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Blow-up, Zero α Limit and the Liouville Type Theorem for the Euler-Poincaré Equations

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Abstract: In this paper we study the Euler-Poincaré equations in $\mathbb{R}^N$. We prove local existence of weak solutions in $W^{2,p}(\mathbb{R}^N)$, $p > N$, and local existence of unique classical solutions in $H^k(\mathbb{R}^N)$, $k > N/2 + 3$, as well as a blow-up criterion. For the zero dispersion equation ($\alpha = 0$) we prove a finite time blow-up of the classical solution. We also prove that as the dispersion parameter vanishes, the weak solution converges to a solution of the zero dispersion equation with sharp rate as $\alpha \to 0$, provided that the limiting solution belongs to $C([0, T); H^k(\mathbb{R}^N))$ with $k > N/2 + 3$. For the *stationary weak solutions* of the Euler-Poincaré equations we prove a Liouville type theorem. Namely, for $\alpha > 0$ any weak solution $\mathbf{u} \in H^1(\mathbb{R}^N)$ is $\mathbf{u} = 0$; for $\alpha = 0$ any weak solution $\mathbf{u} \in L^2(\mathbb{R}^N)$ is $\mathbf{u} = 0$.

1. Introduction

We consider the following Euler-Poincaré equations in $\mathbb{R}^N$:

$$(EP)\begin{cases} \partial_t \boldsymbol{m} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{m} + (\nabla\boldsymbol{u})^\top\boldsymbol{m} + (\operatorname{div}\boldsymbol{u})\boldsymbol{m} = 0, \\ \boldsymbol{m} = (1-\alpha\Delta)\boldsymbol{u}, \\ \boldsymbol{u}_0(x) = \boldsymbol{u}_0, \end{cases}$$

where $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^N$ is the velocity, $\boldsymbol{m} : \mathbb{R}^N \to \mathbb{R}^N$ represents the momentum, constant $\sqrt{\alpha}$ is a length scale parameter, $(\nabla\boldsymbol{u})^\top$ = the transpose of $(\nabla\boldsymbol{u})$. The Euler-Poincaré equations arise in diverse scientific applications and enjoy several remarkable properties both in the one-dimensional and multi-dimensional cases.

The Euler-Poincaré equations were first studied by Holm, Marsden, and Ratiu in 1998 as a framework for modeling and analyzing fluid dynamics [18,19], particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling. There are intensive researches on analogous viscous or inviscid, incompressible Lagrangian averaged models. We refer to [7,12,26] for results on Navier-Stokes-α model in terms of existence and uniqueness, zero α limit to the Navier-Stokes equations, global attractor, etc. We refer to [2,20,23] for results on analysis and simulation of vortex sheets with Birkhoff-Rott-α or Euler-α approximation.

In one dimension, the Euler-Poincaré equations coincide with the dispersion-less case of Camassa-Holm (CH) equation [4]:

$$(CH) \quad \partial_t m + u\partial_x m + 2\partial_x u\, m = 0, \qquad m = (1 - \alpha\partial_{xx})u.$$

The solutions to (CH) are characterized by a discontinuity in the first order derivative at their peaks and are thus referred to as peakon solutions. (CH) is completely integrable with a bi-Hamiltonian structure and their peakon solutions are true solitary waves that emerge from the initial data. Peakons exhibit a remarkable stability–their identity is preserved through nonlinear interactions, see, e.g. [4,22]. We refer to a review paper [25] for a survey of recent results on well-poseness and existence of local and global weak solutions for (CH). The existence of a global weak solution and uniqueness was proven in [3,6,8,10,29]. A class of the so called weak-weak solution was studied in [29]. Breakdown of (CH) solutions was studied in [24].

The Euler-Poincaré equations have many further interpretations beyond fluid applications. For instance, in 2-D, it is exactly the same as the averaged template matching equation for computer vision (see, e.g., [14,17,21]). The Euler-Poincaré equations also have important applications in computational anatomy (see, e.g, [22,30]). The Euler-Poincaré equations can also be regarded as an evolutionary equation for a geodesic motion on a diffeomorphism group and it is associated with Euler-Poincaré reduction via symmetry [1,11,15,22,30]. We refer to a recent book [22] for a comprehensive review on the subject.

There are significant differences in solution behavior between the case of $\alpha > 0$ and the case of $\alpha = 0$. This can be understood from the following dispersion relation for the Camassa-Holm equation / Euler-Poincaré equations

$$\frac{\omega}{k} = u_0 + \frac{2u_0}{1 + \alpha k^2}$$

This dispersion relation indicates the well known fact that long waves travel faster than short ones in shallow water due to gravity. When $\alpha = 0$, the phase velocity is reduced to $3u_0$, i.e. the system is non-dispersive.

The main results obtained in this paper are

1. We provide a theorem on local existence of weak solution in $W^{2,p}(\mathbb{R}^N)$, $p > N$, and local existence of unique classical solutions in $H^k(\mathbb{R}^N)$, $k > N/2 + 3$. Furthermore, when $\alpha = 0$, the Euler-Poincaré equations become a symmetric hyperbolic system of conservation laws with a convex entropy. Consequently, there exists a local unique classical solution if $\boldsymbol{u}_0 \in H^k(\mathbb{R}^N)$ with $k > N/2 + 1$. These results are documented in Sect. 2.
2. For general initial data, the solution to the Euler-Poincaré equations blows up in its derivative. In Sect. 3, we prove a theorem on a blow-up criterion, as well as, a theorem on finite time blow-up of the classical solution for the zero dispersion equation. For classical solutions with reflection symmetry, the divergence $\nabla \cdot \boldsymbol{u}$ satisfy a Riccati equation at the invariant point under the reflection transformation and hence there is a finite time blow up if the divergence is initially negative.
3. The Euler-Poincaré equations can be regarded as a dispersion regularization of the limited equation. In Sect. 4, we prove that as the dispersion parameter α vanishes,

the weak solution to the Euler-Poincaré equations converges to the solution of the zero dispersion equation with a sharp rate as $\alpha \to 0$, provided that the limiting solution belongs to $C([0,T); H^k(\mathbb{R}^N))$ with $k > N/2 + 3$.

4. Finally, for the *stationary weak solutions* of the Euler-Poincaré equations we prove a Liouville type theorem in Sect. 5. For $\alpha > 0$, we prove that any weak solution $\mathbf{u} \in H^1(\mathbb{R}^N)$ is $\mathbf{u} = 0$. For $\alpha = 0$, any weak solution $\mathbf{u} \in L^2(\mathbb{R}^N)$ is $\mathbf{u} = 0$.

2. Preliminaries and Local Existence

In this section, we discuss some mathematical structures of (EP) and then we state a local existence theorem for the weak solution and the classical solution. We refer to [17,22] for more in-depth discussions on (EP).

