Acta Appl Math (2012) 119:43–55
DOI 10.1007/s10440-011-9660-4

# A Note on the Subcritical Two Dimensional Keller-Segel System

**Jose A. Carrillo · Li Chen · Jian-Guo Liu · Jinhuan Wang**

Received: 30 August 2011 / Accepted: 24 November 2011 / Published online: 17 December 2011


**Abstract** The existence of solution for the 2D-Keller-Segel system in the subcritical case, i.e. when the initial mass is less than $8\pi$, is reproved. Instead of using the entropy in the free energy and free energy dissipation, which was used in the proofs (Blanchet et al. in SIAM J. Numer. Anal. 46:691–721, 2008; Electron. J. Differ. Equ. Conf. 44:32, 2006 (electronic)), the potential energy term is fully utilized by adapting Delort's theory on 2D incompressible Euler equation (Delort in J. Am. Math. Soc. 4:553–386, 1991).

**Keywords** Chemotaxis · Critical mass · Global existence · Maximal density function

---

J.A. Carrillo is partially supported by the project MTM2011-27739-C04 DGI-MCI (Spain) and 2009-SGR-345 from AGAUR-Generalitat de Catalunya. L. Chen is partially supported by National Natural Science Foundation of China (NSFC) grant 10871112, 11011130029. The research of J.-G. Liu was partially supported by NSF grant DMS 10-11738. J. Wang is partially supported by Science Foundation of Liaoning Education Department grant L2010146 and China Postdoctoral Science Foundation grant 20110490409. J.-G. Liu wish to acknowledge the hospitality of Mathematical Sciences Center of Tsinghua University where part of this research was performed.

---

J.A. Carrillo (✉)
ICREA and Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra (Barcelona) 08193, Spain
e-mail: carrillo@mat.uab.es

L. Chen · J. Wang
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China

L. Chen
e-mail: lchen@math.tsinghua.edu.cn

J. Wang
e-mail: jhwang@math.tsinghua.edu.cn

J.-G. Liu
Department of Physics and Department of Mathematics, Duke University, Durham, NC 27708, USA
e-mail: jliu@phy.duke.edu

J. Wang
Department of Mathematics, Liaoning University, Shenyang 110036, P.R. China

**Mathematics Subject Classification (2000)** 35K55 · 35B33

## 1 Introduction

The 2D Keller-Segel system

$$\begin{aligned} &u_t = \Delta u - \mathrm{div}(u\nabla c), && x \in \mathbb{R}^2, t \geq 0, \\ &-\Delta c = u, && x \in \mathbb{R}^2, t \geq 0, \\ &u(x,0) = u_0(x), && x \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

has widely been studied in the literature. Keller-Segel-type systems are used to describe the motion of biological cells or organisms in response to chemical gradients. The main feature of this system is that it can describe the mass aggregation phenomena in chemotaxis. The system (1.1) considered here is the simplest version of the Keller-Segel system in the parabolic-elliptic case, one can check more complete models in [12]. The sharp bound on the critical mass was given by Dolbeault and Perthame in [9]. It was announced there that if the initial mass is less than $8\pi$ then weak solutions exist globally, while in the case of initial mass larger than $8\pi$, there must be a mass concentration. Later on, in [1] they completed the global existence of weak solution in the subcritical case, i.e. when the initial mass is less than $8\pi$. A second proof of global weak solutions in the subcritical case was given in [2] where one uses the gradient flow structure of this system and the techniques of variational schemes based on optimal mass transportation.

In the case of critical mass, weak solutions with bounded second moment initial data exist globally leading to aggregation onto a single Dirac Delta in infinite time [3]. There is an in-depth analysis of the unbounded initial second moment case by Blanchet, Carlen and Carrillo in [4]. By making full use of the gradient flow structure of the equation [2] and relative entropy, they give conditions for initial data to belong to the basin of attraction for each of the infinitely many stationary solutions in the critical case. In this note, we focus on the subcritical case $\int_{\mathbb{R}^2} u_0(x)dx < 8\pi$ and give an alternative proof for the global existence of weak solution based on the analogy to Delort's theory of 2D incompressible Euler equation [7, 8, 16].

We will be dealing in this work with measure solutions in most of the arguments. Nevertheless, we will denote our solutions as if they were $L^1$-densities with notation $u(x,t)dx$ by abusing a bit of the notation. The $i$th-moments of the solution $u(x,t)$, $i = 0, 1, 2$, are defined by

$$m_0(t) := \int_{\mathbb{R}^2} u(x,t)dx, \qquad m_1(t) := \int_{\mathbb{R}^2} xu(x,t)dx, \qquad m_2(t) := \int_{\mathbb{R}^2} |x|^2 u(x,t)dx.$$

By a direct computation, we have the following formal conservation relations for these moments:

$$\begin{aligned} m_0'(t) &= \frac{d}{dt}\int_{\mathbb{R}^2} u(x,t)dx = 0, \\ m_1'(t) &= \frac{d}{dt}\int_{\mathbb{R}^2} xu(x,t)dx = 0, \\ m_2'(t) &= \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)dx = 4m_0 - \frac{m_0^2}{2\pi}. \end{aligned}$$

Furthermore, there is a natural free energy of the system,

$$\mathcal{F}\big[u(\cdot,t)\big] := \int_{\mathbb{R}^2} u(x,t)\log u(x,t)dx - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\log\frac{1}{|x-y|}dxdy \quad (1.2)$$

and the formal free energy dissipation relation,

$$\frac{d}{dt}\mathcal{F}\big[u(\cdot,t)\big] + \int_{\mathbb{R}^2} u|\nabla(\log u - c)|^2 dx = 0.$$

Thus the free energy is expected to decay in time, $\mathcal{F}[u(\cdot,t)] \le \mathcal{F}(u_0)$. Notice that in the free energy (1.2), there is a competition between diffusion and nonlocal aggregation exactly matching at $m_0 = 8\pi$, which is the key ingredient of Keller-Segel system.

