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A Note on Aubin-Lions-Dubinskiĭ Lemmas

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Abstract Strong compactness results for families of functions in seminormed nonnegative cones in the spirit of the Aubin-Lions-Dubinskiĭ lemma are proven, refining some recent results in the literature. The first theorem sharpens slightly a result of Dubinskiĭ (in Mat. Sb. 67(109):609–642, 1965) for seminormed cones. The second theorem applies to piecewise constant functions in time and sharpens slightly the results of Dreher and Jüngel (in Nonlinear Anal. 75:3072–3077, 2012) and Chen and Liu (in Appl. Math. Lett. 25:2252–2257, 2012). An application is given, which is useful in the study of porous-medium or fast-diffusion type equations.

Keywords Compactness in Banach spaces · Rothe method · Dubinskii lemma · Seminormed cone

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1 Introduction

The Aubin-Lions lemma states criteria under which a set of functions is relatively compact in $L^p(0, T; B)$, where $p \geq 1$, $T > 0$, and B is a Banach space. The standard Aubin-Lions

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lemma states that if U is bounded in $L^p(0,T;X)$ and $\partial U/\partial t = \{\partial u/\partial t : u \in U\}$ is bounded in $L^r(0,T;Y)$, then U is relatively compact in $L^p(0,T;B)$, under the conditions that

$$X \hookrightarrow B \text{ compactly}, \qquad B \hookrightarrow Y \text{ continuously},$$

and either $1 \le p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Typically, when U consists of approximate solutions to an evolution equation, the boundedness of U in $L^p(0,T;X)$ comes from suitable a priori estimates, and the boundedness of $\partial U/\partial t$ in $L^r(0,T;Y)$ is a consequence of the evolution equation at hand. The compactness is needed to extract a sequence in the set of approximate solutions, which converges strongly in $L^p(0,T;B)$. The limit is expected to be a solution to the original evolution equation, thus yielding an existence result.

In recent years, nonlinear counterparts of the Aubin-Lions lemma were shown [4, 8, 17]. In this note, we aim to collect these results, which are scattered in the literature, and to prove some refinements. In particular, we concentrate on the case in which the set U is bounded in $L^p(0,T;M_+)$, where M_+ is a nonnegative cone (see below). This situation was first investigated by Dubinskiĭ, and therefore, we call the corresponding results Aubin-Lions-Dubinskiĭ lemmas.

Before detailing our main results, let us review the compactness theorems in the literature. The first result on the compact embedding of spaces of Banach space valued functions was shown by Aubin in 1963 [3], extended by Dubinskiĭ in 1965 [11], also see [16, Théorème 5.1, p. 58]. Some unnecessary assumptions on the spaces were removed by Simon in his famous paper [22]. The compactness embedding result was sharpened by Amann [2] involving spaces of higher regularity, and by Roubíček, assuming that the space Y is only locally convex Hausdorff [20] or that $\partial U/\partial t$ is bounded in the space of vector-valued measures [21, Corollary 7.9]. This condition can be replaced by a boundedness hypothesis in a space of functions with generalized bounded variations [15, Prop. 2]. A result on compactness in $L^p(\mathbb{R};B)$ can be found in [23, Theorem 13.2].

The boundedness of U in $L^p(0,T;B)$ can be weakened to tightness of U with respect to a certain lower semicontinuous function; see [19, Theorem 1]. Also the converse of the Aubin-Lions lemma was proved (see [18] for a special situation).

Already Dubinskiĭ [11] observed that the space X can be replaced by a seminormed set, which can be interpreted as a nonlinear version of the Aubin-Lions lemma. (Recently, Barrett and Süli [4] corrected an oversight in Theorem 1 of [11].) Furthermore, the space B can be replaced by $K(X)$, where $K : X \to B$ is a compact operator, as shown by Maitre [17], motivated by the nonlinear compactness result of Alt and Luckhaus [1].