(EP) can be recast as

$$\partial_t \boldsymbol{m} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{m}) + (\nabla \boldsymbol{u})^\top \boldsymbol{m} = 0. \tag{1}$$

The last term above can be written in a conservative/tensor form

$$\begin{aligned}
\sum_{j=1}^N \partial_i u_j m_j &= \sum_{j=1}^N \partial_i u_j u_j - \alpha \sum_{j,k=1}^N \partial_i u_j \partial_k^2 u_j \\
&= \frac{1}{2}\partial_i |\boldsymbol{u}|^2 - \alpha \sum_{j,k=1}^N \partial_k(\partial_i u_j \partial_k u_j) + \alpha \sum_{j,k=1}^N \partial_k \partial_i u_j \partial_k u_j \\
&= \frac{1}{2}\partial_i |\boldsymbol{u}|^2 - \alpha \sum_{j,k=1}^N \partial_j(\partial_i u_k \partial_j u_k) + \frac{\alpha}{2} \sum_{j,k=1}^N \partial_i (\partial_k u_j)^2 \\
&= \sum_{j=1}^N \partial_j \left(\frac{1}{2}\delta_{ij}|\boldsymbol{u}|^2 - \alpha \partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u} + \frac{\alpha}{2}\delta_{ij}|\nabla \boldsymbol{u}|^2 \right).
\end{aligned}$$

Set the stress-tensor

$$T_{ij} = m_i u_j + \frac{\delta_{ij}}{2}|\boldsymbol{u}|^2 - \alpha \partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u} + \frac{\alpha \delta_{ij}}{2}|\nabla \boldsymbol{u}|^2.$$

Then (EP) becomes

$$\partial_t m_i + \sum_{j=1}^N \partial_j T_{ij} = 0. \tag{2}$$

The first term in T_{ij} involves a second order derivative of $\boldsymbol{u}$ and it can be rewritten as

$$m_i u_j = u_i u_j + \alpha \sum_{k=1}^N \partial_k u_i \partial_k u_j - \alpha \sum_{k=1}^N \partial_k \big(u_j \partial_k u_i\big).$$

The symmetric part of tensor T is given by

$$T^a = \boldsymbol{u} \otimes \boldsymbol{u} + \alpha \nabla \boldsymbol{u} \nabla \boldsymbol{u}^\top - \alpha \nabla \boldsymbol{u}^\top \nabla \boldsymbol{u} + \frac{1}{2}(|\boldsymbol{u}|^2 + \alpha |\nabla \boldsymbol{u}|^2)\ \mathrm{Id}\,, \tag{3}$$

and the remainder terms in T are given by

$$T^b_{i,j} = -\alpha \sum_{k=1}^{N} \partial_k \big(u_j \partial_k u_i\big). \tag{4}$$

Hence $T = T^a + T^b$. In view of this, the natural definition of the weak solution of (EP) would be:

Definition 1. *$\boldsymbol{u} \in L^\infty(0, T; H^1_{loc}(\mathbb{R}^N))$ is a weak solution of (EP) with initial data $\boldsymbol{u}_0 \in H^1_{loc}(\mathbb{R}^N)$ if the following equation holds for all vector field $\boldsymbol{\phi}(x, t)$ such that $\boldsymbol{\phi}(\cdot, t) \in C^\infty_0(\mathbb{R}^N)$ for all $t \in [0, T)$ and $\boldsymbol{\phi}(x, \cdot) \in C^1_0([0, T))$ for all $x \in \mathbb{R}^N$,*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \big(\boldsymbol{u} \cdot \boldsymbol{\phi}_t + \alpha \nabla \boldsymbol{u} : \nabla \boldsymbol{\phi}_t\big) dx dt + \int_{\mathbb{R}^N} \big(\boldsymbol{u}_0 \cdot \boldsymbol{\phi}(\cdot, 0) + \alpha \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{\phi}(\cdot, 0)\big)\, dx \\ + \int_0^T \int_{\mathbb{R}^N} T^a : \nabla \boldsymbol{\phi}(x, t)\, dx dt + \alpha \sum_{i,j,k=1}^{N} \int_0^T \int_{\mathbb{R}^N} u_j \partial_k u_i \partial_j \partial_k \phi_i \, dx dt = 0,\end{aligned} \tag{5}$$

where T^a is given by (3).

(EP) also has a natural Hamiltonian structure. Set

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^N} \boldsymbol{u} \cdot \boldsymbol{m}\, d\boldsymbol{x},$$

then $\frac{\delta \mathcal{H}}{\delta \boldsymbol{m}} = \boldsymbol{u}$ and (EP) can be recast as

$$\partial_t \boldsymbol{m} = -\mathcal{A} \frac{\delta \mathcal{H}}{\delta \boldsymbol{m}}, \tag{6}$$

where $\mathcal{A}$ is an anti-symmetric operator defined by

$$\mathcal{A} \boldsymbol{u} = \sum_{j=1}^{N} \partial_j (m_i u_j) + \sum_{j=1}^{N} \partial_i u_j m_j.$$

Consequently, from (2) and (6), there are two conservation laws

$$\frac{d}{dt} \int_{\mathbb{R}^N} \boldsymbol{m}\, d\boldsymbol{x} = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^N} (|\boldsymbol{u}|^2 + \alpha |\nabla \boldsymbol{u}|^2)\, d\boldsymbol{x} = 0.$$

For the one-dimensional case, (EP) coincides with the dispersion-less case of the Camassa-Holm (CH) equation and there is an additional Hamiltonian structure and a Lax-pair which leads to a complete integrability of (CH) [4]. We refer to [13] for a general discussion on bi-Hamiltonian system and complete integrability.

When $\alpha = 0$, the above Hamiltonian structure shows that (EP) is a symmetric hyperbolic system of conservation laws

$$\begin{cases} \partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{2} \nabla |\boldsymbol{u}|^2 = 0 \\ \boldsymbol{u}(x, 0) = \boldsymbol{u}_0 \end{cases} \tag{7}$$

which possess a global convex entropy function

$$\frac{1}{2}\partial_t|\boldsymbol{u}|^2 + \mathrm{div}(|\boldsymbol{u}|^2\boldsymbol{u}) = 0. \tag{8}$$

We refer to (7) as the zero dispersion equation, and we can recast it in the usual form of a symmetric hyperbolic system (we state it in $\mathbb{R}^3$):

$$\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y + C\boldsymbol{u}_z = 0$$

with

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 3u & v & w \\ v & u & 0 \\ w & 0 & u \end{pmatrix}, \ldots$$

A is a symmetric matrix and has three eigenvalues: u, $2u + |\boldsymbol{u}|$, $2u - |\boldsymbol{u}|$, corresponding to one linearly degenerate field, and two genuinely nonlinear fields, respectively, when $\boldsymbol{u} \neq 0$.

We shall remark that although the high dimensional Burgers equation has a similar structure as (7), it does not possess a global convex entropy. In Sect. 5, we will prove a Liouville type theorem for the steady solution of (7). This theorem does not hold true for the high dimensional Burgers equation.

Now we introduce some notations and then we state a theorem on local existence of the weak solution and local existence and uniqueness of the classical solution.

For $s \in \mathbb{R}$ and $p \in [1, \infty]$ we define the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ as follows

$$L^{s,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) \mid \|(1-\Delta)^{\frac{s}{2}} f\|_{L^p} := \|f\|_{L^{s,p}} < \infty\}.$$

For $s \in \mathbb{N} \cup \{0\}$ it is well-known that $L^{s,p}(\mathbb{R}^N)$ is equivalent to the standard Sobolev space $W^{s,p}(\mathbb{R}^N)$(see e.g. [27]). This, in turn, implies immediately that there exist C_1, C_2 such that

$$C_1\|\boldsymbol{u}\|_{W^{k+2,p}} \le \|\boldsymbol{m}\|_{L^{k,p}} \le C_2\|\boldsymbol{u}\|_{W^{k+2,p}} \tag{9}$$

for all $k \in \mathbb{N} \cup \{0\}$, $p \in (1, \infty)$. As usual we denote $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$.

Theorem 1. *(i) Assume $\alpha > 0$ and $\boldsymbol{u}_0 \in W^{2,p}(\mathbb{R}^N)$ with $p > N$. Then, there exists $T = T(\|\boldsymbol{u}_0\|_{W^{2,p}})$ such that a weak solution to (EP) exists, and belongs to $\boldsymbol{u} \in L^\infty(0, T; W^{2,p}(\mathbb{R}^N)) \cap Lip(0, T; W^{1,p}(\mathbb{R}^N))$.*

(ii) Let $\alpha > 0$ and $\boldsymbol{u}_0 \in H^k(\mathbb{R}^N)$ with $k > N/2 + 3$. Then, there exists $T = T(\|\boldsymbol{u}_0\|_{H^k})$ such that a classical solution to (EP) exists uniquely, and belongs to $\boldsymbol{u} \in C([0, T); H^k(\mathbb{R}^N))$.