One of the key ingredients in all the new proofs and improvements for the parabolic-elliptic Keller-Segel system (1.1) in [1–4, 9] is the connection to a functional inequality, the so-called logarithmic Hardy-Littlewood-Sobolev (log-HLS) inequality. We recall the log-HLS inequality, [5, 6, 13, 17], for nonnegative $f \in L^1(\mathbb{R}^2)$, $f\log(e+|x|^2) \in L^1(\mathbb{R}^2)$, and $f\log f \in L^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} f(x)\log f(x)dx - \frac{2}{m_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)f(y)\log\frac{1}{|x-y|}dxdy + C(m_0) \ge 0, \quad (1.3)$$

where $m_0 = \int_{\mathbb{R}^2} f(x)dx$, $C(m_0) := m_0(1+\log\pi - \log m_0)$.

By the log-HLS (1.3), the free energy (1.2) has lower bounds either

$$\mathcal{F}\big[u(\cdot,t)\big] \ge \left(1-\frac{m_0}{8\pi}\right)\int_{\mathbb{R}^2} u(x,t)\log u(x,t)dx - \frac{m_0}{8\pi}C(m_0),$$

or

$$\mathcal{F}\big[u(\cdot,t)\big] \ge 2\left(\frac{1}{m_0}-\frac{1}{8\pi}\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\log\frac{1}{|x-y|^2}dxdy - C(m_0).$$

For convenience, we denote by

$$\begin{aligned}\mathcal{F}_1\big[u(\cdot,t)\big] &:= \int_{\mathbb{R}^2} u(x,t)\log u(x,t)dx,\\ \mathcal{F}_2\big[u(\cdot,t)\big] &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\log\frac{1}{|x-y|}dxdy.\end{aligned} \quad (1.4)$$

Thus in the case $m_0 < 8\pi$, both entropy $\mathcal{F}_1[u(\cdot,t)]$ and the potential energy $\mathcal{F}_2[u(\cdot,t)]$ can be bounded by the initial free energy $\mathcal{F}[u_0]$. In the existence proof in [1], the *a priori* estimates are mainly from the entropy $\mathcal{F}_1[u(\cdot,t)]$, the second moment $m_2(t)$ and the free energy dissipation $\int_{\mathbb{R}^2} u|\nabla(\log u - c)|^2 dx$. Especially, they get some regularity and $L^p$-bounds for the density from the dissipation. In this note, we point out that either $\mathcal{F}_1[u(\cdot,t)]$ and $m_2(t)$ can guarantee the global existence of weak solution or $\mathcal{F}_2[u(\cdot,t)]$ and $m_2(t)$ can be used too. However, the bounds for $\mathcal{F}_1[u(\cdot,t)]$ and $m_2(t)$ are not enough to define $c(x,t)$, while $\mathcal{F}_2[u(\cdot,t)]$ and $m_2(t)$ are enough as we will show in Sect. 5. An estimate on $\nabla c \in L^2_{loc}(\mathbb{R}^2)$ for all times will also be given by using the potential energy $\mathcal{F}_2[u(\cdot,t)]$. On the other hand, the bounds for $\mathcal{F}_1[u(\cdot,t)]$ and $m_2(t)$ are enough to show that the weak solutions are densities, i.e., $u \in L^1(\mathbb{R}^2)$, see [1] and Sect. 3 below, while $\mathcal{F}_2[u(\cdot,t)]$ and $m_2(t)$ allow only to show that $u$ is a positive Radon measure of mass $m_0$ without atomic part for all times. The definition of weak solution of (1.1) that we construct is

**Definition 1.1** A curves of measures $u(x,t) \in L^\infty(0,T:\mathcal{M}_0(\mathbb{R}^2)) \cap Lip(0,T;H^{-m}_{\mathrm{loc}}(\mathbb{R}^2))$ for some $m>0$ and for all $T>0$ is called a global weak solution of (1.1) if $\forall\varphi \in C_0^\infty([0,T)\times\mathbb{R}^2)$ and for all $T>0$, it holds

$$\int_0^T\int_{\mathbb{R}^2}\varphi_t u(x,t)dxdt + \int_{\mathbb{R}^2}\varphi(x,0)u_0(x)dx + \int_0^T\int_{\mathbb{R}^2}\Delta\varphi u(x,t)dxdt$$
$$= \frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}u(x,t)u(y,t)dxdydt, \tag{1.5}$$

where $\mathcal{M}_0(\mathbb{R}^2)$ is the space of nonnegative Radon measures with mass $m_0$.