Instead of boundedness of $\partial U/\partial t$ in $L^r(0,T;Y)$, the condition on the time shifts

$$\|\sigma_h u - u\|_{L^p(0,T-h;Y)} \to 0 \quad \text{as } h \to 0, \quad \text{uniformly in } u \in U,$$

where $(\sigma_h u)(t) = u(t+h)$, can be imposed to achieve compactness [22, Theorem 5]. If the functions u_τ in U are piecewise constant in time with uniform time step $\tau > 0$, this assumption was simplified in [10, Theorem 1] to

$$\|\sigma_\tau u_\tau - u_\tau\|_{L^r(0,T-\tau;Y)} \le C\tau,$$

where $C > 0$ does not depend on τ. This condition avoids the construction of linear interpolations of u_τ (also known as Rothe functions [14]). It was shown in [10, Prop. 2] that the rate τ cannot be replaced by τ^α with $0 < \alpha < 1$. Nonlinear versions were given in [8], generalizing the results of Maitre.

In the literature, discrete versions of the Aubin-Lions lemma were investigated. For instance, compactness properties for a discontinuous and continuous Galerkin time-step scheme were shown in [24, Theorem 3.1]. In [12], compactness of sequences of functions obtained by a Faedo-Galerkin approximation of a parabolic problem was studied.

In this note, we generalize some results of [8, 10] (and [12]) to seminormed nonnegative cones. We call M_+ a *seminormed nonnegative cone* in a Banach space B if the following conditions hold: $M_+ \subset B$; for all $u \in M_+$ and $c \ge 0$, $cu \in M_+$; and if there exists a function $[\cdot] : M_+ \to [0, \infty)$ such that $[u] = 0$ if and only if $u = 0$, and $[cu] = c[u]$ for all $c \ge 0$. We say that $M_+ \hookrightarrow B$ continuously, if there exists $C > 0$ such that $\|u\|_B \le C[u]$ for all $u \in M_+ \subset B$. Furthermore, we write $M_+ \hookrightarrow B$ compactly, if for any bounded sequence (u_n) in M_+ (here the boundedness means that there exists $C > 0$ such that for all $n \in \mathbb{N}$, $[u_n] \le C$), there exists a subsequence converging in B.

Theorem 1 (Aubin-Lions-Dubinskiĭ) *Let B, Y be Banach spaces and M_+ be a seminormed nonnegative cone in B with $M_+ \cap Y \ne \emptyset$. Let $1 \le p \le \infty$. We assume that*

(i) *$M_+ \hookrightarrow B$ compactly.*
(ii) *For all $(w_n) \subset B \cap Y$, $w_n \to w$ in B, $w_n \to 0$ in Y as $n \to \infty$ imply that $w = 0$.*
(iii) *$U \subset L^p(0, T; M_+ \cap Y)$ is bounded in $L^p(0, T; M_+)$.*
(iv) *$\|\sigma_h u - u\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$, uniformly in $u \in U$.*

Then U is relatively compact in $L^p(0, T; B)$ (and in $C^0([0, T]; B)$ if $p = \infty$).

This result generalizes slightly Theorem 3 in [8]. The novelty is that we do *not* require the continuous embedding $B \hookrightarrow Y$. If both B and Y are continuously embedded in a topological vector space (such as some Sobolev space with negative index) or in the space of distributions $\mathcal{D}'$, which is naturally satisfied in nearly all applications, then condition (ii) clearly holds. Therefore, we do not need to check the continuous embedding $B \hookrightarrow Y$, which is sometimes not obvious, like in [9, pp. 1206–1207], where B is an L^1 space with a complicated weight and Y is related to a Sobolev space with negative index. Thus, this generalization is not only interesting in functional analysis but also in applications.