(iii) For $\alpha = 0$, (EP) is a symmetric hyperbolic system of conservation laws with a convex entropy. Consequently, if $\boldsymbol{u}_0 \in H^k(\mathbb{R}^N)$ with $k > N/2 + 1$. Then, there exists $T = T(\|\boldsymbol{u}_0\|_{H^k})$ such that a classical solution to (EP) exists uniquely, and belongs to $\boldsymbol{u} \in C([0, T); H^k(\mathbb{R}^N))$.

Proof. The proof of symmetric hyperbolicity and existence of convex entropy in (iii) are given in (7)–(8). The proof of existence of the unique classical solution for symmetric hyperbolic system is standard, see e.g [16].

The proof of local existence part is standard, and below we derive the key local in time estimate of $\boldsymbol{u}(t) \in L^\infty([0,T); W^{2,p}(\mathbb{R}^N)) \cap Lip(0,T; W^{1,p}(\mathbb{R}^N))$,

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\boldsymbol{m}\|_{L^p}^p &= -\frac{1}{p}\int_{\mathbb{R}^N}(\boldsymbol{u}\cdot\nabla)|\boldsymbol{m}|^p\,dx - \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_j u_i m_i m_j |\boldsymbol{m}|^{p-2}\,dx \\
&\quad - \int_{\mathbb{R}^N}(\operatorname{div}\boldsymbol{u})|\boldsymbol{m}|^p\,dx \\
&= -\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^N} Tr(S)|\boldsymbol{m}|^p\,dx - \sum_{i,j=1}^N \int_{\mathbb{R}^N} S_{ij} m_i m_j |\boldsymbol{m}|^{p-2}\,dx \\
&\le C\|S\|_{L^\infty}\|\boldsymbol{m}\|_{L^p}^p \le C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}\|_{L^p}^p \le C\|\boldsymbol{m}\|_{L^p}^{p+1}, \qquad (10)
\end{aligned}$$

and therefore

$$\frac{d}{dt}\|\boldsymbol{m}\|_{L^p} \le C\|\boldsymbol{m}\|_{L^p}^2.$$

We thus have the following estimate on $L^\infty(0,T; W^{2,p}(\mathbb{R}^N))$:

$$\|\boldsymbol{u}(t)\|_{W^{2,p}} \le \frac{C\|\boldsymbol{u}_0\|_{W^{2,p}}}{1 - Ct\|\boldsymbol{u}_0\|_{W^{2,p}}} \quad \forall t\in[0,T), \qquad (11)$$

where $T = \frac{1}{(C\|\boldsymbol{u}_0\|_{W^{2,p}})}$. In order to have the estimate of $\boldsymbol{u}$ in $Lip(0,T; W^{1,p}(\mathbb{R}^N))$, we take $L^2(\mathbb{R}^N)$ inner product (EP) with the test function $\boldsymbol{\psi} \in W^{1,\frac{p}{p-1}}(\mathbb{R}^N)$ for $p > N$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^N}\partial_t\boldsymbol{m}\cdot\boldsymbol{\psi}\,dx &= \int_{\mathbb{R}^N}\boldsymbol{m}(\boldsymbol{u}\cdot\nabla)\boldsymbol{\psi}\,dx - \int_{\mathbb{R}^N}\boldsymbol{m}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\psi}\,dx \\
&\le C\|\boldsymbol{m}\|_{L^p}\|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{\psi}\|_{L^{\frac{p}{p-1}}} + C\|\boldsymbol{m}\|_{L^p}\|\nabla\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{\psi}\|_{L^{\frac{p}{p-1}}} \\
&\le C\|\boldsymbol{m}\|_{L^p}^2\|\boldsymbol{\psi}\|_{W^{1,\frac{p}{p-1}}},
\end{aligned}$$

which provides us with the estimate,

$$\|\partial_t\boldsymbol{u}\|_{L^\infty(0,T;W^{1,p}(\mathbb{R}^N))} \le C\|\partial_t\boldsymbol{m}\|_{L^\infty(0,T;W^{-1,p}(\mathbb{R}^N))} \le C\|\boldsymbol{m}\|_{L^\infty(0,T);L^p(\mathbb{R}^N)}^2.$$

Hence, for all $0 < t_1 < t_2 < T$ we have

$$\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_{W^{1,p}} \le \int_{t_1}^{t_2}\|\partial_t\boldsymbol{u}(t)\|_{W^{1,p}}\,dt \le C(t_2 - t_1)\|\boldsymbol{m}\|_{L^\infty(0,T;L^p(\mathbb{R}^N))}^2.$$

Namely,

$$\|\boldsymbol{u}\|_{Lip(0,T;W^{1,p}(\mathbb{R}^N))} \le C\|\boldsymbol{m}\|_{L^\infty(0,T;L^p(\mathbb{R}^N))}^2.$$

This gives (i). Next we prove local in time persistency of regularity for $\boldsymbol{u}(t)$ in $H^k(\mathbb{R}^N)$) with $k > N/2+3$. Let $\beta = (\beta_1, \dots, \beta_N)$ be the standard multi-index notation with $|\beta| = \beta_1 + \cdots + \beta_N$. Taking the $H^k(\mathbb{R}^N)$ inner product (EP) with $\boldsymbol{m}$, we find

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{|\beta|\le k}\|D^\beta \boldsymbol{m}\|_{L^2}^2 &= -\sum_{|\beta|\le k}\int_{\mathbb{R}^N} D^\beta\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{m}\}\cdot D^\beta \boldsymbol{m}\,dx \\
&\quad -\sum_{|\beta|\le k}\int_{\mathbb{R}^N} D^\beta\{(\nabla \boldsymbol{u})^\top \boldsymbol{m}\}\cdot D^\beta \boldsymbol{m}\,dx \\
&\quad -\sum_{|\beta|\le k}\int_{\mathbb{R}^N} D^\beta\{(\operatorname{div}\boldsymbol{u})\boldsymbol{m}\}\cdot D^\beta \boldsymbol{m}\,dx \\
&:= I_1 + I_2 + I_3. \qquad (12)
\end{aligned}
$$

We write

$$
\begin{aligned}
I_1 &= -\sum_{|\beta|\le k}\int_{\mathbb{R}^N}\{D^\beta(\boldsymbol{u}\cdot\nabla)\boldsymbol{m} - (\boldsymbol{u}\cdot\nabla)D^\beta \boldsymbol{m}\}\cdot D^\beta \boldsymbol{m}\,dx \\
&\quad + \sum_{|\beta|\le k}\int_{\mathbb{R}^N}(\boldsymbol{u}\cdot\nabla)D^\beta \boldsymbol{m}\cdot D^\beta \boldsymbol{m}\,dx \\
&:= J_1 + J_2,
\end{aligned}
$$

and using the standard commutator estimate, we deduce

$$
\begin{aligned}
J_1 &\le \sum_{|\beta|\le k}\|D^\beta(\boldsymbol{u}\cdot\nabla)\boldsymbol{m} - (\boldsymbol{u}\cdot\nabla)D^\beta \boldsymbol{m}\|_{L^2}\|D^\beta \boldsymbol{m}\|_{L^2} \\
&\le C(\|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}\|_{H^k} + \|\boldsymbol{u}\|_{H^k}\|\nabla \boldsymbol{m}\|_{L^\infty})\|\boldsymbol{m}\|_{H^k} \\
&\le C(\|\boldsymbol{u}\|_{H^{N/2+1+\varepsilon}}\|\boldsymbol{m}\|_{H^k} + \|\boldsymbol{m}\|_{H^{k-2}}\|\boldsymbol{m}\|_{H^{N/2+1+\varepsilon}})\|\boldsymbol{m}\|_{H^k} \quad (\forall\varepsilon>0) \\
&\le C\|\boldsymbol{m}\|_{H^k}^3 \qquad (13)
\end{aligned}
$$

