Commun. Math. Phys. 323, 1017–1070 (2013)
Digital Object Identifier (DOI) 10.1007/s00220-013-1777-z

Communications in
**Mathematical
Physics**

Dynamic and Steady States for Multi-Dimensional Keller-Segel Model with Diffusion Exponent $m > 0$

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Received: 7 June 2012 / Accepted: 1 February 2013
Published online: 1 September 2013 –

Abstract: This paper investigates infinite-time spreading and finite-time blow-up for the Keller-Segel system. For $0 < m \le 2 - 2/d$, the L^p space for both dynamic and steady solutions are detected with $p := \frac{d(2-m)}{2}$. Firstly, the global existence of the weak solution is proved for small initial data in L^p. Moreover, when $m > 1 - 2/d$, the weak solution preserves mass and satisfies the hyper-contractive estimates in L^q for any $p < q < \infty$. Furthermore, for slow diffusion $1 < m \le 2 - 2/d$, this weak solution is also a weak entropy solution which blows up at finite time provided by the initial negative free energy. For $m > 2 - 2/d$, the hyper-contractive estimates are also obtained. Finally, we focus on the L^p norm of the steady solutions, it is shown that the energy critical exponent $m = 2d/(d+2)$ is the critical exponent separating finite L^p norm and infinite L^p norm for the steady state solutions.

1. Introduction

In this paper, we study the Keller-Segel model in spatial dimension $d \ge 3$:

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u \nabla c), \quad x \in \mathbb{R}^d, \ t \ge 0, \\ -\Delta c = u, \quad x \in \mathbb{R}^d, \ t \ge 0, \\ u(x,0) = U_0(x) \ge 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here the diffusion exponent m is taken to be supercritical $0 < m < 2 - 2/d$, critical $m_c := 2 - 2/d$, and subcritical $m > 2 - 2/d$. This model is developed to describe the biological phenomenon chemotaxis [30,37]. In the context of biological aggregation, $u(x,t)$ represents the bacteria density, $c(x,t)$ represents the chemical substance concentration and it is given by the fundamental solution

$$c(x,t) = c_d \int_{\mathbb{R}^d} \frac{u(y,t)}{|x-y|^{d-2}} dy, \tag{1.2}$$

where

$$c_d = \frac{1}{d(d-2)\alpha_d}, \qquad \alpha_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}, \tag{1.3}$$

α_d is the volume of d-dimensional unit ball. The case $m < 1$ is called fast diffusion and the case $m > 1$ is called slow diffusion [14]. The main characteristic of Eq. (1.1) is the competition between the diffusion and the nonlocal aggregation. This is well represented by the free energy for $m > 1$,

$$\begin{aligned} F(u) &= \frac{1}{m-1}\int_{\mathbb{R}^d} u^m(x)dx - \frac{1}{2}\int_{\mathbb{R}^d} uc(x)dx \\ &= \frac{1}{m-1}\int_{\mathbb{R}^d} u^m(x)dx - \frac{c_d}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{u(x)u(y)}{|x-y|^{d-2}}dxdy. \end{aligned} \tag{1.4}$$

For $m = 1$, the first term of the free energy is replaced by $\int_{\mathbb{R}^d} u \log u dx$ [37]. The competition between these two terms leads to finite-time blow-up and infinite time spreading [7–9,13,28,37,40,41]. For the fast diffusion case $0 < m < 1$, the free energy becomes

$$F(u) = -\left[\frac{1}{1-m}\int_{\mathbb{R}^d} u^m dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla c|^2 dx\right], \tag{1.5}$$

both terms are non-positive and we couldn't directly see the competition between the diffusion and the aggregation terms.

On the other hand, (1.1) can be recast as

$$u_t = \nabla\cdot(u\nabla\mu), \tag{1.6}$$

where μ is the chemical potential

$$\mu = \begin{cases} \frac{m}{m-1}u^{m-1} - c, & m \neq 1, \\ \log u - c, & m = 1. \end{cases} \tag{1.7}$$

Indeed μ is the first order variation of $F(u)$, thus multiplying (1.6) by μ and integration in space one has

$$\frac{dF}{dt} + \int_{\mathbb{R}^d} u\,|\nabla\mu|^2\,dx = 0,$$

or

$$\frac{dF}{dt} + \int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u^{m-\frac{1}{2}} - \sqrt{u}\nabla c\right|^2 dx = 0. \tag{1.8}$$

This implies that $F[u(\cdot,t)]$ is non-increasing with respect to t. For simplicity, we will use $\|u\|_r = \|u\|_{L^r(\mathbb{R}^d)}$ through this paper.

Notice that for $d \geq 3$, the PDE (1.1) possesses a scaling invariance which leaves the L^p norm invariant and produces a balance between the diffusion and the aggregation terms where

$$p := \frac{d(2-m)}{2} \in [1, d). \tag{1.9}$$

Indeed, if $u(x,t)$ is a solution, then $u_\lambda(x,t) = \lambda u\left(\lambda^{(2-m)/2}x, \lambda t\right)$ is also a solution and this scaling preserves the L^p norm $\|u_\lambda\|_p = \|u\|_p$. For the critical case $m_c := 2-2/d$, the above scaling becomes the mass invariant scaling $u_\lambda(x,t) = \lambda u\left(\lambda^{1/d}x, \lambda t\right)$. Using the mass invariant scaling, we have that for the supercritical case $m < m_c$, the aggregation dominates the diffusion for high density (large λ) and the density has finite-time blow-up [8,10,25,26,37,40]. While for low density (small λ), the diffusion dominates the aggregation and the density has infinite-time spreading [3,37,40,41]. On the contrary, for the subcritical case $m > m_c$, the aggregation dominates the diffusion for low density and prevents spreading, while for high density, the diffusion dominates the aggregation thus blow-up is precluded [31,40,41]. These behaviors also appear in many other physical systems such as thin film, Hele-Shaw, stellar collapse, etc., and they also bear some similarities to the nonlinear Schrödinger equation [10,15,45,47].

The system (1.1) has been widely studied recently [2,7–9,13,17,28,31,37,40–42] and the references therein, most of the prior estimates for the Keller-Segel model are based on the arguments of Jäger and Luckhaus [28] with fast decay at infinity that for any $q > 1$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^q\,dx = -\frac{4mq(q-1)}{(m+q-1)^2}\int_{\mathbb{R}^d}\left|\nabla u^{(m+q-1)/2}\right|^2 dx + (q-1)\int_{\mathbb{R}^d} u^{q+1}dx. \tag{1.10}$$

When $m > 2-2/d$, there exists a global weak solution without any restriction on the initial data [2,40,41]. For radially symmetric and compactly supported initial data, Kim and Yao [31] showed that the solution remains radially symmetric and compactly supported, and it converges to a compactly supported stationary solution exponentially with the same mass. For non-compactly supported radially initial data, it is still unclear whether all the initial mass are attracted to the steady profile (see Sect. 5.3 for numerical simulations). When $0 < m \le 2-2/d$, Sugiyama, etc. [40,41] considered more general case that the second equation of (1.1) is replaced by $-\Delta c + \gamma c = u,\ \gamma \ge 0$ and proved that for initial data $U_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, $U_0^m \in H^1(\mathbb{R}^d)$, $1 < q < \infty$, if the initial data satisfies $\|U_0\|_p < C$, where C is a positive number depending on q, d, m, then there exists a weak solution with decay property in $L^q(\mathbb{R}^d)$ and they employed Moser's iteration to prove the time global $L^\infty(\mathbb{R}^d)$ bound. When $m = 1$, $d \ge 3$, $p = d/2$, there exists a universal constant $K(d)$ such that if the initial data $\|U_0\|_{d/2} < K(d)$, then the system (1.1) admits a global weak solution with decay property [13,37]. For the critical case $m = 1$, $d = 2$ or $m = 2-2/d$, $d \ge 3$, $p = 1$, the PDE (1.1) is critical with respect to mass, hence there exists a critical mass $\|u\|_1 = \|U_s\|_1$ sharply separating the global existence and finite-time blow-up with some initial regularities [7–9,37] where U_s is the steady solution to Eq. (1.1).

Note that for the supercritical case, all the above results are related to the initial $\|U_0\|_p$ and for critical case it is related to $\|U_s\|_p$. For the general case $0 < m < 2-2/d$, it seems that $\|U_s\|_p$ plays a role as a critical constant separating global existence and finite time blow-up. Based on this, in this work, the L^p norm of both the dynamical solution $u(x,t)$ and the steady solution $U_s(x)$ are explored. For $0 < m \le 2-2/d$, if the initial data $\|U_0\|_p < C_{d,m} \le \|U_s\|_p$, where $C_{d,m}$ is a constant only depending on d, m (see formula (2.27)), then there exists a global weak solution and when $m > 1-2/d$, the hyper-contractive estimates hold that for any $t > 0$, this weak solution is in L^q for any $p < q < \infty$. For $m > 1-2/d$, the weak solution also preserves mass. While when $0 < m < 1-2/d$, this weak solution will vanish at finite time T_{ext} and for $(d-2)/(d+2) < m < 1-2/d$, there exists a $0 < \bar{T} \le T_{ext}$ such that the second moment $\limsup_{t\to\bar{T}}\int_{\mathbb{R}^d} u(x,t)|x|^2dx = \infty$ and $\int_0^{\bar{T}} F[u(\cdot,t)]dt = -\infty$. This difference can also be seen from the free energy (1.5) and (1.4) for $0 < m < 1$ and $m > 1$. Furthermore, for $m > 1$, if the initial second moment is bounded and $U_0 \in L^m(\mathbb{R}^d)$, then this weak solution is also a weak entropy solution satisfying the energy inequality, see Sect. 2. This result also provides a natural blow-up criteria for $1 < m \le 2 - 2/d$: the solution blows up if and only if $\|u\|_{L^r}$ blows up for all $r > p$. In fact, the weak solution blows up at finite time T provided by the negative free energy $F(U_0) < 0$ and this negative free energy implies $\|U_0\|_p > C_{d,m}$ which is consistent with a constant for global existence, see Sect. 3. Since the L^p norm is a scaling invariant, we guess that the $\|U_s\|_p$ is the sharp condition separating infinite-time spreading and finite-time blow-up. Here we only conduct some numerical experiments to verify it in Sect. 5 that for the supercritical case $2d/(d+2) < m < 2-2/d$, the solution spreads globally as $t\to\infty$ by assuming $\|U_0\|_p < ||U_s||_p$ and blows up at finite time with $\|U_0\|_p > ||U_s||_p$.

Let us point out that in the radial context, for $0 < m < 2d/(d+2)$, $\|U_s\|_p$ is infinite. Precisely, in Sect. 4, for $m \ne 1$, the nonnegative steady solutions of the system (1.1) in the sense of distribution satisfy

$$\begin{cases} \frac{m}{m-1}U_s^{m-1} - C_s = \bar{C}, \ \text{ in } \ \Omega, \\ U_s = 0 \ \text{ in } \mathbb{R}^d \setminus \Omega, \quad U_s > 0 \ \text{ in } \Omega, \\ -\Delta C_s = U_s, \ \text{ in } \ \mathbb{R}^d, \end{cases} \tag{1.11}$$

where C_s is given by the Newtonian potential for $m > 1$,

$$C_s(x) = c_d \int_{\mathbb{R}^d} \frac{U_s(y)}{|x-y|^{d-2}}dy, \tag{1.12}$$

where c_d is defined as (1.3). When $m = 1$, the steady equation becomes

$$\begin{cases} \log U_s - C_s = \bar{C}, \ \text{ in } \ \Omega, \\ U_s = 0 \ \text{ in } \mathbb{R}^d \setminus \Omega, \quad U_s > 0 \ \text{ in } \Omega, \\ -\Delta C_s = U_s, \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{1.13}$$

Here $\bar{C}$ is any constant chemical potential and $\Omega = \{x \in \mathbb{R}^d \mid U_s(x) > 0\}$ is a connected open set in $\mathbb{R}^d$. For $m \ge 2d/(d+2)$, every steady solution is a Nash equilibrium, see Proposition 4.3 and Remark 4.4 for more details. We remark that for $0 < m \le 1$, we can't define C_s as the Newtonian potential (1.12), see Theorem 4.8 for more details.

When $m \ne 1$, letting $\phi = \frac{m-1}{m}\left(C_s + \bar{C}\right)$, the steady equation (1.11) is reduced to

$$\begin{cases} -\Delta\phi = \frac{m-1}{m}\phi^k, \ \text{ in } \ \Omega, \quad k = \frac{1}{m-1}, \\ \phi = 0 \ \text{ on } \partial\Omega, \quad \phi > 0, \ \text{ in } \ \Omega. \end{cases} \tag{1.14}$$

When $m = 1$, letting $\phi = \log U_s$ in (1.13) follows

$$-\Delta\phi = e^{\phi} \quad \text{in } \mathbb{R}^d. \tag{1.15}$$

Equation (1.14) is the Lane-Emden equation [15,18,24]. For $\phi \in C^2(\mathbb{R}^d)$ and $m > 1$, it has been widely studied in recent years [12,15,16,18,20,22–24,29,32,35,36,39,49]. When $m = 1$ and $d = 3$, Eq. (1.15) is an isothermal equation and the solution decays to $-\infty$ at far field [15, p. 164], see also Theorem (4.8) (ii) section 4. $m = 2d/(d+2)$ is a critical exponent separating the compact supported solutions with finite mass from non-compact supported solutions with infinite mass. To be precise, for $m > 2d/(d+2)$, there is no positive C^2 solution to Eq. (1.14) in $\mathbb{R}^d$ [20,22]. For $m = 2d/(d+2)$, there is a family of positive radial solutions in $\mathbb{R}^d$ with finite mass [12,16]. For $1 < m < 2d/(d+2)$, there are no finite total mass radial solutions in $\mathbb{R}^d$ [15,38]. For $0 < m < 1$, we have the sharp decay rate at infinity for positive radial solutions U_s, see Lemma 4.7 and Theorem 4.8. By applying the results for Eq. (1.14) and (1.15) to Eqs. (1.11) and (1.13) the results for the steady equation to (1.1) can be summarized as follows:

1. For $0 < m \le 1$, the radial steady solution U_s has the decay rate at infinity $U_s \sim C(d,m)r^{-\frac{2}{2-m}}$ (see [29] for the case $m = 1$) and thus $\|U_s\|_q < \infty$ for $q > p$ and $\|U_s\|_q = \infty$ for $1 \le q \le p$.
2. For $1 < m < 2d/(d+2)$, then $\Omega = \mathbb{R}^d$ and if C_s has the decay rate $C_s(x) = O\left(|x|^{-\frac{2(m-1)}{2-m}}\right)$ as $|x| \to \infty$, then all the steady solutions are radially symmetric and unique up to translation in $\mathbb{R}^d$ [18,24,49]. Furthermore, $\|U_s\|_q < \infty$ for $q > p$ and $\|U_s\|_q = \infty$ for $1 \le q \le p$.
3. For $m = 2d/(d+2)$, then $\Omega = \mathbb{R}^d$ and every positive solution U_s uniquely assumes a radially symmetric form in $\mathbb{R}^d$ up to translation and $\|U_s\|_p$ is a universal constant only depending on d [12,15–17].
4. For $m > 2d/(d+2)$, all the nonnegative solutions U_s are compact supported in $\mathbb{R}^d$ and for any given mass $\|U_s\|_1 = M$ they are unique up to translation. Furthermore, all the solutions C_s, U_s are spherically symmetric up to translation and $\Omega = B(0, R)$ for some $R > 0$ up to translation [12,16,20,22,39]. Particularly, for $m = 2$, R is fixed to be $\sqrt{2}\pi$. Moreover, for $2d/(d+2) < m \le 2 - 2/d$, the L^p norm $\|U_s\|_p$ is also a constant depending only on d, m.
5. When $1 < m < 2d/(d+2)$, it is still open if all positive $C_s \in C^2(\mathbb{R}^d)$ solutions to Eq. (1.11) in $\mathbb{R}^d$ are radially symmetric up to translation [18].

From the above results, $m = 2d/(d+2)$ is a critical exponent under the L^p invariant for both dynamics and steady states. For this reason we refer to it as the energy critical exponent and denote $2d/(d+2)$ as m_{ec}. Indeed, plugging the invariant scaling $u_\lambda(x,t) = \lambda u\left(\lambda^{(2-m)/2}x\right)$ into the free energy (1.4) obtains

$$F(u_\lambda) = \lambda^{\frac{(d+2)m-2d}{2}} F(u),$$

the free energy is invariant when $m = 2d/(d+2)$. We also refer to [17] for the analysis of the energy critical case. For more precise statements, see Sect. 4.2, Theorem 4.8 and Remark 4.9. For simplicity, we denote $\int$ instead of $\int_{\mathbb{R}^d}$ below.

This paper is organized as follows. Section 2 detects the hyper-contractive property of the global weak solutions to Eq. (1.1) with initial data in L^p space for $m > 1 - 2/d$. Furthermore, for $m > 1$, if the initial second moment is bounded in time, this weak solution is also a weak entropy solution. Section 3 considers finite-time blow-up for the supercritical case and the critical case $1 < m \le 2 - 2/d$. It gives a blow-up criteria provided by the initial negative free energy which is consistent with the condition for global existence. Section 4 explores the steady solutions to Eq. (1.1) for the cases $0 < m \le 1$ and $m > 1$. Section 5 is devoted to the numerical study of the global existence and finite time blow-up for the supercritical case $2d/(d+2) < m < 2 - 2/d$. Only numerical experiments verify that the sharp condition $\|U_s\|_p$ separates infinite-time spreading from finite-time blow-up. Numerical experiments for the subcritical case $m > 2 - 2/d$

are also performed that given finite mass, the initial radial solution will converge to the steady compact-supported solution. Finally, Sect. 6 concludes the main work of this paper, some open questions for Eq. (1.1) and its steady equation are also addressed.

2. Existence of Global Weak Entropy Solutions

This section mainly focuses on the global existence of the weak solutions. Starting from the initial data $\|U_0\|_p < C_{d,m}$, where $C_{d,m}$, is a universal constant only depending on d, m, Theorem 2.11 shows the global existence of a weak solution for $0 < m < 2-2/d$, and then the hyper-contractive estimates deduce that the weak solution is bounded in any L^q space for any $t > 0$ when $1-2/d < m < 2-2/d$. This weak solution also satisfies the mass conservation for $m > 1-2/d$ and has finite time extinction for $0 < m < 1-2/d$. In addition, the second moment blows up at finite time before the extinction when $(d-2)/(d+2) < m < 1-2/d$. Using the uniform boundedness of the second moment, Theorem 2.11 also gives the global existence of a weak entropy solution for $1 < m < 2-2/d$. In Theorem 2.17 for the supercritical case and the critical case, if $U_0 \in L^q(\mathbb{R}^d)$ with $q \geq m$ and $q > p$, $p = \frac{d(2-m)}{2}$, then there exists a local in time weak entropy solution, the proof also provides a sharp blow-up criteria that for all $r > p$, $\|u(\cdot,t)\|_r \to \infty$ as t goes to the largest existence time. The global existence of a weak entropy solution for the subcritical case is analyzed in Theorem 2.18 where three initial conditions are presented for the hyper-contractive property when $2-2/d < m < 2$, $m = 2$ and $m > 2$.

Beginning the analysis with the Hardy-Littlewood-Sobolev(HLS) inequality [34],

$$w(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u(x)u(y)}{|x-y|^{d-2}}dxdy \leq C_{HLS}\|u\|^2_{2d/(d+2)}\,, \tag{2.1}$$

where $C_{HLS} = \frac{1}{S_d c_d}$ [34,14] with c_d is given by (1.3) and S_d is given by the Sobolev inequality [34, pp. 202] that for $d \geq 3$,

$$S_d\|u\|^2_{2d/(d-2)} \leq \|\nabla u\|_2^2, \quad S_d = \frac{d(d-2)}{4}2^{\frac{2}{d}}\pi^{1+\frac{1}{d}}\Gamma\left(\frac{d+1}{2}\right)^{-\frac{2}{d}}, \tag{2.2}$$

both terms in the free energy (1.4) make sense if $u \in L^1_+ \cap L^m \cap L^{2d/(d+2)}(\mathbb{R}^d)$. Combining (2.2) with the interpolation inequality gives that for $1 < b/a < \frac{2d}{a(d-2)}$, the interpolation inequality leads to

$$\|w\|_{b/a} \leq \|w\|_1^{1-\theta}\,\|w\|^{\theta}_{\frac{2d}{a(d-2)}} = \|w\|_1^{1-\theta}\left\|w^{1/a}\right\|^{\theta a}_{\frac{2d}{d-2}} \leq S_d^{-\frac{\theta a}{2}}\|w\|_1^{1-\theta}\|\nabla w^{1/a}\|_2^{\theta a}. \tag{2.3}$$

Setting $w = u^{\frac{a(m+q-1)}{2}}$ one has

$$\left\|u^{\frac{a(m+q-1)}{2}}\right\|_{b/a} \leq S_d^{-\frac{\theta a}{2}}\left\|u^{\frac{a(m+q-1)}{2}}\right\|_1^{1-\theta}\left\|\nabla u^{\frac{m+q-1}{2}}\right\|_2^{\theta a}. \tag{2.4}$$

Here $\theta = \left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{d-2}{2d}\right)^{-1}$. By choosing particular a, b in (2.4) and using the Young inequality one has the following three lemmas which will be used in Theorem 2.11.

Lemma 2.1. *Let* $d \geq 3$, $0 < m \leq 2 - 2/d$, $p = \frac{d(2-m)}{2}$, $q \geq p$ *and* $u \in L^1_+(\mathbb{R}^d)$. *Then*

$$\|u\|_{q+1}^{q+1} \leq S_d^{-1} \|\nabla u^{(m+q-1)/2}\|_2^2 \|u\|_p^{2-m}, \tag{2.5}$$

and for $q \geq r > p$,

$$\|u\|_{q+1}^{q+1} \leq S_d^{-\frac{\alpha}{2}} \left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^{\alpha} \|u\|_r^{\beta} \leq \frac{2mq}{(m+q-1)^2} \left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r,d) \left(\|u\|_r^r\right)^{\delta}, \tag{2.6}$$

where

$$\alpha = \frac{2(q-r+1)}{q-r+1+2(r-p)/d} < 2, \qquad \beta = q+1-\frac{m+q-1}{2}\alpha,$$
$$\delta = \frac{\beta}{r(1-\alpha/2)} = 1 + \frac{1+q-r}{r-p},$$
$$C(q,r,d) = \left[\frac{2mq[q-r+1+2(r-p)/d]}{S_d^{-1}(q+m-1)^2(q-r+1)}\right]^{-\frac{d(q-r+1)}{2(r-p)}} \frac{2(r-p)}{d(q-r+1)+2(r-p)}.$$

Lemma 2.2. *Let* $d \geq 3$, $0 < m \leq 2 - 2/d$, $p = \frac{d(2-m)}{2}$, $q > p$ *and* $u \in L^1_+(\mathbb{R}^d)$. *Then one has that for* $q > 1$,

$$\left(\|u\|_q^q\right)^{1+\frac{m-1+2/d}{q-1}} \leq S_d^{-1} \|\nabla u^{(q+m-1)/2}\|_2^2 \|u\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}. \tag{2.7}$$

Proof. For $q \geq p$, by the interpolation inequality and (2.5) one has

$$\|u\|_q^{\frac{q^2}{q-1}} \leq \|u\|_{q+1}^{q+1} \|u\|_1^{\frac{1}{q-1}} \leq S_d^{-1} \|\nabla u^{(q+m-1)/2}\|_{L^2}^2 \|u\|_p^{2-m} \|u\|_1^{\frac{1}{q-1}}$$
$$\leq S_d^{-1} \|\nabla u^{(q+m-1)/2}\|_{L^2}^2 \left(\|u\|_q^{\theta_1} \|u\|_1^{1-\theta_1}\right)^{2-m} \|u\|_1^{\frac{1}{q-1}},$$
$$\theta_1 = \frac{q(p-1)}{p(q-1)}.$$

This ends the proof. □

Lemma 2.3. *Let* $d \geq 3$, $m > 2 - 2/d$, $q > 0$ *and* $u \in L^1_+(\mathbb{R}^d)$. *Then*

$$\|u\|_{q+1}^{q+1} \leq S_d^{-\frac{\alpha}{2}} \left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^{\alpha} \|u\|_1^{\beta} \leq \frac{2mq}{(m+q-1)^2} \left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,m,d)\|u\|_1^{\delta}, \tag{2.8}$$

where

$$\alpha = \frac{2q}{q+m-2+2/d} < 2, \qquad \beta = q+1-\frac{m+q-1}{2}\alpha, \tag{2.9}$$

$$\delta = \frac{\beta}{1-\alpha/2} = 1 + \frac{2q}{d(m-2)+2}. \tag{2.10}$$

Now we define the weak solution which we will deal with through this paper, indeed, we ask for more regularities than needed for the definition and these regularities will be proved in Theorem 2.11.