The main contribution of this note is to show that the potential energy $\mathcal{F}_2[u(\cdot,t)]$ in (1.4) is enough to rule out the concentration phenomena. We also give the rate estimate that eliminate concentration. More precisely, we can provide estimates on the maximal density function defined by DiPerna and Majda [8] associated to a measure $u(x)$ defined by

$$M_r(u) = \sup_{x,t}\int_{B(x,r)} u(y)dy.$$

Our main result is:

**Theorem 1.1** *Assume $u_0 \in L^1_+(\mathbb{R}^2)$ with mass $m_0 < 8\pi$ such that $m_2(0)$ and $\mathcal{F}[u_0]$ are bounded, then (1.1) has a global weak solution in the sense of Definition 1.1 such that its maximal density function satisfies for all $0<r<\frac{1}{4}$*

$$M_r\big(u(\cdot,t)\big) \le C\left(\log\frac{1}{4r}\right)^{-1/2}.$$

*Moreover, the potential*

$$c(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{1}{|x-y|}u(y,t)dy$$

*is well defined and $\nabla c \in L^\infty(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^2))$ for all $T>0$.*

The note is arranged in the following sections. In Sect. 2, we introduce the regularized problem and give the necessary uniform bounds. In Sect. 3, we quickly review part of the results from [1, 17] by using the entropy and show that one can use Dunford-Pettis' criterion to get the existence. In Sect. 4, we show the existence by getting estimates for the positive part of the potential energy and the maximal density function to eliminate concentration. In the last section, we show that $c(x,t)$ is well defined and its gradient is locally in $L^2(\mathbb{R}^2)$.

## 2 Regularized Problem and Uniform Estimates

The regularized problem we start with is

$$\begin{aligned}
&u_t^\varepsilon = \Delta u^\varepsilon - \operatorname{div}(u^\varepsilon\nabla c^\varepsilon), && x\in\mathbb{R}^2, t\ge 0,\\
&c^\varepsilon(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^2}\log\frac{1}{|x-y|^2+\varepsilon^2}u^\varepsilon(y,t)dy, && x\in\mathbb{R}^2, t\ge 0,\\
&u^\varepsilon(x,0) = u_0^\varepsilon(x)\in L^1_+(\mathbb{R}^2), && x\in\mathbb{R}^2,
\end{aligned} \tag{2.1}$$

where $\varepsilon < \frac{1}{2}$. By standard parabolic theory, the system (2.1) admits a unique smooth fast-decaying at infinity solution $(u^\varepsilon,\ c^\varepsilon)$ when initial data are regularized and truncated $u_0^\varepsilon$. It is obvious that the mass $\|u^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^2)}$ is conserved.

**Lemma 2.1** *Let $u^\varepsilon$ be the solution of* (2.1), *then*

$$\int_{\mathbb{R}^2} u^\varepsilon(x,t)dx = m_0.$$

We denote the free energy functional for the regularized problem

$$\begin{aligned}\mathcal{F}_\varepsilon\big[u^\varepsilon(\cdot,t)\big] &= \int_{\mathbb{R}^2} u^\varepsilon \log u^\varepsilon dx - \frac{1}{2}\int_{\mathbb{R}^2} u^\varepsilon c^\varepsilon dx \\ &= \int_{\mathbb{R}^2} u^\varepsilon(x,t)\log u^\varepsilon(x,t)dx \\ &\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy.\end{aligned}$$

Then the energy dissipation relation and the moment bounds still holds for the regularized problem:

**Lemma 2.2** *Let $u^\varepsilon$, $c^\varepsilon$ be the solution of* (2.1), *then*

$$\frac{d}{dt}\mathcal{F}_\varepsilon\big[u^\varepsilon(\cdot,t)\big] = -\int_{\mathbb{R}^2} u^\varepsilon|\nabla\log u^\varepsilon - \nabla c^\varepsilon|^2 dx \le 0 \tag{2.2}$$

*and*

$$\frac{d}{dt}m_2(t) = \frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u^\varepsilon(x,t)dx \le 4m_0. \tag{2.3}$$

*Proof* Since the first equation in (2.1) can be rewritten as $u_t^\varepsilon = \mathrm{div}[u^\varepsilon\nabla(\log u^\varepsilon - c^\varepsilon)]$ where

$$c^\varepsilon = \frac{1}{4\pi}\int_{\mathbb{R}^2}\log\frac{1}{|x-y|^2+\varepsilon^2}u^\varepsilon(y,t)dy,$$

then, by multiplying it by $\log u^\varepsilon - c^\varepsilon + 1$, integrating in $x$ and performing an integration by parts, we directly get

$$\int_{\mathbb{R}^2}[(u^\varepsilon\log u^\varepsilon)_t - u_t^\varepsilon c^\varepsilon]dx = -\int_{\mathbb{R}^2} u^\varepsilon|\nabla\log u^\varepsilon - \nabla c^\varepsilon|^2 dx.$$

The symmetry of the kernel $\log(|x-y|^2+\varepsilon^2)$ implies trivially that

$$\frac{1}{8\pi}\frac{d}{dt}\int_{\mathbb{R}^2\times\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy = \int_{\mathbb{R}^2} u_t^\varepsilon c^\varepsilon dx,$$

from which (2.2) is deduced.