if $k > N/2+1$, where we used the fact $\boldsymbol{u} = (1-\alpha\Delta)^{-1}\boldsymbol{m}$, and therefore $\|\boldsymbol{u}\|_{H^s} \le \|\boldsymbol{m}\|_{H^{s-2}}$ for all $s \in \mathbb{R}$,

$$
\begin{aligned}
J_2 &= \frac{1}{2}\sum_{|\sigma|\le k}\int_{\mathbb{R}^N}(\boldsymbol{u}\cdot\nabla)|D^\sigma \boldsymbol{m}|^2\,dx = -\frac{1}{2}\sum_{|\sigma|\le k}\int_{\mathbb{R}^N}(\operatorname{div}\boldsymbol{u})|D^\beta \boldsymbol{m}|^2\,dx \\
&\le C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}\|_{H^k}^2 \le C\|\boldsymbol{m}\|_{H^{N/2-1+\varepsilon}}\|\boldsymbol{m}\|_{H^k}^2 \quad (\forall\varepsilon>0) \\
&\le C\|\boldsymbol{m}\|_{H^k}^3 \qquad (14)
\end{aligned}
$$

if $k > N/2-1$. The estimates of I_2, I_3 are simpler, and we have

$$
\begin{aligned}
I_2 + I_3 &\le \|(\nabla \boldsymbol{u})^\top \boldsymbol{m}\|_{H^k}\|\boldsymbol{m}\|_{H^k} \le C(\|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}\|_{H^k} + \|\boldsymbol{u}\|_{H^{k+1}}\|\boldsymbol{m}\|_{L^\infty})\|\boldsymbol{m}\|_{H^k} \\
&\le C(\|\boldsymbol{m}\|_{H^{N/2-1+\varepsilon}}\|\boldsymbol{m}\|_{H^k} + \|\boldsymbol{m}\|_{H^{k-1}}\|\boldsymbol{m}\|_{H^{N/2+\varepsilon}})\|\boldsymbol{m}\|_{H^k} \\
&\le C\|\boldsymbol{m}\|_{H^k}^3 \qquad (15)
\end{aligned}
$$

if $k > N/2$. Summarizing the above estimates, we obtain

$$
\frac{d}{dt}\|\boldsymbol{m}\|_{H^k}^2 \le C\|\boldsymbol{m}\|_{H^k}^3
$$

for $k > N/2 + 1$, which implies

$$\|\boldsymbol{u}(t)\|_{H^k} \le \frac{C\|\boldsymbol{u}_0\|_{H^k}}{1 - C\|\boldsymbol{u}_0\|_{H^k} t} \qquad \forall t \in [0, T), \text{ where } T = \frac{1}{C\|\boldsymbol{u}_0\|_{H^k}},$$

where $k > N/2 + 3$.

We now prove uniqueness of the solution in this class. Let $(\boldsymbol{u}_1, \boldsymbol{m}_1)$, $(\boldsymbol{u}_2, \boldsymbol{m}_2)$ be two solution pairs corresponding to initial data $(\boldsymbol{u}_{1,0}, \boldsymbol{m}_{1,0})$, $(\boldsymbol{u}_{2,0}, \boldsymbol{m}_{2,0})$. We set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, and so on. Subtracting the equation for $(\boldsymbol{u}_2, \boldsymbol{m}_2)$ from that of $(\boldsymbol{u}_1, \boldsymbol{m}_1)$, we find that

$$\partial_t \boldsymbol{m} + \operatorname{div}\left(\boldsymbol{u}_1 \otimes \boldsymbol{m}\right) + \operatorname{div}\left(\boldsymbol{u} \otimes \boldsymbol{m}_2\right) + (\nabla \boldsymbol{u}_1)^\top \boldsymbol{m} + (\nabla \boldsymbol{u})^\top \boldsymbol{m}_2 = 0. \tag{16}$$

Let $p > N$. Taking $L^2(\mathbb{R}^N)$ the product of (16) with $\boldsymbol{m}|\boldsymbol{m}|^{p-2}$, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|\boldsymbol{m}(t)\|_{L^p}^p &= -\left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^N} (\operatorname{div} \boldsymbol{u}_1)|\boldsymbol{m}|^p\, dx - \int_{\mathbb{R}^N} (\operatorname{div} \boldsymbol{u})\boldsymbol{m}_2 \cdot \boldsymbol{m}|\boldsymbol{m}|^{p-2}\, dx \\
&\quad - \int_{\mathbb{R}^N} (\boldsymbol{u} \cdot \nabla)\boldsymbol{m}_2 \cdot \boldsymbol{m}|\boldsymbol{m}|^{p-2}\, dx - \int_{\mathbb{R}^N} (\nabla \boldsymbol{u}_1)^\top \boldsymbol{m} \cdot \boldsymbol{m}|\boldsymbol{m}|^{p-2}\, dx \\
&\quad - \int_{\mathbb{R}^N} (\nabla \boldsymbol{u})^\top \boldsymbol{m}_2 \cdot \boldsymbol{m}|\boldsymbol{m}|^{p-2}\, dx \\
&\le C(\|\operatorname{div} \boldsymbol{u}_1\|_{L^\infty}\|\boldsymbol{m}\|_{L^p}^p + \|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}_2\|_{L^p}\|\boldsymbol{m}\|_{L^p}^{p-1} \\
&\quad + \|\boldsymbol{u}\|_{L^p}\|\nabla \boldsymbol{m}_2\|_{L^\infty}\|\boldsymbol{m}\|_{L^p}^{p-1} + \|\nabla \boldsymbol{u}_1\|_{L^\infty}\|\boldsymbol{m}\|_{L^p}^p \\
&\quad + \|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{m}_2\|_{L^p}\|\boldsymbol{m}\|_{L^p}^{p-1}) \\
&\le C(\|\boldsymbol{u}_1\|_{H^k} + \|\boldsymbol{u}_2\|_{H^k})\|\boldsymbol{m}\|_{L^p}^p
\end{aligned}$$

for $k > N/2 + 3$. Hence,

$$\|\boldsymbol{m}(t)\|_{L^p} \le \|\boldsymbol{m}_0\|_{L^p} \exp\left(C\int_0^t (\|\boldsymbol{u}_1(\tau)\|_{H^k} + \|\boldsymbol{u}_2(\tau)\|_{H^k})d\tau\right).$$

This inequality implies the desired uniqueness of solutions in the class $L^1(0, T; H^k(\mathbb{R}^N))$ with $k > N/2 + 3$. This gives (ii). The proof of (iii) was explained at the end of Sect. 2. This completes the proof of Theorem 1. □

3. Finite Time Blow Up

In this section, we first present a theorem on a blow-up criterion and then we prove a theorem on finite time blow up for the zero dispersion equation.

We denote the deformation tensor for $\boldsymbol{u}$ by $S = (S_{ij})$, where $S_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. We recall the Besov space $\dot{B}^0_{\infty,\infty}$, which is defined as follows. Let $\{\psi_m\}_{m\in\mathbb{Z}}$ be the Littlewood-Paley partition of unity, where the Fourier transform $\hat{\psi}_m(\xi)$ is supported on the annulus $\{\xi \in \mathbb{R}^N \mid 2^{m-1} \le |\xi| < 2^m\}$(see e.g. [28]). Then,

$$f \in \dot{B}^0_{\infty,\infty} \quad \text{if and only if} \quad \sup_{m\in\mathbb{Z}} \|\psi_m * f\|_{L^\infty} := \|f\|_{\dot{B}^0_{\infty,\infty}} < \infty.$$

The following is a well-known embedding result,

$$L^\infty(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N). \tag{17}$$

Theorem 2. *For $\alpha \geq 0$, we have the following finite time blow-up criterion of the local solution of (EP) in $\boldsymbol{u} \in C([0, t_*); H^k(\mathbb{R}^N))$, $k > N/2 + 3$.*

$$\limsup_{t \to t_*} \|\boldsymbol{u}(t)\|_{H^k} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty. \tag{18}$$