The proof of Theorem 1 is motivated by Theorem 3.4 in [12] and needs a simple but new idea. Taking the proof of Theorem 5 in [22] as an example, we compare the traditional proof and our new idea. For this, we first list some statements:

$$B \hookrightarrow Y \text{ continuously}, \tag{1}$$

$$X \hookrightarrow B \text{ compactly}, \tag{2}$$

$$X \hookrightarrow Y \text{ compactly}, \tag{3}$$

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall u \in X,\ \|u\|_B \le \varepsilon \|u\|_X + C_\varepsilon \|u\|_Y, \tag{4}$$

$$U \text{ is a bounded subset of } L^p(0, T; X), \tag{5}$$

$$\|\sigma_h u - u\|_{L^p(0,T-h;Y)} \to 0 \text{ as } h \to 0, \text{ uniformly for } u \in U, \tag{6}$$

$$\|\sigma_h u - u\|_{L^p(0,T-h;B)} \to 0 \text{ as } h \to 0, \text{ uniformly for } u \in U, \tag{7}$$

$$U \text{ is relatively compact in } L^p(0, T; Y), \tag{8}$$

$$U \text{ is relatively compact in } L^p(0, T; B). \tag{9}$$

Simon proves (9) [22, Theorem 5] using the steps

I. Theorem 5 in [22]: (1), (2), (5), (6) ⇒ (9).
II. Lemma 8 in [22]: (1), (2) ⇒ (4).
III. Theorem 3 in [22]: (2), (5), (7) ⇒ (9), or (3), (5), (6) ⇒ (8).

More precisely,

Traditional proof of I:

$$\left.\begin{array}{r} (1),(2) \stackrel{\text{II}}{\Longrightarrow} (4)\\ (5) \\ (1),(2)\Longrightarrow \left.\begin{array}{l}(3)\\(5)\\(6)\end{array}\right\} \stackrel{\text{III}}{\Longrightarrow} (8)\end{array}\right\} \Longrightarrow (9);$$

New proof of I:

$$\left.\begin{array}{r}(2)\\(5)\\ (1),(2)\stackrel{\text{II}}{\Longrightarrow}\left.\begin{array}{l}(4)\\(5)\\(6)\end{array}\right\}\Longrightarrow (7)\end{array}\right\}\stackrel{\text{III}}{\Longrightarrow}(9).$$

In the traditional proof of I, the step (1), (2) ⇒ (3) depends on the continuous embedding (1). Hence, in that proof, (1) is essential. In our new proof, only step II: (1), (2) ⇒ (4) depends on (1), which can be replaced by condition (ii) of Theorem 1. This condition follows from (1) and hence, it is weaker than (1).

If U consists of piecewise constant functions in time (u_τ) with values in a Banach space, condition (iv) in Theorem 1 can be simplified. The main feature is that it is sufficient to verify one uniform estimate for the time shifts $u_\tau(\cdot + \tau) - u_\tau$ instead of all time shifts $u_\tau(\cdot + h) - u_\tau$ for $h > 0$.

Theorem 2 (Aubin-Lions-Dubinskiĭ for piecewise constant functions in time) *Let B, Y be Banach spaces and M_+ be a seminormed nonnegative cone in B. Let either $1 \le p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Let $(u_\tau) \subset L^p(0, T; M_+ \cap Y)$ be a sequence of functions, which are constant on each subinterval $((k-1)\tau, k\tau]$, $1 \le k \le N$, $T = N\tau$. We assume that*

(i) *$M_+ \hookrightarrow B$ compactly.*
(ii) *For all $(w_n) \subset B \cap Y$, $w_n \to w$ in B, $w_n \to 0$ in Y as $n \to \infty$ imply that $w = 0$.*
(iii) *(u_τ) is bounded in $L^p(0, T; M_+)$.*
(iv) *There exists $C > 0$ such that for all $\tau > 0$, $\|\sigma_\tau u_\tau - u_\tau\|_{L^r(0,T-\tau;Y)} \le C\tau$.*

Then, if $p < \infty$, (u_τ) is relatively compact in $L^p(0, T; B)$ and if $p = \infty$, there exists a subsequence of (u_τ) converging in $L^q(0, T; B)$ for all $1 \le q < \infty$ to a limit function belonging to $C^0([0, T]; B)$.

