semiconvex, the mass measure generated by a convexified transport potential being absolutely continuous, or there being a finite number of pieces. Also we describe a number of curious and paradoxical examples having fractal structure.

1. Introduction

Imagine that a brittle body, such as a crystal ball, shatters instantaneously into pieces which fly apart from each other with constant velocities. Experience tells us to expect a large number of shards that may be extremely small.

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To model this in a simple way mathematically, we represent the body by a bounded convex open set $\Omega \subset \mathbb{R}^d$, and suppose its mass density is constant and normalized to unity. We suppose that the body shatters into pieces represented by a countable collection of pairwise disjoint open subsets A_i whose union $A = \bigsqcup_i A_i$ has full Lebesgue measure in Ω. For simplicity we presume the pieces travel by rigid translation with no rotation. This means that any point z in A at time $t = 0$ is transported to the point

$$X_t(z) = z + tv(z) \tag{1-1}$$

at time $t > 0$, where the velocity field $v : \Omega \to \mathbb{R}^d$ is a constant v_i on A_i. It is natural to require the pieces to remain pairwise disjoint; thus we require the transport map X_t to be *injective* on A for every $t > 0$. Given such a velocity field v, we will say that v *rigidly breaks* Ω into A_i, $i = 1, 2, \dots$. The number of pieces A_i may be finite or countably infinite.

We imagine that by observations around some time $t > 0$ after shattering occurs, we can determine the mass m_i and the velocity v_i for each piece. Our first result shows that these data suffice to determine all the pieces (and thus the entire flow) in an essentially unique way, provided we happen to know that the velocity is a *gradient of a convex potential.*

Below, we call any function $\varphi : \Omega \to \mathbb{R}$ *locally affine a.e.* if it is affine on some neighborhood of x, for a.e. $x \in \Omega$. Given such a function we associate the set

$$A = \{x \in \Omega : \varphi \text{ is affine on a neighborhood of } x\}. \tag{1-2}$$

This is an open subset of Ω with full Lebesgue measure $\lambda(A) = \lambda(\Omega)$. The set A has countably many components A_i, $i = 1, 2, 3, \dots$, which are open and path-connected. For each i, φ is smooth on A_i and its gradient φ is constant in a neighborhood of each point of A_i, so by path-connectedness there must exist $v_i \in \mathbb{R}^d$ and $h_i \in \mathbb{R}$ such that

$$\varphi(z) = v_i \cdot z + h_i \quad \text{for all } z \in A_i. \tag{1-3}$$

The following characterizes functions that are locally affine a.e. and convex.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex open set, let $v_1, v_2, \dots$ be distinct in $\mathbb{R}^d$, and let $m_1, m_2, \dots$ be positive so that $\sum_i m_i = \lambda(\Omega)$. Then there is a function φ on Ω (unique up to adding a constant) that is locally affine a.e. and convex, so $\nabla\varphi = v_i$ on an open convex set A_i with $\lambda(A_i) = m_i$.*

Theorem 1.1 extends a geometric theorem of Alexandrov [2005] on unbounded convex polytopes to the case of a countably infinite number of faces. Later in this introduction we will discuss this further.

As a consequence of Theorem 1.1, for any given mass-velocity data m_i, v_i, $i = 1, 2, \dots$ as described, there exists a velocity potential φ that is locally affine a.e. and convex and induces a partition of Ω as the data require. Importantly, this map X_t is injective on A for all $t \ge 0$, due to a simple lemma:

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^d$ be open and convex, let $\varphi : \Omega \to \mathbb{R}$ be convex, and let $X_t(z) = z + t\nabla\varphi(z)$ for all $z \in \Omega$. If φ is differentiable at $x, y \in \Omega$, then*

$$|X_t(x) - X_t(y)| \ge |x - y| \quad \textit{for all } t \ge 0. \tag{1-4}$$

It is natural to wonder about a few things at this point. First, under what sort of conditions can we ensure that a rigidly breaking velocity field is a gradient of a convex potential? Second, what is there to say about the difference between having infinitely many pieces versus finitely many? And further, is there a sense in which the flows depend continuously on the mass-velocity data, justifying finite approximation? This paper is aimed at addressing these issues.

Conditions for convexity. Our motivation for considering the first of these questions stems from our work with Dejan Slepčev [Liu et al. 2019]. Certain results in that paper imply, roughly speaking, that any incompressible least-action mass transport flow must have initial velocity which is locally constant on an open set of full measure, equal to the gradient of a potential φ which is locally affine a.e. and *semiconvex*. Saying φ is semiconvex is equivalent to saying that the function

$$\psi_t(z) = \tfrac{1}{2}|z|^2 + t\varphi(z) \tag{1-5}$$

is convex for some $t > 0$. In the immediate context, ψ_t is the potential for the transport map $X_t = \nabla\psi_t$, and the convexity of ψ_t follows from Brenier's theorem in optimal transport theory. (Below, we assume $\psi_t = +\infty$ outside $\bar{\Omega}$.)

In the present paper, we work in a somewhat more general situation. We study flows produced by a.e.-locally affine potentials that start from a convex source domain but need not have least action or even finite action. In this situation, a result of McCann [1997], used to prove uniqueness of energy minimizers, directly implies that for any potential that is locally affine a.e., convexity is equivalent to semiconvexity. From [McCann 1997, Lemma 3.2] we immediately find the following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set, and assume $\varphi : \Omega \to \mathbb{R}$ is locally affine a.e. Then φ is convex if and only if it is semiconvex.*

Note that this result holds even without requiring the transport maps X_t determined by $v = \nabla\varphi$ to be injective a priori. We can list three conditions, different from semiconvexity however, under which the injectivity suffices to entail the convexity of φ (and becomes equivalent to it, due to Lemma 1.2).

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Let $\varphi : \Omega \to \mathbb{R}$ be continuous and locally affine a.e., and define A by* (1-2). *Further, assume any one of the following*:

(i) *The dimension $d = 1$.*

(ii) *The number of components of A is finite.*

(iii) *φ is C^1.*

Then φ is convex if and only if the map $z \mapsto X_t(z) = z + t\nabla\varphi(z)$ is injective on A for all sufficiently small $t > 0$.

Under condition (i), the conclusion is easy to establish, of course. Condition (ii) and the local representation (1-3) together will imply that adjacent pieces must meet along flat faces where both convexity and injectivity reduce to a local monotonicity property for $\nabla\varphi$. For the case of condition (iii) we employ the Hopf–Lax formula which formally provides a solution to the initial-value problem for a Hamilton–Jacobi equation with convex Hamiltonian, namely

$$\partial_t u + \tfrac{1}{2}|\nabla u|^2 = 0, \quad u(x,0) = \varphi(x). \tag{1-6}$$

The maps X_t provide characteristics for this problem.

Our last condition for convexity of φ is related to mass transport associated with the convexification of ψ_t. Below, we let ψ_t^* denote the Legendre transform of ψ_t for $t \geq 0$, taking ψ_t to be defined by (1-5) in the convex domain $\bar{\Omega}$ and $+\infty$ outside. Then the convexification of ψ_t is the double transform ψ_t^{**}.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Let $\varphi : \bar{\Omega} \to \mathbb{R}$ be continuous and locally affine. Then φ is convex if and only if for some $t > 0$, the push-forward of Lebesgue measure under the (a.e.-defined) gradient of the convexification of ψ_t, written*

$$\kappa_t = (\nabla \psi_t^{**})_\sharp \lambda,$$

is absolutely continuous with respect to Lebesgue measure λ on $\mathbb{R}^d$.

The proof of this theorem involves the second Hopf formula for solutions of the initial-value problem for a different Hamilton–Jacobi equation which formally also has characteristics given by X_t. Namely, for the following initial-value problem with convex initial data,

$$\partial_t w + \varphi(\nabla w) = 0, \qquad w(x, 0) = \psi_0^*(x), \tag{1-7}$$

with φ extended continuously to $\mathbb{R}^d$, the Legendre transform $w = \psi_t^*$ is the unique viscosity solution of (1-7), according to a result of Bardi and Evans [1984].

The push-forward measure κ_t in Theorem 1.5 is also described as the *Monge–Ampère measure* determined by ψ_t^*, as we discuss in Section 6. In space dimension $d = 1$, the measure κ_t reduces to a mass measure induced by *sticky particle flow*, due to results of Brenier and Grenier [1998]. When the velocity potential is nonconvex, the velocity is not monotonically increasing, and the sticky particle flow is sure to form mass concentrations. When the dimension $d > 1$, our use of concentrations in κ_t to characterize nonconvexity for locally affine potentials φ is partly motivated by [Brenier et al. 2003; Frisch et al. 2002]. These works describe links between a Monge–Ampère equation, optimal transport, and mass density in the "adhesion model" in cosmology. The adhesion model is used to approximate the formation of mass-concentrating structures in the universe such as sheets and filaments; see, e.g., [Weinberg and Gunn 1990; Vergassola et al. 1994; Gurbatov et al. 2012].

Remark 1.6. It seems reasonable to conjecture that Theorem 1.4 should remain valid in general, without assuming *any* of the additional conditions (i)–(iii), only imposing some mild regularity assumption such as local Lipschitz regularity, perhaps. That is, nonconvexity of φ should imply noninjectivity of X_t. We have been unable to prove or disprove such a result. Thus it appears interesting to investigate various criteria under which injectivity suffices to ensure convexity. Theorem 1.5 shows that nonconvexity yields a measure-theoretic version of noninjectivity, however, insofar as concentrations form instantaneously in κ_t.

Incompressible least-action flows with convex source. Combined with our results from [Liu et al. 2019], Theorems 1.1 and 1.3 provide a classification of action-minimizing mass-transport flows that are incompressible and transport Lebesgue measure in a given bounded open convex set Ω_0 in $\mathbb{R}^d$ to Lebesgue measure in some other bounded open set. A precise description of such flows is provided in Theorem 8.2 of Section 8. There we will show that they correspond in one-to-one fashion with countable sets $\{(m_i, v_i)\}$ of pairs consisting of positive masses m_i and distinct velocities v_i bounded in $\mathbb{R}^d$, such that $\sum_i m_i = \lambda(\Omega_0)$.

Infinitely many vs. finitely many pieces. Characterizing convex and piecewise affine functions by volume and slope data relates to a classic geometric problem. In 1897, Minkowski [Minkowski 1989; Alexandrov 2005] proved that any compact convex polytope is uniquely determined, up to translation, by the *list of face normals and areas*, subject to a natural compatibility condition saying that the integral of the unit outward normal field over all faces must vanish. Alexandrov solved a version of this problem for unbounded convex polytopes whose unbounded edges are parallel, and he presented his solution in his 1950 book *Convex Polyhedra* [Alexandrov 2005] (see Sections 7.3.2 and 6.4.2). We quote Alexandrov's result essentially as reformulated in [Gu et al. 2016] in terms of convex, piecewise affine functions, as follows.

Theorem 1.7 (Alexandrov). *Let Ω be a compact convex polytope with nonempty interior in $\mathbb{R}^d$, let $v_1, \dots, v_k \in \mathbb{R}^d$ be distinct and let $m_1, \dots, m_k > 0$ so that $\sum_{i=1}^k m_i = \lambda(\Omega)$. Then there is convex, piecewise affine function φ on Ω (unique up to adding a constant) so $\nabla\varphi = v_i$ on a convex set A_i with volume $\lambda(A_i) = m_i$.*

Alexandrov's unbounded polyhedra correspond to the supergraph sets

$$\{(z, y) \in \mathbb{R}^d \times \mathbb{R} : z \in \Omega,\ y \geq \varphi(z)\},$$

whose unbounded edges are parallel to the last coordinate axis.

We remark that Gu et al. [2016] provided an elementary self-contained proof for a generalization of Theorem 1.7, essentially equivalent here to minimizing $\int_\Omega \varphi\, d\lambda$ as a function of the constants h_i in the representation (1-3) subject to the given volume constraints on A_i. This is a variant of Minkowski's original proof (presented in [Alexandrov 2005, §7.2]) of the existence of bounded polyhedra with prescribed face areas and normals through a constrained maximization of volume. But this technique does not appear to work in the countably infinite case of Theorem 1.1.

In the case of finitely many pieces, in addition to the conclusions stated in Alexandrov's theorem it is known that:

(i) The velocity field $v = \nabla\varphi$ is discontinuous on Ω if $1 < k < \infty$.

(ii) Each piece A_i is the interior of a convex polytope.

Of course, property (i) is trivial since Ω is connected. Property (ii) is due to the affineness from (1-3) and the convexity of φ, which imply $\varphi(z) \geq v_i \cdot z + h_i$ for all $z \in \Omega$. It follows $z \in A_i$ if and only if $z \in \Omega$ and

$$v_i \cdot z + h_i > v_j \cdot z + h_j \quad \text{for all } j \neq i. \tag{1-8}$$

Equality is not possible since the v_i are distinct and A_i is open. By consequence A_i is the intersection of a finite number of half-spaces, i.e., a polytope.

In the case of infinitely many pieces, it turns out that neither (i) nor (ii) is necessarily true. A rigidly breaking velocity field can be continuous on Ω, and a piece (shard) may assume any convex shape. As the reader may suspect, examples involve fractal structure. We will explore constructions involving Cantor sets, Vitali coverings, and Apollonian gaskets. Figure 1 illustrates the latter: The shaded circles indicate the sets $A_i + t x_i$, where the A_i are Apollonian disks in the unit circle Ω, x_i is the center of A_i, and $t = 0.5$. See Section 9.2 for details.