Remark 1.1. Combining the embedding relation, $W^{1,N}(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N)$ with the inequality $\|D^2\boldsymbol{u}\|_{L^p} \leq C\|\boldsymbol{m}\|_{L^p}$ for $p \in (1, \infty)$(see (22) below), we have

$$\|S\|_{\dot{B}^0_{\infty,\infty}} \leq C\|S\|_{BMO} \leq C\|DS\|_{L^N} \leq C\|D^2\boldsymbol{u}\|_{L^N} \leq C\|\boldsymbol{m}\|_{L^N}.$$

Therefore we obtain the following criterion as an immediate corollary of the above theorem: for all $p > N$,

$$\limsup_{t \to t_*} \|\boldsymbol{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|\boldsymbol{m}(t)\|_{L^N} dt = \infty. \tag{19}$$

Remark 1.2. In the one dimensional case of the Camassa-Holm equation (CH) the above criterion implies that finite time blow-up does not happen if $\int_0^t \|\boldsymbol{u}_{xx}(\tau)\|_{L^1} d\tau < \infty$ for all $t > 0$. Thanks to the conservation law we have $\sup_{0<\tau<t} \|\boldsymbol{u}_x(\tau)\|_{L^2} \leq \|\boldsymbol{u}_0\|_{H^1} < \infty$ for all $t > 0$. Since we have embedding $W^{2,1}(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$, and we do have finite time blow-up for (CH) [24], our criterion is sharp in this one dimensional case.

Proof of Theorem 2. We only give a proof for the case $\alpha > 0$. The proof for the case $\alpha = 0$ is similar and simpler, hence, will be omitted.

Using estimates (12, 13, 14, 15) for I_1, I_2, I_3 in the proof of Theorem 1 in the Appendix, one has

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{m}(t)\|_{H^k} &\leq C(\|\nabla \boldsymbol{u}\|_{L^\infty} + \|\boldsymbol{m}\|_{L^\infty} + \|\nabla \boldsymbol{m}\|_{L^\infty})\|\boldsymbol{m}(t)\|_{H^k} \\ &\leq C(\|\boldsymbol{m}\|_{L^p} + \|D\boldsymbol{m}\|_{L^p} + \|D^2\boldsymbol{m}\|_{L^p})\|\boldsymbol{m}(t)\|_{H^k}.\end{aligned}$$

Hence,

$$\|\boldsymbol{m}(t)\|_{H^k} \leq \|\boldsymbol{m}_0\|_{H^k} \exp\left[C \int_0^t \left\{\|\boldsymbol{m}(\tau)\|_{L^p} + \|D\boldsymbol{m}(\tau)\|_{L^p} + \|D^2\boldsymbol{m}(\tau)\|_{L^p}\right\} d\tau\right] \tag{20}$$

for $k > N/2 + 1$ and $p > N$, where we used the Sobolev embedding. Consequently, blow up of $\|\boldsymbol{m}(t)\|_{H^k}$ as $t \to t^*$ implies that at least one of $\|\boldsymbol{m}(t)\|_{L^p}$, $\|D\boldsymbol{m}(t)\|_{L^p}$ and $\|D^2\boldsymbol{m}(t)\|_{L^p}$ blow up as $t \to t^*$. In the following three steps, we show blow-up criterion for each of them are all given by (18).

Step 1. We first recall the following logarithmic Sobolev inequality (see e.g. [28]),

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{\dot{B}^0_{\infty,\infty}})(\log(1 + \|f\|_{W^{s,p}})), \tag{21}$$

where $s > 0$, $1 < p < \infty$ and $sp > N$. From the estimate in (10) in the Appendix we obtain

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{m}\|_{L^p} &\leq C(1 + \|S\|_{\dot{B}^0_{\infty,\infty}}) \log(1 + \|S\|_{W^{1,p}})\|\boldsymbol{m}\|_{L^p} \quad (\text{for} \quad p > N) \\ &\leq C(1 + \|S\|_{\dot{B}^0_{\infty,\infty}}) \log(1 + \|D^2\boldsymbol{u}\|_{L^p})\|\boldsymbol{m}\|_{L^p} \\ &\leq C(1 + \|S\|_{\dot{B}^0_{\infty,\infty}}) \log(1 + \|\boldsymbol{m}\|_{L^p})\|\boldsymbol{m}\|_{L^p}\end{aligned}$$

for $p > N$, where we used the boundedness on $L^p(\mathbb{R}^N)$ of the pseudo-differential operator

$$\sigma_{ij}(D) := \partial_i\partial_j(1-\alpha\Delta)^{-1} = -R_iR_j\Delta(1-\alpha\Delta)^{-1}$$

with the Riesz transforms $\{R_j\}_{j=1}^N$ on $\mathbb{R}^N$(see Lemma 2.1, pp. 133 [27]), which provides us with

$$\|D^2\boldsymbol{u}\|_{L^p} = \sum_{i,j=1}^N \|\sigma_{ij}(D)\boldsymbol{m}\|_{L^p} \le C\|\boldsymbol{m}\|_{L^p} \tag{22}$$

for all $p \in (1,\infty)$. By Gronwall's Lemma we obtain

$$\log(1+\|\boldsymbol{m}(t)\|_{L^p}) \le \log(1+\|\boldsymbol{m}_0\|_{L^p})\exp\left(C\int_0^t (1+\|S(\tau)\|_{\dot{B}^0_{\infty,\infty}})d\tau\right) \tag{23}$$

for $p > N$. This implies that

$$\limsup_{t\to t_*}\|\boldsymbol{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*}\|S(t)\|_{\dot{B}^0_{\infty,\infty}}dt = \infty. \tag{24}$$

Step 2. Taking the derivative of (EP) and $L^2(\mathbb{R}^N)$ the inner product with $D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}$, we find that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|D\boldsymbol{m}(t)\|_{L^p}^p &= \frac{1}{p}\int_{\mathbb{R}^N}(\operatorname{div}\boldsymbol{u})|D\boldsymbol{m}|^p\,dx - \int_{\mathbb{R}^N}(D\boldsymbol{u}\cdot\nabla)\boldsymbol{m}\cdot D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}\,dx \\
&\quad - \int_{\mathbb{R}^N} D(\nabla\boldsymbol{u})^\top\boldsymbol{m}\cdot D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}\,dx - \int_{\mathbb{R}^N}(\nabla\boldsymbol{u})^\top D\boldsymbol{m}\cdot D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}\,dx \\
&\quad - \int_{\mathbb{R}^N} D(\operatorname{div}\boldsymbol{u})\boldsymbol{m}\cdot D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}\,dx - \int_{\mathbb{R}^N}(\operatorname{div}\boldsymbol{u})\,D\boldsymbol{m}\cdot D\boldsymbol{m}|D\boldsymbol{m}|^{p-2}\,dx \\
&\le \left(3+\frac{1}{p}\right)\int_{\mathbb{R}^N}|D\boldsymbol{u}||D\boldsymbol{m}|^p\,dx + 2\int_{\mathbb{R}^N}|D^2\boldsymbol{u}||\boldsymbol{m}||D\boldsymbol{m}|^{p-1}\,dx \\
&\le \left(3+\frac{1}{p}\right)\|D\boldsymbol{u}\|_{L^\infty}\|D\boldsymbol{m}\|_{L^p}^p + 2\|D^2\boldsymbol{u}\|_{L^{2p}}\|\boldsymbol{m}\|_{L^{2p}}\|D\boldsymbol{m}\|_{L^p}^{p-1} \\
&\le C\|\boldsymbol{m}\|_{L^p}\|D\boldsymbol{m}\|_{L^p}^p + C\|\boldsymbol{m}\|_{L^{2p}}^2\|D\boldsymbol{m}\|_{L^p}^{p-1}
\end{aligned}$$

for $p > N$, where we used the Sobolev embedding and (22) to estimate

$$\|D\boldsymbol{u}\|_{L^\infty} \le C\|D^2\boldsymbol{u}\|_{L^p} \le C\|\boldsymbol{m}\|_{L^p}$$

for $p > N$. Hence, for $p > N$ we have

$$\frac{d}{dt}\|D\boldsymbol{m}(t)\|_{L^p} \le C\|\boldsymbol{m}\|_{L^p}\|D\boldsymbol{m}\|_{L^p} + C\|\boldsymbol{m}\|_{L^{2p}}^2.$$