This result generalizes slightly Theorem 1 in [10] and Theorem 3 in [8] (for piecewise constant functions in time). The proof in [10] is based on a characterization of the norm of Sobolev-Slobodeckii spaces. Our proof just uses elementary estimates for the difference $\sigma_\tau u_\tau - u_\tau$ and thus simplifies the proof in [10]. Note that Theorems 1 and 2 are also valid if M_+ is replaced by a seminormed cone or Banach space. We observe that for functions $u_\tau(t, \cdot) = u_k$ for $t \in ((k-1)\tau, k\tau]$, $1 \le k \le N$, the estimate of (iv) can be formulated in terms of the difference $u_{k+1} - u_k$ since

$$\|\sigma_\tau u_\tau - u_\tau\|^r_{L^r(0,T-\tau;B)} = \sum_{k=1}^{N-1} \int_{(k-1)\tau}^{k\tau} \|u_{k+1} - u_k\|^r_B dt = \tau \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|^r_B.$$

A typical application is the cone of nonnegative functions u with $u^m \in W^{1,q}(\Omega)$, which occurs in diffusion equations involving a porous-medium or fast-diffusion term. Applying Theorem 2, we obtain the following result.

Theorem 3 *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Let (u_τ) be a sequence of nonnegative functions which are constant on each subinterval $((k-1)\tau, k\tau]$, $1 \leq k \leq N$, $T = N\tau$. Furthermore, let $0 < m < \infty$, $\gamma \geq 0$, $1 \leq q < \infty$, and $p \geq \max\{1, \frac{1}{m}\}$.*

(a) *If there exists $C > 0$ such that for all $\tau > 0$,*

$$\tau^{-1}\|\sigma_\tau u_\tau - u_\tau\|_{L^1(0,T-\tau;(H^\gamma(\Omega))')} + \|u_\tau^m\|_{L^p(0,T;W^{1,q}(\Omega))} \leq C,$$

then (u_τ) is relatively compact in $L^{mp}(0,T;L^{mr}(\Omega))$, where $r \geq \frac{1}{m}$ is such that $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

(b) *If additionally $\max\{0, (d-q)/(dq)\} < m < 1 + \min\{0, (d-q)/(dq)\}$ and*

$$\|u_\tau \log u_\tau\|_{L^\infty(0,T;L^1(\Omega))} \leq C \tag{10}$$

for some $C > 0$ independent of $\tau > 0$, then (u_τ) is relatively compact in $L^p(0,T;L^s(\Omega))$ with $s = qd/(qd(1-m)+d-q) > 1$.

Part (a) of this theorem generalizes Lemma 2.3 in [7], in which only relative compactness in $L^{m\ell}(0,T;L^{mr}(\Omega))$ for $\ell < p$ and $q = 2$ was shown. Part (b) improves part (a) for $m < 1$ by allowing for relative compactness in L^p with respect to time instead of the larger space L^{mp}. It generalizes Proposition 2.1 in [13] in which $m = \frac{1}{2}$ and $p = q = 2$ was assumed. Its proof shows that the bound on $u_\tau \log u_\tau$ can be replaced by a bound on $\phi(u_\tau)$, where ϕ is continuous and convex.

The additional estimate (10) is typical for solutions of semidiscrete nonlinear diffusion equations for which $\int_\Omega u_\tau \log u_\tau dx$ is an entropy (Lyapunov functional) with $\int_\Omega |\nabla u_\tau^m|^2 dx$ as the corresponding entropy production (see, e.g., [7, Lemma 3.1]). Theorem 3 improves standard compactness arguments. Indeed, let $\frac{1}{m} \leq q < d$. The additional estimate yields boundedness of (u_τ) in $L^\infty(0,T;L^1(\Omega))$. Hence, $\nabla u_\tau = \frac{1}{m}u_\tau^{1-m}\nabla u_\tau^m$ is bounded in $L^p(0,T;L^\alpha(\Omega))$ with $\alpha = q/(1+q(1-m))$. Thus, (u_τ) is bounded in $L^p(0,T;W^{1,\alpha}(\Omega)) \hookrightarrow L^p(0,T;L^s(\Omega))$. By the Aubin-Lions lemma [10], (u_τ) is relatively compact in $L^p(0,T;L^\beta(\Omega))$ for all $\beta < s$. Part (b) of the above theorem improves this compactness to $\beta = s$ under the condition that $(u_\tau \log u_\tau)$ is bounded in $L^\infty(0,T;L^1(\Omega))$.