Actually, continuity of the velocity is a highly paradoxical property, since it immediately implies that the flow images $X_t(\Omega)$ are connected, so seemingly not "broken" at all! As we will show, this phenomenon generates fat Cantor sets by "expanding" the standard Cantor set in a simple way.

Plan of the paper. Following this introduction, we first provide the proof of Theorem 1.1 and Lemma 1.2 in Section 2. In Section 3 we study and classify rigidly breaking flows in the case of one space dimension, $d = 1$. There we also discuss a paradoxical example with rigidly breaking but continuous velocity given by the Cantor function.

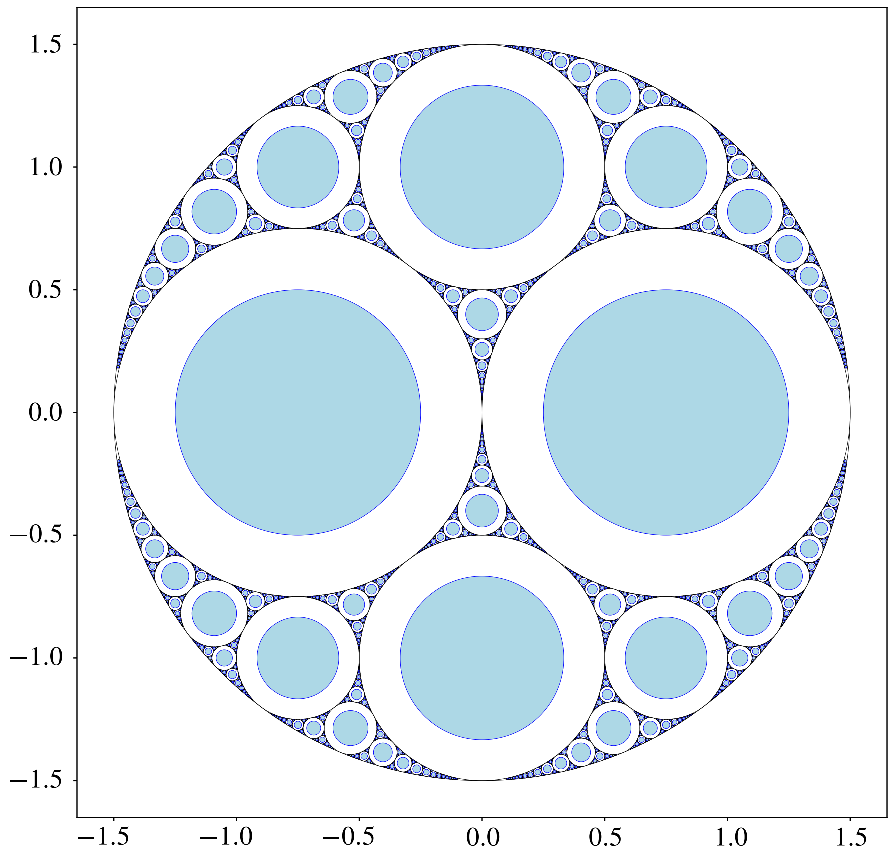

Figure 1. Breaking of an Apollonian gasket at $t = 0.5$

We complete the proof of Theorem 1.4 in Sections 4 and 5. We handle case (ii) in Section 4, where we assume the flow rigidly breaks the convex domain into finitely many pieces. The case (iii), with C^1 potential, is handled in Section 5, making use of the Hopf–Lax formula for the solution of the Hamilton–Jacobi equation (1-6).

We carry out the proof of Theorem 1.5 in Section 6. In particular, if $\varphi : \overline{\Omega} \to \mathbb{R}$ is continuous, locally affine a.e. and *nonconvex*, Theorem 6.5 shows that the Monge–Ampère measure κ_t in Theorem 1.4 has a Lebesgue decomposition with a nontrivial singular part.

We next investigate the stability of rigidly breaking flows with respect to the mass-velocity data, in Section 7. There we show that weak-star convergence of transported Lebesgue measure follows from weak-star convergence of pure point measures naturally associated with the mass-velocity data.

In Section 8 we complete our treatment of incompressible least-action flows with convex source from [Liu et al. 2019], establishing in Theorem 8.2 that these flows are characterized uniquely by their mass-velocity data $\{(m_i, v_i)\}$.

We study the possible shapes that the convex “pieces” A_i may take in Section 9. In particular, we show that all the A_i may be round balls, corresponding to a full packing of Ω (e.g., any Apollonian or osculatory packing), and we show that an individual component A_i can assume any convex shape.

The paper concludes with a discussion that addresses three points. We discuss how the continuity assumption on the potential φ in Theorem 1.4 is ensured by the absence of shear (i.e., symmetry of the distributional gradient ∇v) and a local integrability condition. We complete our Cantor-function example in Section 3 showing how fat Cantor sets are produced in a uniformly expanded way. Finally, although we lack any characterization of rigidly breaking velocity fields that are continuous when the dimension $d > 1$, we discuss some constraints on such fields.

2. Proof of a countable Alexandrov theorem

Here we provide the proofs of Theorem 1.1 and Lemma 1.2. We prove Theorem 1.1 by a straightforward application of a theorem of McCann [1995] which improved Brenier's theorem in optimal transport theory.

Proof of Theorem 1.1. Let the measure μ be given by $\lambda \llcorner \Omega$, Lebesgue measure restricted to the bounded convex open set Ω, and let the measure ν given as a combination of Dirac delta masses concentrated at the distinct points v_i,

$$\nu = \sum_i m_i \delta_{v_i}, \quad \text{where } \sum_i m_i = \lambda(\Omega). \tag{2-1}$$

With no moment assumptions, the main theorem in [McCann 1995] produces a convex function φ on $\mathbb{R}^d$ whose gradient $T = \nabla\varphi$ is determined uniquely a.e. in Ω and pushes μ forward to ν. The push-forward property $T_\sharp\mu = \nu$ has the consequence that the preimage $\hat{A}_i$ of $\{v_i\}$ under the (a.e.-defined) gradient of φ is a Borel set $\hat{A}_i \subset \Omega$ with $\lambda(\hat{A}_i) = m_i$ and $\nabla\varphi = v_i$ on $\hat{A}_i$. Because Ω is connected, this determines φ up to a constant.

Since φ is convex it is not difficult to deduce that φ is affine on the closure of the convex hull of $\hat{A}_i$; see the lemma below. Thus since $\lambda(\hat{A}_i) > 0$, the closed convex hull has convex interior $A_i \subset \hat{A}_i \subset \bar{A}_i$ which is convex and has the same measure $\lambda(A_i) = \lambda(\bar{A}_i) = m_i$. □

Lemma 2.1. *Assume $\Omega \subset \mathbb{R}^d$ is an open convex set and $f : \Omega \to \mathbb{R}$ is convex.*

(i) *If f is differentiable at points $x, y \in \Omega$ with $\nabla f(x) = \nabla f(y)$, then f is affine on the line segment connecting x and y.*

(ii) *If ∇f is constant on a set B, then f is affine on the closed convex hull of B in Ω.*

Proof. To prove (i), restrict f to the line segment connecting x to y, defining $g(\tau) = f(x + \tau(y - x))$. Then g is differentiable at $\tau = 0$ and 1, with

$$g'(0) = \nabla f(x) \cdot (y - x) = \nabla f(y) \cdot (y - x) = g'(1).$$

Then g is affine since it is convex. This proves (i), and we further note that

$$f(x) - f(y) = \nabla f(y) \cdot (x - y). \tag{2-2}$$

To prove (ii), by continuity it suffices to show f is affine on the convex hull of B. By Carathéodory's theorem on convex functions, each point in the convex hull is a convex combination of at most $d + 1$ points in B. Consider a convex combination $x = \sum_{j=1}^k t_j y_j$ with $y_j \in B$ and $t_j \geq 0$ for all j and $\sum t_j = 1$. Invoking convexity and using (2-2), we find that

$$\sum_{j=1}^k t_j f(y_j) \geq f(x) \geq f(y_1) + \nabla f(y_1) \cdot (x - y_1) = \sum_{j=1}^k t_j \big(f(y_1) + \nabla f(y_1) \cdot (y_j - y_1) \big) = \sum_{j=1}^k t_j f(y_j).$$

Hence f is affine on the closed convex hull of B. □

Remark 2.2. Evidently, any arbitrary pure point measure ν on $\mathbb{R}^d$ having total mass $\nu(\mathbb{R}^d) = \lambda(\Omega)$ can be expressed in the form (2-1) for countable mass-velocity data that satisfy the assumptions of Theorem 1.1. Reordering the data yield the same measure; hence there is a bijection between countable sets $\{(m_i, v_i)\}$ of such mass-velocity data and such pure point measures. The main theorem in [McCann 1995] associates a convex potential with any Radon measure ν on $\mathbb{R}^d$ having $\nu(\mathbb{R}^d) = \lambda(\Omega)$. The association of mass-velocity data with potentials in Theorem 1 is obtained by restricting this to pure point measures.

Remark 2.3. In Section 7 we will prove a stability (or continuity) theorem for the flows $X_t = \mathrm{id} + t\nabla\varphi$ determined by mass-velocity data as in the proof of Theorem 1.1 above. In Theorem 7.1 we show that for any sequence of pure point measures ν_n defined as in (2-1), weak-star convergence of ν_n implies weak-star convergence of Lebesgue measure restricted to the transported sets $X_t^n(A^n)$, where A^n is the open set defined as in (1-2) on which φ_n is locally affine.

Proof of Lemma 1.2. Let $\Omega \subset \mathbb{R}^d$ be open and convex, let $\varphi : \Omega \to \mathbb{R}$ be convex, define $X_t(z) = z + t\nabla\varphi(z)$ for $z \in \Omega$, and suppose φ is differentiable at two points $x, y \in \Omega$. Convexity implies the graph of φ lies above the tangent planes at x and y; hence the well-known monotonicity condition follows:

$$(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) \geq 0. \tag{2-3}$$

Thence

$$(X_t(x) - X_t(y)) \cdot (x - y) = |x - y|^2 + t(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) \geq |x - y|^2,$$

and we infer $|X_t(x) - X_t(y)| \geq |x - y|$ by the Cauchy–Schwarz inequality. □

3. One space dimension

In order to develop understanding of rigidly breaking flows with a countably infinite number of components, we consider the case of one space dimension. We provide the easy proof of Theorem 1.4 for this case, and we illustrate and characterize the paradoxical possibility that a rigidly breaking velocity field may be continuous.

3.1. *Convexity in one dimension.*

Proof of Theorem 1.4(i). Make the assumptions of the theorem, including that (i) the dimension $d = 1$. By Lemma 1.2 we know convexity of φ implies injectivity of X_t on A for all $t > 0$. Supposing that X_t is injective on A for all small enough $t > 0$, we claim $\nabla\varphi$ is necessarily increasing on A. Each of the countably many components A_i of the open set A is an open interval. Let v_i be the constant value of $\nabla\varphi$ on A_i. The images $X_t(A_i) = A_i + tv_i$ then remain disjoint and preserve their initial order for all small $t > 0$. Let A_i, A_j be any two component intervals of A and assume $A_i < A_j$, meaning $x < y$ whenever $x \in A_i$ and $y \in A_j$. If A_i and A_j are adjacent, then clearly $v_i \leq v_j$. If they are not adjacent, then the union of all intervals $A_k + tv_k$ with $A_i < A_k < A_j$ preserves its initial Lebesgue measure; hence the interval between $A_i + tv_i$ and $A_j + tv_j$ cannot shrink, and so $v_i \leq v_j$. It follows φ is convex. □

3.2. ***Example: "Cantor's elastic band".*** Take $\Omega = (0, 1) \subset \mathbb{R}$, and consider the velocity field given by $v = c$ in Ω, where $c : [0, 1] \to [0, 1]$ is the standard *Cantor function*. The function c is increasing yet continuous on $[0, 1]$ with $c(0) = 0$ and $c(1) = 1$, and c is locally constant on the open set $A = (0, 1) \setminus \mathcal{C}$, where $\mathcal{C}$ denotes the standard Cantor set.

For each component interval A_i of A, let v_i denote the value of c on A_i. Then the flow in (1-1) is given by rigid transport in A_i, with

$$X_t(z) = z + t v_i, \quad z \in A_i.$$

Note that the distance between $X_t(A_i)$ and $X_t(A_j)$ increases linearly with t, since $v_i < v_j$ for $A_i < A_j$. Thus v rigidly breaks Ω into the A_i, according to our definition at the beginning of the introduction.

Indeed, the velocity potential $\varphi(z) = \int_0^z c(r)\,dr$ is convex and locally affine a.e. Yet $v = \nabla\varphi$ is continuous. This seems paradoxical, for it implies the image $X_t(\Omega)$ remains *connected* under the flow of the "rigidly breaking" velocity field v, and must comprise the full interval $(0, 1+t)$!