By Gronwall's lemma, we have

$$\|D\boldsymbol{m}(t)\|_{L^p} \le \exp\left(C\int_0^t\|\boldsymbol{m}(\tau)\|_{L^p}d\tau\right)\left(\|D\boldsymbol{m}_0\|_{L^p} + C\int_0^t\|\boldsymbol{m}(\tau)\|_{L^{2p}}^2d\tau\right) \tag{25}$$

for $p > N$. From estimate (23), one has

$$\begin{aligned}\int_0^t \|\boldsymbol{m}(s)\|_{L^p} ds &\le t \max_{0\le s\le t} \|\boldsymbol{m}(s)\|_{L^p} \\ &\le t \max_{0\le s\le t} \exp\left(\log(1+\|\boldsymbol{m}(s)\|_{L^p})\right) \\ &\le t \exp\left(\log(1+\|\boldsymbol{m}_0\|_{L^p}) \exp\left(C\int_0^t (1+\|S(\tau)\|_{\dot{B}^0_{\infty,\infty}}) d\tau\right)\right). \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned}&\int_0^t \|\boldsymbol{m}(s)\|_{L^{2p}} ds \\ &\quad\le t \exp\left(\log(1+\|\boldsymbol{m}_0\|_{L^{2p}}) \exp\left(C\int_0^t (1+\|S(\tau)\|_{\dot{B}^0_{\infty,\infty}}) d\tau\right)\right). \end{aligned} \tag{27}$$

Combining (25, 26) and (27), one obtains

$$\limsup_{t\to t_*} \|D\boldsymbol{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty. \tag{28}$$

Step 3. Similarly, taking D^2 of (EP) and $L^2(\mathbb{R}^N)$ the inner product with $D^2\boldsymbol{m}|D^2\boldsymbol{m}|^{p-2}$, we find that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|D^2\boldsymbol{m}(t)\|_{L^p}^p \\ &\quad\le 4\int_{\mathbb{R}^N} |D\boldsymbol{u}||D^2\boldsymbol{m}|^p\, dx + 3\int_{\mathbb{R}^N} |D^2\boldsymbol{u}||D\boldsymbol{m}||D^2\boldsymbol{m}|^{p-1}\, dx \\ &\qquad+2\int_{\mathbb{R}^N} |D^3\boldsymbol{u}||\boldsymbol{m}||D^2\boldsymbol{m}|^{p-1}\, dx \\ &\quad\le 4\|D\boldsymbol{u}\|_{L^\infty}\|D^2\boldsymbol{m}\|_{L^p}^p + 3\|D^2\boldsymbol{u}\|_{L^{2p}}\|D\boldsymbol{m}\|_{L^{2p}}\|D^2\boldsymbol{m}\|_{L^p}^{p-1} \\ &\qquad+2\|D^3\boldsymbol{u}\|_{L^{2p}}\|\boldsymbol{m}\|_{L^{2p}}\|D^2\boldsymbol{m}\|_{L^p}^{p-1} \\ &\quad\le C\|\boldsymbol{m}\|_{L^p}\|D^2\boldsymbol{m}\|_{L^p}^p + C\|\boldsymbol{m}\|_{L^{2p}}\|D\boldsymbol{m}\|_{L^{2p}}\|D^2\boldsymbol{m}\|_{L^p}^{p-1}\end{aligned}$$

for $p > N$, where we used the estimate (22) as follows

$$\begin{aligned}\|D^3\boldsymbol{u}\|_{L^q} &= \|\left\{D^2(1-\alpha\Delta)^{-1}\right\} D(1-\alpha\Delta)\boldsymbol{u}\|_{L^q} \\ &\le \sum_{i,j=1}^N \|\sigma_{ij}(D)D\boldsymbol{m}\|_{L^q} \le C\|D\boldsymbol{m}\|_{L^q},\end{aligned}$$

which holds for all $q \in (1,\infty)$. Hence,

$$\frac{d}{dt}\|D^2\boldsymbol{m}(t)\|_{L^p} \le C\|\boldsymbol{m}\|_{L^p}\|D^2\boldsymbol{m}\|_{L^p} + C\|\boldsymbol{m}\|_{L^{2p}}\|D\boldsymbol{m}\|_{L^{2p}}.$$

By Gronwall's Lemma we have

$$\|D^2\boldsymbol{m}(t)\|_{L^p} \le \exp\left(C\int_0^t \|\boldsymbol{m}(\tau)\|_{L^p}d\tau\right)\Big(\|D^2\boldsymbol{m}_0\|_{L^p} +C\int_0^t \|\boldsymbol{m}(\tau)\|_{L^{2p}}\|D\boldsymbol{m}(\tau)\|_{L^{2p}}d\tau\Big)$$

for $p > N$. Similarly to the estimates in (26) and (27), the right hand side terms in the above inequality can all be controlled

$$\int_0^t (1+\|S(\tau)\|_{\dot{B}^0_{\infty,\infty}})d\tau.$$

Therefore, we have

$$\limsup_{t\to t_*} \|D^2\boldsymbol{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}^0_{\infty,\infty}}dt = \infty. \tag{29}$$

Combination of (20, 24, 28, 29) gives the proof of the theorem. □

We now present a finite time blow-up result for $\alpha = 0$.

Theorem 3. *Let the initial data of the system (7), $\boldsymbol{u}_0 \in H^k(\mathbb{R}^N), k > N/2+2$, has the reflection symmetry with respect to the origin, and satisfies* $\operatorname{div}\boldsymbol{u}_0(0) < 0$. *Then, there exists a finite time blow-up of the classical solution.*

Proof. Taking divergence of (7), we find

$$\partial_t(\operatorname{div}\boldsymbol{u}) + \boldsymbol{u}\cdot\nabla(\operatorname{div}\boldsymbol{u}) + 2\sum_{i,j=1}^N S_{ij}^2 + \sum_{j=1}^N(\Delta u_j)u_j + (\operatorname{div}\boldsymbol{u})^2 + \sum_{i,j=1}^N(\partial_i\partial_j u_i)u_j = 0, \tag{30}$$

where we used $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and the fact

$$\sum_{i,j=1}^N \partial_i u_j\partial_j u_i + \sum_{i,j=1}^N \partial_i u_j\partial_i u_j = 2\sum_{i,j=1}^N \partial_i u_j S_{ij} = \sum_{i,j=1}^N(\partial_i u_j + \partial_j u_i)S_{ij} = 2\sum_{i,j=1}^N S_{ij}^2.$$

Now we consider the reflection transform:

$$R: x \to \bar{x} = -x, \quad \boldsymbol{u}(x,t) \to \bar{\boldsymbol{u}}(x,t) = -\boldsymbol{u}(-x,t).$$

Obviously the system (7) is invariant under this transform. The origin of the coordinate is the invariant point under the reflection transform. We consider the smooth initial data $\boldsymbol{u}_0 \in H^k(\mathbb{R}^N), k > N/2+2$, which has the reflection symmetry. Then, by the uniqueness of the local classical solution in $H^k(\mathbb{R}^N)$, and hence in $C^2(\mathbb{R}^N)$, the reflection symmetry is preserved as long as the classical solution persists. We consider the evolution of the solution at the origin of the coordinates. Then, $\boldsymbol{u}(0,t) = 0$ and $D^2\boldsymbol{u}(0,t) = 0$ for all $t \in [0, T_*)$, where T_* is the maximal time of existence of the classical solution in