Definition 2.4 (Weak solution). *Let $U_0 \in L^1_+(\mathbb{R}^d)$ be the initial data and $T \in (0,\infty)$. c is the concentration associated with u and given by (1.2). u is a weak solution to the system* (1.1) *with initial data U_0 if it satisfies:*

(i) *Regularity:*

$$u \in L^{\max(m,2)}\left(0,T; L^1_+ \cap L^{\max\left(m,\frac{2d}{d+2}\right)}(\mathbb{R}^d)\right), \tag{2.11}$$

$$\partial_t u \in L^{p_2}\left(0,T; W^{-1,p_1}_{loc}(\mathbb{R}^d)\right) \text{ for some } p_1, p_2 \geq 1. \tag{2.12}$$

(ii) *For $\forall\, \psi \in C_0^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,*

$$\int_{\mathbb{R}^d} \psi u(\cdot,t)dx - \int_{\mathbb{R}^d} \psi U_0 dx = \int_0^t \int_{\mathbb{R}^d} \Delta\psi u^m dxds \\ - \frac{c_d(d-2)}{2}\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,s)u(y,s)}{|x-y|^{d-2}}dxdyds. \tag{2.13}$$

Remark 2.5. Notice that the regularity (2.11) is enough to make sense of each term in (2.13). By the HLS inequality one has

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\frac{[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y)}{|x-y|^2}\right|\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy \\ \leq C\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy \\ \leq C\,\|u(x)\|^2_{2d/(d+2)} < \infty.$$

Definition 2.6. *For $m > 1$, the weak solution u in the above definition is also a weak entropy solution to Eqs.* (1.1) *if u satisfies additional regularities that*

$$\nabla u^{m-\frac{1}{2}} \in L^2\left(0,T; L^2(\mathbb{R}^d)\right), \tag{2.14}$$

$$u \in L^3\left(0,T; L^{\frac{3d}{d+2}}(\mathbb{R}^d)\right), \tag{2.15}$$

and $F[u(\cdot,t)]$ is a non-increasing function and satisfies the following energy inequality:

$$F[u(\cdot,t)] + \int_0^t\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u^{m-\frac{1}{2}} - \sqrt{u}\nabla c\right|^2 dxds \leq F(U_0)\,, \quad \textit{for any } t > 0. \tag{2.16}$$

Note that the regularities in (2.14) *and* (2.15) *are enough to make sense of each term in* (2.16).

Lemma 2.7. *If* $u \in L^1_+ \cap L^{\frac{3d}{d+2}}(\mathbb{R}^d)$ *and c is expressed by* (1.2). *Then*

$$\left\| u|\nabla c|^2 \right\|_1 \le C\|u\|_q^3 < \infty, \quad q = \frac{3d}{d+2}. \tag{2.17}$$

Proof. By the Hölder inequality

$$\left\| u|\nabla c|^2 \right\|_{L^1} \le \|u\|_q \left\| |\nabla c|^2 \right\|_{q'}, \quad \left(\frac{1}{q} + \frac{1}{q'} = 1 \right). \tag{2.18}$$

Then by the weak Young inequality [34, formula (9), pp. 107]

$$\left\| |\nabla c|^2 \right\|_{q'} = \|\nabla c\|^2_{L^{2q'}} = C \left\| u(x) * \frac{x}{|x|^d} \right\|^2_{L^{2q'}} \le C\|u\|^2_{L^q} \left\| \frac{x}{|x|^d} \right\|^2_{L_w^{\frac{d}{d-1}}} \le C\|u\|^2_{L^q}, \tag{2.19}$$

where $1 + \frac{1}{2q'} = \frac{1}{q} + \frac{d-1}{d}$. Combining with (2.18) follows $q = \frac{3d}{d+2}$ and completes the proof. □

Consequently, due to (2.15) one has that for any $T > 0$,

$$\int_0^T \left\| \sqrt{u}\nabla c \right\|_2^2 dt \le C \int_0^T \|u\|^3_{3d/(d+2)}\, dt < \infty. \tag{2.20}$$

So the term $\sqrt{u}\nabla c$ in (2.16) makes sense. Before showing the main results for the existence of a weak solution, we need the following lemmas.

Lemma 2.8. *Assume* Ω *is a bounded domain in* $\mathbb{R}^d$, $\bar{p} \ge 2 > m$, $q+1 \ge 2$, $\beta = \min\left(2, \frac{2(q+1)}{4-m}\right)$. *For any* $T > 0$, *if*

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;L^1_+(\Omega))} \le C, \\ &\|u_\varepsilon\|_{L^{q+1}(0,T;L^{q+1}(\Omega))} \le C, \\ &u_\varepsilon \to u \quad in\ L^\beta\left(0, T; L^{\bar{p}}(\Omega)\right), \end{aligned}$$

then there exists a subsequence u_ε *without relabeling such that for* $0 < m \le 1$,

$$u_\varepsilon^m \to u^m \quad in\ L^{\beta/m}\left(0, T; L^{\bar{p}/m}(\Omega)\right). \tag{2.21}$$

For $1 < m < 2 - 2/d$, *one has*

$$u_\varepsilon^m \to u^m \quad in\ L^1\left(0, T; L^{\bar{p}/m}(\Omega)\right). \tag{2.22}$$

Proof. Since $|u_\varepsilon^m - u^m| \le |u_\varepsilon - u|^m$ for $0 < m \le 1$, hence one has

$$\int_0^T \left\| u_\varepsilon^m - u^m \right\|^{\beta/m}_{L^{\bar{p}/m}(\Omega)} dt \le \int_0^T \left\| |u_\varepsilon - u|^m \right\|^{\beta/m}_{L^{\bar{p}/m}(\Omega)} dt \to 0 \text{ as } \varepsilon \to 0. \tag{2.23}$$

On the other hand, for $1 < m < 2 - 2/d$, by the mean value theorem and the Hölder inequality one arrives at

$$\int_\Omega |u_\varepsilon^m - u^m|^{\bar{p}/m} dx \le C(m) \int_\Omega \left[|u_\varepsilon + u|^{m-1}|u_\varepsilon - u|\right]^{\bar{p}/m} dx$$
$$\le C \left(\int_\Omega u_\varepsilon^{\bar{p}} dx\right)^{(m-1)/m} \left[\int_\Omega |u_\varepsilon - u|^{\bar{p}} dx\right]^{1/m},$$

then the Hölder inequality follows

$$\int_0^T \left\|u_\varepsilon^m - u^m\right\|_{L^{\bar{p}/m}(\Omega)} dt \le C \int_0^T \|u_\varepsilon\|_{L^{\bar{p}}(\Omega)}^{m-1} \|u_\varepsilon - u\|_{L^{\bar{p}}(\Omega)} dt$$
$$\le C \left[\int_0^T \|u_\varepsilon\|_{L^{\bar{p}}(\Omega)}^{(m-1)\beta/(\beta-1)} dt\right]^{(\beta-1)/\beta} \left[\int_0^T \|u_\varepsilon - u\|_{L^{\bar{p}}(\Omega)}^{\beta} dt\right]^{1/\beta}$$
$$\le C \left[\int_0^T \|u_\varepsilon - u\|_{L^{\bar{p}}(\Omega)}^{\beta} dt\right]^{1/\beta} \to 0.$$

This ends the proof. □

Lemma 2.9. *Assume $y(t) \ge 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \le \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then*

(i) *For $a > 1$, $y(t)$ has the following hyper-contractive property:*

$$y(t) \le (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad \text{for } t > 0.$$

(ii) *For $a = 1$, $y(t)$ decays exponentially*

$$y(t) \le \alpha/\beta + y(0)e^{-\beta t}.$$

(iii) *For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, which means that there exists a T_{ext} such that $0 < T_{ext} \le \frac{y^{1-a}(0)}{\beta(1-a)}$ and $y(t) = 0$ for all $t > T_{ext}$.*

Proof. Begin with the ODE inequality

$$y'(t) \le \beta(\alpha/\beta - y(t)^a).$$

If for some $t_0 > 0$, $y(t_0) \le (\alpha/\beta)^{1/a}$, then for all $t \ge t_0$, by contradiction one gets

$$y(t) \le (\alpha/\beta)^{1/a}. \tag{2.24}$$

If $y(t) > (\alpha/\beta)^{1/a}$ for $0 < t < t_0$, letting $y(t) = (\alpha/\beta)^{1/a} + z(t)$, then $y(t)^a \ge z(t)^a + \alpha/\beta$ and it follows that

$$z'(t) \le -\beta z(t)^a.$$

Solving this ODE, we arrive at

$$z(t) \le \left[\frac{1}{\beta(a-1)t}\right]^{1/(a-1)}. \tag{2.25}$$

Taking (2.24) and (2.25) together yields

$$y(t) \le (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{1/(a-1)}, \quad t > 0. \tag{2.26}$$

The above lemma directly follows.

Lemma 2.10. *Assume $f(t) \geq 0$ is a non-increasing function for $t > 0$, $y(t) \geq 0$ is a C^1 function for $t > 0$ and satisfies $y'(t) \leq f(t) - \beta y(t)^a$ for some constants $a > 1$ and $\beta > 0$, then for any $t_0 > 0$ one has*

$$y(t) \leq (f(t_0)/\beta)^{1/a} + [\beta(a-1)(t-t_0)]^{-1/(a-1)}, \quad \text{for } t > t_0.$$

Now we consider the global existence of a weak solution. Firstly we define a constant which is related to the initial condition for the existence results:

$$C_{d,m} := \left(\frac{4mp}{(m+p-1)^2 S_d^{-1}} \right)^{\frac{1}{2-m}}, \quad p = \frac{d(2-m)}{2}, \tag{2.27}$$

where S_d is given by (2.2).

Theorem 2.11. *Let $d \geq 3$, $0 < m < 2 - 2/d$ and $p = \frac{d(2-m)}{2}$, $\eta = C_{d,m}^{2-m} - \|U_0\|_p^{2-m}$. Assume $U_0 \in L_+^1(\mathbb{R}^d)$ and $\eta > 0$, then there exists a global weak solution u such that $\|u(\cdot,t)\|_p < C_{d,m}$ for all $t \geq 0$. Furthermore,*

- ***(i)*** *For $0 < m < 1 - 2/d$, there exists a minimal extinction time T_{ext} $(\|U_0\|_1, \eta, p)$ such that the weak solution vanishes a.e. in $\mathbb{R}^d$ for all $t \geq T_{ext}$. Furthermore, for $(d-2)/(d+2) < m < 1 - 2/d$, assume $m_2(0) < \infty$, then there exists a $0 < \bar{T} \leq T_{ext}$ such that*
  - *(a) $m_2(t) < \infty$ and $\|u(\cdot,t)\|_1 = \|U_0\|_1$ for $0 < t < \bar{T}$,*
  - *(b) $\limsup_{t \to \bar{T}} m_2(t) = \infty$, $\int_0^{\bar{T}} \int_{\mathbb{R}^d} u^m dx dt = \infty$ and $\int_0^{\bar{T}} F[u(t)]dt = -\infty$.*
- ***(ii)*** *For $m = 1 - 2/d$, the weak solution decays exponentially*

$$\|u(\cdot,t)\|_p \leq \|U_0\|_p e^{-\frac{\eta}{\|U_0\|_1^{1/(p-1)}} \frac{(p-1)}{p} t}. \tag{2.28}$$

- ***(iii)*** *For $1 - 2/d < m < 2 - 2/d$, the weak solution satisfies mass conservation and the following hyper-contractive estimates hold true that for any $t > 0$ and any $1 \leq q < \infty$:*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq C(\eta, \|U_0\|_1, q) t^{-\frac{q-1}{q(m-1+2/d)}}, \quad 1 \leq q \leq p, \tag{2.29}$$

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq C(\eta, \|U_0\|_1, q) \left(t^{-\frac{(p+\epsilon_0-1)(1+q-p)}{(q+m+2/d-2)\epsilon_0} \frac{q-1}{q(m-1+2/d)}} + t^{-\frac{q-1}{q(m-1+2/d)}} \right), \quad p < q < \infty, \tag{2.30}$$

  where ϵ_0 satisfies $\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} = \frac{\eta}{2}$.
- ***(iv)*** *If the initial data also satisfies $\int_{\mathbb{R}^d} |x|^2 U_0(x)dx < \infty$, then the second moment $\int_{\mathbb{R}^d} |x|^2 u(x,t)dx$ is bounded for any $0 \leq t < \infty$. For $1 < m \leq 2d/(d+2)$, this weak solution $u(x,t)$ is also a weak entropy solution satisfying the energy inequality (2.16). For $2d/(d+2) < m < 2 - 2/d$, assuming also $U_0 \in L^m(\mathbb{R}^d)$ such that this weak solution is also a weak entropy solution.*

Proof. The proof can be divided into 15 steps. Steps 1–7 give a priori estimates for the statement **(ii), (iii)** of Theorem 2.11. In Steps 8–10, a regularized equation is constructed to make these a prior estimates of Steps 1–7 rigorous and obtain the global existence of a weak solution to (1.1). Step 11 shows mass conservation of the weak solution for

$m > 1 - 2/d$ and the boundedness for the second moment, thus proves the statement **(i)** of Theorem 2.11. Steps 12–15 complete the existence of a global weak entropy solution for the slow diffusion $1 < m < 2 - 2/d$ and thus verify the statement **(iv)** of Theorem 2.11.

Following the method of [43], we take a cut-off function $0 \le \psi_1(x) \le 1$,

$$\psi_1(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2, \end{cases} \tag{2.31}$$

where $\psi_1(x) \in C_0^\infty(\mathbb{R}^d)$. Define $\psi_R(x) := \psi_1(x/R)$, as $R \to \infty$, $\psi_R \to 1$, then there exist constants C_1, C_2 such that $|\nabla \psi_R(x)| \le \frac{C_1}{R}$, $|\Delta \psi_R(x)| \le \frac{C_2}{R^2}$ for $x \in \mathbb{R}^d$. This cut-off function will be used to derive the existence of the weak solution.

Beginning with a formal prior estimate, then we deduce the long time decay estimates. Finally we used a regularized system to make the arguments rigorous.

Step 1 (Uniform L^p estimates for $0 < m < 2 - 2/d$). Firstly it's obtained by multiplying Eq. (1.1) with pu^{p-1} leads to

$$\begin{aligned} &\frac{d}{dt}\int u^p dx + \frac{4mp(p-1)}{(m+p-1)^2}\int \left|\nabla u^{(m+p-1)/2}\right|^2 dx \\ &\quad = (p-1)\int u^{p+1} dx \le (p-1) S_d^{-1} \|\nabla u^{(m+p-1)/2}\|_2^2 \|u\|_p^{2-m}, \end{aligned} \tag{2.32}$$

where the last inequality (2.32) follows from (2.5) with $q = p$. Hence one has

$$\frac{d}{dt}\int u^p dx + S_d^{-1}(p-1)\left(C_{d,m}^{2-m} - \|u\|_p^{2-m}\right)\int \left|\nabla u^{(m+p-1)/2}\right|^2 dx \le 0. \tag{2.33}$$

Since $\|U_0\|_p < C_{d,m}$, so the following estimates hold true:

$$\|u(\cdot,t)\|_p < \|U_0\|_p < C_{d,m}, \tag{2.34}$$

$$S_d^{-1}(p-1)\left(C_{d,m}^{2-m} - \|U_0\|_p^{2-m}\right)\int_0^\infty \int \left|\nabla u^{(m+p-1)/2}\right|^2 dxdt \le C_{d,m}. \tag{2.35}$$

Here denote $\eta := C_{d,m}^{2-m} - \|U_0\|_p^{2-m}$, from (2.32), (2.34) and (2.35) one has

$$\int_0^\infty \|u\|_{p+1}^{p+1} dt \le S_d^{-1} C_{d,m}^{2-m} \int_0^\infty \int \left|\nabla u^{(m+p-1)/2}\right|^2 dxdt \le \frac{C_{d,m}^{3-m}}{(p-1)\eta}. \tag{2.36}$$

It leads to the following estimates:

$$u \in L^{p+1}\left(\mathbb{R}_+; L^{p+1}(\mathbb{R}^d)\right), \qquad \nabla u^{\frac{p+m-1}{2}} \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^d)\right). \tag{2.37}$$

Step 2 (L^p decay estimates for $1 - 2/d \le m < 2 - 2/d$). For $1 - 2/d < m < 2 - 2/d$, it follows from (2.7) by using $\|u\|_1 \le \|U_0\|_1$,

$$\frac{\left(\|u\|_p^p\right)^{1+\frac{m-1+2/d}{p-1}}}{S_d^{-1}\|U_0\|_1^{\frac{1}{p-1}}} \le \|\nabla u^{\frac{p+m-1}{2}}\|_2^2. \tag{2.38}$$

It follows from (2.34) by substituting (2.38) into (2.33) that

$$\frac{d}{dt}\int u^p dx + \frac{(p-1)\eta}{\|U_0\|_1^{\frac{1}{p-1}}}\left(\int u^p dx\right)^{\delta} \le 0, \tag{2.39}$$

where $\delta = 1 + \frac{m-1+2/d}{p-1} > 1$ for $1 - 2/d < m < 2 - 2/d$. Now denote $C_p := \frac{(p-1)\eta}{\|U_0\|_1^{1/(p-1)}} > 0$, then one computes

$$\begin{aligned}\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} &\le \left(\frac{1}{(\delta-1)C_p t + \frac{1}{(\|U_0\|_p^p)^{\delta-1}}}\right)^{\frac{p-1}{m-1+2/d}} \\ &\le [(m-1+2/d)\eta]^{-\frac{p-1}{m-1+2/d}}\,\|U_0\|_1^{\frac{1}{m-1+2/d}}\,t^{-\frac{p-1}{m-1+2/d}}. \end{aligned} \tag{2.40}$$

For $m = 1 - 2/d$, the solution decays exponentially

$$\|u(\cdot,t)\|_p \le \|U_0\|_p e^{-\frac{\eta}{\|U_0\|_1^{1/(p-1)}}\frac{(p-1)}{p}t}. \tag{2.41}$$

Thus the proof of **(ii)** of Theorem 2.11 is completed.

Step 3 (Finite time extinction of L^p norm for $0 < m < 1-2/d$). For $0 < m < 1-2/d$, in view of Lemma 2.9 **(iii)**, there exists a finite time $0 < T_{ext} \le \frac{(\|U_0\|_p^p)^{1-\delta}}{C_p(1-\delta)}$ with $0 < \delta = 1 + \frac{m-1+2/d}{p-1} < 1$ such that the norm $\|u(\cdot,t)\|_p$ will vanish a.e. in $\mathbb{R}^d$ for all $t > T_{ext}$, thus the mass can't be preserved.

Step 4 (Uniform L^{r_0} estimate with $r_0 := p + \epsilon_0$ for ϵ_0 small enough for $1-2/d < m < 2-2/d$). Using (2.5) with $q = r_0$ deduces

$$\begin{aligned}&\frac{d}{dt}\int u^{r_0}dx + \frac{4mr_0(r_0-1)}{(r_0+m-1)^2}\int\left|\nabla u^{(m+r_0-1)/2}\right|^2 dx = (r_0-1)\int u^{r_0+1}dx \\ &\quad\le (r_0-1)S_d^{-1}\|\nabla u^{(r_0+m-1)/2}\|_2^2\|u\|_p^{2-m} \le (r_0-1)S_d^{-1}\|\nabla u^{(r_0+m-1)/2}\|_2^2\|U_0\|_p^{2-m}.\end{aligned} \tag{2.42}$$

The last inequality is derived from (2.34). If we choose ϵ_0 such that

$$\frac{\eta}{2} := \frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} < \eta, \tag{2.43}$$

then one has

$$\frac{d}{dt}\int u^{r_0}dx + S_d^{-1}(r_0-1)\frac{\eta}{2}\int\left|\nabla u^{(m+r_0-1)/2}\right|^2 dx \le 0. \tag{2.44}$$

On the other hand, (2.7) leads to

$$\frac{d}{dt}\int u^{r_0}dx + \frac{(r_0-1)\eta}{2\|U_0\|_1^{\frac{1}{r_0-1}}}\left(\int u^{r_0}dx\right)^{\delta} \le 0, \tag{2.45}$$

where $\delta := 1+\frac{m-1+2/d}{r_0-1} > 1$ for $1-2/d < m < 2-2/d$. Now denote $C_{r_0} := \frac{(r_0-1)\eta}{2\|U_0\|_1^{\frac{1}{r_0-1}}}$, then one computes

$$\|u(\cdot,t)\|_{L^{r_0}(\mathbb{R}^d)}^{r_0} \le \left[C_{r_0}(\delta-1)\right]^{-\frac{r_0-1}{m-1+2/d}} t^{-\frac{r_0-1}{m-1+2/d}}. \tag{2.46}$$

Step 5 (Hyper-contractive estimates of L^q norm for $q > r_0$ with $1-2/d < m < 2-2/d$ for any $t > 0$). For $q > r_0$, taking $r = r_0$ in (2.6),

$$\begin{aligned}\frac{d}{dt}\|u\|_q^q + \frac{4qm(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 &= (q-1)\int u^{q+1}dx \\ &\le \frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r_0,d)\left(\|u\|_{r_0}^{r_0}\right)^{\delta},\end{aligned} \tag{2.47}$$

where $\delta = 1+\frac{1+q-r_0}{r_0-p}$. Collecting (2.7) and (2.46) yields

$$\begin{aligned}\frac{d}{dt}\|u\|_q^q \le &-\frac{2mq(q-1)}{S_d^{-1}(m+q-1)^2\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}\left(\|u\|_q^q\right)^{1+\frac{m-1+2/d}{q-1}} \\ &+C(q,r_0,d,\|U_0\|_1)t^{-\frac{(r_0-1)(1+q-p)}{(m-1+2/d)(r_0-p)}},\end{aligned} \tag{2.48}$$

where $C(q,r_0,d,\|U_0\|_1) = C(q,r_0,d)\left(\frac{\eta}{2\|U_0\|_1^{1/(p-1)}}\right)^{-\frac{(r_0-1)(1+q-p)}{(m-1+2/d)(r_0-p)}}$, recalling Lemma 2.10 with $y(t) = \|u\|_q^q,\ a = 1+\frac{m-1+2/d}{q-1} > 1$ and $\beta = \frac{2mq(q-1)}{S_d^{-1}(m+q-1)^2\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}$, $f(t) = C(q,r_0,d,\|U_0\|_1)t^{-\frac{(r_0-1)(1+q-p)}{(m-1+2/d)(r_0-p)}}$, one has that for $t_0 = t/2$ with any $t > 0$ and ϵ_0 satisfies (2.43),

$$\|u\|_q^q \le C(d,q,\|U_0\|_1,\eta)\left(t^{-\frac{(p+\epsilon_0-1)(1+q-p)}{(q+m+2/d-2)\epsilon_0}\frac{q-1}{m-1+2/d}} + t^{-\frac{q-1}{m-1+2/d}}\right). \tag{2.49}$$

Step 6 (Decay estimates on $\|u\|_{L^q}$ for $1-2/d < m < 2-2/d$). In this step, based on the decay of $\|u\|_p$ with time evolution, $\|u(\cdot,t)\|_q$ decays for large time. Divide q into two cases $1 < q < p$ and $p < q < \infty$.