Now, a direct calculation on the regularized problem (2.1) gives

$$\frac{d}{dt}m_2(t) = 4\int_{\mathbb{R}^2} u^\varepsilon(x,t)dx + 2\int_{\mathbb{R}^2} u^\varepsilon(x,t)x\cdot\nabla c_\varepsilon(x,t)dx$$

$$= 4m_0 - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\frac{|x-y|^2}{|x-y|^2+\varepsilon^2}dxdy$$
$$\leq 4m_0. \qquad \square$$

For the regularized nonlocal kernel, we have a log-HLS-type inequality, due to the monotonicity of the logarithmic function and the nonnegativity of $u^\varepsilon$.

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy \leq 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|}dxdy.$$

The log-HLS inequality for $u^\varepsilon(x,t)$ implies that

$$\int_{\mathbb{R}^2} u^\varepsilon(x,t)\log u^\varepsilon(x,t)dx \geq \frac{1}{m_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy - C(m_0), \tag{2.4}$$

where the constant $C(m_0)$ is given in (1.3). Finally, let us assume without loss of generality that the sequence of approximated initial data satisfies that for all $0<\varepsilon<\frac{1}{2}$

$$\mathcal{F}_\varepsilon\big[u_0^\varepsilon\big] \leq C\big(\mathcal{F}[u_0], m_2(0)\big). \tag{2.5}$$

This last assumption can easily be realized by noting that $\mathcal{F}[u_0]<\infty$ and $m_2(0)<\infty$ imply that both terms in the free energy are well-defined and the regularized free energy is decreasing and $\mathcal{F}_\varepsilon[u_0] \searrow \mathcal{F}[u_0]$ as $\epsilon \to 0$ by monotone convergence theorem. Once the logarithmic kernel is regularized then we can use standard cut-off and convolution regularization of the initial data to find $u_0^\varepsilon$ satisfying (2.5).

## 3 Existence by Using Entropy

**Lemma 3.1** *Let $u^\varepsilon$ be the solution of* (2.1), *then for all $T>0$ and all $0\leq t\leq T$ we have*

$$\int_{\mathbb{R}^2} u^\varepsilon|\log u^\varepsilon|dx \leq C\big(\mathcal{F}[u_0], m_2(0), 8\pi-m_0, T\big). \tag{3.1}$$

*Proof* The proof is standard, see [1, 17]. For completeness, we gives some details. Using the refined version of logarithmic Hardy-Littlewood-Sobolev inequality (2.4) and the free energy dissipation relation (2.2), we have

$$\mathcal{F}_\varepsilon\big[u_0^\varepsilon\big] \geq \mathcal{F}_\varepsilon\big[u^\varepsilon(\cdot,t)\big] \geq \int_{\mathbb{R}^2} u^\varepsilon\log u^\varepsilon dx + \frac{1}{4\pi}\left(-C(m_0)\frac{m_0}{2} - \frac{m_0}{2}\int_{\mathbb{R}^2} u^\varepsilon\log u^\varepsilon dx\right)$$
$$= \left(1-\frac{m_0}{8\pi}\right)\int_{\mathbb{R}^2} u^\varepsilon\log u^\varepsilon dx - \frac{m_0}{8\pi}C(m_0),$$

which leads to

$$\int_{\mathbb{R}^2} u^\varepsilon\log u^\varepsilon dx \leq \left(\mathcal{F}_\varepsilon\big[u_0^\varepsilon\big] + \frac{m_0}{8\pi}C(m_0)\right)\frac{8\pi}{8\pi-m_0}. \tag{3.2}$$

We separate the entropy into its positive and negative part in the following,

$$\int_{\mathbb{R}^2} u^\varepsilon\log u^\varepsilon dx = \int_{\mathbb{R}^2} u^\varepsilon\log^+ u^\varepsilon dx - \int_{\mathbb{R}^2} u^\varepsilon\log^- u^\varepsilon dx.$$

The negative part of the entropy can be classically controlled by the second moment, see [2, Lemma 2.2] for instance, as

$$\int_{\mathbb{R}^2} u^\varepsilon \log^- u^\varepsilon dx \le \int_{\mathbb{R}^2} |x|^2 u^\varepsilon dx + \frac{\pi}{e}. \tag{3.3}$$

Combining (3.3) with (3.2), the second moment estimate (2.3) and (2.5), we have

$$\int_{\mathbb{R}^2} u^\varepsilon \log^+ u^\varepsilon dx \le C\big(\mathcal{F}[u_0], m_2(0), 8\pi - m_0, T\big),$$

which leads to (3.1). □

A direct application of the Dunford-Pettis criterion [11] and some measure theory arguments [2, Lemma 2.3] implies the following classical compactness.