Evidently, the injective maps X_t "stretch" the interval $[0, 1]$ to cover the longer interval $[0, 1+t]$ by countably many rigidly translated images $X_t(A_i)$ together with the image of the Cantor set $X_t(\mathcal{C})$. The union of the rigid images is the set $X_t(A)$, which is open and dense in $(0, 1+t)$. Of course the Lebesgue measure $\lambda((0, 1+t)) = 1+t$, yet evidently

$$\lambda(X_t(A)) = \sum_i \lambda(X_t(A_i)) = \sum_i \lambda(A_i) = \lambda(A) = 1.$$

What we infer from this is that the image $\mathcal{C}_t := X_t(\mathcal{C})$ is a *fat Cantor set*. It is closed and nowhere dense in $(0, 1+t)$, and has Lebesgue measure $\lambda(\mathcal{C}_t) = t$. The map X_t has "stretched" the Cantor set $\mathcal{C}$ with Lebesgue measure zero to a set with positive Lebesgue measure.

In terms of physical intuition, we might fancifully imagine $\mathcal{C}$ as consisting of an ephemeral kind of matter having zero mass and always nowhere dense, but infinitely stretchable so it can cover a set of positive Lebesgue measure. The body $\Omega = (0, 1)$ might be considered to model an elastic band made of a mixture of such stretchy stuff and ordinary rigid matter. In this interpretation, deforming Ω to $X_t(\Omega)$ stretches the band but it does not disconnect it.

Less fancifully, we wish to describe what is "broken" in a mathematically natural way. For this we can focus on matter that has positive mass density. The rigid translation of the connected open pieces A_i induces a mass measure ν_t on the image domain $X_t(\Omega)$ that is *not* the restriction of Lebesgue measure to $X_t(\Omega)$. Instead, ν_t is the restriction of Lebesgue measure to the disconnected open (yet dense) set $X_t(A) = \bigsqcup_i X_t(A_i)$. We can say the body Ω is broken into the disconnected components $X_t(A_i)$ that carry all the mass. This induced mass measure ν_t is nothing but the push-forward under X_t of $\lambda \llcorner \Omega$, Lebesgue measure restricted to Ω. We have $(X_t)_\sharp(\lambda \llcorner \Omega) = \lambda \llcorner X_t(A)$ in the present example, and this *differs* from $\lambda \llcorner X_t(\Omega)$. While one can make different choices of the set A with this property, it seems natural to take A to be the open set in (1-2) on which the velocity potential is locally affine.

In Figure 2 we illustrate this example by plotting the velocity $v = c$ as a function of transported position $x = X_t(z) = z + tc(z)$. The transported pieces $X_t(A_i)$ are (nonsingleton) level sets of the transported velocity $v = f(x, t)$, which is constant along the flow lines $x = z + tc(z)$. As a side remark, it is interesting to note that while the partial derivative $\partial f/\partial x = 0$ in every translated component $X_t(A_i)$, it turns out that $\partial f/\partial x = 1/t$ a.e. in the fat Cantor set $\mathcal{C}_t$, meaning these sets expand uniformly in time. We defer proof to the discussion below; see Proposition 10.2.

3.3. ***Characterization of continuity in one dimension.*** The Cantor-function example generalizes to provide necessary and sufficient conditions for a rigidly breaking velocity field to be continuous when $d = 1$. Recall that by Theorem 1.4(i), such a velocity field must be the derivative of a C^1 potential φ that is convex and locally affine a.e.

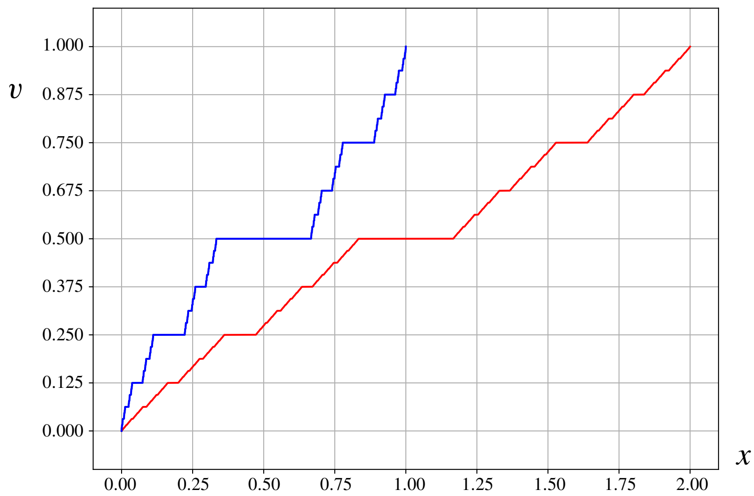

Figure 2. Cantor expansion wave: $v = c(z)$ *vs.* $x = z + tc(z)$ at $t = 0$ and 1.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval, and let φ be convex and locally affine a.e. on Ω, with φ' taking the distinct values $\{v_i\}$ on an open set of full measure in Ω. Then φ is C^1 if and only if the sequence $\{v_i\}$ is dense in an interval.*

Proof. Suppose φ is convex and locally affine a.e., so φ' is defined and constant on each component of an open set A of full measure in Ω. If φ is C^1, then the continuous image $\varphi'(\Omega)$ must be connected, and hence an interval, and $\varphi'(A) = \{v_i\}$ must be dense in it. On the other hand, if $\varphi'(A)$ is dense in an interval I, then, because φ' is increasing on A, we have $\varphi = \int v\,dx$ where the function given by

$$v(x) = \lim_{z\uparrow x,\, z\in A} \varphi'(z), \qquad x \in \Omega$$

is increasing with no jump discontinuities. So v is continuous, and φ is C^1. □

Remark 3.2. By Theorem 1.1, for any sequence $\{v_i\}$ of distinct values dense in an interval, such C^1 potentials exist and are specified uniquely by any positive sequence $\{m_i\}$ with $\sum_i m_i = \lambda(\Omega)$. In this case $v = \varphi'$ is a Cantor-like function, continuous and increasing on Ω and constant on an interval A_i with $\lambda(A_i) = m_i$.

4. Finitely many pieces

In this section we prove Theorem 1.4 under condition (ii) which states that the number of components A_i of A is finite. Recall that convexity of φ implies injectivity of X_t by Lemma 1.2. Briefly, our strategy for proving the converse will be to show that if φ is nonconvex, then two adjacent components must have velocities that force their images under the flow X_t to overlap immediately for $t > 0$. We do this by finding a line segment along which the restriction of φ is nonconvex and intersects ∂A only at finitely many points on flat "faces" between adjacent components.

Throughout this section we work under the basic assumptions of Theorem 1.4, and assume the dimension $d > 1$. Recall we assume A is given by (1-2) and its components A_i are open and connected and their number N is finite. The case $N = 1$ is trivial, so assume $N > 1$. Given that φ is locally affine on A and continuous on Ω, there exist $v_1, \dots, v_N \in \mathbb{R}^d$ and $h_1, \dots, h_N$ such that the representation (1-3) extends by continuity to say

$$\varphi(z) = v_i \cdot z + h_i, \quad z \in \bar{A}_i, \quad i \in [N] = \{1, \dots, N\}. \tag{4-1}$$

Repeated values are possible. By (4-1) and (1-2), each point in the interior of $\bar{A}_i$ must be in A. Since $\bar{A}_i$ is disjoint from A_j for $j \neq i$, the interior of $\bar{A}_i$ is A_i and $\bar{A}_i$ is the disjoint union of A_i and ∂A_i.

4.1. ***Geometry of the pieces.*** We begin by precisely describing some of the geometric structure of the dense open set A and its boundary (or complement) in Ω. Define an "adjacency function" by

$$\mathcal{I}(z) = \{i \in [N] : z \in \bar{A}_i\} \quad \text{for each } z \in \Omega. \tag{4-2}$$

Evidently the cardinality $\#\mathcal{I}(z) = 1$ if $z \in A$. Define "face" and "edge" sets respectively by

$$F = \{z \in \Omega : \#\mathcal{I}(z) = 2\}, \qquad E = \{z \in \Omega : \#\mathcal{I}(z) \geq 3\}. \tag{4-3}$$

Lemma 4.1. *Make the assumptions of Theorem 1.4 including condition* (ii). *Let* $A^c = \Omega \setminus A$. *Then* $A^c = \partial A \cap \Omega$ *and we have*

$$A = \{z \in \Omega : \#\mathcal{I}(z) = 1\}, \qquad A^c = \{z \in \Omega : \#\mathcal{I}(z) \geq 2\} = F \cup E.$$

Proof. Because A is open and dense, $\partial A \cap \Omega = A^c$. The finite union $\bigcup_i \bar{A}_i$ is closed and contains A; hence $\bar{A} = \bar{\Omega}$, so necessarily $\#\mathcal{I}(z) \geq 1$ for all $z \in \Omega$.

Now, let $z \in A^c$. It remains to show $\#\mathcal{I}(z) \geq 2$. Fix i with $z \in \bar{A}_i$. Necessarily $z \in \partial \bar{A}_i$ since $z \notin A_i$. For any $k > 0$ there exists $y_k \in \Omega \setminus \bar{A}_i$ with $|y_k - z| < 1/k$. Then since N is finite, some subsequence of the y_k lie in $\bar{A}_j$ for some fixed $j \neq i$. It follows that $z \in \bar{A}_j$; hence $\#\mathcal{I}(z) \geq 2$ as required. ☐

Next, for all $i, j \in [N]$ with $i \neq j$ we define

$$H_{ij} = \{z \in \mathbb{R}^d : v_i \cdot z + h_i = v_j \cdot z + h_j\}. \tag{4-4}$$

Provided $v_i \neq v_j$ this set is a hyperplane of codimension 1. Let $\mathcal{H}$ denote the collection of these codimension–1 sets.

Proposition 4.2. *Make the assumptions of Theorem 1.4 including condition* (ii). *Then*:

(a) *The set A^c is contained in a finite union of codimension–1 hyperplanes.*

(b) *For any $z \in A^c$, $z \in F$ if and only if z lies in $H_{ij} \cap B$ for some hyperplane H_{ij} in $\mathcal{H}$ and some open ball B with $B \subset A_i \cup A_j \cup H_{ij}$.*

(c) *The set E is contained in a finite union of codimension–2 hyperplanes.*

Proof. (a) Let $z \in A^c$. Then $\#\mathcal{I}(z) \geq 2$. For each pair of indices $i, j \in \mathcal{I}(z)$, we must have

$$v_i \cdot z + h_i = v_j \cdot z + h_j. \tag{4-5}$$

Some such pair exists with $v_i \neq v_j$, for φ is not affine in any neighborhood of z since $z \notin A$. Then z lies in the codimension–1 hyperplane H_{ij}. This proves (a).

(b) Let $z \in A^c$ and assume $z \in F$. Then $\mathcal{I}(z) = \{i, j\}$ with $v_i \neq v_j$, so $z \in H_{ij}$, and the distance from z to $\bar{A}_k$ is positive for any $k \notin \mathcal{I}(z)$. Since A has only finitely many components by (ii), there are only finitely many such k. Then for any small enough open ball B containing z, $B \subset \bar{A}_i \cup \bar{A}_j$, while both $B \cap \partial A_i$ and $B \cap \partial A_j$ lie in H_{ij}. Hence $B \subset A_i \cup A_j \cup H_{ij}$.

Conversely, suppose $z \in A^c$ and $z \in H_{ij} \cap B$ for some hyperplane H_{ij} in the finite collection $\mathcal{H}$ and some open ball $B \subset A_i \cup A_j \cup H_{ij}$. Then for each $k \in \mathcal{I}(z)$, $B \cap A_k$ is a nonempty open set. Whenever $k \notin \{i, j\}$, however, since $A_k \cap A_i$ and $A_k \cap A_j$ are empty, necessarily $B \cap A_k = B \cap H_{ij}$. This set must be empty since it is open and H_{ij} has codimension 1. It follows $\mathcal{I}(z) = \{i, j\}$ since $\#\mathcal{I}(z) \geq 2$. Hence $z \in F$.

(c) If $z \in E$, then $z \in A^c$ but $z \notin F$. It follows from part (a) that z must lie in some hyperplane H_{ij} of $\mathcal{H}$, and from part (b) that $B \setminus H_{ij}$ intersects A^c for every sufficiently small ball B. Then since $\mathcal{H}$ is finite, necessarily z must lie in the intersection of two different (i.e., noncoinciding) hyperplanes of $\mathcal{H}$. Such intersections form a finite collection of hyperplanes of codimension 2. □

4.2. ***Convexity for finitely many pieces.*** If the transport map $X_t(z) = z + t\nabla\varphi(z)$ is injective on A for small $t > 0$, Proposition 4.2 allows us to prove the following local monotonicity property.

Lemma 4.3. *Assume X_t is injective on A for all sufficiently small $t > 0$. Suppose $\bar{A}_i \cap \bar{A}_j$ contains a point $z \in F$. Then in any sufficiently small open ball containing z,*

$$(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) > 0 \quad \textit{for all } x \in A_i \textit{ and } y \in A_j.$$

Proof. Necessarily $\mathcal{I}(z) = \{i, j\}$ and $z \in H_{ij}$. Let B be an open ball as given by Proposition 4.2(b). Let u be a unit vector orthogonal to the hyperplane H_{ij} pointing from A_j toward A_i. By the definition of $\mathcal{H}$, we have that $v_i \neq v_j$ and $v_i - v_j = au$ for some nonzero $a \in \mathbb{R}$. For all small enough $b > 0$, $z_i := z + bu \in A_i$ and $z_j := z - bu \in A_j$. The injectivity hypothesis on X_t implies

$$0 \neq X_t(z_i) - X_t(z_j) = z_i - z_j + t(v_i - v_j) = (2b + ta)u$$

for all sufficiently small positive b and t. This necessitates $a > 0$, and implies $(v_i - v_j) \cdot (z_i - z_j) = 2ab > 0$. This entails the result, since both $u \cdot (z_i - z_j)$ and $u \cdot (x - y)$ are positive for $x, y \in B$ with $x \in A_i$, $y \in A_j$. □

Now we are able to complete the proof of Theorem 1.4 under condition (ii).