(1) $1 < q < p$. For $1-2/d < m < 2-2/d$, it follows from (2.40) by applying the interpolation inequality that for any $t > 0$,

$$\begin{aligned}\|u\|_q^q &\le \|u\|_p^{\frac{p(q-1)}{q(p-1)}}\|u\|_1^{\frac{p-q}{(p-1)q}} \\ &\le [(m-1+2/d)\eta]^{-\frac{q-1}{m-1+2/d}}\|U_0\|_1^{\frac{q-1}{(p-1)(m-1+2/d)}+\frac{p-q}{p-1}}t^{-\frac{q-1}{m-1+2/d}}.\end{aligned} \tag{2.50}$$

(2) $p < q < \infty$. For $1-2/d < m < 2-2/d$, since $\|u\|_p$ decays to zero as time goes to infinity, then for t larger than some T_q one has

$$(q-1)S_d^{-1}\|u\|_p^{2-m} \le \frac{2mq(q-1)}{(m+q-1)^2} =: C(q,m). \tag{2.51}$$

So due to (2.5), the following estimates hold:

$$\frac{d}{dt}\int u^q dx + C(q,m)\left\|\nabla u^{(m+q-1)/2}\right\|_2^2 \le 0, \qquad \text{for } t > T_q. \tag{2.52}$$

Combining with (2.7) and $\|u\|_1 \le \|U_0\|_1$ gives

$$\frac{d}{dt}\int u^q dx + \frac{C(q,m)}{S_d^{-1}\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}\left(\|u\|_q^q\right)^{1+\frac{m-1+2/d}{q-1}} \le 0, \qquad \text{for } t > T_q. \tag{2.53}$$

Denote $\alpha := 1 + \frac{m-1+2/d}{q-1} > 1$ and $C_q = \frac{C(q,m)}{S_d^{-1}\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}} = O(1)$. Solving (2.53) gives

$$\|u\|_q^q \le \left(\frac{S_d^{-1}(m+q-1)^2}{2mq(m-1+2/d)}\right)^{\frac{q-1}{m-1+2/d}} \|U_0\|_1^{\frac{2q+d(m-1)}{d(m-1)+2}}(t-T_q)^{-\frac{q-1}{m-1+2/d}}, \quad t > T_q. \tag{2.54}$$

By virtue of (2.49), (2.50) and (2.54), the statement **(iii)** of Theorem 2.11 hold true.

Step 7 (Mass conservation for u when $m > 1-2/d$). Using (2.37) and $m < p+1$ for $0 < m < 2-2/d$ one has that for any $t > 0$,

$$\int_0^t \|u\|_{2d/(d+2)}^2 ds \le \int_0^t \|u\|_{p+1}^{2\theta}\|u\|_1^{2(1-\theta)} ds \le C(t)\int_0^t \|u\|_{p+1}^{p+1} ds < C(t), \tag{2.55}$$

and for $m > 1$,

$$\int_0^t \|u\|_m^m ds \le \int_0^t \|u\|_{p+1}^{m\lambda}\|u\|_1^{m(1-\lambda)} ds \le C(t)\int_0^t \|u\|_{p+1}^{p+1} ds \le C(t), \quad 0<\lambda<1. \tag{2.56}$$

For $1-2/d < m < 1$, recalling (2.31) and letting $\psi(x) = \psi_R(x)$ in (2.13) one has

$$\begin{aligned}
&\left|\frac{d}{dt}\int_{\mathbb{R}^d} u(\cdot,t)\psi_R(x)dx\right| \\
&= \left|\int_{\mathbb{R}^d} u^m \Delta\psi_R dx - \frac{c_d(d-2)}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\right. \\
&\quad \left.\times \frac{[\nabla\psi_R(x)-\nabla\psi_R(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy\right| \\
&\le \frac{C(\|U_0\|_1)}{R^{2-d(1-m)}}\left(\int_{B_{2R}/B_R} udx\right)^m + C\frac{\|u\|_{2d/(d+2)}^2}{R^2},
\end{aligned} \tag{2.57}$$

where the Hölder inequality is applied that

$$\begin{aligned}
\int_{\mathbb{R}^d} u^m \Delta\psi_R dx &\le \frac{C}{R^2}\int_{B_{2R}} u^m dx \le \frac{C}{R^2}\left[\int_{B_{2R}} udx\right]^m \left[\int_{B_{2R}} 1dx\right]^{(1-m)} \\
&\le \frac{C}{R^{2-d(1-m)}}\|u\|_1^m,
\end{aligned} \tag{2.58}$$

denote $\epsilon := 2 - d(1-m)$ and (2.55) implies

$$
\begin{aligned}
-C(\|U_0\|_1)\frac{t}{R^\epsilon} - \frac{C(t)}{R^2} &\le \int_{\mathbb{R}^d} \psi_R(x)u(x,t)dx - \int_{\mathbb{R}^d} \psi_R(x)U_0(x)dx \\
&\le C(\|U_0\|_1)\frac{t}{R^\epsilon} + \frac{C(t)}{R^2}. \qquad (2.59)
\end{aligned}
$$

Since $1-2/d < m < 1$ such that $\epsilon > 0$, thus as $R \to \infty$ by the dominated convergence theorem one has

$$
\lim_{R\to\infty}\int_{B_R} U_0(x)dx \le \lim_{R\to\infty}\int_{\mathbb{R}^d} \psi_R(x)u(x,t)dx \le \lim_{R\to\infty}\int_{B_{2R}} U_0(x)dx. \quad (2.60)
$$

For $1 \le m < 2 - 2/d$, one obtains

$$
\left|\frac{d}{dt}\int_{\mathbb{R}^d} u(\cdot,t)\psi_R(x)dx\right| \le C\frac{\|u\|_m^m + \|u\|_{\frac{2d}{d+2}}^2}{R^2}, \qquad (2.61)
$$

combining with (2.55) and (2.56) also derives (2.60).

Hence for $1 - 2/d < m < 2 - 2/d$,

$$
\lim_{R\to\infty}\int_{B_R} u(x,t)dx \le \int_{\mathbb{R}^d} U_0(x)dx \le \lim_{R\to\infty}\int_{B_{2R}} u(x,t)dx,
$$

such that $\int_{\mathbb{R}^d} u(x,t)dx = \int_{\mathbb{R}^d} U_0 dx$.

Step 8 (Regularization for $m > 0$). In order to show the existence of a weak solution with the above properties and make the proof rigorous, we consider the regularized problem for $\varepsilon > 0$,

$$
\begin{cases}
\partial_t u_\varepsilon = \Delta u_\varepsilon^m + \varepsilon\Delta u_\varepsilon - \nabla\cdot(u_\varepsilon\nabla c_\varepsilon), & x \in \mathbb{R}^d, t \ge 0, \\
-\Delta c_\varepsilon = J_\varepsilon * u_\varepsilon, & x \in \mathbb{R}^n, t \ge 0, \\
u_\varepsilon(x,0) = u_{0\varepsilon}, & x \in \mathbb{R}^d.
\end{cases} \qquad (2.62)
$$

Here $J_\varepsilon(x) = \frac{1}{\varepsilon^d}J(\frac{x}{\varepsilon})$, $J(x) = \frac{1}{\alpha_d}\left(1+|x|^2\right)^{-(d+2)/2}$ and $\int_{\mathbb{R}^d} J_\varepsilon(x)dx = 1$. Simple computations show that c_ε can be expressed by

$$
c_\varepsilon(x,t) = c_d\int_{\mathbb{R}^d}\frac{u_\varepsilon(y,t)}{\left(|x-y|^2+\varepsilon^2\right)^{(d-2)/2}}dy, \qquad (2.63)
$$

where c_d is the same as in (1.3). Here $u_{0\varepsilon} \in C^\infty(\mathbb{R}^d)$ is a sequence of approximation for U_0 and can be constructed and satisfies that there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$,

$$
u_{0\varepsilon} > 0, \qquad (2.64)
$$

$$
u_{0\varepsilon} \in L^r(\mathbb{R}^d) \quad \text{for all } r \ge 1, \qquad (2.65)
$$

$$
\|u_\varepsilon(x,0)\|_1 = \|U_0\|_1, \qquad (2.66)
$$

$$
|x|^2 u_{0\varepsilon}dx \to \int |x|^2 U_0(x)dx \quad \text{as } \varepsilon \to 0. \qquad (2.67)
$$

If $U_0 \in L^q$ for some q, then

$$u_{0\varepsilon} \to U_0 \quad \text{in } L^q \text{ as } \varepsilon \to 0. \tag{2.68}$$

From parabolic theory, for any fixed $\varepsilon > 0$, the above regularized problem has a global smooth positive solution u_ε with the regularity for all $r \geq 1$,

$$u_\varepsilon \in L^\infty\left(0, T; L^r(\mathbb{R}^d)\right) \cap L^{r+1}\left(0, T; L^{r+1}(\mathbb{R}^d)\right). \tag{2.69}$$

Taking similar arguments as Step 7 arrives at the mass conservation of u_ε for $m > 1-2/d$.

For simplicity in presentation, we omit all the ε dependents and use u instead of u_ε in Steps 1–7 and all the formal calculations in Steps 1–7, mainly, (2.32), (2.42), (2.47) shall be justified below. For the rigorous proof, for any $q \geq p$, multiplying Eq. (2.62) with $qu_\varepsilon^{q-1}\psi_R(x)$, where $\psi_R(x)$ is defined in (2.31) and integrating in space one has

$$\begin{aligned}
&\frac{d}{dt}\int u_\varepsilon^q \psi_R(x)dx + \frac{4mq(q-1)}{(m+q-1)^2}\int \left|\nabla u_\varepsilon^{(m+q-1)/2}\right|^2 \psi_R(x)dx \\
&\quad + \varepsilon\frac{4(q-1)}{q}\int \left|\nabla u_\varepsilon^{q/2}\right|^2 \psi_R(x)dx \\
&= (q-1)\int u_\varepsilon^q J_\varepsilon * u_\varepsilon \psi_R(x)dx + \frac{mq}{m+q-1}\int u_\varepsilon^{m+q-1}\Delta\psi_R(x)dx \\
&\quad + \int u_\varepsilon^q \nabla c_\varepsilon \cdot \nabla\psi_R(x)dx.
\end{aligned} \tag{2.70}$$

Integrating (2.70) from 0 to t in time yields that

$$\begin{aligned}
&\int u_\varepsilon(t)^q \psi_R(x)dx - \int u_{0\varepsilon}^q \psi_R(x)dx \\
&+ \frac{4mq(q-1)}{(m+q-1)^2}\int_0^t \int \left|\nabla u_\varepsilon^{(m+q-1)/2}\right|^2 \psi_R(x)dxds \\
&+ \varepsilon\frac{4(q-1)}{q}\int \left|\nabla u_\varepsilon^{q/2}\right|^2 \psi_R(x)dx = (q-1)\int_0^t \int u_\varepsilon^q J_\varepsilon * u_\varepsilon \psi_R(x)dxds \\
&+ \frac{mq}{m+q-1}\int_0^t \int u_\varepsilon^{m+q-1}\Delta\psi_R(x)dxds + \int_0^t \int u_\varepsilon^q \nabla c_\varepsilon \cdot \nabla\psi_R(x)dxds.
\end{aligned} \tag{2.71}$$

Below we will show that the last two terms of (2.71) will vanish as $R \to \infty$. It holds from (2.31) by using the Young inequality that the last term of (2.71) satisfies

$$\begin{aligned}
\int u_\varepsilon^q \nabla c_\varepsilon \cdot \nabla\psi_R(x)dx &\leq \frac{C}{R}\int u_\varepsilon^q \left|\nabla c_\varepsilon\right| dx \leq \frac{C}{R}\|u_\varepsilon^p\|_{r_1}\|\nabla c_\varepsilon\|_{r_2} \\
&\leq \frac{C}{R}\|u_\varepsilon^p\|_{r_1}\|u_\varepsilon\|_{r_3}\left\|\frac{x}{|x|^d}\right\|_{L_w^{d/(d-1)}} \leq \frac{C}{R}\|u_\varepsilon\|_{d(q+1)/(d+1)}^{q+1},
\end{aligned} \tag{2.72}$$

where the exponents satisfy $\frac{1}{r_1}+\frac{1}{r_2}=1$, $\frac{1}{r_3}+\frac{d-1}{d}=1+\frac{1}{r_2}$ and $qr_2 = r_3$. Thus

$$\int_0^t \int u_\varepsilon^q \nabla c_\varepsilon \cdot \nabla\psi_R(x)dxds \leq \frac{C(\|u_{0\varepsilon}\|_1, t)}{R}\int_0^t \|u_\varepsilon\|_{q+1}^{q+1}ds.$$

Moreover, using the Hölder inequality with $m+q-1 \geq m+p-1 \geq 1$ one has

$$\int_0^t \int_{\mathbb{R}^d} u_\varepsilon^{m+q-1} \Delta \psi_R dx ds \leq \frac{C}{R^2} \int_0^t \|u_\varepsilon\|_{m+q-1}^{m+q-1} ds.$$

By virtue of (2.69) and the dominated convergence theorem that taking $R \to \infty$ in (2.71) one has

$$\int u_\varepsilon(t)^q dx - \int u_{0\varepsilon}^q dx + \frac{4mq(q-1)}{(m+q-1)^2} \int_0^t \int \left|\nabla u_\varepsilon^{(m+q-1)/2}\right|^2 dx ds$$
$$+\varepsilon \frac{4(q-1)}{q} \int \left|\nabla u_\varepsilon^{q/2}\right|^2 dx = (q-1) \int_0^t \int u_\varepsilon^q J_\varepsilon * u_\varepsilon dx ds.$$

Now taking a time derivative of the above equation, one arrives, for any $t > 0$,

$$\frac{d}{dt} \int u_\varepsilon^q dx + \frac{4mq(q-1)}{(m+q-1)^2} \int \left|\nabla u_\varepsilon^{(m+q-1)/2}\right|^2 dx$$
$$+\varepsilon \frac{4(q-1)}{q} \int \left|\nabla u_\varepsilon^{q/2}\right|^2 dx = (q-1) \int u_\varepsilon^q J_\varepsilon * u_\varepsilon dx. \tag{2.73}$$

All the estimates in Steps 1–6 holds true since

$$\int u_\varepsilon^q J_\varepsilon * u_\varepsilon \leq \|u_\varepsilon^q\|_{(q+1)/q} \|J_\varepsilon * u_\varepsilon\|_{q+1} \leq \|u_\varepsilon\|_{q+1}^q \|u_\varepsilon\|_{q+1}$$
$$\leq S_d^{-1} \left\|\nabla u_\varepsilon^{(q+m-1)/2}\right\|_2^2 \|u_\varepsilon\|_p^{2-m}. \tag{2.74}$$

Next we will show the compactness and convergence of u_ε to a weak solution.

For initial data satisfies $u_{0\varepsilon} \in L^p(\mathbb{R}^d)$, the following basic estimates are obtained:

$$\|u_\varepsilon\|_{L^\infty(0,T;L_+^1 \cap L^p(\mathbb{R}^d))} \leq C, \tag{2.75}$$

$$\left\|\nabla u_\varepsilon^{\frac{m+r-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C, \quad 1 < r \leq p, \tag{2.76}$$

$$\|u_\varepsilon\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))} \leq C. \tag{2.77}$$

On the other hand, applying the weak Young inequality yields

$$\int_0^T \|\nabla c_\varepsilon\|_{L^2(\mathbb{R}^d)}^{p+1} dt \leq C \int_0^T \left\|(J_\varepsilon * u_\varepsilon) * \frac{1}{|x|^{d-1}}\right\|_{L^2(\mathbb{R}^d)}^{p+1} dt$$
$$\leq C \int_0^T \|u_\varepsilon\|_{L^{2d/(d+2)}(\mathbb{R}^d)}^{p+1} \left\|\frac{1}{|x|^{d-1}}\right\|_{L_w^{\frac{d}{d-1}}(\mathbb{R}^d)}^{p+1} dt$$
$$\leq C \int_0^T \|u_\varepsilon\|_{L^{2d/(d+2)}(\mathbb{R}^d)}^{p+1} dt \leq C, \tag{2.78}$$

thus there exists a subsequence u_ε without relabeling such that for any $T > 0$,

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^{p+1}\left(0, T; L^{p+1}(\mathbb{R}^d)\right), \tag{2.79}$$

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0, T; L_+^1 \cap L^p(\mathbb{R}^d)\right), \tag{2.80}$$

$$\nabla c_\varepsilon \overset{*}{\rightharpoonup} \nabla c \quad \text{in } L^{p+1}\left(0, T; L^2(\mathbb{R}^d)\right). \tag{2.81}$$

Now we will show that the a priori bounds in the theorem hold true uniformly in ε and thus we can pass to the limit.

Step 9 (Time regularity and applications of Lions-Aubin lemma for $m > 0$). This step firstly gives the following uniform bounds, for any $T > 0$:

$$\|u_\varepsilon \nabla c_\varepsilon\|_{L^2\left(0,T;L^{\frac{2p}{p+2}}(\mathbb{R}^d)\right)} \leq C, \tag{2.82}$$

$$\left\|\nabla u_\varepsilon^m\right\|_{L^2\left(0,T;L^{\min\left(2,\frac{2p}{m+1}\right)}(\mathbb{R}^d)\right)} \leq C, \tag{2.83}$$

$$\|\nabla u_\varepsilon\|_{L^{r_2}(0,T;L^{r_2}(\mathbb{R}^d))} \leq C, \quad r_2 := \min\left\{2, \frac{2(p+1)}{4-m}\right\}. \tag{2.84}$$

Due to the initial data $U_0 \in L^p(\mathbb{R}^d)$, by the above estimates, one gets that for any $T > 0$ and any bounded domain Ω,

$$\|(u_\varepsilon)_t\|_{L^{\min\left(2,\frac{2(p+1)}{4-m}\right)}\left(0,T;W^{-1,\frac{2p}{p+2}}(\Omega)\right)} \leq C, \tag{2.85}$$

which verifies the time regularity in Definition 2.4.

Next we study the compactness of u_ε. Let $r_1 = \frac{2p}{p+2}$, if $\bar{p}$ satisfies $\frac{dr_1}{d+r_1} \leq \bar{p} < \frac{dr_2}{d-r_2}$, then the following compact embedding holds:

$$W^{1,r_2}(\Omega) \hookrightarrow\hookrightarrow L^{\bar{p}}(\Omega) \hookrightarrow W^{-1,r_1}(\Omega). \tag{2.86}$$

By the Lions-Aubin Lemma and combining with (2.84) and (2.85) one arrives at

$$u_\varepsilon \text{ is compact in } L^{r_2}\left(0, T; L^{\bar{p}}(\Omega)\right). \tag{2.87}$$

Letting $q' = \frac{2(p+1)}{4-m}$, some computations derive that $\frac{dr_1}{d+r_1} < 2$, and $\frac{dr_2}{d-r_2} = \min\left\{\frac{2d}{d-2}, \frac{dq'}{d-q'}\right\} > 2$ which implies that it can be chosen $\bar{p} = 2$. Consequently, there exists a subsequence u_ε without relabeling such that

$$u_\varepsilon \to u \qquad \text{in } L^{r_2}\left(0, T; L^{\bar{p}}(\Omega)\right). \tag{2.88}$$

Let $\{B_k\}_{k=1}^\infty \subset \mathbb{R}^d$ be a sequence of balls centered at 0 with radius R_k, $R_k \to \infty$. By a standard diagonal argument, there is a subsequence of u_ε. Without relabeling the following uniform strong convergence holds true:

$$u_\varepsilon \to u \qquad \text{in } L^{r_2}\left(0, T; L^{\bar{p}}(B_k)\right), \quad \forall k, \tag{2.89}$$

where $r_2 = \min\left\{2, \frac{2(p+1)}{4-m}\right\}$ and $\frac{dr_1}{d+r_1} \leq \bar{p} < \frac{dr_2}{d-r_2} = \min\left\{\frac{2d}{d-2}, \frac{dq'}{d-q'}\right\}$ with $q' = \frac{2(p+1)}{4-m}$.

Now we will show (2.82), (2.83), (2.84). It follows from (2.75) and (2.77) by applying the Hölder inequality that

$$\begin{aligned}\int_0^T \|u_\varepsilon \nabla c_\varepsilon\|_{\frac{2p}{p+2}}^2 dt &\leq \int_0^T \|\nabla c_\varepsilon\|_2^2 \|u_\varepsilon\|_p^2 dt \leq C \int_0^T \|u_\varepsilon\|_{2d/(d+2)}^2 dt \\ &\leq C(T) \int_0^T \|u_\varepsilon\|_{p+1}^{p+1} dt \leq C.\end{aligned} \tag{2.90}$$

To estimate ∇u_ε^m, we split it into two cases:

For $p \ge m+1$, taking $r = m+1$ in (2.76) leads to $\left\|\nabla u_\varepsilon^m\right\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))} = \left\|\nabla u_\varepsilon^{\frac{m+r-1}{2}}\right\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))} \le C.$

For $p < m+1$ one has

$$\left\|\nabla u_\varepsilon^m\right\|_{\frac{2p}{m+1}} \le C \left\|u_\varepsilon^{\frac{m-p+1}{2}}\right\|_{\frac{2p}{m-p+1}} \left\|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\right\|_2 = C\, \|u_\varepsilon\|_p^{\frac{m-p+1}{2}} \left\|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\right\|_2. \quad (2.91)$$

Then using the fact that $\|u_\varepsilon\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \le C$, it follows that $\left\|\nabla u_\varepsilon^m\right\|_{L^2\left(0,T;L^{\frac{2p}{m+1}}(\mathbb{R}^d)\right)} \le C$, hence one has $\|\nabla u_\varepsilon^m\|_{L^2\left(0,T;L^{\min\left(2,\frac{2p}{m+1}\right)}(\mathbb{R}^d)\right)} \le C.$

Now we show (2.84). We split it into two cases: $p < 3-m$ and $p \ge 3-m$. For $p < 3-m$, recast ∇u_ε as

$$\nabla u_\varepsilon = \frac{2}{m+p-1} u_\varepsilon^{\frac{3-m-p}{2}} \nabla u_\varepsilon^{\frac{m+p-1}{2}}. \quad (2.92)$$

By the Hölder inequality one has

$$\int_{\mathbb{R}^d} |\nabla u_\varepsilon|^r dx = C \int_{\mathbb{R}^d} \left| u_\varepsilon^{\frac{3-m-p}{2}} \nabla u_\varepsilon^{\frac{m+p-1}{2}} \right|^r dx \le C \left\| u_\varepsilon^{r\frac{3-m-p}{2}} \right\|_{p_1} \left\| \left|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\right|^r \right\|_{q_1}, \quad (2.93)$$

where $p_1 = \frac{2(p+1)}{r(3-m-p)}$, $q_1 = \frac{2}{r}$ satisfying $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then $r = \frac{2(p+1)}{4-m}$. Moreover, by using the Hölder inequality in time and combining (2.76) and (2.77) one has

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |\nabla u_\varepsilon|^r dxdt &\le C \left(\int_0^T \left\| u_\varepsilon^{r\frac{3-m-p}{2}} \right\|_{p_1}^{p_1} dt \right)^{1/p_1} \left(\int_0^T \left\| \left|\nabla u_\varepsilon^{\frac{m+p-1}{2}}\right|^r \right\|_{q_1}^{q_1} dt \right)^{1/q_1} \\ &\le C \left(\int_0^T \|u_\varepsilon\|_{p+1}^{p+1} dt \right)^{1/p_1} \left(\int_0^T \left\| \nabla u_\varepsilon^{\frac{m+p-1}{2}} \right\|_2^2 dt \right)^{1/q_1} \le C.\end{aligned} \quad (2.94)$$

Hence it follows $\|\nabla u_\varepsilon\|_{L^{\frac{2(p+1)}{4-m}}\left(0,T;L^{\frac{2(p+1)}{4-m}}(\mathbb{R}^d)\right)} \le C$. For $p \ge 3-m$, taking $r = 3-m$ in (2.76) one has $\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^d))} = \left\|\nabla u_\varepsilon^{\frac{m+r-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C$. Taking the two cases yields (2.84).