This note is organized as follows. In Sect. 2, Theorems 1–3 are proved. Section 3 is concerned with additional results.

2 Proofs

2.1 Proof of Theorem 1

The proof of Theorem 1 is based on the following Ehrling type lemma.

Lemma 4 *Let B, Y be Banach spaces and M_+ be a seminormed nonnegative cone in B. We assume that*

(i) *$M_+ \hookrightarrow B$ compactly.*
(ii) *For all $(w_n) \subset B \cap Y$, $w_n \to w$ in B, $w_n \to 0$ in Y as $n \to \infty$ imply that $w = 0$.*

Then for any $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that for all* $u, v \in M_+ \cap Y$,

$$\|u - v\|_B \le \varepsilon([u] + [v]) + C_\varepsilon \|u - v\|_Y.$$

Proof The proof is performed by contradiction. Suppose that there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there exist $u_n, v_n \in M_+ \cap Y$ such that

$$\|u_n - v_n\|_B > \varepsilon_0([u_n] + [v_n]) + n\|u_n - v_n\|_Y. \tag{11}$$

This implies that $[u_n] + [v_n] > 0$ for all $n \in \mathbb{N}$ since otherwise, $[u_m] = [v_m] = 0$ for a certain $m \in \mathbb{N}$ would lead to $u_m = v_m = 0$ which contradicts (11). Define

$$\tilde{u}_n = \frac{u_n}{[u_n] + [v_n]}, \qquad \tilde{v}_n = \frac{v_n}{[u_n] + [v_n]}.$$

Then $\tilde{u}_n, \tilde{v}_n \in M_+ \cap Y$ and $[\tilde{u}_n] \le 1$, $[\tilde{v}_n] \le 1$. Taking into account the compact embedding $M_+ \hookrightarrow B$, there exist subsequences of $(\tilde{u}_n)$ and $(\tilde{v}_n)$, which are not relabeled, such that $\tilde{u}_n \to u$ and $\tilde{v}_n \to v$ in B and hence,

$$\tilde{u}_n - \tilde{v}_n \to u - v \quad \text{in } B. \tag{12}$$

We infer from (11) that $\|\tilde{u}_n - \tilde{v}_n\|_B > \varepsilon_0 + n\|\tilde{u}_n - \tilde{v}_n\|_Y$. This shows, on the one hand, that $\|\tilde{u}_n - \tilde{v}_n\|_B > \varepsilon_0$ and, by (12), $\|u - v\|_B \ge \varepsilon_0$. On the other hand, using the continuous embedding $M_+ \hookrightarrow B$,

$$\|\tilde{u}_n - \tilde{v}_n\|_Y \le \frac{1}{n}\|\tilde{u}_n - \tilde{v}_n\|_B \le \frac{C}{n}([\tilde{u}_n] + [\tilde{v}_n]) \le \frac{2C}{n}.$$

Consequently, $\tilde{u}_n - \tilde{v}_n \to 0$ in Y. Together with (12), condition (ii) implies that $u - v = 0$, contradicting $\|u - v\| \ge \varepsilon_0$. □

Proof of Theorem 1 First, we prove that

$$\|\sigma_h u - u\|_{L^p(0,T-h;B)} \to 0 \quad \text{as } h \to 0, \quad \text{uniformly in } u \in U. \tag{13}$$

Indeed, by condition (iii), there exists $C > 0$ such that $\|u\|_{L^p(0,T;M_+)} \le C$ for all $u \in U$. Lemma 4 shows that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $0 < h < T$, $u \in U$, and $t \in [0, T - h]$,

$$\|u(t+h) - u(t)\|_B \le \frac{\varepsilon}{4C}\big([u(t+h)] + [u(t)]\big) + C_\varepsilon \|u(t+h) - u(t)\|_Y.$$

Integration over $t \in (0, T - h)$ then gives

$$\begin{aligned}\|\sigma_h u - u\|_{L^p(0,T-h;B)} &\le \frac{\varepsilon}{2C}\|u\|_{L^p(0,T;M_+)} + C_\varepsilon \|\sigma_h u - u\|_{L^p(0,T-h;Y)} \\ &\le \frac{\varepsilon}{2} + C_\varepsilon \|\sigma_h u - u\|_{L^p(0,T-h;Y)}.\end{aligned}$$