**Proposition 3.1** *Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^1_+(\mathbb{R}^n)$, such that $\{f_k|\log f_k|\}_{k\in\mathbb{N}}$ and $\{|x|^2 f_k\}_{k\in\mathbb{N}}$ are bounded in $L^1_+(\mathbb{R}^n)$. Then $\{f_k\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(\mathbb{R}^n)$. As a consequence, it holds that $\{f_k(x)\}_{k\in\mathbb{N}}$ and $\{f_k(x)f_k(y)\}_{k\in\mathbb{N}}$ are $L^1$-weakly compact sets and for any convergent subsequence (without relabeled) we have*

$$\begin{aligned} &f_k(x) \rightharpoonup f(x) \quad \text{in } L^1\big(\mathbb{R}^n\big),\\ &f_k(x)f_k(y) \rightharpoonup f(x)f(y) \quad \text{in } L^1\big(\mathbb{R}^n\times\mathbb{R}^n\big). \end{aligned}$$

The existence of weak solution in Definition 1.5 is a direct consequence of this last proposition. We refer to [1, 2] for more details of the proof and properties of the solutions.

## 4 Existence by Using Potential Energy

*Step 1. Limit functions:* By the conservation of mass Lemma 2.1 and the control of mass at infinity Lemma 2.2, we can apply Prohorov compactness theorem to show that there exists $u(x,t) \in L^\infty(0,T:\mathcal{M}_0(\mathbb{R}^2))$ such that $u^\varepsilon(\cdot,t) \rightharpoonup^* u(\cdot,t)$, weakly-∗ as measures in $\mathcal{M}_0(\mathbb{R}^2)$ for a.e. $t \in (0,T)$ as $\varepsilon \to 0$, for some not relabeled sequence of $\varepsilon$ by abuse of notation.

*Step 2. Estimate for positive part of potential energy:* The positive part of the potential energy can be controlled in the following lemma, see similar arguments in [18].

**Lemma 4.1** *Let $u^\varepsilon$ be the solution of (2.1), then for all $T > 0$ and all $0 \le t \le T$ we get*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log^+\frac{1}{|x-y|^2+\varepsilon^2}dxdy \le C\big(\mathcal{F}[u_0], m_2(0), 8\pi - m_0, T\big). \tag{4.1}$$

*Proof* Using the log-HLS-type inequality (2.4) and the free energy dissipation relation (2.2), we have

$$\begin{aligned} \mathcal{F}_\varepsilon\big[u_0^\varepsilon\big] &\ge \mathcal{F}_\varepsilon\big[u^\varepsilon(\cdot,t)\big]\\ &\ge \left(\frac{1}{m_0} - \frac{1}{8\pi}\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy - C(m_0). \end{aligned}$$

Or equivalently we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log\frac{1}{|x-y|^2+\varepsilon^2}dxdy \le \frac{(\mathcal{F}_\varepsilon[u_0^\varepsilon]+C(m_0))8\pi m_0}{8\pi - m_0}. \tag{4.2}$$

The positive part of the potential energy can be bounded using (4.2) and (2.5) in the following way,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u^\varepsilon(x,t)u^\varepsilon(y,t)\log^+\frac{1}{|x-y|^2+\varepsilon^2}dxdy \\
&\quad= \int\int_{\mathbb{R}^2\times\mathbb{R}^2} u^\varepsilon(x,t)\log\frac{1}{|x-y|^2+\varepsilon^2}u^\varepsilon(y,t)dxdy \\
&\qquad - \int\int_{|x-y|^2+\varepsilon^2\ge 1} u^\varepsilon(x,t)\log\frac{1}{|x-y|^2+\varepsilon^2}u^\varepsilon(y,t)dxdy \\
&\quad\le C\big(\mathcal{F}_\varepsilon\big[u_0^\varepsilon\big],m_0\big) + \int\int_{|x-y|^2+\varepsilon^2\ge 1} u^\varepsilon(x,t)\log\big(|x-y|^2+\varepsilon^2\big)u^\varepsilon(y,t)dxdy \\
&\quad\le C\big(\mathcal{F}_\varepsilon\big[u_0^\varepsilon\big],m_0\big) + \int\int_{|x-y|^2+\varepsilon^2\ge 1} u^\varepsilon(x,t)\big(|x|^2+|y|^2+C\big)u^\varepsilon(y,t)dxdy \\
&\quad\le C\big(\mathcal{F}[u_0],m_0,m_2(0),T\big).
\end{aligned}$$

□

*Step 3. Estimate on maximal density function:*

**Lemma 4.2** *Let $u^\varepsilon$ be a solution to the regularized problem (2.1) and let $u(x,t)\in L^\infty(0,T:\mathcal{M}_0(\mathbb{R}^2))$ be a weak-$*$ adherence point of $u^\varepsilon$ for some not relabeled sequence. It holds that for all $T>0$, a.e. $t\in(0,T)$, and $0<r<\frac{1}{4}$*

$$M_r\big(u^\varepsilon(\cdot,t)\big)\le C\left(\log\frac{1}{(2r)^2+\varepsilon^2}\right)^{-1/2} \quad \text{and} \quad M_r\big(u(\cdot,t)\big)\le C\left(\log\frac{1}{4r}\right)^{-1/2}. \tag{4.3}$$