Step 10 (Existence of a global weak solution for $m > 0$). The weak formulation for u_ε is that for $\forall \psi \in C_0^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi u_\varepsilon(\cdot,t) dx - \int_{\mathbb{R}^d} \psi U_0 dx = \int_0^t \int_{\mathbb{R}^d} \Delta\psi (u_\varepsilon^m + \varepsilon u_\varepsilon) dxds - \frac{c_d(d-2)}{2} \\ &\times \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[\nabla\psi(x) - \nabla\psi(y)]\cdot(x-y)}{|x-y|^2+\varepsilon^2} \frac{u_\varepsilon(x,s)u_\varepsilon(y,s)}{(|x-y|^2+\varepsilon^2)^{(d-2)/2}} dxdyds.\end{aligned} \quad (2.95)$$

In order to prove the existence of a weak solution, firstly for $0 < m < 2 - 2/d$ and any bounded domain Ω one has

$$u_\varepsilon^m \to u^m \quad \text{in } L^1\left(0, T; L^1(\Omega)\right). \tag{2.96}$$

For $0 < m < 1$, Lemma 2.8 with $m < p + 1$ and (2.89) follow that $u_\varepsilon^m \to u^m$ in $L^1\left(0, T; L^1(\Omega)\right)$. For $1 \le m < 2 - 2/d < 2$, (2.96) is obtained by taking $\bar{p} = 2$ in (2.89). This directly derives that

$$\int_0^T \int_{\mathbb{R}^d} \Delta\psi u_\varepsilon^m dxdt \to \int_0^T \int_{\mathbb{R}^d} \Delta\psi u^m dxdt, \quad \varepsilon \to 0. \tag{2.97}$$

As to the second term of the right side of (2.95). Notice that

$$\frac{1}{|x-y|^d} - \frac{1}{(|x-y|^2+\varepsilon^2)^{d/2}} \le \frac{d}{2}\frac{\varepsilon}{|x-y|^{d+1}}, \tag{2.98}$$

hence by (2.77) the following estimates hold true:

$$\begin{aligned}
&\left|\int_0^T \iint [\nabla\psi(x) - \nabla\psi(y)] \cdot (x-y)\right.\\
&\quad \left. \times \left(\frac{1}{|x-y|^d} - \frac{1}{(|x-y|^2+\varepsilon^2)^{d/2}}\right) u_\varepsilon(x) u_\varepsilon(y) dxdydt\right|\\
&\le C\varepsilon \int_0^T \iint \frac{u_\varepsilon(x)u_\varepsilon(y)}{|x-y|^{d-1}} dxdydt \le C\varepsilon \int_0^T \|u_\varepsilon\|^2_{L^{2d/(d+1)}} dt \le C(T)\varepsilon.
\end{aligned} \tag{2.99}$$

In addition, for any $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} [\nabla\psi(x) - \nabla\psi(y)] \cdot (x-y) \left(\frac{u_\varepsilon(x)u_\varepsilon(y)}{|x-y|^d} - \frac{u(x)u(y)}{|x-y|^d}\right) dxdy\right|\\
&\le C \iint_{\Omega\times\Omega} \frac{|u_\varepsilon(x) - u(x)| u_\varepsilon(y)}{|x-y|^{d-2}} dxdy + \iint_{\Omega\times\Omega} \frac{|u_\varepsilon(y) - u(y)| u(x)}{|x-y|^{d-2}} dxdy\\
&=: I_1 + I_2.
\end{aligned} \tag{2.100}$$

For $\int_0^T I_1 dt$, taking $\bar{p} = 2$ in (2.89) one has

$$\begin{aligned}
\int_0^T I_1 dt &= C \int_0^T \int_\Omega |u_\varepsilon(x) - u(x)| \left[\int_{\mathbb{R}^d} \frac{u_\varepsilon(y)}{|x-y|^{d-2}} dy\right] dxdt\\
&\le C \int_0^T \|u_\varepsilon - u\|_{L^{2d/(d+2)}(\Omega)} \|u_\varepsilon\|_{L^{2d/(d+2)}(\mathbb{R}^d)} dt\\
&\le C \int_0^T \|u_\varepsilon - u\|^\theta_{L^2(\Omega)} \|u_\varepsilon - u\|^{1-\theta}_{L^1(\Omega)} \|u_\varepsilon\|^{1-\theta}_{L^1(\mathbb{R}^d)} \|u_\varepsilon\|^\theta_{L^2(\mathbb{R}^d)} dt\\
&\le C \left[\int_0^T \|u_\varepsilon - u\|^{\theta\frac{p+1}{p}}_{L^2(\Omega)} dt\right]^{p/(p+1)} \left[\int_0^T \|u_\varepsilon\|_2^{\theta(p+1)} dt\right]^{1/(p+1)},
\end{aligned}$$

where (2.77) has been used with $\theta\frac{p+1}{p}=\frac{d-2}{d}\frac{p+1}{p}<r_2$, which is defined in (2.89) in the last inequality. The estimates for $\int_0^T I_2dt$ is exactly the same as that for $\int_0^T I_1dt$, thus taking the limit $\varepsilon\to 0$ and combining with (2.99), (2.100) conclude that

$$\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y) \times\left(\frac{u_\varepsilon(x)u_\varepsilon(y)}{(|x-y|^2+\varepsilon^2)^{d/2}}-\frac{u(x)u(y)}{|x-y|^d}\right)dxdydt\to 0. \tag{2.101}$$

Owing to (2.97) and (2.101), passing to the limit $\varepsilon\to 0$ in (2.95) one has that for any $0<t<T$,

$$\int_{\mathbb{R}^d}\psi u(\cdot,t)dx-\int_{\mathbb{R}^d}\psi U_0dx=\int_0^t\int_{\mathbb{R}^d}\Delta\psi u^m dxds -\frac{c_d(d-2)}{2}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,s)u(y,s)}{|x-y|^{d-2}}dxdyds. \tag{2.102}$$

This gives the existence of a global weak solution and similar arguments as Step 7 lead to the mass conservation for $m>1-2/d$. The following steps will show that this weak solution is also a weak entropy solution with the energy inequality. Next we will consider the statement **(iv)** of Theorem 2.11.

Step 11 (Mass conservation for any $t\geq 0$ when $m>1-2/d$ or mass conservation in finite time when $(d-2)/(d+2)<m<1-2/d$ and the second moment is bounded in finite time). This step firstly shows a claim for mass conservation and then completes the statement **(i)** of Theorem 2.11.

Claim. (1) When $m>1-2/d$, the mass conservation holds for any $t\geq 0$.
(2) For $(d-2)/(d+2)<m<1-2/d$, if there exists a $\bar{T}$ such that $\bar{C}:=\sup_{0\leq t\leq\bar{T}}m_2(t)<\infty$, then we have

$$\int_{\mathbb{R}^d}u(x,t)dx\equiv\int_{\mathbb{R}^d}U_0(x)dx,\quad \text{for } 0\leq t\leq\bar{T}.$$

Proof of claim. Similar arguments as in Step 7 establishes the mass conservation of a weak solution for $1-2/d<m<2-2/d$, here we have used $\psi_R(x)=\psi(x)$ in (2.102), where $\psi_R(x)$ is defined by (2.31), due to $u_\varepsilon\in L^1(\mathbb{R}^d)$ and the facts in (2.56) and (2.57) in Step 7 one has

$$\int_0^t\|u\|_{p+1}^{p+1}ds\leq\liminf_{\varepsilon\to 0}\int_0^t\|u_\varepsilon\|_{p+1}^{p+1}ds,\quad 0<\lambda<1, \tag{2.103}$$

and also using Fatou's Lemma,

$$\int_0^t\|u\|_{2d/(d+2)}^2ds\leq\liminf_{\varepsilon\to 0}\int_0^t\|u_\varepsilon\|_{2d/(d+2)}^2ds. \tag{2.104}$$

Hence passing to the limit as $R\to\infty$ by using the dominated convergence theorem in (2.102) gives the case (1).

For the case (2), it follows from (2.57) that for $0 \le t \le \bar{T}$,

$$\left|\frac{d}{dt}\int_{\mathbb{R}^d} u(\cdot,t)\psi_R(x)dx\right| \le \frac{C(\|U_0\|_1)}{R^{2-d(1-m)}}\left(\int_{B_{2R}/B_R} udx\right)^m + C\frac{\|u\|_{2d/(d+2)}^2}{R^2}$$
$$\le \frac{C(\|U_0\|_1)}{R^{2-d(1-m)}}\left(\frac{m_2(t)}{R^2}\right)^m + C\frac{\|u\|_{2d/(d+2)}^2}{R^2} = \frac{C(\|U_0\|_1)}{R^{(d+2)m-(d-2)}}\bar{C}^m$$
$$+C\frac{\|u\|_{2d/(d+2)}^2}{R^2} \to 0 \quad \text{as } R\to\infty, \tag{2.105}$$

in the last limit we have used the condition $m > (d-2)/(d+2)$. This gives the mass conservation for $0 \le t \le \bar{T}$. This proves the case (2) and completes the claim.

Now we will prove the second moment is bounded in time provided the bounded initial second moment $\int_{\mathbb{R}^d}|x|^2U_0(x)dx$. Consider a test function $\psi_R(x) \in C_0^\infty(\mathbb{R}^d)$ and $\psi_R(x) = |x|^2$ for $|x| < R$, $\psi_R(x) = 0$ for $|x| \ge 2R$, letting $\psi = \psi_R(x)$ in (2.102) and similar arguments for mass conservation follow that

$$\int_{\mathbb{R}^d}\psi_R u(\cdot,t)dx - \int_{\mathbb{R}^d}\psi_R U_0 dx = \int_0^t\int_{\mathbb{R}^d}\Delta\psi_R u^m dxds$$
$$-\frac{c_d(d-2)}{2}\int_0^t\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{[\nabla\psi_R(x)-\nabla\psi_R(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,s)u(y,s)}{|x-y|^{d-2}}dxdyds. \tag{2.106}$$

As before, since $\Delta\psi_R(x)$ and $\frac{[\nabla\psi_R(x)-\nabla\psi_R(y)]\cdot(x-y)}{|x-y|^2}$ are bounded, thus both terms in the right-hand side of (2.106) are bounded. As a consequence, as $R\to\infty$ we can pass to the limit using the Lebesgue monotone convergence theorem with $u \in L^1(\mathbb{R}^d)$ and obtain that for any $t > 0$,

$$\int_{\mathbb{R}^d}|x|^2u(x,t)dx = \int_{\mathbb{R}^d}|x|^2U_0(x)dx + 2d\int_0^t\|u\|_{L^m(\mathbb{R}^d)}^m ds$$
$$-(d-2)\int_0^t\int_{\mathbb{R}^d} ucdxds. \tag{2.107}$$

Next we proceed to show the statement **(i)** of Theorem 2.11. Firstly we use method of contradiction to show that when $(d-2)/(d+2) < m < 1-2/d$,

$$\sup_{0<t\le T_{ext}} m_2(t) = \infty.$$

If not, then we take $\bar{T} = T_{ext}$ in the claim that we have

$$\int_{R^d} u(x,T_{ext})dx = \int_{R^d} U_0(x)dx.$$

This contradicts with the fact that $u(x,T_{ext}) = 0$ a.e. for $m < 1-2/d$. Now we take $0 < \bar{T} \le T_{ext}$ to be the first time such that

$$m_2(t) < \infty, \text{ for } 0 < t < \bar{T}, \quad \limsup_{t\to\bar{T}} m_2(t) = \infty, \tag{2.108}$$

then for any $0 < t < \bar{T}$ one has

$$m_2(t) = m_2(0) + 2d \int_0^t \int_{\mathbb{R}^d} u^m dxds - (d-2) \int_0^t \int_{\mathbb{R}^d} |\nabla c|^2 dxds, \quad (2.109)$$

when $(d-2)/(d+2) < m < 1 - 2/d$, taking $t \to \bar{T}$ one has

$$\limsup_{t \to \bar{T}} m_2(t) = m_2(0) + 2d \int_0^{\bar{T}} \int_{\mathbb{R}^d} u^m(x,t) dxdt - (d-2) \int_0^{\bar{T}} \int_{\mathbb{R}^d} |\nabla c|^2 dxdt. \quad (2.110)$$

Since $m_2(0) < \infty$, therefore (2.108) leads to $\int_0^{\bar{T}} \int_{\mathbb{R}^d} u^m(x,t)dxdt = \infty$. Recalling (1.5), this also gives that for $(d-2)/(d+2) < m < 1-2/d$, the free energy $\int_0^{\bar{T}} F(u(\cdot,t)dt = -\infty$. This completes the proof for the statement **(i)** of Theorem 2.11.

Step 12 (Strong convergence for the weak solution). For $1 < m < 2 - 2/d$, firstly the second moment estimate is applied to establish the uniform integrability of u_ε at far field. From (2.62) and (2.63) one has

$$\begin{aligned}
\frac{d}{dt} m_2(u_\varepsilon(\cdot,t)) &= 2d \int u_\varepsilon^m dx - (d-2) \int u_\varepsilon c_\varepsilon dx + 2\varepsilon d \int u_\varepsilon dx \\
&\quad + \varepsilon^2 (d-2) c_d \iint \frac{u_\varepsilon(x) u_\varepsilon(y)}{\left(|x-y|^2+\varepsilon^2\right)^{d/2}} dxdy \\
&\le 2d \int_{\mathbb{R}^d} u_\varepsilon^m dx + 2C_{HLS}(d-2) \|u_\varepsilon\|_{2d/(d+2)}^2 + 2\varepsilon d \int u_\varepsilon(x,0) dx \\
&\quad + \varepsilon (d-2) c_d C_{HLS} \|u_\varepsilon\|_{2d/(d+1)}^2
\end{aligned} \quad (2.111)$$

where we have used that $\varepsilon \iint \frac{u_\varepsilon(x)u_\varepsilon(y)}{\left(|x-y|^2+\varepsilon^2\right)^{d/2}} dxdy \le \iint \frac{u_\varepsilon(x)u_\varepsilon(y)}{\left(|x-y|^2+\varepsilon^2\right)^{(d-1)/2}} \le C_{HLS}$ $\|u_\varepsilon\|_{2d/(d+1)}^2$. Then one gets from integrating from 0 to t in time

$$\begin{aligned}
\int_{\mathbb{R}^d} |x|^2 u_\varepsilon(x,t) dx \le& \int_{\mathbb{R}^d} |x|^2 u_\varepsilon(x,0) dx + 2d \int_0^t \|u_\varepsilon\|_m^m ds \\
&+ 2C_{HLS}(d-2) \int_0^t \|u_\varepsilon\|_{2d/(d+2)}^2 ds + 2\varepsilon d \int_0^t \|u_\varepsilon(x,0)\|_1 ds \\
&+ \varepsilon (d-2) c_d C_{HLS} \int_0^t \|u_\varepsilon\|_{2d/(d+1)}^2 ds < C,
\end{aligned}$$

the last inequality follows from (2.77) by using the interpolation inequality for $1 < m < p+1$, that's

$$\begin{aligned}
\int_0^t \|u_\varepsilon\|_{2d/(d+1)}^2 ds &\le \int_0^t \|u_\varepsilon\|_{p+1}^{2\theta} \|u_\varepsilon\|_1^{2(1-\theta)} ds \\
&\le C \int_0^t \|u_\varepsilon\|_{p+1}^{2\theta} ds \le C(t) \int_0^t \|u_\varepsilon\|_{p+1}^{p+1} ds < C,
\end{aligned} \quad (2.112)$$

and

$$\int_0^t \|u_\varepsilon\|_m^m ds \le \int_0^t \|u_\varepsilon\|_{p+1}^{m\lambda} \|u_\varepsilon\|_1^{m(1-\lambda)} ds \le C(t) \int_0^t \|u_\varepsilon\|_{p+1}^{p+1} ds \le C(t), \quad 0 < \lambda < 1. \tag{2.113}$$

So for $1 \le r_0 < p+1$ using the interpolation inequality and (2.77) one has

$$\begin{aligned} \int_0^T \|u_\varepsilon\|_{L^{r_0}(|x|>R)}^{p+1} dt &\le C \int_0^T \|u_\varepsilon\|_{L^1(|x|>R)}^{(p+1)(1-\theta)} \|u_\varepsilon\|_{L^{p+1}(|x|>R)}^{(p+1)\theta} dt, \quad 0 \le \theta < 1, \\ &\le C \frac{[m_2(u_\varepsilon(\cdot,t))]^{(p+1)(1-\theta)}}{R^{2(p+1)(1-\theta)}} \int_0^T \|u_\varepsilon\|_{L^{p+1}(\mathbb{R}^d)}^{(p+1)\theta} dt \to 0 \quad \text{as } R \to \infty, \end{aligned} \tag{2.114}$$

hence the weak semi-continuity of $L^{p+1}(0,T; L^{r_0}(|x|>R))$ yields

$$\int_0^T \|u\|_{L^{r_0}(|x|>R)}^{p+1} dt \le \liminf_{\varepsilon\to 0} \int_0^T \|u_\varepsilon\|_{L^{r_0}(|x|>R)}^{p+1} dt \to 0 \quad \text{as } R \to \infty. \tag{2.115}$$

Let $r_2' = \min\{r_2, p+1\} = r_2$ and $1 \le r_0 \le \bar{p}$, $1 \le r_0 < p+1$, where r_2 and $\bar{p}$ are defined as in (2.89), the following inequality is derived that as $R \to \infty$, $\varepsilon \to 0$,

$$\begin{aligned} \int_0^T \|u_\varepsilon - u\|_{L^{r_0}(\mathbb{R}^d)}^{r_2'} dt &= \int_0^T \left[\|u_\varepsilon - u\|_{L^{r_0}(|x|>R)}^{r_0} + \|u_\varepsilon - u\|_{L^{r_0}(|x|\le R)}^{r_0} \right]^{r_2'/r_0} dt \\ &\le C(r_0, r_2') \left[\int_0^T \|u_\varepsilon\|_{L^{r_0}(|x|>R)}^{r_2'} dt + \int_0^T \|u\|_{L^{r_0}(|x|>R)}^{r_2'} dt + \int_0^T \|u_\varepsilon - u\|_{L^{r_0}(|x|\le R)}^{r_2'} dt \right] \to 0. \end{aligned} \tag{2.116}$$

In the last inequality, the first term goes to zero due to (2.114), the second term is due to (2.115) with $r_2' \le p+1$ and the third term is due to (2.89) with $r_0 \le \bar{p}$ and $r_0 < p+1$, thus one has the following strong convergence in $\mathbb{R}^d$ that

$$\begin{aligned} &u_\varepsilon \to u \quad \text{in } L^{r_2}\left(0,T; L^{r_0}(\mathbb{R}^d)\right), \quad 1 \le r_0 < \min\left(\frac{dr_2}{d-r_2}, p+1\right), \\ &r_2 = \min\left\{2, \frac{2(p+1)}{4-m}\right\}, \end{aligned} \tag{2.117}$$

after some computations one has $\min\left(\frac{dr_2}{d-r_2}, p+1\right) > 2$ for $0 < m < 2 - 2/d$.

Step 13 (Convergence of the free energy for $m > 1$). This step takes the strong convergence of the free energy into account. Firstly the following estimate holds true:

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 \to \|\nabla c(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 \quad \text{a.e. in } (0,T). \tag{2.118}$$

By the Hölder inequality one has

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\left||\nabla c_\varepsilon|^2-|\nabla c|^2\right|dxdt &= \int_0^T\int_{\mathbb{R}^d}|\nabla c_\varepsilon+\nabla c|\,|\nabla c_\varepsilon-\nabla c|dxdt\\ &\le 2\int_0^T\|\nabla c_\varepsilon\|_2\|\nabla c_\varepsilon-\nabla c\|_2dt\\ &\le 2\left[\int_0^T\|\nabla c_\varepsilon\|_2^{p+1}dt\right]^{1/(p+1)}\left[\int_0^T\|\nabla c_\varepsilon-\nabla c\|_2^{\frac{p+1}{p}}dt\right]^{p/(p+1)}\\ &\le C\left[\int_0^T\|\nabla c_\varepsilon-\nabla c\|_2^{\frac{p+1}{p}}dt\right]^{p/(p+1)},\end{aligned} \tag{2.119}$$

where (2.77) follows the last inequality, then by the interpolation inequality one has

$$\begin{aligned}\int_0^T\|\nabla c_\varepsilon-\nabla c\|_2^{\frac{p+1}{p}}dt &\le C\int_0^T\|u_\varepsilon-u\|_{2d/(d+2)}^{\frac{p+1}{p}}dt\\ &\le \int_0^T\|u_\varepsilon-u\|_1^{\frac{p+1}{p}(1-\theta)}\|u_\varepsilon-u\|_2^{\frac{p+1}{p}\theta}dt\to 0,\end{aligned} \tag{2.120}$$

where $\frac{p+1}{p}\theta=\frac{p+1}{p}\frac{d-2}{d}<r_2$, thus combining (2.119) leads to (2.118).

On the other hand, since $1<m<2-2/d<2$ and $\min\left(\frac{dr_2}{d-r_2},p+1\right)>2$, thus taking $r_0=m$ in (2.117) such that there exists a subsequence u_ε without relabeling such that

$$\|u_\varepsilon(\cdot,t)\|_{L^m(\mathbb{R}^d)}^m\to\|u(\cdot,t)\|_{L^m(\mathbb{R}^d)}^m\quad\text{a.e. in }(0,T). \tag{2.121}$$

Hence taking (2.118), (2.121) together one has that as $\varepsilon\to 0$,

$$\begin{aligned}F[u_\varepsilon(\cdot,t)] &= \int\frac{u_\varepsilon^m}{m-1}dx-\frac{1}{2}\int|\nabla c_\varepsilon|^2dx\\ &\to\int\frac{u^m}{m-1}dx-\frac{1}{2}\int|\nabla c|^2dx=F[u(\cdot,t)]\quad\text{a.e. in }(0,T).\end{aligned} \tag{2.122}$$

Since $F[u_\varepsilon(\cdot,t)]$ is decreasing in $(0,T)$, then $F[u(\cdot,t)]$ is also a decreasing function a.e. in $(0,T)$.

Step 14 (Lower semi-continuity of the dissipation term for $m>1$). For $1<m\le 2d/(d+2)\le p$, due to the initial data $U_0\in L^p(\mathbb{R}^d)$, it follows that $U_0\in L^m(\mathbb{R}^d)$ by interpolation. For $2d/(d+2)<m<2-2/d$, we have the additional assumption $U_0\in L^m(\mathbb{R}^d)$. Hence $U_0\in L_+^1\cap L^m\cap L^p(\mathbb{R}^d)$.