We deduce from condition (iv) that for $\varepsilon_1 = \varepsilon/(2C_\varepsilon)$, there exists $\eta > 0$ such that for all $0 < h < \eta$ and $u \in U$, $\|\sigma_h u - u\|_{L^p(0,T-h;Y)} \le \varepsilon_1$. This shows that $\|\sigma_h u - u\|_{L^p(0,T-h;B)} \le \varepsilon/2 + \varepsilon/2 = \varepsilon$, proving the claim.

Because of condition (iii) and (13), the assumptions of Lemma 6 in [8] are satisfied, and the desired compactness result follows. In Lemma 6, only the (compact) embedding $M_+ \hookrightarrow B$ is needed. Let us mention that this lemma is a consequence of a nonlinear Maitre-type compactness result [8, Theorem 1] (see Proposition 7), which itself uses Theorem 1 in [22]. □

2.2 Proof of Theorem 2

The proof of Theorem 2 is based on an estimate of the time shifts $\sigma_h u_\tau - u_\tau$.

Lemma 5 *Let* $1 \le p \le \infty$ *and let* $u_\tau \in L^p(0,T;B)$ *be piecewise constant in time, i.e.,* $u_\tau(t) = u_k$ *for* $(k-1)\tau < t \le k\tau$, $k = 1,\dots,N$, $T = N\tau$. *Then, for* $0 < h < T$,

$$\|\sigma_h u_\tau - u_\tau\|_{L^p(0,T-h;B)} \le h^{1/p} \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|_B = \frac{h^{1/p}}{\tau} \|\sigma_\tau u_\tau - u_\tau\|_{L^1(0,T-\tau;B)}.$$

Proof Denoting by H the Heaviside functions, defined by $H(t) = 0$ for $t \le 0$ and $H(t) = 1$ for $t > 0$, we find that

$$u_\tau(t) = u_1 + \sum_{k=1}^{N-1} (u_{k+1} - u_k) H(t - k\tau), \quad 0 < t < T.$$

This gives

$$u_\tau(t+h) - u_\tau(t) = \sum_{k=1}^{N-1} (u_{k+1} - u_k) \big(H(t+h-k\tau) - H(t-k\tau) \big), \quad 0 < t < T-h,$$

and

$$\|\sigma_\tau u_\tau - u_\tau\|_{L^p(0,T-h;B)} \le \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|_B \|H(t+h-k\tau) - H(t-k\tau)\|_{L^p(0,T-h)}. \quad (14)$$

If $1 \le p < \infty$, we have for $1 \le k \le N-1$,

$$\begin{aligned}
&\|H(t+h-k\tau) - H(t-k\tau)\|^p_{L^p(0,T-h)} \\
&\quad \le \int_{-\infty}^{\infty} \big| H(t+h-k\tau) - H(t-k\tau) \big|^p dt \\
&\quad = \left(\int_{-\infty}^{k\tau - h} + \int_{k\tau-h}^{k\tau} + \int_{k\tau}^{\infty} \right) \big| H(t+h-k\tau) - H(t-k\tau) \big|^p dt = \int_{k\tau-h}^{k\tau} dt = h.
\end{aligned}$$

If $p = \infty$, we infer that

$$\begin{aligned}
\|H(t+h-k\tau) - H(t-k\tau)\|_{L^\infty(0,T-h)} &\le \|H(t+h-k\tau) - H(t-k\tau)\|_{L^\infty(\mathbb{R})} \\
&= \|H(t+h-k\tau) - H(t-k\tau)\|_{L^\infty(k\tau-h,k\tau)} = 1.
\end{aligned}$$

Hence,

$$\|H(t+h-k\tau) - H(t-k\tau)\|_{L^\infty(0,T-h)} \le h^{1/p}, \quad 1 \le p \le \infty.$$

Together with (14), this finishes the proof. □

Theorem 1 above and Lemma 4 in [22] imply the following proposition involving the time derivative instead of the time shifts.