*Proof* A direct computation using (4.1) implies that

$$\begin{aligned}
&\left(\int_{B(x,r)} u^\varepsilon(y,t)dy\right)^2 \log\left(\frac{1}{(2r)^2+\varepsilon^2}\right) \\
&\quad\le \int_{B(x,r)\times B(x,r)} \log\frac{1}{|y-z|^2+\varepsilon^2}u^\varepsilon(y,t)u^\varepsilon(z,t)dydz \\
&\quad\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \log^+\frac{1}{|y-z|^2+\varepsilon^2}u^\varepsilon(y,t)u^\varepsilon(z,t)dydz \le C,
\end{aligned}$$

giving the desired estimate on $M_r(u^\varepsilon(\cdot,t))$. For $M_r(u(\cdot,t))$, let us take a "mesa" function $0\le\theta(x)\le 1$ being 1 in $B(x,r)$ and 0 outside $B(x,2r)$, then by the convergence in Step 2 we get

$$\begin{aligned}
\int_{B(x,r)} u(y,t)dy &\le \int_{\mathbb{R}^2}\theta(y)u(y,t)dy = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\theta(y)u^\varepsilon(y,t)dy \\
&\le \lim_{\varepsilon\to 0}\int_{B(x,2r)} u^\varepsilon(y,t)dy \le \lim_{\varepsilon\to 0} C\left(\log\frac{1}{(4r)^2+\varepsilon^2}\right)^{-1/2} \\
&= C\left(\log\frac{1}{4r}\right)^{-1/2}.
\end{aligned}$$

□

**Lemma 4.3** *Let us define the measure* $d\mu = u(x,t)u(y,t)dxdy$, *then for all* $T > 0$, *a.e.* $t \in (0,T)$, *and* $0 < r < \frac{1}{4}$

$$\mu(\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2, |x-y| \le r\}) \le C\left(\log \frac{1}{4r}\right)^{-1/2}.$$

*As a consequence, we conclude that* $\mu(\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2, x = y\}) = 0$.

*Proof* With the help of (4.3), we deduce

$$\int_{|x-y|\le r} u^\varepsilon(x)u^\varepsilon(y)dxdy = \int_{\mathbb{R}^2} u^\varepsilon(x) \int_{B(x,r)} u^\varepsilon(y)dxdy \le m_0 C\left(\log \frac{1}{(2r)^2+\varepsilon^2}\right)^{-1/2}. \tag{4.4}$$

Proceeding as in the proof of the previous Lemma using the auxiliary function $\theta$, we get from (4.4) that

$$\begin{aligned}\mu\big((x,y) \in \mathbb{R}^2 \times \mathbb{R}^2, |x-y| \le r\big) &\le \lim_{\varepsilon\to 0} \int_{|x-y|\le 2r} u^\varepsilon(x)u^\varepsilon(y)dxdy \\ &= Cm_0\left(\log \frac{1}{4r}\right)^{-1/2}.\end{aligned}$$

□

*Step 4. Proof of the existence:* Let $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^2)$ be a test function for (2.1), then simple computations lead to

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^2} \varphi_t(x,t)u^\varepsilon(x,t)dxdt + \int_{\mathbb{R}^2} \varphi(x,0)u_0^\varepsilon(x)dx + \int_0^T \int_{\mathbb{R}^2} \Delta\varphi(x,t)u^\varepsilon(x,t)dxdt \\ &= \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\nabla\varphi(x,t) - \nabla\varphi(y,t)) \cdot (x-y)}{|x-y|^2+\varepsilon^2} u^\varepsilon(x,t)u^\varepsilon(y,t)dxdydt.\end{aligned} \tag{4.5}$$

Passing to the limit as $\varepsilon \to 0$ in the left hand side of (4.5) is obvious using the compactness of Step 2. In the next discussions, we will focus on the integral on the right hand side of (4.5). Let us define for all $0 \le \varepsilon < \frac{1}{2}$ the kernel

$$K_\varepsilon(x,y,t) := \frac{(\nabla\varphi(x,t) - \nabla\varphi(y,t)) \cdot (x-y)}{|x-y|^2+\varepsilon^2}. \tag{4.6}$$

Taking into account that $\varphi$ is a smooth function combined with the last part of Lemma 4.3, we know that as $\varepsilon \to 0$, $K_\varepsilon(x,y,t) \to K_0(x,y,t)$ a.e. in $\mu$, where $K_0(x,y,t)$ is given by (4.6) with $\varepsilon = 0$. Since the kernel verifies for all $t \in [0,T]$ that $|K_\varepsilon(x,y,t)| \le \|D^2\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$, then the dominated convergence theorem shows that

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} K_\varepsilon(x,y,t)d\mu \to \int_{\mathbb{R}^2\times\mathbb{R}^2} K_0(x,y,t)d\mu.$$

On the other hand, let us consider the difference of the right-hand sides as

$$I := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u^\varepsilon(x,t)u^\varepsilon(y,t) - u(x,t)u(y,t))K_\varepsilon(x,y,t)dxdy = I_1^\varepsilon + I_2^\varepsilon,$$

where for some $0 < r < \frac{1}{4}$, we define

$$I_1^\varepsilon = \int\int_{|x-y|\le r} (u^\varepsilon(x,t)u^\varepsilon(y,t) - u(x,t)u(y,t))K_\varepsilon(x,y,t)dxdy$$

and $I_2^\varepsilon = I - I_1^\varepsilon$. By Lemma 4.2, we have

$$|I_1^\varepsilon| \le C\|D^2\varphi\|_{L^\infty([0,T]\times\mathbb{R}^2)}\left(\log\frac{1}{(2r)^2+\varepsilon^2}\right)^{-1/2}.$$

Notice that as $\varepsilon \to 0$, we have

$$\sup_{|x-y|>r}|K_\varepsilon(x,y,t) - K_0(x,y,t)| = \sup_{|x-y|>r}\left|K_0(x,y,t)\frac{\varepsilon^2}{|x-y|^2+\varepsilon^2}\right| \to 0,$$

from which we obtain that $|I_2^\varepsilon| \to 0$ as $\varepsilon \to 0$. A similar argument can be found in [10, 15].