Denote $q:=\max(m,p)$ and similar arguments as Steps 1-6 give that for any $T>0$,

$$\|u_\varepsilon\|_{L^\infty(0,T;L_+^1\cap L^r(\mathbb{R}^d))}\le C,\ \text{ for } 1<r\le q, \tag{2.123}$$

$$\left\|\nabla u_\varepsilon^{\frac{m+r-1}{2}}\right\|_{L^2(0,T;L^2(\mathbb{R}^d))}\le C,\ \text{ for } 1<r\le q, \tag{2.124}$$

$$\|u_\varepsilon\|_{L^{r+1}(0,T;L^{r+1}(\mathbb{R}^d))}\le C,\ \text{ for } 1<r\le q. \tag{2.125}$$

Firstly for any $T > 0$, the dissipation term is uniformly bounded,

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-\frac{1}{2}} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \right|^2 dxdt \leq C. \tag{2.126}$$

Actually, the dissipation term can be recast as:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-\frac{1}{2}} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \right|^2 dxdt \\ &= \int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} \right|^2 dxdt + \int_0^T \int_{\mathbb{R}^d} u_\varepsilon |\nabla c_\varepsilon|^2 dxdt \\ &-2 \int_0^T \int_{\mathbb{R}^d} u_\varepsilon^{m+1} dxdt. \end{aligned} \tag{2.127}$$

Taking $r = m$ in (2.124) yields $\left\| \nabla u_\varepsilon^{m-1/2} \right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C$.

As to the second term, by Lemma 2.7 and using (2.123) and (2.125) with $q = \max(m, p) \geq 2d/(d+2)$ one has

$$\int_0^T \left\| u_\varepsilon |\nabla c_\varepsilon|^2 \right\|_1 dt \leq C \int_0^T \|u_\varepsilon\|_{3d/(d+2)}^3 dt \leq C \int_0^T \|u_\varepsilon\|_{2d/(d+2)}^{3(1-\theta)} \|u_\varepsilon\|_{q+1}^{3\theta} dt \leq C, \tag{2.128}$$

where $3\theta = \frac{d+2}{d+2-2d/(q+1)} \leq q+1$. Taking $q = m$ in (2.125) also derives that the third terms u_ε^{m+1} are bounded in $L^1\left(0, T; L^1(\mathbb{R}^d)\right)$. Thus (2.126) holds. Then there exists a subsequence $\frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon$ and $v \in L^2\left(0, T; L^2(\mathbb{R}^d)\right)$ such that

$$\frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \rightharpoonup v \quad \text{in } L^2\left(0, T; L^2(\mathbb{R}^d)\right). \tag{2.129}$$

By the lower semi-continuity of L^2 norm one has

$$\|v\|_{L^2(0,T:L^2(\mathbb{R}^d))} \leq \liminf_{\varepsilon \to 0} \left\| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \right\|_{L^2(0,T:L^2(\mathbb{R}^d))} \leq C. \tag{2.130}$$

So for any $T > 0$, one has

$$\int_0^T \int_{\mathbb{R}^d} |v|^2 dxdt \leq \liminf_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \right|^2 dxdt \leq C. \tag{2.131}$$

Now we will show that there exists a subsequence $\frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon$ without relabeling such that the weak limit $v = \frac{2m}{2m-1} \nabla u^{m-1/2} - \sqrt{u} \nabla c$. Since $C_0^\infty\left((0, T) \times \mathbb{R}^d\right)$ is dense in $L^2\left((0, T) \times \mathbb{R}^d\right)$, one only needs to show that for $\forall \psi \in C_0^\infty\left((0, T) \times \mathbb{R}^d\right)$,

$$\int_0^T \int_{\mathbb{R}^d} \left(\frac{2m}{2m-1} u_\varepsilon^{m-1/2} \nabla \psi + \sqrt{u_\varepsilon} \nabla c_\varepsilon \psi \right) dxdt \\ \to \int_0^T \int_{\mathbb{R}^d} \left(\frac{2m}{2m-1} u^{m-1/2} \nabla \psi + \sqrt{u} \nabla c \psi \right) dxdt. \quad (2.132)$$

For any bounded domain Ω and for $1 < m < 2 - 2/d$ by Lemma 2.8 one has

$$u_\varepsilon^{m-1/2} \to u^{m-1/2} \quad \text{in } L^1\left(0, T; L^1(\Omega)\right). \quad (2.133)$$

It leads to

$$\int_0^T \int_{\mathbb{R}^d} \frac{2m}{2m-1} u_\varepsilon^{m-1/2} \nabla \psi dxdt \to \int_0^T \int_{\mathbb{R}^d} \frac{2m}{2m-1} u^{m-1/2} \nabla \psi dxdt, \quad (2.134)$$

hence $\nabla u^{m-1/2} \in L^2\left((0,T) \times \mathbb{R}^d\right)$. This proved the regularity in Definition 2.6.

Next for any bounded domain Ω, $\sqrt{u_\varepsilon} \nabla c_\varepsilon \to \sqrt{u} \nabla c$ in $L^1\left(0, T; L^1(\Omega)\right)$ holds true. This can be shown by the following estimate:

$$\|\sqrt{u_\varepsilon} \nabla c_\varepsilon - \sqrt{u} \nabla c\|_{L^1(\Omega)} \le \left\|\sqrt{u_\varepsilon} - \sqrt{u}\right\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\mathbb{R}^d)} \\ + \left\|\sqrt{u}\right\|_{L^2(\mathbb{R}^d)} \|\nabla (c_\varepsilon - c)\|_{L^2(\mathbb{R}^d)} =: I_1 + I_2.$$

Firstly $\int_0^T I_1 dt \to 0$ follows from (2.123) by the Young inequality, and by Lemma 2.8 one has

$$\int_0^T I_1 dt \le C \int_0^T \|\sqrt{u_\varepsilon} - \sqrt{u}\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} dt \\ \le C \left(\int_0^T \|\sqrt{u_\varepsilon} - \sqrt{u}\|_{L^2(\Omega)}^{r_2} dt \right)^{\frac{1}{r_2}} \to 0.$$

As to $\int_0^T I_2 dt$, (2.123) yields

$$\int_0^T I_2 dt = \int_0^T \left\|\sqrt{u}\right\|_{L^2(\mathbb{R}^d)} \|\nabla (c_\varepsilon - c)\|_{L^2(\mathbb{R}^d)} dt \\ \le C \int_0^T \|u_\varepsilon - u\|_{L^{2d/(d+2)}(\mathbb{R}^d)} dt \to 0. \quad (2.135)$$

So one has

$$\int_{t_0}^T \int_{\mathbb{R}^d} \sqrt{u_\varepsilon} \nabla c_\varepsilon \psi dxdt \to \int_{t_0}^T \int_{\mathbb{R}^d} \sqrt{u} \nabla c \psi dxdt. \quad (2.136)$$

Combining (2.134) with (2.136) deduces (2.132). Then plugging $v = \frac{2m}{2m-1} \nabla u^{m-1/2} - \sqrt{u} \nabla c$ into (2.131) gives that for any $T > 0$,

$$\int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u^{m-1/2} - \sqrt{u} \nabla c \right|^2 dxdt \\ \le \liminf_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\varepsilon^{m-1/2} - \sqrt{u_\varepsilon} \nabla c_\varepsilon \right|^2 dxdt. \quad (2.137)$$

Step 15 (Weak entropy solution with the energy inequality). For $1 < m < 2 - 2/d$, multiplying $\mu_\varepsilon = \frac{m}{m-1}u_\varepsilon^{m-1} - c_\varepsilon$ to Eqs. (2.62) gives

$$\frac{d}{dt}F[u_\varepsilon(\cdot,t)] + \int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\varepsilon^{m-\frac{1}{2}} - \sqrt{u_\varepsilon}\nabla c_\varepsilon\right|^2 dx + \frac{4\varepsilon}{m}\int\left|\nabla u_\varepsilon^{m/2}\right|^2 dx = \varepsilon\int u_\varepsilon J_\varepsilon * u_\varepsilon dx, \tag{2.138}$$

and integrating (2.138) in time from 0 to t it follows:

$$F[u_\varepsilon(\cdot,t)] + \int_0^t\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\varepsilon^{m-\frac{1}{2}} - \sqrt{u_\varepsilon}\nabla c_\varepsilon\right|^2 dxds \le F[u_\varepsilon(\cdot,0)] + \varepsilon\int_0^t\int u_\varepsilon J_\varepsilon * u_\varepsilon dxds. \tag{2.139}$$

Recalling (2.125) and using the Hölder inequality one obtains

$$\int_0^t\int u_\varepsilon J_\varepsilon * u_\varepsilon dxds \le \int_0^t \|u_\varepsilon\|_{L^2}^2 ds \le \int_0^t \|u_\varepsilon\|_{L^{q+1}}^{2\theta}\|u_\varepsilon\|_{L^1}^{2(1-\theta)} ds \le C, \quad 0 < \theta < 1. \tag{2.140}$$

Hence combining with (2.122), (2.137) and (2.68), letting $\varepsilon \to 0$ in (2.139) it follows that

$$F[u(\cdot,t)] + \int_0^t\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u^{m-\frac{1}{2}} - \sqrt{u}\nabla c\right|^2 dxds \le F[U_0], \quad a.e.\ t > 0,$$

hence the existence of a global weak entropy solution with energy inequality is derived and thus completes the proof for the statement **(iv)** of Theorem 2.11. □

Remark 2.12. For the critical case $m = 2 - 2/d$, $p = 1$, similarly assume $\eta = \left(\frac{2dS_d}{d-1}\right)^{d/2} - \|U_0\|_1 > 0$, then similar arguments as Step 4 and Step 5 in Theorem 2.11 derive the following hyper-contractive estimate

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C(\|U_0\|_1, \eta, q)t^{-(q-1)/q}, \ 1 \le q < \infty, \quad \text{for any } t > 0. \tag{2.141}$$

Using the above inequality and the fact $m < 2$ one has

$$\int_0^t\int_{\mathbb{R}^d} u^m dxds \le C(\|U_0\|_1, \eta, q)\int_0^t t^{-(m-1)} ds \le C(t), \tag{2.142}$$

so as in Step 11 in Theorem 2.11, the second moment is bounded for all $0 \le t < \infty$ and similar arguments as Theorem 2.11 arrive at the global existence of a weak solution.

Remark 2.13. For $1 - 2/d < m \le 2 - 2/d$, $p = \frac{d(2-m)}{2}$, let U_s be the steady solutions of (1.1), multiplying pU_s^{p-1} for $p > 1$ or $\log U_s$ for $p = 1$ to (4.1) and taking $q = p$ in (2.5) one obtains

$$\frac{4mp}{(p+m-1)^2}\int\left|\nabla U_s^{(m+p-1)/2}\right|^2 dx = \int U_s^{p+1} dx \le S_d^{-1}\|\nabla U_s^{(p+m-1)/2}\|_2^2\|U_s\|_p^{2-m}; \tag{2.143}$$

this leads to $C_{d,m} \le \|U_s\|_p$.

Remark 2.14. In a series of papers [40–42], Sugiyama, etc. proved that for initial data $U_0 \in L^1 \cap L^\infty(\mathbb{R}^d), U_0^m \in H^1(\mathbb{R}^d), 0 < m \le 2-2/d, 1 < q < \infty$, if the initial data satisfies $\|U_0\|_p < C$, where C is a positive number depending on q, d, m, then there exists a weak solution with decay property in L^q and they employed Moser's iteration technique developed in Alikakos [1] to prove the time global L^∞ bound. For $0 < m < 1-2/d$, the $\|u(t)\|_q$ will vanish at finite time and for $1-2/d \le m < 2-2/d$, the solution satisfies mass conservation. Compared with their results, this paper reduces the initial regularity to $U_0 \in L^1_+(\mathbb{R}^d)$ and gives a universal constant $C_{d,m}$ such that there exists a global weak solution bounded in any $L^q(\mathbb{R}^d)$ space for $\|U_0\|_p < C_{d,m} \le \|U_s\|_p$.

Remark 2.15. For the existence of the weak entropy solution, when $1 < m < 2-2/d$, if the initial second moment is finite, then there exists a global weak entropy solution by showing the weak lower continuity of the dissipation term. As to some related results, in [40, Proposition 6.1], Sugiyama proved the non-increasing of the free energy. For $m = 1, d = 2$ in [9], Blanchet, Dolbeault and Perthame proved the existence of a global weak entropy solution provided $\|U_0\|_1 < 8\pi$.

Remark 2.16. For $m = 1$ and $d \ge 3$, [37,13] proved the global existence and decay property of weak solutions for $\|u(t)\|_{L^{d/2}}$ with small $\|U_0\|_{L^{d/2}}$ and blow-up for small initial second moment which implies large value for $\|U_0\|_{L^{d/2}}$. and their method can be adapted to prove the energy inequality in the above Step 6 for the case $m = 1, d \ge 3$.

For general initial data, the following local in time existence and blow-up criteria hold true.

Theorem 2.17. *Let* $1 < m \le 2-2/d,\ p = \frac{d(2-m)}{2}$. *Assume* $U_0 \in L^1_+ \cap L^q(\mathbb{R}^d)$ *for some* $q \ge m$ *and* $q > p$ *and the initial second moment* $\int_{\mathbb{R}^d} |x|^2 U_0(x)dx < \infty$. *Then there are* $T > 0$ *and a weak entropy solution* $u(\cdot,t)$ *in* $0 < t < T$ *to Eqs.* (1.1) *with mass conservation.*

Let $T_{\max}$ be the largest existence time for the weak entropy solution, i.e. for all $0 < t < T_{\max}$, $\|u(\cdot,t)\|_q < \infty$ and $\limsup_{t\to T_{\max}} \|u(\cdot,t)\|_q = \infty$. If $T_{\max} < \infty$, then for all $r > p$, $\limsup_{t\to T_{\max}} \|u(\cdot,t)\|_r = \infty$.

Proof. ***Step 1*** (Existence of a local in time weak entropy solution). Taking $q = r > p$ in (2.6) yields

$$\begin{aligned} &\frac{d}{dt}\|u\|_q^q + \frac{4qm(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 \\ &= (q-1)\int u^{q+1}dx \\ &\le \frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,d)\left(\|u\|_q^q\right)^\delta, \end{aligned} \tag{2.144}$$

where $\delta = 1 + \frac{1}{q-p}$. Hence the local in time estimates are followed

$$\|u\|_q^q \le \left(\frac{C(q,d)}{T_q - t}\right)^{q-p}, \quad T_q = C(q,d)\|U_0\|_q^{\frac{q}{p-q}}. \tag{2.145}$$

The proof for the regularization, compactness, existence of a weak solution and energy inequality are the same as that in Steps 8–15 of the proof for Theorem 2.11.

Step 2 (L^r estimate for $r > p$ at the largest existence time $T_{\max}$).

Claim. If $T_{\max} < \infty$ and for some $r > p$ such that $A := \limsup_{t\to T_{\max}} \|u\|_r < \infty$, then $\limsup_{t\to T_{\max}} \|u(\cdot,t)\|_q \le C(A, T_{\max}, \|U_0\|_q)$.

Proof of claim. If $1 < q < r$, the claim is directly derived by the interpolation inequality. If $p < r < q$, it follows from (2.6) that

$$\begin{aligned}
&\frac{d}{dt}\|u\|_q^q + \frac{4qm(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 \\
&= (q-1)\int u^{q+1}dx \\
&\le \frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r,d)\left(\|u\|_r^r\right)^\delta,
\end{aligned} \tag{2.146}$$

where $\delta = 1 + \frac{1+q-r}{r-p}$. Thus the upper bound of $\|u\|_q$ is obtained

$$\frac{d}{dt}\|u\|_q^q \le C(q,d)\left(\|u\|_r^r\right)^\delta \le C(A,d,r,q). \tag{2.147}$$

Hence the claim is completed. As a direct consequence of the claim one has $\limsup_{t\to T_{\max}} \|u(\cdot,t)\|_r = \infty$ for all $r > p$. This ends the proof. □

For the subcritical case, the hyper-contractive estimates also hold true:

Theorem 2.18. *For $m > 2 - 2/d$, assume $U_0 \in L^1_+(\mathbb{R}^d)$. Assume also $U_0 \log U_0 \in L^1(\mathbb{R}^d)$ for $m = 2$ and $U_0 \in L^{m-1}(\mathbb{R}^d)$ for $m > 2$, then there exists a weak solution globally in time satisfying the following hyper-contractive property that for all $q > 1$,*

$$\|u\|_q \le C(\|U_0\|_1, q, m, d) + \left[\frac{q-1}{t}\right]^{(q-1)/q}, \quad \textit{for any } t > 0. \tag{2.148}$$

In addition, if $\int_{\mathbb{R}^d} |x|^2 U_0(x)dx < \infty$ and $U_0 \in L^m(\mathbb{R}^d)$, then this weak solution is also a global weak entropy solution satisfying mass conservation.

Proof. For all $q > 1$, from (2.8) one arrives at

$$\frac{d}{dt}\|u\|_q^q \le -\frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{m+q-1}{2}}\right\|_2^2 + C(m,q)(q-1)\|u\|_1^\epsilon, \tag{2.149}$$

where $\epsilon = 1 + \frac{2q}{d(m-2+2/d)}$. Due to

$$\begin{aligned}
\left(\|u\|_q^q\right)^{\frac{q}{q-1}} &\le S_d^{-\frac{bq}{2(q-1)}}\left\|\nabla u^{\frac{m+q-1}{2}}\right\|_2^b \|u\|_1^\beta \le \frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{(m+q-1)/2}\right\|_2^2 \\
&\quad + C(m,q)\|U_0\|_1^{\frac{\frac{q^2}{q-1}-\frac{m+q-1}{2}b}{1-b/2}},
\end{aligned}$$

where $b = \frac{2dq}{dm+dq-2d+2} < 2$ for $m > 2 - 2/d$, substituting the above inequality into (2.149) one has

$$\frac{d}{dt}\|u\|_q^q \le -\left(\|u\|_q^q\right)^{q/(q-1)} + C(\|U_0\|_1, q, m, d). \tag{2.150}$$

Using Lemma 2.9 for $a = q/(q-1) > 1,\ \ \alpha = C(\|U_0\|_1, q, m, d)$ one has

$$\|u\|_q^q \le \alpha^{(q-1)/q} + \left[\frac{q-1}{t}\right]^{q-1}, \quad \text{for any } t > 0, \tag{2.151}$$

hence for $2 - 2/d < m < 2,\ U_0 \in L^1_+$, one has $\int_0^t \|u(t)\|_m^m ds < C(t)$.

For $m = 2$, by Lemma 2.3 with $q = 1$ and $m = 2$ one computes

$$\begin{aligned} &\frac{d}{dt}\left[\int_{\mathbb{R}^d} u \log u dx + 5m_2(t)\right] + \int_{\mathbb{R}^d} u^2 dx \\ &= -2\int_{\mathbb{R}^d} |\nabla u|^2 dx + 2\int_{\mathbb{R}^d} u^2 dx + 10d\int_{\mathbb{R}^d} u^2 dx - 5/2\int_{\mathbb{R}^d} ucdx \\ &\le -2\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} |\nabla u|^2 dx + C(d)\|u\|_1^2, \end{aligned} \tag{2.152}$$

then integrating from 0 to t follows

$$\begin{aligned} &\int_{\mathbb{R}^d} u \log u dx + 5m_2(t) + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxds + \int_0^t \int_{\mathbb{R}^d} u^2 dxds \\ &\le C(d)\|u\|_1^2 t + \int_{\mathbb{R}^d} U_0 \log U_0 dx + 5m_2(0). \end{aligned} \tag{2.153}$$

In addition, plugging in the fact

$$\int_{\mathbb{R}^d} u(x)|\log u(x)|dx \le \int_{\mathbb{R}^d} u(x)\log u(x)dx + 2\int_{\mathbb{R}^d} e^{-|x|^2}dx + 4\int_{\mathbb{R}^d} |x|^2 u(x)dx, \tag{2.154}$$

into (2.153) leads to

$$\begin{aligned} &\int_{\mathbb{R}^d} u(x)|\log u(x)|dx - 2\int_{\mathbb{R}^d} e^{-|x|^2}dx + m_2(t) + \int_0^t \int_{\mathbb{R}^d} u^2 dxds \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxds \\ &\le C(d)\|u\|_1^2 t + \int_{\mathbb{R}^d} U_0 \log U_0 dx + 5m_2(0), \end{aligned} \tag{2.155}$$

hence $m_2(t) + \int_0^t \int_{\mathbb{R}^d} |\nabla u|^2 dxds < C(t)$.

For $m > 2$, using the condition $U_0 \in L^{m-1}(\mathbb{R}^d)$ in (2.149) leads to $\int_0^t \|u\|_m^m dt < C(d, m, \|U_0\|_1)t + \|U_0\|_{m-1}^{m-1} < C(t)$. Combining the three cases of $m > 2 - 2/d$ and similar arguments as the proof of Step 11 for Theorem 2.11 $m_2(t) \le m_2(0) + C(t)$ follow.

The proof for the regularization, compactness, existence of a weak solution and energy inequality are similar to that in Steps 8–15 of the proof for Theorem 2.11. Here we omit the details. □

Remark 2.19. In [40], they proved that for the initial data $U_0 \in L^1_+ \cap L^\infty$, $U_0^m \in H^1(\mathbb{R}^d)$, there exists a global weak solution without any restriction on the initial data. Here it can be reduced to $U_0 \in L^1_+(\mathbb{R}^d)$ to obtain the hyper-contractive estimates in $L^q(\mathbb{R}^d)$ for all $1 < q < \infty$.

3. Blow up Behavior for the Supercritical Case

When $1 \le m < 2 - 2/d$, the aggregation is dominant at high concentration, and in this case, the solution can blow up at finite time [10,25,26,37,40,41,48]. This part presents some blow-up behaviors provided by the initial negative free energy and this blow-up condition will derive $\|U_0\|_p > C_{d,m}$ which coincides with the condition for global existence.

Actually, the blow up behavior can be analyzed through the decreasing of the second moment. The following result is more or less standard. Here we give a more detailed behavior.