Proof of Theorem 1.4(ii). 1. Assume X_t is injective on A for all sufficiently small $t > 0$, but φ is not convex. Then there must exist distinct $x, y \in \Omega$ and $\hat{\tau} \in (0, 1)$ such that

$$\varphi(x\hat{\tau} + y(1 - \hat{\tau})) > \varphi(x)\hat{\tau} + \varphi(y)(1 - \hat{\tau}). \tag{4-6}$$

We may take $x, y \in A$, since φ is continuous and A is dense. Let $u = x - y$ and let $u^\perp$ be the hyperplane of codimension 1 through the origin and orthogonal to u. The orthogonal projection P_u of $\mathbb{R}^d$ onto $u^\perp$ maps the line segment $\overline{xy}$ to a point, where

$$\overline{xy} = \{x\tau + y(1 - \tau) : \tau \in [0, 1]\}.$$

The same projection maps the set E of Proposition 4.2 into a finite union of hyperplanes of relative codimension 1 in $u^\perp$. The same is true for any codimension–1 hyperplanes H_{ij} in $\mathcal{H}$ that happen to have u in their tangent space. There exist arbitrarily small $v \in u^\perp$ such that $P_u x + v = P_u(x + v)$ does not lie on any of these projected hyperplanes. Since $P_u(x + v) = P_u(y + v)$, we may then replace x, y

by $x+v$, $y+v$ and ensure that the line $\overline{xy}$ is disjoint from E and transverse to every hyperplane $H_{ij} \in \mathcal{H}$ that it intersects, and (4-6) still holds. The line $\overline{xy}$ then intersects A^c only at points of F, and only at finitely many of those. As the line $\overline{xy}$ cannot be contained in a single component of A, at least one such intersection point exists.

2. The function $\hat{\varphi}(\tau) = \varphi(x\tau + y(1-\tau))$ defined for $\tau \in [0, 1]$ satisfies

$$\frac{d\hat{\varphi}}{d\tau} = \nabla\varphi(x\tau + y(1-\tau)) \cdot (x-y)$$

whenever $x\tau + y(1-\tau) \in A$. Then $d\hat{\varphi}/d\tau$ is locally constant on $(0, 1)$, with a jump at any value of τ where $z = x\tau + y(1-\tau) \in A^c$. Necessarily $z \in F$ by step 1, and by applying Lemma 4.3 we can conclude that $d\hat{\varphi}/d\tau$ makes a *positive* jump at such a value of τ. This implies $\hat{\varphi}$ is convex on $(0, 1)$, contradicting (4-6). Hence φ is convex in Ω. □

5. Continuously differentiable potentials

In order to prove Theorem 1.4 under condition (iii), it suffices to prove the following proposition. The proof is motivated by the idea that the transport maps X_t are related to characteristic curves for the Hamilton–Jacobi initial-value problem

$$\partial_t u + \tfrac{1}{2}|\nabla u|^2 = 0, \quad u(x, 0) = \varphi(x),$$

whose solution, under suitable conditions, is given by the Hopf–Lax formula

$$u(x, t) = \min_y \left(\frac{|x-y|^2}{2t} + \varphi(y) \right). \tag{5-1}$$

The proof will make use of Theorem 1.3 in order to ensure that a certain needed minimizer exists inside Ω.

Proposition 5.1. *Let Ω be a bounded open convex set in $\mathbb{R}^d$. Let $\varphi : \Omega \to \mathbb{R}$ be C^1 on Ω and locally affine a.e. Let A be the open set in* (1-2). *Suppose φ is not convex. Then X_t is noninjective on A for all sufficiently small $t > 0$.*

Proof. 1. Suppose φ is not convex. Then it is not convex in some nonempty subset

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

for some $\varepsilon > 0$ (fixed). The set Ω_ε is convex itself, as is easily shown. Let $L = \sup_{\overline{\Omega_\varepsilon}} |\nabla\varphi|$ and $M = \sup_{\overline{\Omega_{\varepsilon/4}}} |\varphi|$. Fix $t > 0$ so that $Lt < \varepsilon/2$ and $Mt < \varepsilon^2/64$.

2. By Theorem 1.3, φ is not semiconvex on Ω_ε; hence $\psi_t(z) = \frac{1}{2}|z|^2 + t\varphi(z)$ is not convex on Ω_ε, and cannot coincide with its convexification ψ_t^{**}. Since A is dense in Ω_ε, there exists $z_0 \in \Omega_\varepsilon \cap A$ such that $\psi_t(z_0) > \psi_t^{**}(z_0)$. Then $v_0 = \nabla\varphi(z)$ is constant for all z in some small neighborhood of z_0 contained in A. Let $x = z_0 + tv_0$. Then $|x - z_0| \le tL < \varepsilon/2$, so $x \in \Omega_{\varepsilon/2}$.

3. Taking the min over $y \in \overline{\Omega_{\varepsilon/4}}$ in the Hopf–Lax formula (5-1), we have $u(x, t) \le M$ by taking $y = x$. When $y \in \partial\Omega_{\varepsilon/4}$ we have $|x - y| > \varepsilon/4$, whence

$$\frac{|x-y|^2}{2t} + \varphi(y) \ge \frac{\varepsilon^2}{32t} - M > M.$$

Hence any minimizer y_1 in $\overline{\Omega_{\varepsilon/4}}$ lies in the open set $\Omega_{\varepsilon/4}$, and it follows that

$$x = y_1 + t\nabla\varphi(y_1) = z_0 + t\nabla\varphi(z_0), \quad \text{i.e.,} \quad X_t(y_1) = X_t(z_0).$$

Moreover, with $h = tu(x,t) - \frac{1}{2}|x|^2$, we have $h + x\cdot y \le \psi_t(y)$ for all $y \in \Omega_{\varepsilon/4}$, with equality at y_1. Since the affine function $h + x\cdot y \le \psi_t^{**}(y) \le \psi_t(y)$ for all y, we infer $\psi_t(y_1) = \psi_t^{**}(y_1)$. Hence $y_1 \ne z_0$.

4. Note $z_0 = x - tv_0 = X_t(y_1) - tv_0$. Because A is dense and X_t is continuous, we can find $\tilde{y}_1 \in A$ such that $\tilde{z}_0 := X_t(\tilde{y}_1) - tv_0 \in A$ with $\tilde{z}_0 \ne \tilde{y}_1$. Yet $v_0 = \nabla\varphi(\tilde{z}_0)$ and $\tilde{x} := X_t(\tilde{z}_0) = X_t(\tilde{y}_1)$. This contradicts the assumed injectivity of X_t on A. □

Remark 5.2. This proposition handles locally affine functions φ that resemble the Cantor expansion example in one dimension in that they have continuous gradient.

Remark 5.3. We suspect that if φ is C^1, locally affine and nonconvex then X_t is noninjective for *every* $t > 0$. But we leave this issue aside for the present.

6. Mass concentrations in convexified transport

The main goal of this section is to prove Theorem 1.5. As mentioned in the Introduction, the measure κ_t is related to the second Hopf formula for the solution to the following initial value problem with convex initial data:

$$\partial_t w + \varphi(\nabla w) = 0, \quad w(x,0) = \psi_0^*. \tag{6-1}$$

Here $f^*(x) = \sup_{z\in\mathbb{R}^d} x\cdot z - f(z)$ denotes the Legendre transform of f, and

$$\psi_0(y) = \tfrac{1}{2}|y|^2 + \mathbb{I}_{\overline{\Omega}}(y), \tag{6-2}$$

where $\mathbb{I}_S$ is the indicator function of the set S:

$$\mathbb{I}_S(z) = \begin{cases} 0 & \text{if } z \in S, \\ +\infty & \text{if } z \notin S. \end{cases}$$

Since $\psi_0^*(x) = \sup_{z\in\overline{\Omega}} x\cdot z - \frac{1}{2}|z|^2$ and this is Lipschitz, results of Bardi and Evans [1984] imply that (regarding φ as extended continuously to all of $\mathbb{R}^d$) the unique viscosity solution of (6-1) is given by the second Hopf formula, which states

$$w_t = \psi_t^*, \quad \text{where } \psi_t = \psi_0 + t\varphi. \tag{6-3}$$

We will make no direct use of this fact. Instead, we will focus attention on what is known as the *Monge–Ampère* measure for the convex function ψ_t^*. This is the Borel measure whose value on each Borel set B in $\mathbb{R}^d$ is given by

$$\kappa_t(B) = \lambda(\partial\psi_t^*(B)) = \lambda\Big(\bigcup_{x\in B} \partial\psi_t^*(x)\Big). \tag{6-4}$$

See [Figalli 2017, p. 7]. Results to be quoted below show that this agrees with the pushforward formula for κ_t stated in Theorem 1.5.

Remark 6.1. For fixed t, the fact that the function w_t has Monge–Ampère measure given by κ_t simply means that $u = w_t$ is the *Alexandrov solution* to the Monge–Ampère equation

$$\det D^2 u = \kappa_t.$$

6.1. *Convex mass transport.* First, we establish that absolute continuity of κ_t is a necessary consequence when φ is convex. Indeed, κ_t is given by locally rigid transport in this case.

Proposition 6.2. *Let Ω be a bounded open convex set in $\mathbb{R}^d$, and let $\varphi : \overline{\Omega} \to \mathbb{R}$ be continuous. Assume φ is locally affine a.e.*, *and let A be the open set defined in* (1-2). *Further assume φ is convex. Then for all $t > 0$, the Monge–Ampère measure in* (6-4) *is given by*

$$\kappa_t = \lambda \llcorner X_t(A),$$

Lebesgue measure on the set $X_t(A)$ whose each component is translated rigidly.

Proof. To begin we note that for each $t > 0$, $\psi_t : \mathbb{R}^d \to (-\infty, \infty]$ is convex, lower semicontinuous, and finite on $\overline{\Omega}$. Then $\psi_t = \psi_t^{**}$ by the Fenchel–Moreau theorem; see [Brezis 2011, §1.4]. Several further basic facts regarding the subgradients $\partial\psi_t$ in this context are the following (see [Liu et al. 2019, Appendix A] for simple proofs of (2) and (3)):

(1) The inverse $(\partial\psi_t)^{-1}$ is equal to $\partial\psi_t^*$, according to [Rockafellar 1970, Theorem 23.5].

(2) $x \in \partial\psi_t(y)$ if and only if $x = y + z$ with $z \in \partial(\mathbb{I}_{\overline{\Omega}} + t\varphi)(y)$.

(3) $\partial\psi_t$ has range $\mathbb{R}^d$.

Let B be a Borel set in $\mathbb{R}^d$ and let $x \in B$, $y \in (\partial\psi_t^*)(x)$. Then $x \in \partial\psi_t(y)$ by (1), whence necessarily $y \in \overline{\Omega}$, for otherwise $\partial\psi_t(y)$ is empty. As the set $\overline{\Omega} \setminus A$ has Lebesgue measure zero, by (1) it follows

$$\kappa_t(B) = \lambda(A \cap (\partial\psi_t)^{-1}(B)).$$

Let the components of A be denoted by A_i and let v_i be the value of $v = \nabla\varphi$ in A_i. We claim that, for each i,

$$A_i \cap (\partial\psi_t)^{-1}(B) = A_i \cap (B - tv_i).$$

Indeed, if $y \in A_i \cap (\partial\psi_t)^{-1}(B)$, there exists $x \in B$ with $x \in \partial\psi_t(y) = \{y + tv_i\}$ so $y \in B - tv_i$. And if $y \in A_i \cap (B - tv_i)$, then $x := y + tv_i = \nabla\psi_t(y) \in B$ so $y \in (\partial\psi_t)^{-1}(B)$.

Recalling that $X_t(y) = y + t\nabla\varphi(y)$ is injective on A by Lemma 1.2, by translation invariance of Lebesgue measure it follows that

$$\kappa_t(B) = \sum_i \lambda(A_i \cap (B - tv_i)) = \sum_i \lambda(X_t(A_i) \cap B) = \lambda(X_t(A) \cap B).$$

Hence $\kappa_t = \lambda \llcorner X_t(A)$. □

Remark 6.3. The situation in Proposition 6.2 provides a particularly simple special solution of a Monge–Ampère equation of the general form

$$\rho'(\nabla\psi(y)) \det D^2\psi(y) = \rho(y), \tag{6-5}$$

in which $\psi = \psi_t$, $\rho = \mathbb{1}_\Omega$ and $\rho' = \mathbb{1}_{X_t(A)}$. McCann [1997, §4] proved that for any convex ψ, if ρ is the density of an absolutely continuous probability measure also denoted by ρ in the interior of the domain of ψ, and if $\rho' = \nabla\psi_\sharp\rho$ is absolutely continuous with density also denoted by ρ', then the Monge–Ampère equation (6-5) holds a.e. in Ω, where the Hessian $D^2\psi(y)$ is interpreted in the Alexandrov sense.