Finally, we can take limit in (4.5) and get the existence of weak solution except the time regularity $u \in Lip(\mathbb{R}_+; H^{-4}_{\mathrm{loc}}(\mathbb{R}^2))$ which will be given at the end of Sect. 5.

## 5 Drift Potential $c(x,t)$

We know that the limit $u$ in Sect. 4 belongs to $L^\infty(0,T:\mathcal{M}_0(\mathbb{R}^2))$ and we will show in this section that its potential energy is bounded. Consequently, we can use these facts to define $c$, give a uniform estimate on $\nabla c$ and obtain the time regularity of $u$. This will complete the proof of the main Theorem 1.1.

*Step 1. Uniform local estimate for $\nabla c^\varepsilon$ in the regularized problem:*

**Lemma 5.1** *Let $c^\varepsilon$ be the solution of* (2.1), *then for any fixed $R > 0$, for all $T > 0$ and a.e. $t \in (0,T)$*

$$\int_{|x|\le R}|\nabla c^\varepsilon(x,t)|^2dx \le C(R,\mathcal{F}[u_0],m_2(0),8\pi - m_0,T).$$

*Proof* The drift in (2.1) is given by

$$\nabla c^\varepsilon(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2+\varepsilon^2}u^\varepsilon(y,t)dy.$$

By choosing a smooth cut-off function $0 \le \theta(x) \le 1$ such that $\theta(x) = 1$ for $|x| \le R$, $\theta(x) = 0$ for $r \ge 2R$, we obtain

$$\begin{aligned}
&\int_{|x|\le R}|\nabla c^\varepsilon(x,t)|^2dx \\
&\quad\le \int_{\mathbb{R}^2}\theta(x)|\nabla c^\varepsilon(x,t)|^2dx \\
&\quad= \int_{\mathbb{R}^2\times\mathbb{R}^2}u^\varepsilon(y,t)u^\varepsilon(z,t)\left(\int_{\mathbb{R}^2}\theta(x)\frac{(x-y)\cdot(x-z)}{(|x-y|^2+\varepsilon^2)(|x-z|^2+\varepsilon^2)}dx\right)dydz.
\end{aligned}$$

Let us define for all $0 \le \varepsilon < \frac{1}{2}$

$$\mathcal{K}_\varepsilon(\theta, y, z) := \int_{\mathbb{R}^2} \frac{\theta(x)(x-y)\cdot(x-z)}{(|x-y|^2+\varepsilon^2)(|x-z|^2+\varepsilon^2)} dx. \tag{5.1}$$

By [14, Lemma 2.2], the integral kernel $K_\varepsilon(\theta, y, z)$ can be controlled by

$$\mathcal{K}_\varepsilon(\theta, y, z) \le C(R) \log^+ \frac{1}{|y-z|^2+\varepsilon^2}. \tag{5.2}$$

Then it follows that

$$\begin{aligned}\int_{|x|\le R} |\nabla c^\varepsilon(x,t)|^2 dx &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathcal{K}_\varepsilon(\theta, y, z) u^\varepsilon(y,t) u^\varepsilon(z,t) dy dz \\ &\le C(R) \int_{\mathbb{R}^2\times\mathbb{R}^2} \log^+ \frac{1}{|y-z|^2+\varepsilon^2} u^\varepsilon(y,t) u^\varepsilon(z,t) dy dz \le C,\end{aligned}$$

where in the last inequality we have used Lemma 4.1. □

*Step 2. Estimates on the potential energy:* Adapting some ideas in [15], we obtain bounds for the potential energy of the weak solution.

**Lemma 5.2** *Let $u \in L^\infty(\mathbb{R}_+; \mathcal{M}_0(\mathbb{R}^2))$ be a weak solution of* (1.1), *then for all $T > 0$ and a.e. $t \in (0, T)$*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\log|x-y|| u(x,t) u(y,t) dx dy \le C\big(\mathcal{F}[u_0], m_2(0), 8\pi - m_0, T\big). \tag{5.3}$$

*Proof* The negative part of the potential energy can be controlled by the second moment

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log^- \frac{1}{|x-y|} u(x,t)u(y,t) dx dy \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|x|^2+|y|^2) u(x,t)u(y,t) dx dy \le C.$$

We only need to show that the positive part of the potential energy is bounded, i.e.