Theorem 3.1. *Assume* $1 < m < 2 - 2/d$, $p = \frac{d(2-m)}{2}$, $U_0 \in L^1_+ \cap L^q(\mathbb{R}^d)$ *for some* $q \ge m$ *and* $q > p$ *and* $\int_{\mathbb{R}^d} |x|^2 U_0(x)dx < \infty$. *Let* $u(x,t)$ *be a weak entropy solution to Eqs.* (1.1)*, then it satisfies*

$$\int_{\mathbb{R}^d} |x|^2 u(x,t)dx \le \int_{\mathbb{R}^d} |x|^2 U_0(x)dx + 2(d-2)F(U_0)t. \tag{3.1}$$

If $F(U_0) < 0$, *then there exists a* $0 < T < \infty$ *such that*

$$\frac{1}{m-1}\|u\|_m^m < \frac{1}{2}\|\nabla c\|_2^2 < \infty, \quad 0 < t < T, \tag{3.2}$$

$$\limsup_{t\to T} \|\nabla c(t)\|_2^2 = \infty. \tag{3.3}$$

$$\limsup_{t\to T} \|u(t)\|_r = \infty, \quad \text{for all } r > p. \tag{3.4}$$

$$\limsup_{t\to T} \frac{d}{dt} m_2(t) = -\infty. \tag{3.5}$$

Proof. Equations (3.2) and (3.3) can be shown by contradiction. Firstly the second moment can be estimated by using the non-increasing of the free energy

$$\begin{aligned}\frac{d}{dt} m_2(u(\cdot,t)) &= 2d\int_{\mathbb{R}^d} u^m dx - (d-2)\int_{\mathbb{R}^d} ucdx \\ &= \left(2d + \frac{2(2-d)}{m-1}\right)\int_{\mathbb{R}^d} u^m dx + 2(d-2)F[u(\cdot,t)] \\ &\le \left(2d + \frac{2(2-d)}{m-1}\right)\int_{\mathbb{R}^d} u^m dx + 2(d-2)F(U_0).\end{aligned} \tag{3.6}$$

Integrating in time from 0 to t gives

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2 u(x,t)dx \le& \int_{\mathbb{R}^d} |x|^2 U_0(x)dx \\ &+ \left(2d + \frac{2(2-d)}{m-1}\right)\int_0^t \int_{\mathbb{R}^d} u^m dxds + 2(d-2)F(U_0)t,\end{aligned}$$

combining $1 < m < 2 - 2/d$ follows (3.1). Since $F(U_0) < 0$ implies that $F[u(\cdot,t)] \le F(U_0) < 0$ such that $\|u(t)\|_m^m < \frac{m-1}{2}\|\nabla c\|_2^2$, if T doesn't exist, it means that the solution exists globally for all $t > 0$ and

$$\|u\|_m^m < \frac{m-1}{2}\|\nabla c\|_2^2 < \infty \quad \text{for all } t > 0. \tag{3.7}$$

On the other hand, $F(U_0) < 0$ in (3.1) means that there is a $\hat{T} > 0$ such that $\lim_{t\to\hat{T}} m_2(t) = 0$, and using the Hölder inequality one has

$$\int_{\mathbb{R}^d} u(x)dx = \int_{|x|\le R} u(x)dx + \int_{|x|>R} u(x)dx \le C R^{(m-1)d/m}\|u\|_{L^m} + \frac{1}{R^2} m_2(t).$$

Choosing $R = \left(\frac{Cm_2(t)}{\|u\|_m}\right)^{1/(a+2)}$ with $a = (m-1)d/m$ one has $\|U_0\|_1 = \|u\|_{L^1} \le C\|u\|_m^{\frac{2}{a+2}} m_2(t)^{\frac{a}{a+2}}$, so

$$\limsup_{t\to\hat{T}} \|u\|_m \ge \lim_{t\to\hat{T}} \frac{\|U_0\|_{L^1}^{\frac{a+2}{2}}}{Cm_2(t)^{\frac{a}{2}}} = \infty, \tag{3.8}$$

which induces that

$$\limsup_{t\to\hat{T}} \|\nabla c\|_2^2 > C \limsup_{t\to\hat{T}} \|u\|_m^m = \infty. \tag{3.9}$$

This contradiction with (3.7) implies that there exists $T > 0$ such that (3.2) and (3.3) hold true.

Now we deduce (3.4) by contradiction. If for all $r > p$, $\limsup_{t\to T} \|u(t)\|_r < \infty$ as $t \to T$, then by Theorem 2.17, one has $\limsup_{t\to T} \|\nabla c(t)\|_2^2 \le C \limsup_{t\to T} \|u\|_{2d/(d+2)}^2 \le C \limsup_{t\to T} \|u\|_q^2 < \infty$ for all $q \ge m$ and $q > p$. This contradicts with (3.3). So (3.4) is proved.

By the HLS inequality and the interpolation inequality with $p < 2d/(d+2) < m$ or $m \le 2d/(d+2) < p$,

$$\frac{1}{c_d}\|\nabla c\|_2^2 \le C_{HLS}\|u\|_{\frac{2d}{d+2}}^2 \le C_{HLS}\|u\|_p^{2-m}\|u\|_m^m. \tag{3.10}$$

Besides, $\frac{d}{dt}m_2(t) = \gamma\|u\|_m^m + 2(d-2)F(u) < \gamma\|u\|_m^m < 0$ follows

$$\limsup_{t\to T} \frac{d}{dt}m_2(t) < \gamma \limsup_{t\to T} \|u\|_m^m = -\infty. \tag{3.11}$$

Thus the proof is completed. □

Remark 3.2. For $1 < m < 2-2/d$, by the HLS inequality and the interpolation inequality with $p < 2d/(d+2) < m$ or $m \le 2d/(d+2) < p$,

$$F(U_0) = \frac{1}{m-1}\|U_0\|_m^m - \frac{c_d}{2}w(U_0) \ge \|U_0\|_m^m \left(\frac{1}{m-1} - \frac{C_{HLS}c_d}{2}\|U_0\|_p^{2-m}\right); \tag{3.12}$$

combining with $F(U_0) < 0$ yields

$$\|U_0\|_p > \left[\frac{2}{(m-1)C_{HLS}c_d}\right]^{\frac{1}{2-m}} = \left[\frac{2}{(m-1)S_d^{-1}}\right]^{\frac{1}{2-m}}$$
$$\ge \left(\frac{4mp}{(p+m-1)^2 S_d^{-1}}\right)^{\frac{1}{2-m}} = C_{d,m}. \tag{3.13}$$

4. Qualitative Properties of the Steady Profiles

This section is primarily devoted to the analysis on the steady solution of (1.1). The steady equation to (1.1) is followed in the sense of distribution

$$\begin{cases} \Delta U_s^m(x) - \nabla \cdot [U_s(x)\nabla C_s(x)] = 0,\ x \in \mathbb{R}^d, \\ -\Delta C_s(x) = U_s(x),\ x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

When $m > 1$, we take C_s as the Newtonian potential,

$$C_s(x) = c_d \int_{\mathbb{R}^d} \frac{U_s(y)}{|x-y|^{d-2}} dy, \tag{4.2}$$

where c_d is defined as (1.3). It was known that for $C_s \in C^2(\mathbb{R}^d)$ with some decay properties at infinity unless U_s is compact supported, all the steady solutions U_s are radially symmetric [12,16,23,24,39,49], see Theorem 4.8 for details. When $0 < m \le 1$, we consider radial solutions. The radial solution U_s has a slow decay rate and C_s can't be defined by (4.2), thus we use $-\Delta C_s = U_s$ in $\mathbb{R}^d$ directly, see Lemma 4.7 and Theorem 4.8 for detailed derivations.

Section 4.1 gives some general properties for the steady solution. Firstly a Pohozaev-Rellich type identity is shown in Lemma 4.1 and this identity will be used to decide the constant chemical potential inside the support of the density. In Proposition 4.3, four equivalent statements hold for the steady solutions that (i) equilibrium, (ii) no dissipation, (iii) the critical point of the free energy, (iv) the chemical potential equals a constant in the support of steady density (Nash equilibrium) which is its minimum in $\mathbb{R}^d$. Section 4.2 focuses on radially symmetric properties and radial solutions. For $m > 1$, the results are well-known [12,16,23,24,39,49]. For $0 < m \le 1$, we study the radial solution and obtain the sharp decay rate for the radial steady solutions U_s, the results are new. As a consequence, the $L^{d(2-m)/2}$ norm of all the radial solutions $\|U_s\|_{d(2-m)/2}$ is finite only for $2d/(d+2) \le m \le 2 - 2/d$, while for $0 < m < 2d/(d+2)$, the L^q norm for all the radial solutions $\|U_s\|_q < \infty$ for all $q > d(2-m)/2$. We summarize the details for the radial solutions in Theorem 4.8.

4.1. Equivalent properties of steady solutions.

Lemma 4.1. (A Pohozaev-Rellich type identity for steady solutions). *Let $m > 1$. Assume $U_s \in L^m \cap L^{2d/(d+2)}(\mathbb{R}^d)$ satisfying (4.1),(4.2), then the steady solutions satisfies the following identity in the sense of distribution,*

$$\int_{\mathbb{R}^d} U_s^m dx = \frac{d-2}{2d} \int_{\mathbb{R}^d} C_s U_s dx. \tag{4.3}$$

Proof. A similar argument was conducted in Step 7 of Theorem 2.11. Consider a cut-off function $\psi_R(x) \in C_0^\infty(\mathbb{R}^d)$ and $\psi_R(x) = |x|^2$ for $|x| < R$, $\psi_R(x) = 0$ for $|x| \ge 2R$. Multiplying $\psi_R(x)$ to (4.1) we compute as before

$$\int_{\mathbb{R}^d} \Delta U_s^m \psi_R(x) dx = \int_{\mathbb{R}^d} U_s^m \Delta\psi_R(x) dx, \tag{4.4}$$

and also

$$\int_{\mathbb{R}^d} \psi_R(x)\nabla\cdot(U_s\nabla C_s)dx = \int_{\mathbb{R}^d} \nabla\psi_R(x)U_s\nabla C_s dx$$
$$= \frac{c_d(d-2)}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[\nabla\psi_R(x)-\nabla\psi_R(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy. \quad (4.5)$$

Both terms in the right-hand side of (4.4) and (4.5) are bounded $\big($because $\Delta\psi_R(x)$ and $\frac{[\nabla\psi_R(x)-\nabla\psi_R(y)]\cdot(x-y)}{|x-y|^2}$ are bounded and $U_s\in L^m\cap L^{2d/(d+2)}(\mathbb{R}^d)\big)$. Therefore as $R\to\infty$, we may pass to the limit in each term using the Lebesgue monotone convergence theorem and obtain the identity. □

Remark 4.2. For the radially symmetric case, when $1<m<2d/(d+2)$, $\|U_s\|_m=\infty$ and also $\|U_s\|_{2d/(d+2)}=\infty$. On the other hand, for $m\ge 2d/(d+2)$, $U_s\in L^{2d/(d+2)}\cap L^m(\mathbb{R}^d)$, see Theorem 4.8 and Statement 2 of Remark 4.9.

Next four equivalent statements for the steady solutions are shown and using the above identity one obtains the constant chemical potential inside the support of the steady solution.

Proposition 4.3 *(Four equivalent statements for the steady state). Let $m\ge\frac{2d}{d+2}$ and $\Omega\subset\mathbb{R}^d$ be a connected open set. Assuming that $U_s\in L^1_+\cap L^m(\mathbb{R}^d)$ is bounded with $\int_{\mathbb{R}^d}U_s dx=M$, $U_s\in C(\bar\Omega)$ and $U_s>0$ in Ω, $U_s=0$ in $\mathbb{R}^d\setminus\Omega$.*

Assume also $C_s\in C^2(\mathbb{R}^d)$ is the Newtonian potential (4.2) satisfying the following equation in the sense of distribution

$$\Delta U_s^m-\nabla\cdot(U_s\nabla C_s)=0 \quad in\ \mathbb{R}^d, \quad (4.6)$$
$$\mu_s=\frac{m}{m-1}U_s^{m-1}-C_s \quad in\ \mathbb{R}^d. \quad (4.7)$$

Moreover, if Ω is unbounded, assume that U_s decays at infinity. Then the following four statements are equivalent:

- *(i) Equilibrium (definition of weak steady solutions): $\mu_s\in H^1(\mathbb{R}^d)$ and $\nabla\cdot[U_s\nabla\mu_s]=0$ in $H^{-1}(\mathbb{R}^d)$.*
- *(ii) No dissipation: $\int_\Omega U_s\left|\nabla\mu_s\right|^2dx=0$.*
- *(iii) U_s is a critical point of $F(u)$.*
- *(iv) Define a constant*

$$\bar C=\frac{1}{M}\left[\left(\frac{1}{m-1}-\frac{d+2}{d-2}\right)\|U_s\|_m^m\right]\le 0. \quad (4.8)$$

Then one has the chemical potential

$$\begin{cases}\mu_s(x)=\bar C & \forall x\in Supp(U_s),\\ \mu_s(x)\ge\bar C & \forall x\in\mathbb{R}^d.\end{cases} \quad (4.9)$$

For $m=2d/(d+2)$, $\bar C=0$ and then Ω is unbounded. If $m>2d/(d+2)$, then U_s is compactly supported.

Proof. (i) $\Rightarrow$ (ii): Since $\nabla \cdot [U_s \nabla \mu_s] = 0$ in $H^{-1}(\mathbb{R}^d)$ and $\mu_s \in H^1(\mathbb{R}^d)$, by virtue of $C_0^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and U_s is bounded, one has

$$0 = \int_{\mathbb{R}^d} \mu_s \nabla \cdot (U_s \nabla \mu_s) dx = -\int_{\mathbb{R}^d} U_s |\nabla \mu_s|^2 dx = -\int_\Omega U_s |\nabla \mu_s|^2 dx. \quad (4.10)$$

(ii) is directly from (iv).

Since μ_s is a harmonic function in $\mathbb{R}^d \setminus \Omega$, $\mu_s \to 0$ as $|x| \to \infty$, from (4.8), if $\mu_s = \bar{C} \le 0$ on $\partial\Omega$, by maximum principle one knows that $\mu_s \ge \bar{C}$ in $\mathbb{R}^d$. Therefore next we will only show (ii) and (iii) can derive $\mu_s = \bar{C}$ in Ω.

Before showing (iii) $\Leftrightarrow$ (iv), define the critical point of $F(u)$: For $\forall \varphi \in C_0^\infty(\Omega)$, let $\overline{\Omega}_0 =$ supp φ with $\int_\Omega \varphi(x) dx = 0$, $\Omega_0 \subset\subset \Omega$. There exists

$$\varepsilon_0 := \frac{\min_{y \in \overline{\Omega}_0} U_s(y)}{\max_{y \in \overline{\Omega}_0} |\varphi(y)|} > 0,$$

such that $U_s + \varepsilon\varphi \ge 0$ in Ω for $0 < \varepsilon < \varepsilon_0$. Now U_s is a critical point of $F(u)$ in Ω if and only if

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(U_s + \varepsilon\varphi) = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (4.11)$$

The above definition derives

$$\int_\Omega \left(\frac{m}{m-1} U_s^{m-1} - C_s \right) \varphi dx = 0, \quad \text{for } \forall \varphi \in C_0^\infty(\Omega). \quad (4.12)$$

Then one has in the support of U_s,

$$\frac{m}{m-1} U_s^{m-1} - C_s = \bar{C}, \quad a.e. \text{ in } \Omega. \quad (4.13)$$

Taking the inner product to (4.13) by U_s yields

$$\bar{C} = \frac{1}{M} \int_\Omega \left(\frac{m}{m-1} U_s^m - U_s C_s \right) dx. \quad (4.14)$$

On the other hand, using (4.3) we obtain the constant $\bar{C}$ is

$$\bar{C} = \frac{1}{M} \left(\frac{m}{m-1} \|U_s\|_m^m - \int_{\mathbb{R}^d} C_s U_s dx \right) = \frac{1}{M} \left[\left(\frac{1}{m-1} - \frac{d+2}{d-2} \right) \|U_s\|_m^m \right] \text{ in } \Omega. \quad (4.15)$$

When $m > 2d/(d+2)$, then $\bar{C} < 0$ by (4.15) and Ω is bounded. If Ω is unbounded, using the fact $C_s \to 0$ at infinity, it follows from (4.14) that $\bar{C} = 0$ in Ω. This contradicts with $\bar{C} < 0$, thus U_s is compactly-supported. For $m = 2d/(d+2)$, $\bar{C} = 0$ from (4.15). If Ω is bounded, then it again follows from (4.14) that $\bar{C} < 0$ at the boundary of Ω, and this contradiction with $\bar{C} = 0$ implies Ω is unbounded.

(ii) $\Rightarrow$ (iv): Suppose $\int_\Omega U_s \left| \nabla \mu_s \right|^2 dx = 0$. It follows from $U_s > 0$ at any point $x_0 \in \Omega$ that $\nabla \mu_s = 0$ in a neighborhood of x_0 and thus μ_s is constant in this neighborhood. By the connectedness of Ω one has $\mu_s \equiv \bar{C}$ in Ω.

Hence we complete the proof for (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv).

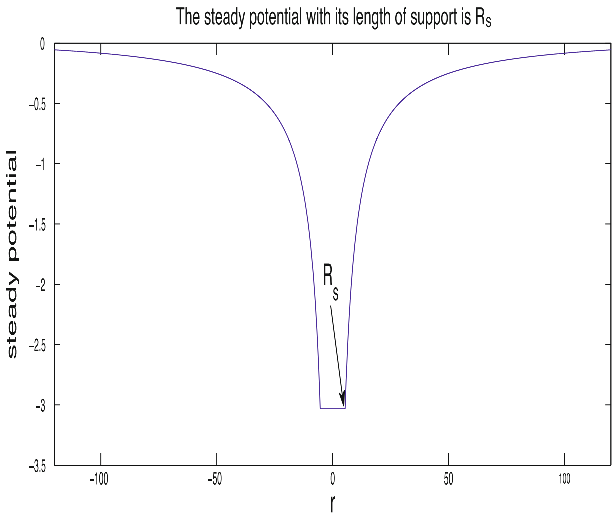


Fig. 1. The steady state chemical potential for $m > 2d/(d+2)$: a constant inside the support of the density and a Newtonian potential outside the support

(iv) $\Rightarrow$ (i): From (2.1), $\|\nabla C_s\|_2^2 = Cw(U_s) \le C\|U_s\|^2_{L^{2d/(d+2)}(\mathbb{R}^d)} < \infty$, due to $2d/(d+2) \le m < \infty$ and $U_s \in L^1_+ \cap L^m(\mathbb{R}^d)$. Hence $\nabla C_s \in L^2(\mathbb{R}^d)$. On the other hand, one obtains

$$\mu_s = -C_s \ \text{ in } \mathbb{R}^d \setminus \Omega, \quad \mu_s \in C(\mathbb{R}^d), \quad \mu_s = \text{constant} \ \text{ in } \bar{\Omega}.$$

Consequently (i) follows from $\nabla \mu_s \in L^2(\mathbb{R}^d)$ and $U_s \nabla \mu_s = 0$ in $L^2(\mathbb{R}^d)$. This ends the proof. □

Remark 4.4. The steady solution satisfying (4.9) gives a connection to the Nash equilibrium [19]. Indeed in the mean field potential games theory with the chemical potential μ_s is a constant function for all individual player. Equation (4.9) gives an equivalent definition of Nash equilibrium, see Fig. 1 for the radial chemical potential which exhibits this property.

Remark 4.5. The free energy of steady state solutions follows from the identity (4.3),

$$F(U_s) = \left(\frac{1}{m-1} - \frac{d}{d-2}\right)\|U_s\|_m^m \begin{cases} >0, \ 1<m<2-2/d, \\ =0, \ m = 2-2/d, \\ <0. \ m > 2-2/d. \end{cases} \tag{4.16}$$

Particularly, for $m = 2d/(d+2)$ and $m = 2-2/d$, the steady state free energy is an invariant which only depends on d, m. Steady state solutions for thin film equation also have similar properties, refer to [32].

Remark 4.6. Let R_s be the support of the radially steady solution, then the constant chemical potential can be derived by the mass M,

$$\mu_s(r) = \begin{cases} \frac{-c_d M}{R_s^{d-2}}, & r \le R_s, \\ \frac{-c_d M}{r^{d-2}}, & r > R_s, \end{cases}$$

where c_d is the same as defined in (1.3).

4.2. Existence and uniqueness of steady state solutions. This section explores the existence and uniqueness of the steady solutions to the system (1.1) as well as its radially symmetry. Firstly define

$$\Omega = \{x \in \mathbb{R}^d \mid U_s(x) > 0\}. \tag{4.17}$$

For simplicity, assume Ω is a connected set. For the results of the general open set with a countable number of connected components, see [39]. By virtue of Proposition 4.3, one knows that $U_s \in C(\bar{\Omega})$ satisfies that for $m \neq 1$,

$$\begin{cases} \frac{m}{m-1} U_s^{m-1} - C_s = \bar{C}, \quad \text{in } \Omega, \\ U_s = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega, \quad U_s > 0 \quad \text{in } \Omega, \\ -\Delta C_s = U_s, \quad \text{in } \mathbb{R}^d. \end{cases} \tag{4.18}$$

As mentioned at the beginning of this section, when $m > 1$, C_s is given by the Newtonian potential (4.2) in $\mathbb{R}^d$. When $m = 1$, the steady equation becomes

$$\begin{cases} \log U_s - C_s = \bar{C}, \quad \text{in } \Omega, \\ U_s = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega, \quad U_s > 0 \quad \text{in } \Omega, \\ -\Delta C_s = U_s, \quad \text{in } \mathbb{R}^d. \end{cases} \tag{4.19}$$

Letting $\phi = \log U_s$ in (4.19), the steady equation (4.19) reduces to

$$-\Delta\phi = e^{\phi} \text{ in } \mathbb{R}^d. \tag{4.20}$$

While, for $m \neq 1$, taking $\phi = \frac{m-1}{m}\left(C_s + \bar{C}\right)$ and plugging it into (4.18) yields that

$$\begin{cases} -\Delta\phi = \frac{m-1}{m}\phi^k, \quad \text{in } \Omega, \quad k = \frac{1}{m-1}, \\ \phi = 0 \quad \text{on } \partial\Omega, \quad \phi > 0, \quad \text{in } \Omega. \end{cases} \tag{4.21}$$

When $\Omega = \mathbb{R}^d$, the second boundary condition in the second line above is removed and ϕ can be unbounded at far field. Note that the sign changes on the right-hand side of (4.21) from $0 < m < 1$ to $m > 1$. That's ϕ is sub-harmonic for $0 < m < 1$ and super-harmonic for $m \geq 1$, and ϕ increases to infinity at infinity for $0 < m < 1$ while when $m > 1$, it goes to zero at finite R or infinity. When $m = 1$, one can see from (4.20) that ϕ goes to negative infinity at infinity. See Lemma 4.7 below for more detailed behaviors.

When $\Omega = \mathbb{R}^d$, it's well-known [16, Thm. 2] that for $\phi \in C^2(\mathbb{R}^d)$ in (4.21), there is no positive $\phi(x)$ in $\mathbb{R}^d$ when $m > 2d/(d+2)$, therefore all nonnegative solutions U_s are compact supported and radially symmetric up to translation. In this case, Ströhmer [39, Thm. 4] showed that U_s, C_s are both spherically symmetric and the domain Ω is a ball centered at zero up to translation. Notice that Eq. (4.21) isn't equivalent to (4.18) for the non-radially symmetric domain. Indeed, for any bounded regular domain Ω, there exists a positive solution to Eq. (4.21) for $m > 2d/(d+2)$. See [35, Thm. 1.1] for the case $2d/(d+2) < m < 2$ and [35, Rem. 1.9] or [5] for $m > 2$. When Ω is not a ball, these positive solutions are not radially symmetric, hence they are not the Newtonian potential as it's mentioned for Ströhmer's results [39] above.

Now for $m > 2d/(d+2)$, without loss of generality assuming the center is 0 such that the domain $\Omega = B(0, R)$ for some $R > 0$. Hence Eq. (4.18) is equivalent to the following equation

$$\begin{cases} -\Delta\phi = \frac{m-1}{m}\phi^{\frac{1}{m-1}}, \quad \text{in } B(0, R), \\ \phi > 0, \quad \text{in } B(0, R), \\ \phi = 0, \quad \text{on } \partial B(0, R). \end{cases} \tag{4.22}$$

Set the radial symmetric solution $U_s = \phi^{\frac{1}{m-1}}$ for $|x| \le R$ and $U_s(x) = 0$ for $|x| > R$. Let C_s be the Newtonian potential solving $-\Delta C_s = U_s$ in $\mathbb{R}^d$ and thus C_s is also radially symmetric. When $R < \infty$, C_s is a constant on $|x| = R$ denoted as $-\bar{C}$. Thus

$$\Delta\left(\frac{m}{m-1}\phi - C_s - \bar{C}\right) = 0 \quad \text{in } |x| < R,$$
$$\frac{m}{m-1}\phi - C_s - \bar{C} = 0, \quad \text{on } |x| = R.$$

Hence $\frac{m}{m-1}\phi - C_s - \bar{C} = 0$ in $B(0, R)$ by the maximum principle and thus (4.22) is equivalent to (4.18).