$H^k(\mathbb{R}^N)$. If $T_* = \infty$, we will show that this leads to a contradiction. The system (30) at the origin is reduced to

$$\partial_t(\operatorname{div} \boldsymbol{u}) + 2\sum_{i,j=1}^{N} S_{ij}^2 + (\operatorname{div} \boldsymbol{u})^2 = 0,$$

which implies

$$\partial_t(\operatorname{div} \boldsymbol{u}) = -2\sum_{i,j=1}^{N} S_{ij}^2 - (\operatorname{div} \boldsymbol{u})^2 \le -(\operatorname{div} \boldsymbol{u})^2, \tag{31}$$

and therefore

$$\operatorname{div} \boldsymbol{u}(0,t) \le \frac{\operatorname{div} \boldsymbol{u}_0(0)}{1 + \operatorname{div} \boldsymbol{u}_0(0)t}.$$

Since $\operatorname{div} \boldsymbol{u}_0(0) < 0$ by hypothesis, we have $T_* \le \frac{1}{|\operatorname{div} \boldsymbol{u}_0(0)|}$ and hence contradicts to the assumption of $T_* = \infty$. $\square$

4. Zero α Limit for Weak Solutions

In this section, we show the following theorem on the zero dispersion limit $\alpha \to 0$ for the weak solutions.

Theorem 4. *Let $\boldsymbol{u}^\alpha \in L^\infty((0,T); H^1(\mathbb{R}^N))$ be a weak solution with initial data $\boldsymbol{u}_0^\alpha$ to (EP) with $\alpha > 0$, and $\boldsymbol{u} \in L^\infty((0,T); H^k(\mathbb{R}^N)) \cap Lip((0,T); H^2(\mathbb{R}^N))$, $k > N/2+3$, be the classical solution with initial data $\boldsymbol{u}_0$ to (EP) with $\alpha = 0$, i.e., (7). Then, we have*

$$\begin{aligned}
&\sup_{0\le t\le T} \{\|\boldsymbol{u}^\alpha(t) - \boldsymbol{u}(t)\|_{L^2} + \sqrt{\alpha}\|\nabla(\boldsymbol{u}^\alpha(t) - \boldsymbol{u}(t))\|_{L^2}\} \\
&\qquad \le C\left(\alpha + \|\boldsymbol{u}_0^\alpha - \boldsymbol{u}_0\|_{L^2} + \sqrt{\alpha}\|\nabla(\boldsymbol{u}_0^\alpha - \boldsymbol{u}_0)\|_{L^2}\right),
\end{aligned} \tag{32}$$

where $C = C(\|u\|_{L^\infty(0,T;H^k(\mathbb{R}^N))}, \|u\|_{Lip(0,T;H^2(\mathbb{R}^N))})$ is a constant.

Proof. We denote $\boldsymbol{m} := \boldsymbol{u} - \alpha\Delta\boldsymbol{u}$. Then $(\boldsymbol{u}, \boldsymbol{m})$ satisfy (EP) with a truncation term as below

$$\partial_t \boldsymbol{m} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{m}) + (\nabla \boldsymbol{u})^\top \boldsymbol{m} = -\alpha\left\{\Delta \boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u} \otimes \Delta\boldsymbol{u}) + (\nabla\boldsymbol{u})^\top \Delta\boldsymbol{u}\right\}. \tag{33}$$

Subtracting (33) from the first equation of (EP), and setting $\bar{\boldsymbol{m}} := \boldsymbol{m}^\alpha - \boldsymbol{m}$ and $\bar{\boldsymbol{u}} := \boldsymbol{u}^\alpha - \boldsymbol{u}$, we find

$$\begin{aligned}
&\partial_t\bar{\boldsymbol{m}} + \operatorname{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{m}}) + \operatorname{div}(\bar{\boldsymbol{u}} \otimes \boldsymbol{m}) + \operatorname{div}(\boldsymbol{u} \otimes \bar{\boldsymbol{m}}) + (\nabla\bar{\boldsymbol{u}})^\top\bar{\boldsymbol{m}} + (\nabla\bar{\boldsymbol{u}})^\top\boldsymbol{m} + (\nabla\boldsymbol{u})^\top\bar{\boldsymbol{m}} \\
&\quad = \alpha\left\{\Delta\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u} \otimes \Delta\boldsymbol{u}) + (\nabla\boldsymbol{u})^\top\Delta\boldsymbol{u}\right\}.
\end{aligned} \tag{34}$$

Taking the $L^2(\mathbb{R}^N)$ inner product (34) with $\bar{\boldsymbol{u}}$, and integrating by part, and observing

$$\begin{aligned}
\int_{\mathbb{R}^N} \operatorname{div}(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{m}}) \cdot \bar{\boldsymbol{u}}\, dx &= -\int_{\mathbb{R}^N} \bar{\boldsymbol{u}} \cdot (\nabla\bar{\boldsymbol{u}})^\top\bar{\boldsymbol{m}}\, dx \\
\int_{\mathbb{R}^N} \operatorname{div}(\bar{\boldsymbol{u}} \otimes \boldsymbol{m}) \cdot \bar{\boldsymbol{u}}\, dx &= -\int_{\mathbb{R}^N} \bar{\boldsymbol{u}} \cdot (\nabla\bar{\boldsymbol{u}})^\top\boldsymbol{m}\, dx,
\end{aligned}$$

we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\left(|\bar{\boldsymbol{u}}|^2+\alpha|\nabla\bar{\boldsymbol{u}}|^2\right)\,dx=-\int_{\mathbb{R}^N}\mathrm{div}(\boldsymbol{u}\otimes\bar{\boldsymbol{m}})\cdot\bar{\boldsymbol{u}}\,dx-\int_{\mathbb{R}^N}\bar{\boldsymbol{u}}\cdot(\nabla\boldsymbol{u})^\top\bar{\boldsymbol{m}}\,dx$$
$$+\alpha\int_{\mathbb{R}^N}\left[\bar{\boldsymbol{u}}\cdot\left\{\Delta\boldsymbol{u}_t+\mathrm{div}(\boldsymbol{u}\otimes\Delta\boldsymbol{u})+(\nabla\boldsymbol{u})^\top\Delta\boldsymbol{u}\right\}\right]\,dx$$
$$:=I_1+I_2+I_3.$$

We estimate

$$I_1=-\sum_{i,j=1}^N\int_{\mathbb{R}^N}\partial_i u_i(\bar{u}_j-\alpha\Delta\bar{u}_j)\bar{u}_j\,dx-\sum_{i,j=1}^N\int_{\mathbb{R}^N}u_i\partial_i(\bar{u}_j-\alpha\Delta\bar{u}_j)\bar{u}_j\,dx$$
$$=J_1+J_2,$$

where

$$J_1=-\sum_{i,j=1}^N\int_{\mathbb{R}^N}\partial_i u_i|\bar{u}_j|^2dx+\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_i\partial_k u_i(\partial_k\bar{u}_j)\bar{u}_j dx$$
$$+\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_i u_i(\partial_k\bar{u}_j)\partial_k\bar{u}_j dx$$
$$\le C\|u(t)\|_{C^2}(\|\bar{\boldsymbol{u}}\|_{L^2}^2+\alpha\|\nabla\bar{\boldsymbol{u}}\|_{L^2}^2),$$