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log^+ \frac{1}{|x-y|} u(x,t)u(y,t) dx dy \le C, \tag{5.4}$$

where we will use the uniform bounds for the regularized potential energy in Lemma 4.1. Notice that

$$\log^+ \frac{1}{|x-y|^2+\varepsilon^2} \nearrow \log^+ \frac{1}{|x-y|^2}, \quad \text{as } \varepsilon \to 0, \quad \text{a.e.} \quad \text{in } \mu.$$

By the monotone convergence theorem, we get

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log^+ \frac{1}{|x-y|^2+\varepsilon^2} d\mu = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log^+ \frac{1}{|x-y|^2} d\mu.$$

This together with Lemma 4.1 gives (5.4). □

*Step 3. Well-defined potential:* Due to (5.3) and Fubini-Tonelli's theorem, $c$ can be well defined by $u$ as

$$c(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|} u(y)dy.$$

Moreover, the proof of Lemma 5.1 implies

$$\int_{|x|\le R} |\nabla c|^2 \le \int_{\mathbb{R}^2\times\mathbb{R}^2} K_0(\theta, y, z) u(y,t) u(z,t) dydz$$

with $K_0(\theta, y, z)$ given by (5.1) with $\varepsilon = 0$. As in (5.2), one also has bounds on the integral kernel $K_0(\theta, y, z)$ [7, 14]

$$K_0(\theta, y, z) \le C(R)\log^+ \frac{1}{|y-z|}.$$

Then it follows from (5.3) that

$$\int_{|x|\le R} |\nabla c|^2 \le C. \tag{5.5}$$

Thus we have $\nabla c \in L^\infty(0,T; L^2_{\text{loc}}(\mathbb{R}^2))$ for all $T > 0$.

*Step 4. Time regularity of $u$:* Take $\varphi(x) \in C_0^2 \cap H^4(\mathbb{R}^2)$ with compact support as test function for the first equation in (1.1), then we have by using the second equation of (1.1) and a simple computation that

$$\int_{\mathbb{R}^2} \varphi u_t dx = \int_{\mathbb{R}^2} \Delta\varphi u dx + \int_{\mathbb{R}^2} \nabla^2\varphi : \nabla c \otimes \nabla c dx - \frac{1}{2}\int_{\mathbb{R}^2} \Delta\varphi |\nabla c|^2 dx.$$

Hence, we get for all $T > 0$ from (5.5) and taking into account the compact support of $\varphi$ that

$$\left| \int_{\mathbb{R}^2} \varphi u_t dx \right| \le C\|D^2\varphi\|_{L^\infty(\mathbb{R}^2)} \le C\|\varphi\|_{H^4(\mathbb{R}^2)}$$

and by the duality $u_t \in H^{-4}_{\text{loc}}(\mathbb{R}^2)$ yielding the time regularity $u \in Lip(0,T; H^{-4}_{\text{loc}}(\mathbb{R}^2))$.

## References

1. Blanchet, A., Dolbeault, J., Perthame, B.: Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. Electron. J. Differ. Equ. Conf. **44**, 32 (2006) (electronic)
2. Blanchet, A., Calvez, V., Carrillo, J.A.: Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model. SIAM J. Numer. Anal. **46**, 691–721 (2008)
3. Blanchet, A., Carrillo, J.A., Masmoudi, N.: Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. Commun. Pure Appl. Math. **61**, 1449–1481 (2008)
4. Blanchet, A., Carlen, E.A., Carrillo, J.A.: Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model. arXiv:1009.0134v1
5. Carlen, E., Loss, M.: Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on $S^n$. Geom. Funct. Anal. **2**, 90–104 (1992)
6. Carlen, E., Carrillo, J.A., Loss, M.: Hardy-Littlewood-Sobolev inequalities via fast diffusion flows. Proc. Natl. Acad. Sci. USA **107**, 19696–19701 (2010)
7. Delort, J.M.: Existence de nappes de tourbillon en dimension deux. J. Am. Math. Soc. **4**, 386–553 (1991)
8. DiPerna, R., Majda, A.: Concentrations in regularizations for 2D incompressible flow. Commun. Pure Appl. Math. **40**, 301–345 (1987)

9. Dolbeault, J., Perthame, B.: Optimal critical mass in the two dimensional Keller-Segel model in $\mathbb{R}^2$. C.R. Acad. Sci. Paris, Ser. I **339**, 611–616 (2004)
10. Dolbeault, J., Schmeiser, C.: The two-dimensional Keller-Segel model after blow-up. Discrete Contin. Dyn. Syst. **25**, 109–121 (2009)
11. Dunford, N., Schwartz, J.T.: Linear Operators: General Theory. Wiley-Interscience, New York (1988)
12. Hillen, T., Painter, K.J.: A user's guide to PDE models for chemotaxis. J. Math. Biol. **58**, 183–217 (2009)
13. Lieb, E.H., Loss, M.: Analysis, Graduate Studies in Mathematics, vol. 14, 2nd edn. Am. Math. Soc., Providence (2001)
14. Liu, J.G., Xin, Z.P.: Convergence of vortex methods for weak solution to the 2-D Euler equations with vortex sheet data. Commun. Pure Appl. Math. **48**, 611–628 (1995)
15. Liu, J.G., Xin, Z.P.: Convergence of point vortex method for 2-D vortex sheet. Math. Comput. **70**, 565–606 (2001)
16. Majda, A.J.: Remarks on weak solution for vortex sheets with a distinguished sign. Indiana Univ. Math. J. **42**, 921–939 (1993)
17. Perthame, B.: Transport Equations in Biology. Birkhäuser, Basel (2007)
18. Poupaud, F.: Diagonal defect measures, adhesion dynamics and Euler equation. Methods Appl. Anal. **9**, 533–562 (2002)