When $m \neq 1$, the nonnegative radial classical solution of (4.21) can be written in the form $\phi(x) = \phi(r)$, thus for any $a > 0$, letting $L = \{r \mid \phi(r) \ge 0\}$, $\phi(r) \in C^2(L)$ satisfies the following initial value equation

$$\begin{cases} \phi_{rr} + \frac{d-1}{r}\phi_r = -\frac{m-1}{m}\phi^k, \quad r > 0, \quad k = \frac{1}{m-1}, \\ \phi'(0) = 0, \quad \phi(0) = a > 0. \end{cases} \tag{4.23}$$

Notice that $\phi(r)^k$ is meaningful before it reaches zero.

When $m = 1$, from (4.20), the radial solution $\phi(r)$ satisfies the following initial value equation

$$\begin{cases} \phi_{rr} + \frac{d-1}{r}\phi_r = -e^{\phi}, \quad r > 0, \\ \phi'(0) = 0, \quad \phi(0) = a. \end{cases} \tag{4.24}$$

Indeed, the uniqueness and existence of the solutions to Eq. (4.23) for $m > 1$ are widely studied [20,22,32,36]. For $d = 3$, Eq. (4.23) is relevant to the stellar structure in astrophysics [15], and Chandrasekhar derived the sharp upper and lower bound at infinity by phase-plane analysis for $1 < m < 6/5$ [15, p. 143, formula (308)] and [15, p. 164, formula (438)] for $m = 1$. For higher dimensions and $1 < m < 2d/(d+2)$, these solutions are also similarly obtained by analyzing the phase-space [29,22]. In the following Lemma 4.7, the results for the fast diffusion case $0 < m < 1$ are new. While the results for $m > 1$ are well known, here it gives an elementary proof for the decay property when $m = 1$ and $m > 1$.

Lemma 4.7. ***(i)*** *For $0 < m < 1$ and any $a > 0$, there is a unique positive strictly increasing solution $\phi(r) \in C^2[0, \infty)$ to ODE (4.23). Furthermore, $\phi(r)$ has the sharp lower and upper bounds*

$$C_1(m, d, a)\left(1 + r^2\right)^{\frac{1-m}{2-m}} \le \phi(r) \le C_2(d, m, a)\left[1 + r^{\frac{2(1-m)}{2-m}}\right] \text{ for all } r \ge 0. \tag{4.25}$$

(ii) *For $m = 1$ and any $a \in \mathbb{R}$, there is a unique decreasing solution $\phi \in C^2([0, \infty))$ to ODE (4.24) possessing a sharp upper bound and lower bound*

$$\phi(r) \le -\ln\left[e^{-a} + \frac{r^2}{2d}\right] \quad \text{for all } r \ge 0, \tag{4.26}$$

$$\phi(r) \ge a - e^a/(2d) - \frac{2d}{d-2}(\ln r)_+ \quad \text{for all } r \ge 0. \tag{4.27}$$

(iii) *For $m > 1$ and any $a > 0$, there is a unique positive strictly decreasing solution $\phi(r)$ to ODE (4.23) before $\phi(r)$ reaches zero at finite R or $\phi(r) \to 0$ as $r \to \infty$. Furthermore, if $\phi(r)$ reaches zero at finite R, then $\phi(r) \in C^2[0, R]$, otherwise $\phi(r) \in C^2([0, \infty))$. Moreover,*

(a) *For $1 < m < 2 - 2/d$, $\phi(r)$ has a sharp upper bound $\phi(r) \le C_1 r^{-\frac{2(m-1)}{2-m}}$ with $C_1 = \left[\frac{2dm}{2-m}\right]^{\frac{m-1}{2-m}}$ for all $r \ge 0$.*

(b) *For $1 < m \le 2d/(d+2)$, $\phi(r)$ is positive for all $r \ge 0$ and it has a lower bound $\phi(r) \ge C(a, m) r^{-(d-2)}$ for large r.*

Proof. The proof can be divided into 4 steps.

Step 1 (Existence and uniqueness for $m > 0$). At $r = 0$, the ODE (4.23) is not continuous in r. Hence we convert the ODE (4.23) to an integral equation and then applying the fixed point theorem prove that there is a unique solution near $r = 0$. Denote $C(m) := \frac{m-1}{m}$.

Here we take the case $m > 1$ for example, when $0 < m \le 1$, similar arguments with that for the case $m > 1$ can derive the existence and uniqueness of $\phi(r)$ for $r \ge 0$ and $\phi(r) \in C^2([0, \infty)$. Equation (4.23) can be recast as

$$\phi'(r) = -\frac{C(m) \int_0^r \phi(t)^k t^{d-1} dt}{r^{d-1}} < 0. \tag{4.28}$$

Hence ϕ is a decreasing function of r. Integrating the above equation gives

$$\begin{aligned} \phi(r) &= a - C(m) \int_0^r \frac{1}{s^{d-1}} \int_0^s \phi(t)^k t^{d-1} dt ds \\ &= a - \frac{C(m)}{d-2} \int_0^r \left(1 - \left(\frac{t}{r}\right)^{d-2}\right) t \phi(t)^k dt. \end{aligned} \tag{4.29}$$

Define $F(\phi) = a - \frac{C(m)}{d-2} \int_0^r \left(1 - \left(\frac{t}{r}\right)^{d-2}\right) t\phi(t)^k dt$, and $B = \left\{\phi \in C[0, r_0] : \frac{1}{2}a \le \phi(r) \le a\right\}$, B is a subset of Banach space $C[0, r_0]$, where r_0 is a constant to be determined. For $\phi \in B$, one has

$$a \ge F(\phi) = a - \frac{C(m)}{d-2} \int_0^r \left(1 - \left(\frac{t}{r}\right)^{d-2}\right) t\phi(t)^k dt \ge a - \frac{C(m)a^k}{2d} r_0^2. \tag{4.30}$$

Choosing $r_0 < \sqrt{\frac{a^{1-k}d}{C(m)}}$, then $F(\phi)$ is a mapping from B to B.

Next for any $\varphi, \psi \in B$,

$$F(\varphi) - F(\psi) = -\frac{C(m)}{d-2} \int_0^r \left(1 - \left(\frac{t}{r}\right)^{d-2}\right) t \left(\varphi(t)^k - \psi(t)^k\right) dt, \tag{4.31}$$

it follows that

$$\|F(\varphi) - F(\psi)\|_{C[0,r_0]} \le \frac{C(m)2ka^{k-1}}{2d} r_0^2 \|\varphi - \psi\|_{C[0,r_0]}.$$

Taking $r_0 < \min\left(\sqrt{\frac{2d}{C(m)2ka^{k-1}}}, \sqrt{\frac{a^{1-k}d}{C(m)}}\right)$ such that $F(\phi)$ is a contraction mapping from B to B. Consequently there is a unique solution for $F(\phi) = \phi$, i.e., ϕ solves ODE (4.23) in $(0, r_0)$. Since ODE (4.23) has Lipschitz continuity as long as $\phi(r) > 0$ and $r > 0$. By the extension theorem of ODE, there is a unique solution to (4.23) when $\phi(r) > 0$.

On the other hand, L'Hopital rule and (4.23) lead to $\phi''(0) = -\frac{C(m)a^k}{d}$. If $\phi(r)$ reaches zero at finite R, then from (4.23) and (4.28), $\phi''(R) = -\frac{d-1}{R}\phi'(R) = \frac{d-1}{R^d}C(m)\int_0^R \phi(t)^k t^{d-1}dt < \infty$, thus $\phi(r) \in C^2([0, R])$. Furthermore, if $\phi(r)$ doesn't reach zero at finite R, then

$$\phi(r) \to 0 \quad \text{as } r \to \infty. \tag{4.32}$$

This result can be argued by contradiction. Suppose $\phi(r)$ has a limit at infinity, denote $\lim_{r\to\infty}\phi(r)$ as ϕ_∞, if $\phi_\infty > 0$, by (4.28) and taking the decreasing of $\phi(r)$ into account it follows that as $r \to \infty$,

$$\phi'(r) = -\frac{C(m)\int_0^r \phi(t)^k t^{d-1}dt}{r^{d-1}} \le -C(m)\phi_\infty^k r/d. \tag{4.33}$$

Hence $\phi'(r) \to -\infty$ as $r \to \infty$ this contradicts with $\lim_{r\to\infty}\phi(r) = \phi_\infty > 0$ and thus (4.32) holds true.

Step 2 (The sharp upper bound and sharp lower bound for $0 < m < 1$). It follows from (4.28) by similar computations that

$$\phi'(r) = \frac{1-m}{m}\frac{\int_0^r \phi^{1/(m-1)}r^{d-1}dr}{r^{d-1}} > 0, \tag{4.34}$$

then by the increasing of $\phi(r)$ one has

$$\begin{aligned}\phi'(r) &= \frac{1-m}{m}\frac{\int_0^r \phi(t)^{1/(m-1)}t^{d-1}dt}{r^{d-1}}\\ &\ge \frac{1-m}{m}\frac{\phi(r)^{1/(m-1)}\int_0^r t^{d-1}dt}{r^{d-1}} = \frac{1-m}{dm}\phi(r)^k r.\end{aligned} \tag{4.35}$$

It's equivalent to the following ODE

$$[\phi^{1-k}]' \ge (1-k)\frac{1-m}{dm}r, \tag{4.36}$$

therefore integrating (4.36) from 0 to r follows

$$\phi(r) \ge \left[\phi(0)^{\frac{2-m}{1-m}} + \frac{2-m}{2dm}r^2\right]^{\frac{1-m}{2-m}}, \quad \text{for } r \ge 0. \tag{4.37}$$

On the other hand, substituting (4.37) into (4.34) one has

$$\begin{aligned}\phi' &\le \left(\frac{2-m}{2dm}\right)^{1/(2-m)}\frac{1-m}{m}\frac{\int_0^r \frac{r^{d-1}}{r^{2/(2-m)}}dr}{r^{d-1}}\\ &= \left(\frac{2-m}{2dm}\right)^{1/(2-m)}\frac{1-m}{m(2d-dm-2)}r^{-m/(2-m)};\end{aligned} \tag{4.38}$$

integrating from 0 to r one has

$$\phi(r) \le \left(\frac{2-m}{2dm}\right)^{\frac{3-m}{2-m}} \frac{1}{2-2/d-m} r^{\frac{2(1-m)}{2-m}} + \phi(0), \quad \text{for } r \ge 0. \tag{4.39}$$

This completes the case for $0 < m < 1$.

Step 3 (The sharp upper bound for $m = 1$). We consider the ODE (4.24). Similarly from (4.24) one has

$$\phi'(r) = -\frac{\int_0^r e^{\phi} r^{d-1} dr}{r^{d-1}} < 0, \tag{4.40}$$

using the decreasing of $\phi(r)$ follows

$$\phi'(r) = -\frac{\int_0^r e^{\phi} r^{d-1} dr}{r^{d-1}} \le -e^{\phi} r/d. \tag{4.41}$$

Letting $\varphi(r) = e^{\phi(r)}$ yields $\frac{\varphi'}{\varphi^2} \le -r/d$, and after some computations one gets

$$e^{\phi} = \varphi \le \frac{1}{e^{-\phi(0)} + \frac{r^2}{2d}}, \quad \text{for } r \ge 0. \tag{4.42}$$

On the other hand, plugging (4.42) into (4.40) one has

$$\phi'(r) = -\frac{\int_0^r \varphi(t) t^{d-1} dt}{r^{d-1}} \ge -\frac{2d}{d-2}\frac{1}{r}. \tag{4.43}$$

Integrating from 1 to r leads to $\phi(r) \ge -\frac{2d}{d-2}\log r + \phi(1)$. Using (4.40) again we have $\phi(1) \ge a - \frac{e^a}{2d}$. For $0 < r \le 1$, $\phi(r) \ge \phi(1)$, combining the two cases yields the lower bound (4.27).

Step 4 (The lower bound and sharp upper bound for $m > 1$). The upper bound of $\phi(r)$ is derived similarly as [12]. By (4.28) and the decreasing of $\phi(r)$ one has

$$\phi'(r) = -\frac{C(m)\int_0^r \phi(t)^k t^{d-1} dt}{r^{d-1}} \le -C(m)\phi(r)^k r/d. \tag{4.44}$$

Therefore $\phi(r)^{-k}\phi'(r) \le -C(m)r/d$. Integrating this from 0 to r follows $\phi^{1-k} \ge \phi(0)^{1-k} + \frac{C(m)(k-1)}{2d} r^2$. This shows

$$\phi(r) \le C_1 r^{-2/(k-1)} \quad \text{for all } r \ge 0. \tag{4.45}$$

Part **(ii)**: For the lower bound, (4.29) can be written as

$$\phi(r) = a - \frac{C(m)}{d-2}\int_0^r t\phi(t)^k dt + \frac{C(m)}{d-2}\int_0^r \left(\frac{t}{r}\right)^{d-2} t\phi(t)^k dt, \tag{4.46}$$

and in view of (4.45) one has

$$\int_0^r \left(\frac{t}{r}\right)^{d-2} t\phi(t)^k dt \le C\frac{1}{r^{d-2}}\int_0^r t^{d-1} t^{-2k/(k-1)} dt = C\frac{1}{r^{2/(k-1)}}. \tag{4.47}$$

Besides, simple computations yield the following type of Pohozaev identity [12] for $\phi(r) \geq 0$:

$$\frac{d-2}{d}\phi(r)\phi'(r)r^{d-1} + \frac{1}{d}r^d\phi'(r)^2 + \frac{2C(m)}{d(k+1)}r^d\phi^{k+1}(r)$$
$$= C(m)\frac{m(d+2)-2d}{dm}\int_0^r \phi^{k+1}s^{d-1}ds. \tag{4.48}$$

In fact, Caffarelli etc. [12] used this identity to show that ϕ reaches zero at finite R when $m > 2d/(d+2)$. This identity can also derive (4.32). For $1 < m < 2d/(d+2)$, if $\phi(r)$ reaches zero at finite R, then the left-hand side of (4.48) is nonnegative, while when the right-hand side is negative this contradiction implies (4.32). Similarly, using (4.28) by contradiction also gives that (4.32) doesn't hold true for $m > 2d/(d+2)$.

Hence for $1 < m \leq 2d/(d+2)$, ϕ satisfies (4.32), thus taking $r \to \infty$ and combining (4.32), (4.46) and (4.47) derives $a = \frac{C(m)}{d-2}\int_0^\infty t\phi(t)^k dt$. Thus for $r \geq r_0$, where r_0 is the same as (4.30),

$$\phi(r) = \frac{C(m)}{d-2}\int_r^\infty t\phi(t)^k dt + \frac{C(m)}{d-2}\int_0^r \left(\frac{t}{r}\right)^{d-2} t\phi(t)^k dt$$
$$\geq \frac{C(m)}{d-2}\frac{1}{r^{d-2}}\int_0^{r_0} t^{d-1}\phi(t)^k dt = \frac{(a/2)^k C(m)}{(d-2)d}\frac{r_0^d}{r^{d-2}}.$$

Thus completes this lemma. □

Now applying the results of Lemma 4.7 and the well-known results for Eq. (4.21) with $m > 1$ we summarize the results for (4.18) and (4.19) into one theorem:

Theorem 4.8. *Let $m > 0$ and $p = \frac{d(2-m)}{2}$. Assuming $U_s \in C^0(\mathbb{R}^d)$, $C_s \in C^2(\mathbb{R}^d)$ satisfy Eqs. (4.18) and (4.19) in the sense of distribution. When $m > 1$, we also assume C_s is the Newtonian potential given by (4.2). Then for any $U_s(0) > 0$,*

- ***(i)*** *If $0 < m < 1$, then $\Omega = \mathbb{R}^d$ and every positive radial solution U_s has a sharp decay rate up to translation,*

$$U_s(r) \sim C(d,m)\left(1+r^2\right)^{-\frac{1}{2-m}} \quad \textit{for all } r \geq 0. \tag{4.49}$$

Thus $\|U_s\|_q < \infty$ for $q > p$ and $\|U_s\|_q = \infty$ for $1 \leq q \leq p$. Furthermore, C_s can't be defined by the Newtonian potential (4.2) and

$$C_s(x) \sim -C(d,m)r^{\frac{2(1-m)}{2-m}}, \quad \textit{for large } r. \tag{4.50}$$

Moreover, the steady free energy is negative infinity.

- ***(ii)*** *If $m = 1$, then $\Omega = \mathbb{R}^d$ and every positive radial solution U_s of (4.19) has a lower bound and a sharp upper bound: $U_s(0)e^{-U_s(0)/(2d)}\min\left(1, r^{-2d/(d-2)}\right) \leq U_s(r) \leq \frac{1}{\frac{1}{U_s(0)}+\frac{r^2}{2d}}$. Thus $\|U_s\|_q < \infty$ for $q > p$, and when $d \geq 4$, one has $\|U_s\|_1 = \infty$. Furthermore,*

$$C_s(x) \sim -C(d,m)\ln r, \quad \textit{for large } r. \tag{4.51}$$

(iii) *If* $1 < m < 2d/(d+2)$*, then* $\Omega = \mathbb{R}^d$*. Moreover, let* $k = \frac{1}{m-1}$*,* $C_1(m) = \left(\frac{m-1}{m}\right)^k$ *and*

$$C_0 = \left[\frac{2(d-2)}{C_1(m)(k-1)^2}\left(k - \frac{d}{d-2}\right)\right]^{1/(k-1)}. \tag{4.52}$$

(a) *When* $d \geq 3$ *and* $\frac{2(d-1)}{d+1} < m < 2d/(d+2)$*, if* C_s *has the decay rate* $C_s(x) = O\left(|x|^{-2/(k-1)}\right)$ *as* $|x| \to \infty$*, then all the positive solutions* C_s, U_s *of* (4.18) *are radially symmetric up to translation.*

(b) *When* $d \geq 5$ *and* $1 < m \leq 2 - 4/d$*, every positive solution of* (4.18) *is radially symmetric up to translation if and only if* C_s *satisfies* $\lim_{|x|\to\infty} |x|^{2/(k-1)} C_s(x) = C_0$*.*

(c) *When* $d \geq 4$ *and* $2 - 4/d < m \leq \frac{2(d-1)}{d+1}$*, all the positive solutions* C_s, U_s *are radially symmetric up to translation if and only if when* $\alpha = 4/(k-1) + 4 - 2d$*,*

$$\lim_{|x|\to\infty} |x|^{2/(k-1)} C_s(x) - C_0 = 0 \quad \textit{and}$$
$$\lim_{|x|\to\infty} |x|^{1-(\alpha+d)/2}\left(|x|^{2/(k-1)} C_s(x) - C_0\right) = 0. \tag{4.53}$$

Furthermore, for $d \geq 3$*, all the radial solutions of* (4.18) *are unique up to translation and decay near infinity at the rate of*

$$U_s(r) \sim C_1(m) C_0^k \frac{1}{r^{2/(2-m)}}. \tag{4.54}$$

Hence $\|U_s\|_q < \infty$ *for* $q > p$*,* $\|U_s\|_q = \infty$ *for* $1 \leq q \leq p$ *and then* $F(U_s) = \infty$*.*

(iv) *If* $m = 2d/(d+2)$*, then* $\Omega = \mathbb{R}^d$ *and every positive solution* U_s *uniquely assumes the radially symmetric form in* $\mathbb{R}^d$ *up to translation,*

$$U_s(r) = 2^{(d+2)/4} d^{(d+2)/2} \left[\frac{\lambda}{\lambda^2 + r^2}\right]^{\frac{2}{2-m}}, \quad \lambda > 0. \tag{4.55}$$

Thus $\|U_s\|_q < \infty$ *for all* $q \geq 1$*. Moreover,* $\|U_s\|_p$ *is a universal constant only depending on* d *and* $F(U_s)$ *is also a constant.*

(v) *If* $m > 2d/(d+2)$*, then all the nonnegative solutions* U_s *are compact supported and for any given mass* $\|U_s\|_1 = M$ *they are unique up to translation. Furthermore,*

(a) *all the solutions* C_s, U_s *are spherically symmetric up to translation and* $\Omega = B(0, R)$ *for some* $R > 0$ *up to translation. Particularly, for* $m = 2$*,* R *is fixed to be* $\sqrt{2}\pi$*,*

(b) *for* $2d/(d+2) < m \leq 2 - 2/d$*, the* L^p *norm* $\|U_s\|_p$ *is a constant depending on* d, m*.*

Proof. ***Step 1*** $\Big($Proof of Part ***(i)*** and ***(ii)***$\Big)$. For $0 < m < 1$, we consider the radial case. Firstly the sharp decay rate (4.25) and the fact $U_s = \phi^{\frac{-1}{1-m}}$ directly follows (4.49). Then the L^q norm $\|U_s\|_q = C(d,m)\int_0^\infty \frac{r^{d-1}}{(1+r^2)^{\frac{q}{2-m}}} dr < \infty$ for all $q > d(2-m)/2$ and is infinite for all $1 \leq q \leq d(2-m)/2$.

Now we prove that for $0 < m < 1$, the concentration C_s can't be expressed by the Newtonian potential (4.2). If C_s is the Newtonian potential (4.2), then using the decay rate (4.49) one has for $0 < m < 1$,

$$\begin{aligned} C_s(x) &= c_d \int_{\mathbb{R}^d} \frac{U_s(y)}{|x-y|^{d-2}} dy \geq c_d \int_{|x-y|\geq 2|x|} \frac{U_s(y)}{|x-y|^{d-2}} dy \\ &\geq \frac{c_d}{2^{d-2}} \int_{|y|\geq 3|x|} \frac{U_s(y)}{|y|^{d-2}} dy = \infty. \end{aligned} \tag{4.56}$$

On the other hand, $-\phi = \frac{1-m}{m}\left(C_s + \bar{C}\right) = -C(d,m)\left(1 + r^{\frac{2(1-m)}{2-m}}\right)$ follows (4.50), and thus C_s will go to $-\infty$ as r goes to ∞. This contradiction with (4.56) for large r implies C_s can't be expressed by (4.2) for $0 < m < 1$. Furthermore, simple computations can obtain that $F(U_s) = -\infty$ from the free energy for fast diffusion (1.5) and the U_s estimate (4.49). Thus completes the proof of Part **(i)**.

Similarly, for $m = 1$, using the bounds (4.26) and (4.27) one has for large r, $\phi(r) = C_s + \bar{C}$ leads to $C_s \to -\infty$ as $r \to \infty$ which contradicts with the positivity of C_s by (4.2), therefore C_s can't be expressed by the Newtonian potential either. Furthermore, the upper bound (4.26) and the fact $U_s = e^{\phi(r)}$ yield that

$$\|U_s\|_q \leq C\left(d, m, U_s(0)\right) \int_0^\infty \frac{r^{d-1}}{\left(1+r^2\right)^q} dr < \infty \quad \text{for all } q > d/2.$$

Moreover, when $d \geq 4$, using the lower bound one has

$$\int_{\mathbb{R}^d} U_s dx \geq C(d) \int_{r\geq 1} \frac{r^{d-1}}{r^{2d/(d-2)}} dr = \infty.$$

This completes the proof of Part **(ii)**.

Step 2 (Proof of Part ***(iii)***). When $1 < m < 2d/(d+2)$, it has been proved by Caffarelli, Gidas, Spruck [12] and Chen, Li [16] that all the positive solutions ϕ for (4.21) are not compact supported and $\Omega = \mathbb{R}^d$. Next the radially symmetry of $\phi(r)$ for (4.21) are proved by Zou [49] for $\frac{2(d-1)}{d+1} < m < 2d/(d+2)$ and Guo [24] for $1 < m \leq \frac{2(d-1)}{d+1}$, and then the uniqueness of radial solutions is proved by Gui, Ni and Wang [23] that for any $a > 0$, Eq. (4.23) admits a unique positive radial solution $\phi(r)$ satisfies $\lim_{|x|\to\infty} |x|^{2(m-1)/(2-m)}\phi(x) = C_0$, where C_0 is defined by (4.52).