6.2. ***Nonconvex mass transport.*** Our next goal is to associate nonconvexity of φ with formation of singular concentrations in κ_t, as follows. Recall we assume φ is continuous on $\bar{\Omega}$, and ψ_t is given by (6-3), taking the value $+\infty$ outside the convex set $\bar{\Omega}$. Then ψ_t is lower semicontinuous on $\mathbb{R}^d$ and its convexification is ψ_t^{**}, which is also finite only on $\bar{\Omega}$. The Legendre transform of ψ_t^{**} is $\psi_t^{***} = \psi_t^*$ by the Fenchel–Moreau theorem, and by [Rockafellar 1970, Theorem 23.5] cited in (1) above we have the inverse relation

$$(\partial\psi_t^{**})^{-1} = \partial\psi_t^*.$$

Hence the Monge–Ampère measure $\kappa_t = (\partial\psi_t^{**})_\sharp(\lambda \llcorner \Omega)$, for this simply means

$$\kappa_t(B) = \lambda((\partial\psi_t^{**})^{-1}(B)), \tag{6-6}$$

which is the same as (6-4). This is not different from the formula in Theorem 1.5, saying $\kappa_t = (\nabla\psi_t^{**})_\sharp\lambda$, because any set $(\partial\psi_t^{**})^{-1}(B)$ is contained in $\bar{\Omega}$ and can differ from $(\nabla\psi_t^{**})^{-1}(B)$ only at points where ψ_t^{**} is not differentiable, which form a Lebesgue null set. (A similar point is made in [McCann 1997, Lemma 4.1] in a more general context.)

Our main result in this section is the following theorem which completes the proof of Theorem 1.5 by establishing the sufficiency of the absolute continuity of κ_t for the convexity of φ. It shows that when φ is nonconvex, the mass evolution determined by the Monge–Ampère measure κ_t decomposes into a part given by rigid translation $z \mapsto \nabla\psi_t(z) = z + t\nabla\varphi(z)$ locally, and a nontrivial remainder that instantaneously concentrates on a null set. We comment on the relationship of this result with the adhesion model of cosmology at the end of this section.

Definition 6.4. For each $t > 0$ we define the "touching set"

$$\Theta_t = \{y \in \bar{\Omega} : \psi_t(y) = \psi_t^{**}(y)\}, \tag{6-7}$$

and for $t = 0$ we define $\Theta_0 = \bar{\Omega}$. We let Θ_t° denote the interior of Θ_t.

Theorem 6.5. *Let Ω be a bounded open convex set in $\mathbb{R}^d$, and let $\varphi : \bar{\Omega} \to \mathbb{R}$ be continuous. Assume φ is locally affine a.e., and let A be the open set defined in* (1-2). *Also assume φ is nonconvex. Let $t > 0$ and define the sets*

$$\mathcal{B}_t = \nabla\psi_t(A \cap \Theta_t^{\circ}), \quad \mathcal{S}_t = \nabla\psi_t^{**}(A \setminus \Theta_t^{\circ}).$$

Then the Monge–Ampère measure κ_t for ψ_t^* has the (*Lebesgue*) *decomposition*

$$\kappa_t = \mu_t + \nu_t, \quad \text{where } \mu_t = \lambda \llcorner \mathcal{B}_t, \ \ \nu_t = \kappa_t \llcorner \mathcal{S}_t.$$

In addition,

(i) *the sets $\mathcal{B}_t$ and $\mathcal{S}_t$ are disjoint*,

(ii) *the map $\nabla\psi_t : A \cap \Theta_t^{\circ} \to \mathcal{B}_t$ is bijective and locally rigid translation*,

(iii) $\lambda(\mathcal{S}_t) = 0$ *and* $\kappa_t(\mathcal{S}_t) > 0$.

To proceed toward the proof of the Theorem 6.5 we relate ψ_t^* to the function u_t given by the Hopf–Lax formula (1st Hopf formula)

$$u_t(x) = \min_{z\in\bar{\Omega}} \frac{|x-z|^2}{2t} + \varphi(z). \tag{6-8}$$

We relate the touching sets to minimizers in this formula as follows. First, note that by expanding the quadratic, we have

$$tu_t(x)+\psi_t^*(x)=\tfrac{1}{2}|x|^2 \quad \text{for all } x\in\mathbb{R}^d. \tag{6-9}$$

Lemma 6.6. *Let $t>0$ and $x\in\mathbb{R}^d$. Then in the Hopf–Lax formula* (6-8), *a point $y\in\bar\Omega$ is a minimizer if and only if $y\in\Theta_t\cap\partial\psi_t^*(x)$.*

Proof. Recall the Young inequality says

$$x\cdot z\le \psi_t^*(x)+\psi_t^{**}(z)=\tfrac{1}{2}|x|^2-tu_t(x)+\psi_t^{**}(z)$$

for all x and z, with equality when $z\in\partial\psi_t^*(x)$ or equivalently $x\in\partial\psi_t^{**}(z)$. Since $\psi_t^{**}\le\psi_t=\frac{1}{2}|\cdot|^2+t\varphi$, we find that, for all $z\in\bar\Omega$,

$$\begin{aligned}tu_t(x)&\le\tfrac{1}{2}|x-z|^2-\tfrac{1}{2}|z|^2+\psi_t^{**}(z)\\&\le\tfrac{1}{2}|x-z|^2+t\varphi(z).\end{aligned}$$

If $z=y$ is a minimizer in (6-8) then equality holds in both inequalities here; hence $\psi_t^{**}(y)=\psi_t(y)$ and $y\in\partial\psi_t^*(x)$. And the converse holds: If $z=y\in\Theta_t\cap\partial\psi_t^*(x)$ then equality holds in the Young inequality above, and $\psi_t^{**}(z)=\psi_t(z)=\frac{1}{2}|z|^2+t\varphi(z)$, and this implies that y is a minimizer in (6-8). □

Lemma 6.7. *Let $t>0$. If $y\in\Theta_t\cap\Omega$ and φ is differentiable at y then $\partial\psi_t^{**}(y)$ is a singleton set containing only $x=y+t\nabla\varphi(y)$.*

Proof. Let $y\in\Theta_t\cap\Omega$. Then $\psi_t(z)\ge\psi_t^{**}(z)$ for all z with equality for $z=y$, so given any $x\in\partial\psi_t^{**}(y)$, it follows that

$$\psi_t(z)-x\cdot z+\tfrac{1}{2}|x|^2\ge\psi_t(y)-x\cdot y+\tfrac{1}{2}|x|^2$$

for all z with equality for $z=y$. This means that $\frac{1}{2}|z-x|^2+t\varphi(z)$ is minimized at $z=y$. Since φ is differentiable at y, necessarily $x=y+t\nabla\varphi(y)$. □

The touching set Θ_t is a closed subset of $\bar\Omega$. Its (relative) complement is the nontouching set $\Theta_t^c=\bar\Omega\setminus\Theta_t$, which is (relatively) open. Then their common boundary $\partial\Theta_t=\partial\Theta_t^c$ is nowhere dense.

Proposition 6.8. *Let $t>0$, $y\in A$ and $x\in\partial\psi_t^{**}(y)$. Then there are three cases*:

(i) *If $y\in\Theta_t^c$ then $\partial\psi_t^*(x)$ is not a singleton.*

(ii) *$y\in\Theta_t^{\circ}$ if and only if $\partial\psi_t^{**}(y)$ is a singleton set containing $x=y+t\nabla\varphi(y)$ and $\partial\psi_t^*(x)$ is a singleton containing y.*

(iii) *If $y\in\partial\Theta_t$ then $\partial\psi_t^{**}(y)$ is a singleton set containing $x=y+t\nabla\varphi(y)$ and $\partial\psi_t^*(x)$ is not a singleton.*

Proof. 1. Suppose $y\in A\cap\Theta_t^c$ and $x\in\partial\psi_t^{**}(y)$. Let $y_*\in\bar\Omega$ be a minimizer in the Hopf–Lax formula (6-8). Then by Lemma 6.6, $y_*\in\Theta_t\cap\partial\psi_t^*(x)$. But since $y\in\partial\psi_t^*(x)$ also, $\partial\psi_t^*(x)$ is not a singleton. This proves (i).

2. For both parts (ii) and (iii), note that if $y\in A\cap\Theta_t$ then φ is differentiable at y, so by Lemma 6.7 we have $\partial\psi_t^{**}(y)=\{x\}$ with $x=y+t\nabla\varphi(y)$.

3. Suppose next that $y \in A \cap \Theta_t^{\mathrm{o}}$. Note that in some neighborhood of y, φ is affine and we have that

$$\psi_t^{**}(z) = \psi_t(z) = \tfrac{1}{2}|z|^2 + t\varphi(z), \tag{6-10}$$

which is strictly convex and quadratic. Thus hyperplanes with slope x that support the graph of ψ_t^{**} at y cannot touch it at any other point, so $\partial\psi_t^*(x)$ must be a singleton, and the singleton is $\{y\}$.

4. Now assuming that $y \in A_i$ (so $\nabla\varphi(y) = v_i$), that $\partial\psi_t^{**}(y) = \{x\}$ where $x = y + tv_i$, and that $\partial\psi_t^*(x) = \{y\}$, we wish to show $y \in \Theta_t^{\mathrm{o}}$.

By part (i), necessarily $y \in \Theta_t$, and by Lemma 6.6, $z = y$ is the unique minimizer in the Hopf–Lax formula (6-8). For any $p \in \mathbb{R}^d$ given, define

$$H_p(z) := \frac{|p-z|^2}{2t} + \varphi(z).$$

Then $z = y$ is the unique minimizer of H_x in $\overline{\Omega}$, and $H_x(y) = u_t(x)$. Choosing $\delta > 0$ so that $z \in A_i$ whenever $|z - y| < \delta$, since H_x is continuous on the compact set $\overline{\Omega}$ with unique minimizer at y, we necessarily have

$$\min\{H_x(z) : |z-y| \ge \delta,\ z \in \overline{\Omega}\} = u_t(x) + \gamma, \quad \text{where } \gamma > 0. \tag{6-11}$$

We claim that if $|p| > 0$ is sufficiently small then $H_{x+p}(z)$ is globally minimized at $y + p$. By Lemma 6.6 this means $y + p \in \Theta_t$, and $y \in \Theta_t^{\mathrm{o}}$ will follow. To prove the claim, note that, for all z,

$$H_{x+p}(z) = H_x(z) + \frac{p\cdot(x-z)}{t} + \frac{|p|^2}{2t}. \tag{6-12}$$

Note that φ takes the form $\varphi(z) = v_i \cdot z + h_i$ in the open set A_i. Thus we have

$$H_{x+p}(z) = \widetilde{H}(z) \quad \text{for all } z \in A_i,$$

where we define $\widetilde{H}$ to be the quadratic function given by

$$\widetilde{H}(z) := \frac{|x-z+p|^2}{2t} + v_i \cdot z + h_i \quad \text{for all } z \in \mathbb{R}^d.$$

The global minimum of $\widetilde{H}$ is at $z = x - v_i t + p = y + p$. Provided $|p| < \delta$, this point lies in A_i, so the minimum of $H_{x+p}(z)$ within A_i takes the value

$$\min_{z\in A_i} H_{x+p}(z) = \widetilde{H}(y+p) = H_{x+p}(y+p) = H_x(y) + v_i \cdot p.$$

Provided $|v_i||p| < \frac{1}{2}\gamma$ also, this value $H_x(y) + v_i \cdot p < u_t(x) + \frac{1}{2}\gamma$. On the other hand, since $x = y + tv_i$, from (6-12) and (6-11) we find that whenever $z \in \overline{\Omega} \setminus A_i$,

$$H_{x+p}(z) \ge u_t(x) + \gamma + \frac{p\cdot(y-z)}{t}.$$

Thus, if $|p| \operatorname{diam}\Omega < \frac{1}{2}\gamma t$ also, then

$$H_{x+p}(z) \ge u_t(x) + \tfrac{1}{2}\gamma \quad \text{for all } z \in \overline{\Omega} \setminus A_i.$$

This proves the claim, and finishes the proof of (ii).

5. Part (iii) follows from parts (i) and (ii) as the remaining case. □

Before beginning the proof of Theorem 6.5 we recall that a function f, convex on $\mathbb{R}^d$ and finite at x, is differentiable at x if and only if $\partial f(x)$ is a singleton [Rockafellar 1970, Theorem 25.1].

Proof of Theorem 6.5. 1. On each component A_i of A, recall $\nabla\psi_t$ is given by rigid translation, $\nabla\psi_t(y) = y + tv_i$. Moreover, on $A \cap \Theta_t^o$, $\psi_t^{**} = \psi_t$ is strictly convex so $\nabla\psi_t^{**} = \nabla\psi_t$ on this open set and is injective there. The set $\mathcal{B}_t = \nabla\psi_t(A \cap \Theta_t^o)$ is then a disjoint union of open sets

$$\mathcal{B}_t = \bigsqcup_i B_i, \quad \text{where } B_i = \nabla\psi_t(A_i \cap \Theta_t^o) = (A_i \cap \Theta_t^o) + tv_i.$$

For each $x \in B_i$, $x = y + tv_i = \nabla\psi_t^{**}(y)$, where $y \in A_i \cap \Theta_t^o$. So by part (ii) of Proposition 6.8, $\partial\psi_t^*(x)$ is the singleton $\{y\}$; hence ψ_t^* is differentiable at x with $\nabla\psi_t^*(x) = x - tv_i$. Given any Borel set $B \subset \mathcal{B}_t$, we have

$$\kappa_t(B) = \sum_i \lambda(\nabla\psi_t^*(B \cap B_i)) = \sum_i \lambda(B \cap B_i - tv_i) = \lambda(B).$$

Thus $\kappa_t \llcorner \mathcal{B}_t = \lambda \llcorner \mathcal{B}_t$.