and

$$J_2=-\sum_{i,j=1}^N\int_{\mathbb{R}^N}u_i(\partial_i\bar{u}_j)\bar{u}_j\,dx+\alpha\sum_{i,j=1}^N\int_{\mathbb{R}^N}u_i\partial_i(\Delta\bar{u}_j)\bar{u}_j\,dx$$
$$=-\frac{1}{2}\sum_{i,j=1}^N\int_{\mathbb{R}^N}u_i\partial_i|\bar{u}_j|^2\,dx-\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_k u_i\partial_i(\partial_k\bar{u}_j)\bar{u}_j\,dx$$
$$-\frac{\alpha}{2}\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}u_i\partial_i|\partial_k\bar{u}_j|^2\,dx$$
$$=\frac{1}{2}\sum_{i,j=1}^N\int_{\mathbb{R}^N}\partial_i u_i|\bar{u}_j|^2\,dx+\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_i\partial_k u_i(\partial_k\bar{u}_j)\bar{u}_j\,dx$$
$$+\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_k u_i(\partial_k\bar{u}_j)\partial_i\bar{u}_j\,dx+\frac{\alpha}{2}\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_i u_i|\partial_k\bar{u}_j|^2\,dx$$
$$\le C\|u(t)\|_{C^2}(\|\bar{\boldsymbol{u}}\|_{L^2}^2+\alpha\|\nabla\bar{\boldsymbol{u}}\|_{L^2}^2).$$

$$I_2=\sum_{i,j=1}^N\int_{\mathbb{R}^N}\bar{u}_i\partial_i u_j(\bar{u}_j-\alpha\Delta\bar{u}_j)\,dx$$
$$=\sum_{i,j=1}^N\int_{\mathbb{R}^N}\bar{u}_i\partial_i u_j\bar{u}_j\,dx+\alpha\sum_{i,j,k=1}^N\int_{\mathbb{R}^N}\partial_k\bar{u}_i\partial_i u_j\partial_k\bar{u}_j\,dx$$

$$
\begin{aligned}
&+\alpha \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^N} \bar{u}_i \partial_i \partial_k u_j \partial_k \bar{u}_j \, dx \\
&\leq C \|u(t)\|_{C^2} (\|\bar{\boldsymbol{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\boldsymbol{u}}\|_{L^2}^2).
\end{aligned}
$$

One can estimate I_3 immediately as

$$
I_3 \leq \|\bar{\boldsymbol{u}}\|_{L^2}^2 + \alpha^2 C (\|\boldsymbol{u}\|^2_{Lip(0,T;H^2(\mathbb{R}^N))} + \|\boldsymbol{u}\|^4_{L^\infty(0,T;H^3(\mathbb{R}^N))}).
$$

Summarizing the above estimates, we obtain

$$
\begin{aligned}
\frac{d}{dt} (\|\bar{\boldsymbol{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\boldsymbol{u}}\|_{L^2}^2) \leq C \|u(t)\|_{C^2} (\|\bar{\boldsymbol{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\boldsymbol{u}}\|_{L^2}^2) \\
+ \alpha^2 C (\|\boldsymbol{u}\|^2_{Lip(0,T;H^2(\mathbb{R}^N))} + \|\boldsymbol{u}\|^4_{L^\infty(0,T;H^3(\mathbb{R}^N))}),
\end{aligned}
$$

which implies by Gronwall's Lemma that

$$
\|\bar{\boldsymbol{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\boldsymbol{u}}\|_{L^2}^2 \leq C_1 (\alpha^2 + \|\bar{\boldsymbol{u}}(0)\|_{L^2}^2 + \alpha \|\nabla \bar{\boldsymbol{u}}(0)\|_{L^2}^2),
$$

where constant C_1 depended only on $\|\boldsymbol{u}\|_{Lip(0,T;H^2(\mathbb{R}^N))}$ and $\|\boldsymbol{u}\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}$. This completes the proof of theorem. □

5. Liouville Type Theorem for Stationary Solutions

In this section, we prove a Liouville type theorem for stationary solutions. Recall that the stationary weak solution defined in Definition 1 reduces to

Definition 2. $\boldsymbol{u} \in H^1(\mathbb{R}^N)$ *is a stationary weak solution to (EP) on* $\mathbb{R}^N$, *if the following holds:*

$$
\begin{aligned}
&\sum_{j=1}^{N} \int_{\mathbb{R}^N} \left\{ u_i u_j + \alpha \nabla u_i \cdot \nabla u_j \right\} \partial_j \varphi_i \, dx + \alpha \sum_{j=1}^{N} \int_{\mathbb{R}^N} u_j \nabla u_i \cdot \nabla \partial_j \varphi_i \, dx \\
&+ \sum_{j=1}^{N} \int_{\mathbb{R}^N} \left\{ \frac{\delta_{ij}}{2} |\boldsymbol{u}|^2 - \alpha \partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u} + \frac{\alpha \delta_{ij}}{2} |\nabla \boldsymbol{u}|^2 \right\} \partial_j \varphi_i \, dx = 0
\end{aligned}
\tag{35}
$$

for $i = 1, \ldots, N$ *and for all* $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}^N)$.

Theorem 5. *(i) Let* $\boldsymbol{u} \in H^1(\mathbb{R}^N)$ *be a stationary weak solution to (EP) with* $\alpha > 0$. *Then,* $\boldsymbol{u} = 0$.
(ii) Let $\boldsymbol{u} \in L^2(\mathbb{R}^N)$ *be a stationary weak solution to (EP) with* $\alpha = 0$. *Then,* $\boldsymbol{u} = 0$.

Proof. For $\alpha > 0$, one can write (35) in the following form:

$$
\sum_{j=1}^{N} \int_{\mathbb{R}^N} T^a_{ij} \partial_j \varphi_i \, dx + \sum_{j,k=1}^{N} \int_{\mathbb{R}^N} \tilde{T}^b_{ijk} \partial_j \partial_k \varphi_i \, dx = 0,
\tag{36}
$$

where T^a_{ij} is defined in (3) and we recall here

$$
T^a_{ij} = u_i u_j + \alpha \nabla u_i \cdot \nabla u_j + \frac{\delta_{ij}}{2} |\boldsymbol{u}|^2 - \alpha \partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u} + \frac{\alpha \delta_{ij}}{2} |\nabla \boldsymbol{u}|^2,
$$

and

$$\tilde{T}^b_{ijk} = \alpha u_j \partial_k u_i.$$

corresponding to T^b_{ij} in (4).

Let us consider the radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and $0 \le \sigma(x) \le 1$ for $1 < |x| < 2$. Then, for each $R > 0$, we define

$$\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).$$

Choosing $\varphi_i(x) = x_i \sigma_R(x)$ in (36), we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^N \int_{\mathbb{R}^N} T^a_{ii} \sigma_R(x)\, dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} T^a_{ij} x_j \partial_i \sigma_R(x)\, dx + \sum_{i,k=1}^N \int_{\mathbb{R}^N} \tilde{T}^b_{iik} \partial_k \sigma_R(x)\, dx \\ &\quad + \sum_{i,j=1}^N \int_{\mathbb{R}^N} \tilde{T}^b_{iji} \partial_j \sigma_R(x)\, dx + \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \tilde{T}^b_{ijk} x_i \partial_j \partial_k \sigma_R(x)\, dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{37}$$

The hypothesis $u \in H^1(\mathbb{R}^N)$ implies that $T \in L^1(\mathbb{R}^N)$. Thus, we obtain

$$|I_2| \le \frac{1}{R} \int_{\{R \le |x| \le 2R\}} |T^a||x||\nabla\sigma|\, d\boldsymbol{x} \le 2\|\nabla\sigma\|_{L^\infty} \int_{\{R \le |x| \le 2R\}} |T^a|\, d\boldsymbol{x} \to 0$$

as $R \to \infty$ by the dominated convergence theorem. Similarly, $I_3, I_4, I_5 \to 0$ as $R \to \infty$.

Thus, passing $R \to \infty$ in (37), we have

$$\begin{aligned} 0 &= \lim_{R\to\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} T^a_{ii} \sigma_R(x)\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} \left\{ \frac{(N+2)}{2} |\boldsymbol{u}|^2 + \frac{\alpha N}{2} |\nabla \boldsymbol{u}|^2 \right\} d\boldsymbol{x}, \end{aligned}$$

which implies $\boldsymbol{u} = 0$. This gives (i).

For the case $\alpha = 0$. All the terms involving α drop and (ii) holds true. This completes the proof of the theorem □

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