Hence due to (4.21) one has that $U_s = \phi^{1/(m-1)} \sim C(d,m)|x|^{-2/(2-m)}$ at infinity and thus $\|U_s\|_p = \infty$. Moreover, the Newtonian potential (4.2) and Eq. (4.18) yield that $\bar{C} = 0$ and thus $\frac{m-1}{m} C_s = \phi$. Moreover, (4.16) and $\|U_s\|_m = \infty$ for $1 < m < 2d/(d+2)$ can deduce that $F(U_s) = \infty$. Thus ends the proof of Part ***(iii)***.

Step 3 (Proof of Part ***(iv)***). For $m = 2d/(d+2)$, it has been proved [12,16] that all the solutions $\phi(x) \in C^2(\mathbb{R}^d)$ of (4.21) uniquely assume the radial form in $\mathbb{R}^d$ up to translation

$$\phi(r) = \left(\frac{m-1}{m}\right)^{\frac{m-1}{m-2}} [d(d-2)]^{(d-2)/4} \left[\frac{\lambda}{\lambda^2 + r^2}\right]^{\frac{d-2}{2}}, \ \lambda > 0.$$

On the other hand, by the statement (iv) of Proposition 4.3 one has that $\bar{C} = 0$, hence using the fact $U_s = \phi^{1/(m-1)}$ obtains (4.55) and some simple computations follow $\|U_s\|_q < \infty$ for all $q \geq 1$; this completes Part ***(iv)***.

Step 4 (Proof of Part ***(v)***). For $m > 2d/(d+2)$, it was proved in [16] that all the $\phi \in C^2(\mathbb{R}^d)$ for Eq. (4.21) are compact supported (see also Proposition 4.3). Thus $U_s = \phi^{1/(m-1)}$ is also compact supported and then the radially symmetry of C_s and U_s for Eq. (4.18) was proved by Ströhmer [39]. Hence denote the support of the radial solution U_s as R_s, using the rescaling method one has $\phi(r) = R_s^{\frac{2(m-1)}{m-2}} \tilde{\phi}(z)$ with $r = R_s z$ $(0 \leqslant z \leqslant 1)$ is also the solution of (4.23), then for $\frac{2d}{d+2} < m < 2 - 2/d$, $\|U_s\|_p$ is a constant only depending on d, m follows from

$$\begin{aligned}\int_{\mathbb{R}^d} U_s(x)^p dx &= d\alpha(d) \int_0^{R_s} \phi(r)^{\frac{p}{m-1}} r^{d-1} dr \\ &= R_s^{\frac{bp}{m-1}+d} d\alpha(d) \int_0^1 \tilde{\phi}(z)^{\frac{p}{m-1}} z^{d-1} dz = M(m,p). \qquad (4.57)\end{aligned}$$

In addition, for $m = 2$, the explicit solutions for Eq. (4.23) can be expressed

$$\phi(r) = \begin{cases} \lambda \frac{J_\alpha(\frac{\sqrt{2}}{2} r)}{r^\alpha}, & 0 < r < \sqrt{2}\pi, \\ 0, & r \geq \sqrt{2}\pi, \end{cases} \qquad (4.58)$$

where $\alpha = \frac{d-2}{2}$ and

$$J_\alpha(z) = (-1)^n \left(\sqrt{\frac{2}{\pi}}\right) z^{n+\frac{1}{2}} \left(\frac{d}{zdz}\right)^n \left\{\frac{\sin z}{z}\right\}, \quad \alpha = n + 1/2, \quad n = 0, 1, 2, \ldots,$$

$$J_\alpha(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\alpha+1)} \left(\frac{1}{2} z\right)^{2s+\alpha}, \quad \alpha = n, \quad n = 0, 1, 2, \ldots.$$

From the above expression, for $m = 2$ all radial solutions of (4.23) have fixed support $R_s = \sqrt{2}\pi$ which completes ***(v)*** of Theorem 4.8.

Remark 4.9. 1. Notice that for $1 < m < 2d/(d+2)$ the radial positive solutions had slow decay $r^{-2/(2-m)}$ compared to $r^{-4/(2-m)}$ for $m = 2d/(d+2)$ and then the solution becomes compact supported for $m > 2d/(d+2)$. Furthermore, for $m > 2d/(d+2)$, the mass can change from 0 to an arbitrarily large quantity, while for $m = 2d/(d+2)$, the mass was finite but for $1 < m < 2d/(d+2)$, the mass becomes also infinite [15, p.144]. The above behaviors show that $m = 2d/(d+2)$ is a critical exponent for the steady state solutions [12]. Indeed, there are some deep reasons for the differences of these three cases, see Chandrasekhar [15, Chap. IV, Sect. 17–20] for detailed 3 dimensional phase-plane analysis.

2. When $m = 1$, the lower bound we obtained is not sharp. The asymptotic result is $U_s(r) \sim \frac{2(d-2)}{r^2}$ as $r \to \infty$, see [15, formula (438)] and [29]. Hence the radial solution $\|U_s\|_p = \infty$.
3. When m is the supercritical $1 < m < 2d/(d+2)$, it is still open if all positive $C_s \in C^2(\mathbb{R}^d)$ solutions to Eq. (4.18) in $\mathbb{R}^d$ are radially symmetric up to translation [18].

5. Numerical Results on Infinite-Time Spreading, Finite-Time Blow-up and Convergence to the Steady Profiles

Through this section, we assume the initial mass $\|U_0\|_1 < \infty$ and $m > \frac{2d}{d+2}$ such that $\|U_s\|_p < \infty$. The original equation can be written as

$$\begin{cases} u_t = (u^m)_{rr} + \frac{d-1}{r}(u^m)_r - (uc_r)_r - \frac{d-1}{r}uc_r, & r > 0,\ t \geq 0, \\ -(c_{rr} + \frac{d-1}{r}c_r) = u, & r > 0,\ t \geq 0, \\ c_r(0) = 0, \quad u_r(0) = 0, \quad u, c \to 0,\ \text{as } r \to \infty, \\ u(r, 0) = U_0(r). \end{cases} \tag{5.1}$$

For simplicity, we will consider the calculation of radial solutions using the fully implicit difference method on a large but finite domain $0 \leq r \leq L$ with $L \gg 1$ in the spirit of [46]. The discretized solution at each discrete time is presented as a vector $u^n \in \mathbb{R}^{N+1}$, where $u(r_i, t^n) = u_i^n,\ \ c(r_i, t^n) = c_i^n,\ \ r_i = i\Delta r$, where $\Delta r = L/N$ and $i = 0, 1, 2, .., N$, thus the right boundary condition $c_N = u_N = 0$, for boundary condition at zero, we use the second order one sided difference.

Using the central difference method to discretize $-\Delta$ operator and representing the discretized matrix as $A \in \mathbb{R}^{N\times N}$, the second equation of (5.1) can be expressed as

$$Ac^{n+1} = u^{n+1}. \tag{5.2}$$

For the first equation of (5.1), we use the fully implicit method with the backward Euler scheme,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t^n} = R_i\left(c^{n+1}, u^{n+1}\right), \quad i = 0, 1, 2, \ldots N, \quad \Delta t^n = t^{n+1} - t^n, \tag{5.3}$$

with the initial condition $u_i^0 = U_0(r_i)$. Here $R\left(c^{n+1}, u^{n+1}\right)$ is used to represent an appropriate discretization to the spatial operator in the first equation of (5.1). This method is only first order accurate in time, but it's sufficient for our purpose. Hence the discretized system of (5.1) can be followed by

$$\begin{cases} Ac^{n+1} - u^{n+1} = 0, \quad n = 0, 1, 2, \ldots, \\ u^{n+1} - u^n - \Delta t^n R\left(c^{n+1}, u^{n+1}\right) = 0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{5.4}$$

collecting the unknown values to a vector

$$W^{n+1} := \left(c_0^{n+1}, \ldots, c_{N-1}^{n+1}, u_0^{n+1}, \ldots, u_{N-1}^{n+1}\right) \in \mathbb{R}^{2N}. \tag{5.5}$$

Solutions of (5.1) at each time-step involves a system of $2N$ nonlinear equations for W^{n+1}, namely $F\left(W^{n+1}\right) = 0$. We use Newton's method to solve $F\left(W^{n+1}\right) = 0$ starting from the initial guess $W_{(0)}^{n+1} = W^n$, that is we calculate successive correction $\epsilon_{(k)} = W_{(k+1)}^{n+1} - W_{(k)}^{n+1}$ for $k = 0, 1, 2, \ldots$ to an initial guess $W_{(0)}^{n+1}$ from

$$J_{(k)}\epsilon_{(k)} = -F\left(W_{(k)}^{n+1}\right). \tag{5.6}$$

Here J is the Jacobian matrix for the system (5.4) which is given in terms of a discretization of Eq. (5.1),

$$J_{(k)} = \frac{\delta F\left(W_{(k)}^{n+1}\right)}{\delta W^{n+1}} = \begin{pmatrix} A & -I \\ -\Delta t^n \frac{\delta R(c^{n+1}, u^{n+1})}{\delta c^{n+1}} & I - \Delta t^n \frac{\delta R(c^{n+1}, u^{n+1})}{\delta u^{n+1}} \end{pmatrix}. \tag{5.7}$$

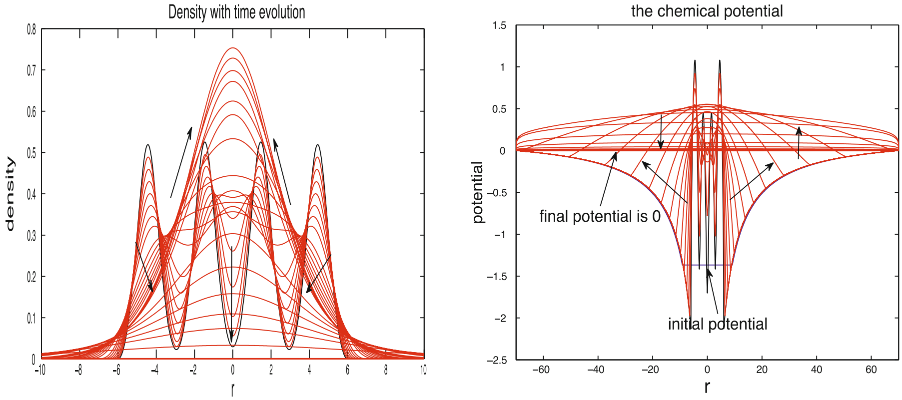


Fig. 2. Time evolution of the density (left) and the chemical potential (right). Spreading for the supercritical case $2d/(d+2) < m < 2-2/d$, $p = \frac{d(2-m)}{2}$. Here the initial data has two maximums and satisfies $C_{d,m} < \|U_0\|_p < \|U_s\|_p$, the solution will spread out to the whole space and decays to zero, and its chemical potential also tend to zero as $t \to \infty$

Here Δt^n changes in each time step in order to guarantee the Newton method is convergent. The correction $\epsilon_{(k)}$ of (5.6) will yield quadratic convergence to the solution of $F(W) = 0$.

5.1. Simulation for infinite-time spreading. For the supercritical case $1 < m < 2-2/d$ with infinite-time spreading, it has been shown in Sect. 2 that for initial data $\|U_0\|_p < C_{d,m} \le \|U_s\|_p$, the solution exists globally and decays to zero. For initial data $C_{d,m} < \|U_0\|_p < \|U_s\|_p$, it's believed that the solution has the same behavior. This is verified numerically in Fig. 2. An example is shown in Fig. 2(a) where initial data has two maximums with its total L^p norm is chosen to be $C_{d,m} < \|U_0\|_p < \|U_s\|_p$, as we can see that the solution merges into 'a single bump' and then spreads out to the whole space, and the chemical potential tends to zero as $t \to \infty$, see Fig. 2(b). The numerical results also show that the free energy decays to zero as time goes to infinity.

5.2. Simulation for finite-time blow-up. For the supercritical case with finite-time blow-up, it is demonstrated in Sect. 3 that if $F(U_0) < 0$, then $\|u(t)\|_q \to \infty$ as $t \to T$ for all $q > p = \frac{d(2-m)}{2}$. An interesting question is whether $\|u(\cdot,t)\|_p$ also blows up. For $d = 3$ and $m = 1$, there exists the self-similar solutions whose L^p norm doesn't blow-up as $t \to T$ [10]. Nevertheless, the following numerical simulation indicates that for some range of m, $\|u(\cdot,t)\|_p$ also blows up as $t \to T$.

Figure 3(a) shows a simulation starting from non-negative solutions given by two maximums with $F(U_0) < 0$ and its total L^p norm $\|U_0\|_p > \|U_s\|_p$. It is interesting to note that the two bumps merge into one bump, then the one bump blows up and only one singularity can be seen in Fig. 3(a), rather than two bumps occur. In Fig. 3(b) we can see that the chemical potential squeezes to a narrow deep needle and has a negative minimum at the blow-up position. The numerical computations also verify that the free energy goes to $-\infty$ dramatically at the blow-up time. While the second moment steeply decreases and finally reaches a positive number at the blow-up time T.

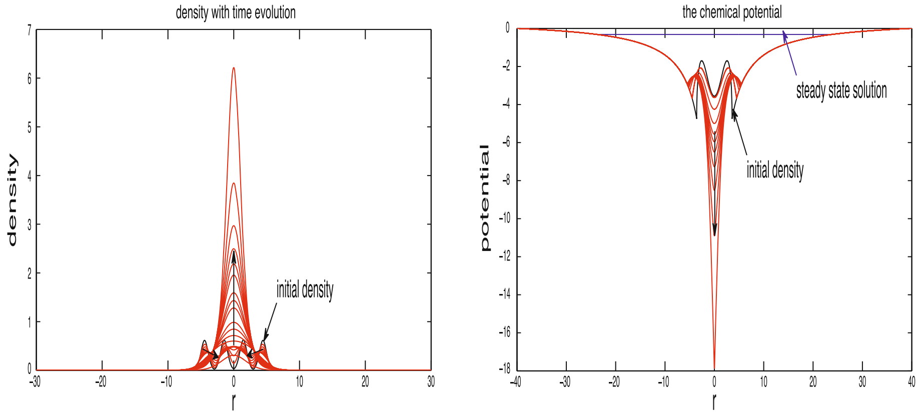


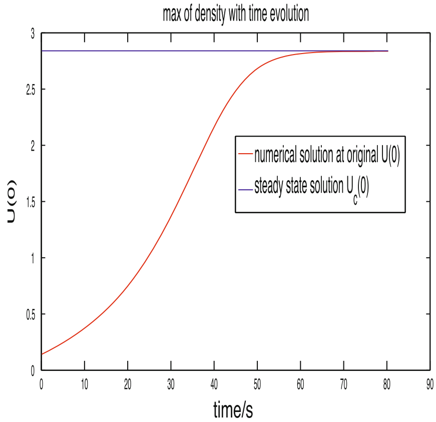
Fig. 3. Time evolution of the density (left) and the chemical potential (right). Blow-up for the supercritical case $2d/(d+2) < m < 2-2/d,\ p = \frac{d(2-m)}{2}$, the initial data is non-decreasing with two maximums and satisfies $\|U_0\|_p > \|U_s\|_p$, the solution will blow up at a finite time T and its chemical potential has a negative minimum at the blow-up position as $t \to T$

5.3. Simulation for the convergence to steady state solutions. Figure 4(a) monitors the time evolution of the density at its center with the initial data compactly supported, and it converges to $U_s(0)$ which is the maximum height of the steady state solution. As in Fig. 4(b), the chemical potential converges to the steady state solution which is a constant within the support of the density connected with a Newtonian potential outside of the support by a vertical angle. From Fig. 4(c) where the time evolution of the contact angle is plotted, it can be seen that the contact angle converges to the steady state contact angle which is positive at the support location. It's more evident to plot the density in log-scale as is shown in Fig. 4(d) where the density converges to the steady state solution for the density larger than 10^{-15}.

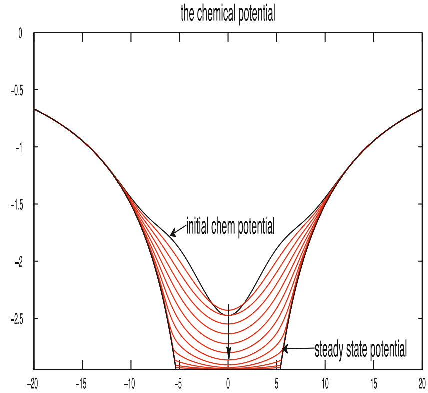
We also believe that if the initial data is non-compactly supported, the solution will be attracted to the steady profile which is compactly supported. For example taking $U_0 = \frac{1}{(1+r^2)^{(d+1)/2}}$, it can be seen in Fig. 5(b) where log-scale in density is plotted that the solution converges to the steady profile with its support going to R_s and the solution converges to the steady profile for the density larger than 10^{-10}. Figure 5(d) also shows that the free energy converges to the steady free energy $F(U_s)$.

6. Conclusions

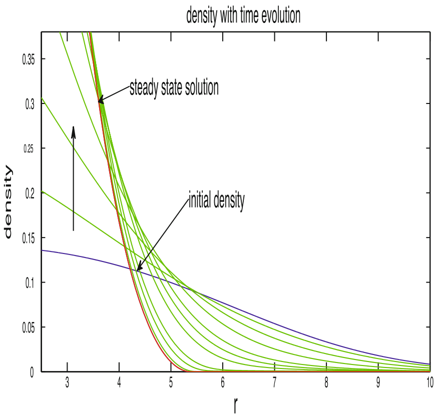
This paper concerns Eq. (1.1) in terms of different diffusion exponents m. For $0 < m < 2-2/d$, the global existence of a weak solution to (1.1) is analyzed. When $\|U_0\|_p < C_{d,m} \le \|U_s\|_p,\ p = \frac{d(2-m)}{2}$, where $C_{d,m}$ is a universal constant depending only on d, m and $\|U_s\|_p$ is the L^p norm of the radially steady solutions, there exists a global weak solution and when $m > 1-2/d$, this weak solution satisfies the hyper-contractive estimates that for any $t > 0$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ is bounded for any $p < q < \infty$. For slow diffusion $1 < m < 2-2/d$, this weak solution $u(x,t)$ is also a weak entropy solution provided by $U_0 \in L^m(\mathbb{R}^d)$ and bounded initial second moment. On the other hand, the weak solution blows up at finite time T provided by the initial negative free energy, and the negative free energy implies $\|U_0\|_p > C_{d,m}$



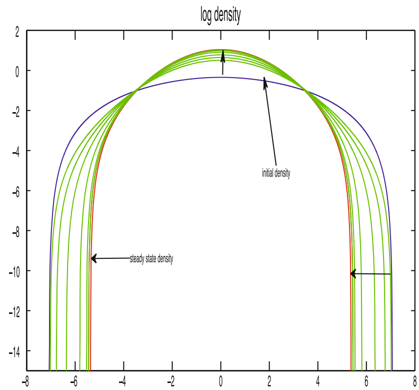
(a) maximum height of density with time evolution



(b) chemical potential with time evolution



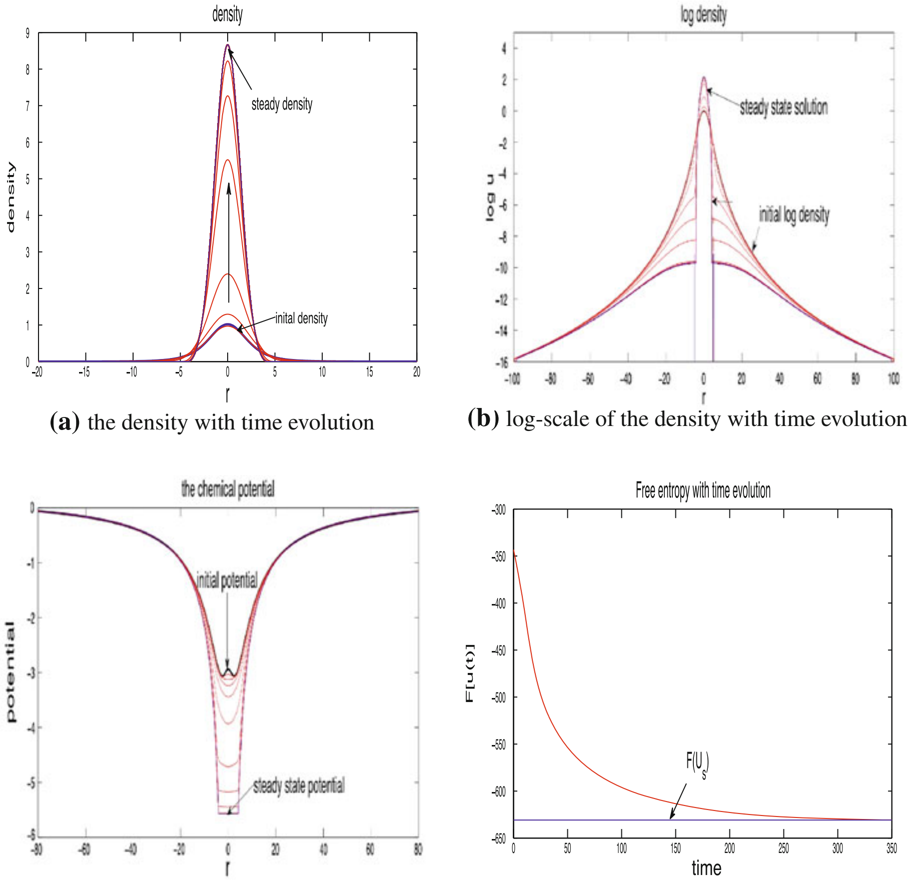
(c) zoom in near R_s



(d) zoom in log-plot

Fig. 4. Convergence to the steady state solution for the subcritical case $m > 2 - 2/d$, $p = \frac{d(2-m)}{2}$. If the initial data is radially symmetric decreasing and compactly supported, the solution will converge to the compactly supported steady state solution with the same mass and its corresponding chemical potential also converges to μ_s which is a constant within the support of the steady state solution and a newtonian potential outside of the support of the density

which is consistent with the condition for global existence. Our numerical analysis shows that for $2d/(d+2) < m < 2 - 2/d$, the L^p norm for the steady solution $\|U_s\|_p$ is the sharp condition separating infinite-time spreading from finite-time blow-up. Indeed, for $2d/(d+2) \leq m \leq 2 - 2/d$, $\|U_s\|_p$ is a constant only depending on d, m, while for $0 < m < 2d/(d+2)$, $\|U_s\|_p$ is unbounded which is discussed in Sect. 4 for steady solutions. When $1 < m < 2d/(d+2)$, there are still some open questions presented in Sect. 4.2 for the radial symmetry of the steady solutions. When $2d/(d+2) < m < 2 - 2/d$, the rigorous proof for the sharp condition $\|U_s\|_p$ separating global existence and finite time blow-up is also a challenging open question. When $m > 2 - 2/d$, the convergence to steady solutions for general initial data is also unknown.



(a) the density with time evolution

(b) log-scale of the density with time evolution

(c) chemical potential with time evolution

(d) free energy with time evolution

Fig. 5. Convergence to the steady state solution for the subcritical case $m > 2 - 2/d$, time evolution of the density in terms of its chemical potential and free energy. The initial data $U_0 = \frac{1}{(1+r^2)^{(d+1)/2}}$ is non-compactly supported and all the mass will attract to the steady profile as its corresponding free energy goes to the steady free energy $F(U_s)$

Acknowledgement We would like to acknowledge some helpful discussions with Tom Witelski. We also thank the anonymous referee for his/her insightful and constructive suggestions which improved the manuscript significantly, particularly for suggesting us to study hyper-contractivity for the global existence of the weak solution and to use the terminology energy critical exponent for $2d/(d+2)$. This research was partially supported by NSF grant DMS 10-11738.

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Communicated by P. Constantin