2. For each point $x \in \widehat{\mathcal{S}}_t := \partial\psi_t^{**}(A \setminus \Theta_t^o)$ we have $x \in \partial\psi_t^{**}(y)$ for some $y \in A \setminus \Theta_t^o$. By parts (i) and (iii) of Proposition 6.8, $\partial\psi_t^*(x)$ is not a singleton. Thus ψ_t^* is not differentiable at any point of $\widehat{\mathcal{S}}_t$. As ψ_t^* is convex, and hence locally Lipschitz, we must have $\lambda(\widehat{\mathcal{S}}_t) = 0$ by Rademacher's theorem.

3. By step 1, $\partial\psi_t^*$ is single-valued on $\mathcal{B}_t$. Since $\partial\psi_t^*(x)$ cannot be both singleton and nonsingleton, $\mathcal{B}_t$ and $\widehat{\mathcal{S}}_t$ are disjoint. Moreover,

$$\partial\psi_t^*(\mathcal{B}_t) = A \cap \Theta_t^o \quad \text{and} \quad \partial\psi_t^*(\widehat{\mathcal{S}}_t) \supset A \setminus \Theta_t^o.$$

Since $\partial\psi_t^*(x) \subset \overline{\Omega}$ for any $x \in \mathbb{R}^d$ and A has full measure in $\overline{\Omega}$,

$$\begin{aligned}\kappa_t(\mathbb{R}^d) &\le \lambda(\overline{\Omega}) = \lambda(A \cap \Theta_t^o) + \lambda(A \setminus \Theta_t^o)\\ &\le \lambda(\partial\psi_t^*(\mathcal{B}_t)) + \lambda(\partial\psi_t^*(\widehat{\mathcal{S}}_t))\\ &= \kappa_t(\mathcal{B}_t) + \kappa_t(\widehat{\mathcal{S}}_t) \le \kappa_t(\mathbb{R}^d).\end{aligned}$$

Hence equality holds throughout, whence we get the Lebesgue decomposition

$$\kappa_t = \kappa_t \llcorner \mathcal{B}_t + \kappa_t \llcorner \widehat{\mathcal{S}}_t.$$

4. Let $B = \widehat{\mathcal{S}}_t \setminus \mathcal{S}_t$ with $\mathcal{S}_t = \nabla\psi_t^{**}(A \setminus \Theta_t^o)$ and let $\hat{A} = (\partial\psi_t^{**})^{-1}(B) = \partial\psi_t^*(B)$. By definition of $\mathcal{S}_t$, ψ_t^{**} is not differentiable at any point of $\hat{A} \cap (A \setminus \Theta_t^o)$. Further, by step 2, ψ_t^* is not differentiable at any point of B, so $\hat{A} \cap (A \cap \Theta_t^o)$ is empty due to Proposition 6.8(ii). Hence $\hat{A} \cap A = \hat{A} \cap (A \setminus \Theta_t^o)$. Then each point of $\hat{A}$ is either a point where ψ_t^{**} is not differentiable or is in A^c. Hence by (6-6),

$$\lambda(\hat{A}) = \kappa_t(\widehat{\mathcal{S}}_t \setminus \mathcal{S}_t) = 0,$$

whence $\kappa_t \llcorner \widehat{\mathcal{S}}_t = \kappa_t \llcorner \mathcal{S}_t$.

Moreover, $\kappa_t(\mathcal{S}_t) > 0$ since the nontouching set Θ_t^c is relatively open in $\overline{\Omega}$ and so the open set $A \cap \Theta_t^c \subset A \setminus \Theta_t^o$ is nonempty. □

Remark 6.9. Our results in this section can be compared to works in cosmology [Frisch et al. 2002; Brenier et al. 2003] which use the adhesion model for cosmological reconstruction. In these works the authors use optimal transportation to determine an initial velocity potential for matter flow in a large region of the universe, from presumed mass distributions at two epochs of a time-like variable. Without getting into details, the adhesion model takes the velocity potential essentially as the viscosity solution u of (1-6), the zero-viscosity limit of the potential Burgers equation, and the primordial mass density as uniform. The present distribution of cold dark matter is inferred from observations and exhibits concentrations such as mass sheets, filaments and nodes, and appears to be taken to correspond to the Monge–Ampère measure κ_t.

As discussed in [Brenier et al. 2003], optimal transport in principle can determine only the convexified potential (ψ_t^{**} here) whose gradient pushes the initial uniform distribution forward to κ_t, and the original velocity can be inferred only at points outside of mass concentrations at the present time.

In Theorem 1.5 above, this compares to points in $\mathcal{B}_t$, the set where the absolutely continuous part of κ_t is concentrated. Naturally, our assumption that the initial velocity potential is locally affine is not suitable for cosmology.

Remark 6.10. A more general related result exists that describes rigorously how the Lebesgue decomposition of Monge–Ampère-like measures is determined in terms of the Alexandrov Hessian of the transport potential. See Remark 7.4 in the lecture notes of Ambrosio et al. [2021]. One can alternatively prove Theorem 6.5 by using the result of that remark together with the results of Proposition 6.8 above, but we retain the arguments above for simplicity.

7. Stability and approximation of rigidly breaking flows

For the rigidly breaking potential flows provided by Theorem 1.1, the countable Alexandrov theorem, a natural question that arises is whether and in what sense the flow produced depends continuously on the mass-velocity data, particularly in the absence of a moment assumption. In this section we provide a stability theorem that addresses this issue.

Recall from Remark 2.2 that sets of mass-velocity data $\{(m_i, v_i)\}$ for which Theorem 1.1 applies are in bijective correspondence with pure point measures ν on $\mathbb{R}^d$ having $\nu(\mathbb{R}^d) = \lambda(\Omega)$. A natural notion of stability of the flows determined by such data involves weak-star convergence of measures in $\mathcal{M}(\mathbb{R}^d) = C_0(\mathbb{R}^d)^*$, the space of finite signed Radon measures on $\mathbb{R}^d$.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex open set with $\lambda(\Omega) = 1$. For each $n \in \mathbb{N} \cup \{\infty\}$ let ν_n be a pure point probability measure on $\mathbb{R}^d$. Let φ_n be the potential associated with ν_n in the proof of Theorem 1.1, and let A^n be the open set in Ω given by* (1-2) *with φ_n replacing φ. Let $X_t^n = \mathrm{id} + t\nabla\varphi_n$ be the corresponding flow map, and also let $\kappa_t^n = \lambda \llcorner X_t^n(A^n)$.*

If $\nu_n \xrightarrow{\star} \nu_\infty$ as $n \to \infty$ weak-$*$ in $\mathcal{M}(\mathbb{R}^d)$, then $\kappa_t^n \xrightarrow{\star} \kappa_t^\infty$ weak-$*$ in $\mathcal{M}(\mathbb{R}^d)$ for each $t > 0$.

The basis of the proof is the following result, which provides a stability theorem for the transport maps provided by the main theorem in [McCann 1995]. This result is unlikely to be new, but we were unable to locate a precise reference. It is closely related to well-known stability results for transport maps in optimal transport theory; see [Villani 2009, Corollary 5.23], for example. The result of that corollary does not apply here, however, because we make no assumptions regarding optimality or bounded moments for the measures ν_n.

Theorem 7.2. *Let μ be a probability measure on $\mathbb{R}^d$ absolutely continuous with respect to Lebesgue measure λ. For each $n \in \mathbb{N} \cup \{\infty\}$, let ν_n be a probability measure on $\mathbb{R}^d$, and let $\varphi_n : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a convex function as given by the main theorem in* [McCann 1995] *. If $\nu_n \xrightarrow{\star} \nu_\infty$ weak-$*$ as $n \to \infty$, then $\nabla\varphi_n$ converges to $\nabla\varphi_\infty$ in μ-measure on $\mathbb{R}^d$.*

Proof. The coupling defined by $\gamma_n = (\mathrm{id} \times \nabla\varphi_n)_\sharp\mu$ has marginals μ and ν_n. These couplings are probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, so by the Banach–Alaoglu theorem, any subsequence has a further subsequence that converges weak-$*$ to some measure $\gamma \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$. Since we assume that ν_n converges weak-$*$ to ν_∞, by Lemma 9(ii) of [McCann 1995] we infer that the limit measure γ is a probability measure coupling μ and ν_∞. Lemma 9(i) of [McCann 1995] implies the support of γ is cyclically monotone in the sense of McCann's Definition 3; hence, as McCann states, a theorem of Rockafellar implies that the support of γ is contained in the subdifferential of some convex function ψ on $\mathbb{R}^d$. Next, by Proposition 10 of [McCann 1995], the gradient of ψ pushes μ forward to ν_∞, i.e., $\nabla\psi_\sharp\mu = \nu_\infty$.

By the uniqueness part of the main theorem in [McCann 1995], it follows that $\nabla\psi = \nabla\varphi_\infty$ μ-a.e. in $\mathbb{R}^d$. Thus we can say the coupling $\gamma = (\mathrm{id} \times \nabla\varphi_\infty)_\sharp\mu$. Since this limit measure γ is unique, the full sequence γ_n converges to it.

The last step of the proof is to invoke Theorem 6.12 on stability of transport maps in [Ambrosio et al. 2021], which states that in this situation, the weak-$*$ convergence of γ_n to γ is equivalent to the convergence of $\nabla\varphi_n$ to $\nabla\varphi_\infty$ in the sense of μ-measure on $\mathbb{R}^d$. □

Proof of Theorem 7.1. Make the assumptions stated in the Theorem. For each $n \in \mathbb{N} \cup \{\infty\}$, the transport map $X_t^n = \mathrm{id} + t\nabla\varphi_n$ is well-defined on the set A^n. Let $\mu = \lambda \llcorner \Omega$, and recall from the proof of Theorem 1.1 that φ_n is a potential associated with ν_n by the main theorem in [McCann 1995]. For any $t > 0$ fixed, evidently it follows from Theorem 7.2 that X_t^n converges to X_t^∞ in μ-measure as $n \to \infty$.

Next, recall from Proposition 6.2 that the pushforward measure

$$(X_t^n)_\sharp\mu = \lambda \llcorner X_t^n(A^n) = \kappa_t^n.$$

In order to prove κ_t^n converges to κ_t^∞ weak-$*$ on $\mathbb{R}^d$, we should prove that for any continuous function f on $\mathbb{R}^d$ that vanishes at ∞,

$$\int_{\mathbb{R}^d} f(x)\, d\kappa_t^n(x) \to \int_{\mathbb{R}^d} f(x)\, d\kappa_t^\infty(x) \quad \text{as } n \to \infty. \tag{7-1}$$

Since the measures κ_t^n are uniformly bounded in the space $\mathcal{M}(\mathbb{R}^d)$, it suffices to prove this for functions f of compact support. But in this case we have

$$\int_{\mathbb{R}^d} f(x)\, d\kappa_t^n(x) = \int_\Omega f(X_t^n(z))\, d\lambda(z), \qquad n \in \mathbb{N} \cup \{\infty\}.$$

For any subsequence of these quantities, there is a further subsequence along which X_t^n converges to X_t^∞ a.e. in Ω. We conclude that (7-1) holds by using the dominated convergence theorem and the uniqueness of the limit. □

Remark 7.3. Consider a countably infinite set $\{(m_i, v_i)\}$ of mass-velocity data with $\sum_i m_i = \lambda(\Omega) = 1$ and arbitrary v_i. A natural way to approximate the pure point measure $\nu = \sum_i m_i\delta_{v_i}$ is by truncating to a finite sum of Dirac masses and normalizing, taking $\nu_n = \tilde{\nu}_n/\tilde{\nu}_n(\mathbb{R}^d)$, where $\tilde{\nu}_n = \sum_{i=1}^n m_i\delta_{v_i}$ are the partial sums. The Alexandrov theorem (Theorem 1.7) then can be used to provide the velocity potential φ_n, instead of McCann's theorem which is based on cyclic monotonicity for couplings. Theorem 7.1 then implies that the piecewise-rigidly breaking flows X_t^n converge to X_t in the sense that the restricted Lebesgue measures $\lambda \llcorner X_t^n(A^n)$ converge weak-$*$ to $\lambda \llcorner X_t(A)$.

Evidently this still relies on cyclic monotonicity and Rockafellar's theorem, however, through the proof of Theorem 7.1 above. It could be interesting to seek a stability proof that avoids this reliance and proceeds completely in the spirit of Minkowski and Alexandrov, perhaps using a standard stability theorem for Monge–Ampère measures like Proposition 2.6 in [Figalli 2017].

8. Incompressible optimal transport flows with convex source

In this section we complete our characterization of incompressible optimal transport flows with convex source as was mentioned in the Introduction. Our paper with Dejan Slepčev [Liu et al. 2019] mainly concerned transport distance along volume-preserving paths of set deformations. In terms of optimal transport, effectively this means studying paths $t \mapsto \rho_t = \lambda \llcorner \Omega_t$ comprising Lebesgue measure on a family of sets Ω_t having the same measure. One of the main results of [Liu et al. 2019] was that, given two bounded measurable sets Ω_0 and Ω_1 of equal measure, the infimum of the Benamou–Brenier action

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} |v|^2 \, d\rho_t \, dt,$$

subject to the transport equation $\partial_t \rho + \nabla \cdot (\rho v) = 0$, but further constrained by the requirement that the measures ρ_t have the form

$$\rho_t = \lambda \llcorner \Omega_t, \quad t \in [0, 1], \tag{8-1}$$

is the *same* as $d_W(\mu, \nu)^2$, the squared Monge–Kantorovich (Wasserstein) distance between the measures

$$\mu = \lambda \llcorner \Omega_0, \quad \nu = \lambda \llcorner \Omega_1. \tag{8-2}$$

The squared distance $d_W(\mu, \nu)^2$ is the infimum of $\mathcal{A}$ *without* the constraint (8-1), and the minimum is achieved for a unique minimizing path $(\mu_t)_{t \in [0,1]}$ known as the Wasserstein geodesic path.

Assume Ω_0 and Ω_1 are open, for the rest of this section. Let $(\mu_t)_{t \in [0,1]}$ be the Wasserstein geodesic path connecting the measures μ and ν in (8-2). Theorem 1.4 of [Liu et al. 2019] says that if the infimum of $\mathcal{A}$ is achieved as described above at some path $(\rho_t)_{t \in [0,1]}$ satisfying the constraint (8-1), then $\rho_t = \mu_t$. That is, any minimizing path satisfying the incompressibility constraint (8-1) must be the Wasserstein geodesic path.

We refer to such minimizers as *incompressible optimal transport paths*. Let (ρ_t) be such an incompressible optimal transport path. Let ψ be the convex Brenier potential whose gradient pushes $\mu = \rho_0$ to $\nu = \rho_1$: $\nabla \psi_\sharp \mu = \nu$. Then $\rho_t = (\nabla \psi_t)_\sharp \rho_0$ for each $t \in (0, 1)$, where

$$\psi_t(z) = \tfrac{1}{2}|z|^2 + t\varphi(z) \quad \text{with} \quad \varphi(z) = \psi(z) - \tfrac{1}{2}|z|^2. \tag{8-3}$$

At points of differentiability of ψ, the transport flow is given by

$$X_t(z) = \nabla \psi_t(z) = z + t v(z) \quad \text{with} \quad v = \nabla \varphi.$$

This velocity potential φ is semiconvex, by (8-3).

Because Ω_0, Ω_1 are bounded open sets and the characteristic functions on Ω_0 and Ω_1 are smooth, according to the regularity theory of [Caffarelli 1991; Figalli 2010; Figalli and Kim 2010], $\nabla \psi$ is a smooth diffeomorphism $\nabla \psi : A_0 \to A_1$, where $A_0 \subset \Omega_0$ and $A_1 \subset \Omega_1$ are open sets of full measure.

In this situation, we call the flow given by X_t an *incompressible optimal transport flow* taking Ω_0 to Ω_1. Corollary 5.8 of [Liu et al. 2019] states that necessarily the velocity v of such a flow is constant on each component of the open set A_0 of full measure in Ω_0. Therefore φ is locally affine a.e. and semiconvex.

Then the range of $v = \nabla\varphi$ is a countable set $\{v_i\}$ of distinct vectors in $\mathbb{R}^d$, $v = v_i$ on an open subset A_i with positive measure $m_i = \lambda(A_i) > 0$, and $\sum_i m_i = \lambda(\Omega_0)$. Recall that we refer to the set $\{(m_i, v_i)\}$ as the *mass-velocity data* of the incompressible optimal transport flow.

Definition 8.1. Let $MV(\Omega_0)$ denote the collection of countable sets of pairs (m_i, v_i) such that the v_i are uniformly bounded and distinct in $\mathbb{R}^d$ ($v_i = v_j$ implies $i = j$), the m_i are positive, and $\sum_i m_i = \lambda(\Omega_0)$.

As we have just seen, each incompressible optimal transport flow determines some set of mass-velocity data in $MV(\Omega_0)$. The result we are aiming at asserts that this association is *bijective* if the source domain is convex.

Theorem 8.2. *Let Ω_0 be a convex bounded open set in $\mathbb{R}^d$. Given any incompressible optimal transport flow taking Ω_0 to some other bounded open set, let $\{(m_i, v_i)\} \in MV(\Omega_0)$ be the mass-velocity data of the flow as described above. Then this map from flows to data is bijective.*

Proof. Let an incompressible optimal transport flow be given as above, taking Ω_0 to some bounded open set Ω_1 with the same measure. Such a flow, and its associated mass-velocity data $\{(m_i, v_i)\} \in MV(\Omega_0)$, is determined uniquely by the a.e.-locally affine and semiconvex velocity potential φ. Since Ω_0 is convex, the potential φ is necessarily convex by Theorem 1.3. Then Theorem 1.1 applies. Because of the invariance of φ under reordering of the data as discussed in Remark 2.2, the set of pairs $\{(m_i, v_i)\}$ determines φ (up to a constant), and hence the flow, uniquely.

Conversely, given any countable set $\{(m_i, v_i)\}$ in $MV(\Omega_0)$, Theorem 1.1 provides velocity potential φ that is convex and locally affine a.e. on $\Omega = \Omega_0$ The velocity field $v = \nabla\varphi$ defined a.e. is bounded, rigidly breaks Ω_0, and the ensuing flow is an incompressible optimal transport flow. □

9. Shapes of shards

In Section 4, we have seen that when the number of pieces A_i is finite, the pieces are bounded by hyperplanes, like polytopes. And in general, with infinitely many pieces possible, the pieces are convex. It is interesting to investigate what shapes the pieces may have. In this section we will discuss constructions that show a given piece may take an arbitrary convex shape, for example, or that all pieces can be round balls.

9.1. ***Power diagrams.*** Recall that in the case of finitely many pieces, the A_i are determined by the condition (6). This means that, with $\varphi(x) = v_i \cdot x + h_i$ in A_i as in (1-3),

$$A_i = \{x \in \Omega : v_i \cdot x + h_i > v_j \cdot x + h_j \text{ for all } j \neq i\}. \tag{9-1}$$

Through completing the square, this provides the equivalent description

$$A_i = \{x \in \Omega : |x - v_i|^2 - w_i < |x - v_j|^2 - w_j \text{ for all } i \neq j\}, \tag{9-2}$$

where $w_i = 2h_i + |v_i|^2$. This realizes the decomposition of Ω into the pieces A_i as a *power diagram* determined by the points v_i and weights w_i. Power diagrams are a generalization of Voronoi tessellations (for which the $w_i = 0$) and which have many uses in computational geometry and other subjects; see [Aurenhammer 1987; Aurenhammer et al. 2013].

In the general case here, when φ is convex and locally affine a.e. with countably many pieces possible, the pieces A_i satisfy

$$A_i = \operatorname{int}\bigl\{x \in \Omega : v_i \cdot x + h_i \geq \sup_{j \neq i} v_j \cdot x + h_j\bigr\}, \tag{9-3}$$

(int denotes the interior) or with $w_i = 2h_i + |v_i|^2$ as before,

$$A_i = \operatorname{int}\bigl\{x \in \Omega : |x - v_i|^2 - w_i \leq \inf_{j \neq i} |x - v_j|^2 - w_j\bigr\}. \tag{9-4}$$

Thus the decomposition of Ω into the A_i can be considered as a countable power diagram determined by the countably many points v_i and weights w_i.

9.2. ***Full packings by balls.*** The power-diagram description motivates the possibility that with countably many pieces, the pieces can assume some convex shape different from a polytope, such as a ball. We will describe three ways that optimal breaking can produce pieces that are *all* ball-shaped.

Take $\Omega \subset \mathbb{R}^d$ as any bounded open convex set. By a *full packing* of Ω by balls we mean a countable collection of disjoint open balls $B_i = \{x : |x - x_i| < r_i\}$ in Ω with centers x_i and radii r_i, such that the union $B = \bigsqcup_i B_i$ is an open set of full measure in Ω.

Lemma 9.1. *Given any full packing $\{B_i\}$ of Ω by balls, there exists a function φ convex and locally affine a.e., with pieces $A_i = B_i$, such that $\nabla\varphi$ maps B_i to the center of B_i.*

Proof. Since $\bigcup_{j \neq i} B_j$ is dense in $\Omega \setminus B_i$, we can say

$$B_i = \{x \in \Omega : |x - x_i|^2 - r_i^2 < \inf_{j \neq i} |x - x_j|^2 - r_j^2\}. \tag{9-5}$$

Comparing this with (9-4), we see that the B_i constitute a power diagram determined by the ball centers x_i and squared radii $w_i = r_i^2$. We infer that the convex function defined by

$$\varphi(x) = \sup_i v_i \cdot x + h_i \quad \text{with } v_i = x_i, \quad h_i = \tfrac{1}{2}(r_i^2 - |x_i|^2), \tag{9-6}$$

is locally affine a.e., with pieces $A_i = B_i$ and $\nabla\varphi = x_i$ in A_i. □

Any velocity potential φ produced by this lemma cannot be C^1, for each point in the set of ball centers $\{x_i\}$ is isolated, so $\nabla\varphi(\Omega)$ cannot be connected.

Full packings by balls can be produced in a variety of ways. Three that are interesting to discuss are:

(i) *Using Vitali's covering theorem.* The collection of all open balls in Ω constitutes a *Vitali covering* of Ω, so a full packing of Ω by balls exists by the Vitali covering theorem [Dunford and Schwartz 1958, Theorem III.12.3]. Actually, one can specify a finite number of the balls at will: Take $B_1, \dots, B_k$ to be given disjoint balls in Ω. Then apply the Vitali covering theorem to the collection of open balls in $\Omega \setminus \bigcup_{i=1}^k \bar{B}_i$.

(ii) *Osculatory packings.* A sequence $\{B_j\}$ of disjoint balls in Ω is called *osculatory* if B_i is a ball of largest possible radius in $\Omega \setminus \bigcup_{j=1}^{i-1} B_j$ whenever i is greater than some k. Boyd [1970] elegantly proved that an osculatory sequence in any open set $\Omega \subset \mathbb{R}^d$ of finite measure is a full packing. Earlier, Melzak [1966] had proved this for the case of dimension $d = 2$ and when Ω itself is a disk.

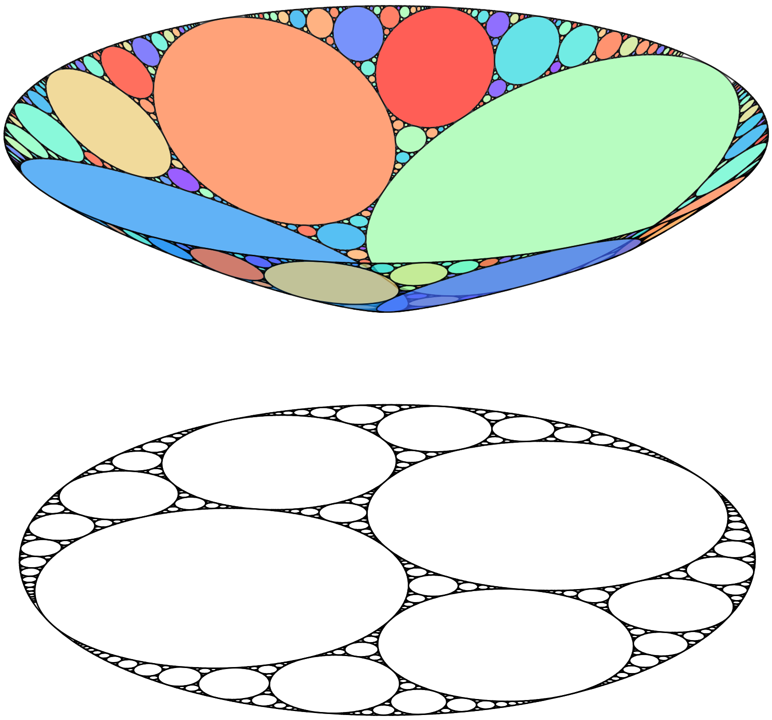

Figure 3. Apollonian bowl: graph of velocity potential locally affine a.e.

(iii) *Apollonian packings of disks.* A classic and beautiful tree construction that produces an osculatory packing in case Ω is the unit disk in $\mathbb{R}^2$ is associated with the name of Apollonius of Perga, who in antiquity classified all configurations of circles tangent to three given ones.

Start with two circles bounding disjoint disks B_1, B_2 in Ω, tangent to each other and tangent to the unit circle. These circles determine two curvilinear triangles. At stage 1, inscribe a circle in each of the curvilinear triangles. These circles bound new disks B_3, B_4 and divide each curvilinear triangle into three smaller ones. At each subsequent stage we continue by inscribing a circle in each of the curvilinear triangles created at the previous stage, adding the disks they bound to the collection, and subdividing the curvilinear "parent" triangle into three "children." From the two triangles and disks we start with at stage 1, upon completing stage k we have $2 \cdot 3^k$ disks at stage k.

Rearranged in order of decreasing radii, the sequence of disks produced in this way is osculatory. A proof that this Apollonian sequence produces a full packing of Ω was provided in [Kasner and Supnick 1943]. The closed set $\overline{\Omega} \setminus \bigcup_i B_i$, determined by removing the open disks in an Apollonian packing from the unit disk, is known as an *Apollonian gasket*. It has measure zero and is nowhere dense.

Apollonian packings can be generated algorithmically using the generalized Descartes circle theorem due to Lagarias et al. [2002]. If parent circles C_1, C_2, C_3 (possibly including the unit circle) are mutually tangent and tangent to children C_4 and C_5, and C_i has complex center z_i and curvature $b_i = 1/r_i$ (with $b_i = -1$ for the outer unit circle), this theorem implies

$$b_4 + b_5 = 2(b_1 + b_2 + b_3),$$
$$b_4 z_4 + b_5 z_5 = 2(b_1 z_1 + b_2 z_2 + b_3 z_3).$$

From the data b_i and z_i for three parent circles and one child, these equations determine the entire packing. Famously, all curvatures b_i are integers if the initial four are. Possibly, this property was first noticed by the chemist Soddy [1937]. In Figure 3 we plot the graph of the convex and a.e.-locally affine velocity potential generated by Lemma 9.1 in this case. Recall that Figure 1 illustrates the rigidly separated disks $X_t(B_i)$ at time $t = 0.5$ as shaded in blue.

9.3. ***Shards with arbitrary convex shape.*** As promised, we will show here that it is possible for some piece to assume an arbitrary convex shape. Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set, and let U be any convex open subset of Ω. Without loss of generality, for convenience we translate and scale coordinates so that $0 \in U$ and Ω is contained in the unit ball $\{x : |x| < 1\}$.

To begin, we construct a sequence of approximations to the distance function

$$\Phi(x) := \operatorname{dist}(x, U) = \inf\{|x - y| : y \in U\}.$$

The function Φ is convex, and of course, $\overline{U} = \{x \in \overline{\Omega} : \Phi(x) = 0\}$. Let $\{\sigma_i\}_{i\in\mathbb{N}}$ be a sequence of unit vectors in $\mathbb{R}^d$ dense in the sphere $\mathbb{S}^{d-1}$, and for each i choose $x_i \in \partial U$ to maximize $\sigma_i \cdot x$ on $\overline{U}$. Then it is simple to show that

$$\Phi(x) = 0 \vee \sup_{i\in\mathbb{N}} \sigma_i \cdot (x - x_i). \tag{9-7}$$

For each $n \in \mathbb{N}$, put

$$\Phi_n(x) = 0 \vee \max_{1\le i\le n} \sigma_i \cdot (x - x_i).$$

Then $\Phi_n(x)$ increases as $n \to \infty$ to the limit $\Phi(x)$ for all x, with

$$0 \le \Phi_n(x) \le \Phi(x) \le 1. \tag{9-8}$$

Moreover, Φ_n is convex and piecewise affine, and since $|\nabla\Phi_n| \le 1$ a.e. we have $|\Phi_n(x)-\Phi_n(y)| \le |x-y|$ for all $x, y \in \Omega$. Invoking the Arzela–Ascoli theorem we can conclude that Φ_n converges uniformly to Φ. Thus, for any $k \in \mathbb{N}$ there exists N_k such that for all $n \ge N_k$,

$$\sup_{x\in\Omega} |\Phi_n(x) - \Phi(x)| < \frac{1}{k}. \tag{9-9}$$

With these preliminaries, we can construct a convex function, locally affine a.e., having U as one of its pieces, as follows.

Proposition 9.2. *Let $\{a_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers satisfying $a_{k+1} \le a_k/k$ for all k. Let*

$$\varphi(x) = \sup_k a_k \Phi_{N_k}(x), \quad x \in \Omega. \tag{9-10}$$

Then φ is nonnegative, convex, and locally affine a.e., with $\varphi(x) = 0$ if and only if $x \in \overline{U}$.

Proof. Let $\varphi_k(x) = \max_{1\le i\le k} a_i \Phi_{N_i}(x)$. Then φ_k is nonnegative and piecewise affine, and it vanishes on a polytope containing U. If $\operatorname{dist}(x, U) = \Phi(x) > 2/k$ and $x \in \Omega$, then by (9-9) and (9-8),

$$\varphi_k(x) \ge a_k \Phi_{N_k}(x) > \frac{a_k}{k} \ge a_{k+1} \ge a_{k+1} \Phi_{N_{k+1}}(x).$$

Hence $\varphi_{k+1}(x) = \varphi_k(x)$. By consequence, φ is piecewise affine outside any open neighborhood of $\overline{U}$. We can conclude it is locally affine a.e. in Ω and vanishes only on $\overline{U}$. ☐

10. Discussion

In this paper we have focused attention on flows that rigidly break a convex domain, flows of a type that permits a classification in terms of mass-velocity data for the pieces. In particular, we have investigated conditions under which rigidly breaking potential flows must arise from a *convex* potential. As mentioned in Remark 1.6, it may be reasonable to conjecture that the conditions (i)–(iii) in Theorem 1.4 which ensure the potential's convexity may be weakened or discarded. We have also investigated and illustrated several differences between flows that break a domain into finitely many vs. infinitely many pieces.

We conclude this paper with a discussion of a few points, concerning

(a) conditions that ensure the velocity field can be realized as the gradient of a continuous potential;

(b) in our one-dimensional example of Section 3.2, the fat Cantor sets expand *uniformly* in time;

(c) some necessary criteria for a rigidly breaking velocity field to be continuous in dimensions $d > 1$.

10.1. ***Sufficient conditions for continuity of the potential.*** In Theorem 1.4 we assume the velocity field is the gradient of a potential φ that is locally affine a.e. in the convex set Ω, and we assume a priori that φ is continuous. In this subsection we briefly investigate conditions on v that are sufficient to ensure these properties.

In order that some $\varphi \in L^1_{\text{loc}}(\Omega)$ should exist with $v = \nabla\varphi$ in the sense of distributions, it is simple to check that necessarily the distributional Jacobian matrix $(\partial_j v_k)$ should be symmetric. In physical terms, this means that the velocity field should generate *no shear*.

Some integrability condition on v appears needed as well. Note, however, that Theorem 1.1, our countable Alexandrov theorem, provides a rigidly breaking velocity field v that fails to be in $L^1(\Omega)$ if the mass-velocity data is such that $\sum_i m_i|v_i| = \infty$. However, since $v = \nabla\varphi$ with φ convex, necessarily v is *locally* bounded a.e. in Ω.

In order to ensure that a velocity field $v = \nabla\varphi$ with φ continuous, then, we should require v is curl-free and it is reasonable to require some local boundedness or integrability in Ω. We find the following conditions are indeed sufficient.

Proposition 10.1. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and convex. For some $p > d$, suppose that $v \in L^p_{\text{loc}}(\Omega, \mathbb{R}^d)$ and that its (matrix-valued) distributional derivative is symmetric. Then $v = \nabla\varphi$ a.e. in Ω, for some locally Hölder continuous function $\varphi : \Omega \to \mathbb{R}$.*

Proof. By a standard cutoff and mollification argument we find a sequence of smooth velocity fields v^k converging to v in $L^p_{\text{loc}}(\Omega)$. Fix $z_0 \in \Omega$. Inside any convex subdomain $\Omega' \subset \Omega$ with compact closure in Ω and containing z_0, we can ensure that for k sufficiently large, the v^k are curl-free, having symmetric Jacobian matrices ∇v^k inside Ω'. By path integration along line segments from z_0, we can define smooth φ^k on Ω such that $\varphi^k(z_0) = 0$ and on Ω' we have $\nabla\varphi^k = v^k$. Then the sequence $(\nabla\varphi^k)$ is bounded in $L^p(\Omega')$ and by Morrey's inequality, (φ^k) is bounded in C^α norm on Ω' for $\alpha = 1 - d/p$. Then it follows that φ^k converges locally uniformly in Ω to a Hölder continuous limit φ. □

Finally, we comment on what might happen with rigidly breaking flows if shear is allowed. Without the potential flow assumption, it is easy to imagine a great variety of rigidly breaking flows that appear difficult to classify. E.g., as a simple example consider Ω to be the unit ball in $\mathbb{R}^2$, let f be any function whose graph $x_2 = f(x_1)$ disconnects Ω in two pieces, and let v be the velocity field that sends the upper piece moving rigidly upward and the lower piece downward at speed 1. If the graph is not a horizontal line, however, then the distributional curl of v will be concentrated on the graph and nonzero.

10.2. ***Uniform expansion of the Cantor set.*** Here we provide a proof of our comment in Section 3.2 regarding the uniform expansion of the Cantor set under the transported velocity field plotted in Figure 2. This figure plots the Cantor-function velocity $v = c(z)$ *vs.* the transported location $x = z + tc(z) = X_t(z)$, which is a continuous and strictly increasing function of z for $t > 0$. Define this velocity as a function of $x \in \mathbb{R}$ and $t \geq 0$ by

$$f(x,t) = c(z), \qquad \text{where } x = z + tc(z). \tag{10-1}$$

Here

$$c(z) = \begin{cases} 0 & \text{for } z \leq 0, \\ 1 & \text{for } z \geq 1. \end{cases}$$

This is the Lax implicit formula for a solution of the inviscid Burgers equation $\partial_t f + \partial_x\left(\frac{1}{2}f^2\right) = 0$. The function $f(\,\cdot\,,t)$ is increasing. As discussed in Section 3.2, $f(\,\cdot\,,t)$ is constant on each component interval of the complement of the "expanded" set $\mathcal{C}_t = \{z + tc(z) : z \in \mathcal{C}\}$, which is a fat Cantor set of Lebesgue measure $\lambda(\mathcal{C}_t) = \lambda(\mathcal{C}_t) = t$. Indeed, this set expands Lebesgue measure *uniformly*, as we now show.

Proposition 10.2. *For $t > 0$, the function in* (10-1) *is given by*

$$f(x,t) = \frac{1}{t}\int_0^x \mathbb{1}_{\mathcal{C}_t}(s)\,d\lambda(s).$$

Thus $\partial f/\partial x = 0$ on $\mathcal{C}_t^c$, and $\partial f/\partial x = 1/t$ at each Lebesgue point of $\mathcal{C}_t$.

Proof. Fix $t > 0$. The function $x \mapsto f(x,t)$ satisfies a one-sided Lipschitz bound (Oleinik inequality), with a simple proof: Say $\hat{x} = X_t(\hat{z}) > x = X_t(z)$. Then

$$\hat{x} - x = \hat{z} - z + t(c(\hat{z}) - c(z)) \geq \hat{z} - z.$$

Hence

$$0 \leq \frac{f(\hat{x},t) - f(x,t)}{\hat{x} - x} = \frac{c(\hat{z}) - c(z)}{\hat{x} - x} = \frac{1}{t}\left(1 - \frac{\hat{z} - z}{\hat{x} - x}\right) \leq \frac{1}{t}.$$

Since f is increasing in x, it is Lipschitz, and hence differentiable a.e., whence $0 \leq \partial f/\partial x \leq 1/t$. We infer from Lebesgue's version of the fundamental theorem of calculus that

$$1 = c(1) = f(1+t,t) = \int_{\mathcal{C}_t} \frac{\partial f}{\partial x}(s,t)\,d\lambda(s) \leq \frac{1}{t}\lambda(\mathcal{C}_t) = 1.$$

Then indeed $\partial f/\partial x(\,\cdot\,,t) = 1/t$ a.e. in $\mathcal{C}_t$, and

$$f(x,t) = \frac{1}{t}\int_0^x \mathbb{1}_{\mathcal{C}_t}(s)\,d\lambda(s).$$

Moreover this shows $\partial f/\partial x = 1/t$ at every Lebesgue point of $\mathcal{C}_t$. □

Remark 10.3. The function f is in fact the entropy solution to the inviscid Burgers equation with initial data $f(x,0) = c(x)$; see [Evans 2010, §3.4].

10.3. ***On continuous velocities in multidimensions.*** We lack any characterization like the one in Proposition 3.1 for describing rigidly breaking velocity fields that are continuous when $d > 1$. So here we confine ourselves to discuss some necessary constraints.

Suppose $v = \nabla\varphi$ is rigidly breaking and continuous, where φ is C^1, convex and locally affine a.e. on a bounded open convex set $\Omega \subset \mathbb{R}^d$. Let $\{v_i\}$ be the distinct values of v on the open set A in (1-2) where φ is locally affine. Since A is dense in Ω, necessarily the set $\{v_i\}$ is dense in the continuous image $v(\Omega)$, which must be connected, as in the case $d = 1$ treated in Proposition 3.1.

Recall that for all $t \geq 0$, the flow map X_t is a continuous injection from Ω onto $X_t(\Omega)$. Indeed, it is a homeomorphism, since the inequality proved in Lemma 1.2,

$$|X_t(z) - X_t(y)| \geq |z - y|,$$

implies the inverse is a contraction. Brouwer's domain invariance theorem (see [Kulpa 1998] or [Tao 2014, §1.6.2]) implies $X_t(\Omega)$ is open in $\mathbb{R}^d$. Topologically $X_t(\Omega)$ is the same as Ω, not disconnected in any way nor having "holes." Instead it is contractible to a point. Moreover we can deform Ω into $v(\Omega)$ through the homotopy defined by

$$S(x, \tau) = (1 - \tau)x + \tau v(x),$$

noting that S is continuous on $\Omega \times [0, 1]$. Thus the image $v(\Omega)$ is a limit of homeomorphic images $S_\tau(\Omega) = X_t(\Omega)/(1 + t)$, $\tau = t/(1 + t)$.

But we have been unable to determine whether $v(\Omega)$ must be homotopy equivalent to Ω, or whether this property, say, would suffice to ensure φ be C^1. The monotonicity of the velocity (as in (2-3)) should be relevant, since for example, the smooth but nonmonotone map $v(x_1, x_2) = (\cos 8x_1, \sin 8x_1)$ maps the square $\Omega = (0, 1)^2$ surjectively onto the unit circle.

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