Proposition 6 *Let B, Y be Banach spaces and M_+ be a seminormed nonnegative cone in B. Let either $1 \le p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Assume that conditions* (i)–(iii) *of Theorem 1 hold and*

$$\frac{\partial U}{\partial t} \quad \text{is bounded in } L^r(0, T; Y).$$

Then U is relatively compact in $L^p(0, T; B)$ (and in $C^0([0, T]; B)$ if $p = \infty$).

Proof of Theorem 2 The case $1 \le p < \infty$ is a consequence of Theorem 1 and Lemma 5. Therefore, let $p = \infty$. We define the linear interpolations

$$\tilde{u}_\tau(t) = \begin{cases} u_1 & \text{if } 0 < t \le \tau, \\ u_k - (k\tau - t)(u_k - u_{k-1})/\tau & \text{if } (k-1)\tau < t \le k\tau,\ 2 \le k \le N. \end{cases}$$

Since $(k\tau - t)/\tau \le 1$ for $(k-1)\tau < t \le k\tau$, we have

$$\|\tilde{u}_\tau\|_{L^\infty(0,T;M_+)} \le 2\|u_\tau\|_{L^\infty(0,T;M_+)} \le C,$$

using condition (iii). Furthermore, by condition (iv),

$$\left\| \frac{\partial \tilde{u}_\tau}{\partial t} \right\|_{L^r(0,T;Y)} = \frac{1}{\tau} \|\sigma_\tau u_\tau - u_\tau\|_{L^r(0,T-\tau;Y)} \le C.$$

By Proposition 6, there exists a subsequence $(\tilde{u}_{\tau'})$ of $(\tilde{u}_\tau)$ such that $\tilde{u}_{\tau'} \to \tilde{u}$ in $C^0([0, T]; B)$ (and $\tilde{u} \in C^0([0, T]; B)$). Applying Theorem 2 with $p = 1$ and $r = 1$, there exists a subsequence $(u_{\tau''})$ of $(u_{\tau'})$ such that $u_{\tau''} \to u$ in $L^1(0, T; B)$. Since

$$\|\tilde{u}_\tau - u_\tau\|_{L^1(0,T;B)} \le \|\sigma_\tau u_\tau - u_\tau\|_{L^1(0,T-\tau;B)} \le C\tau,$$

it follows that $(\tilde{u}_{\tau''})$ and $(u_{\tau''})$ converge to the same limit, implying that $\tilde{u} = u$. By the boundedness of (u_τ) in $L^\infty(0, T; M_+) \subset L^\infty(0, T; B)$ and interpolation, we infer that for $1 \le q < \infty$, as $\tau \to 0$,

$$\|u_{\tau''} - u\|_{L^q(0,T;B)} \le \|u_{\tau''} - u\|_{L^1(0,T;B)}^{1/q} \|u_{\tau''} - u\|_{L^\infty(0,T;B)}^{1-1/q} \le C\|u_{\tau''} - u\|_{L^1(0,T;B)}^{1/q} \to 0.$$

This shows that a subsequence of (u_τ) converges in $L^q(0, T; B)$ to a limit function $u \in C^0([0, T]; B)$. □

2.3 Proof of Theorem 3

(a) We apply Theorem 2 to $B = L^{mr}(\Omega)$, $Y = (H^s(\Omega))'$, and $M_+ = \{u \ge 0 : u^m \in W^{1,q}(\Omega)\}$ with $[u] = \|u^m\|_{W^{1,q}(\Omega)}^{1/m}$ for $u \in M_+$. Then M_+ is a seminormed nonnegative cone in B. We claim that $M_+ \hookrightarrow B$ compactly. Indeed, it follows from the continuous embedding $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ that for any $u \in M_+$,

$$\|u\|_{L^{mr}(\Omega)} = \|u^m\|_{L^r(\Omega)}^{\frac{1}{m}} \le C\|u^m\|_{W^{1,q}(\Omega)}^{\frac{1}{m}} = C[u].$$

Then $M_+ \hookrightarrow B$ continuously. Let (v_n) be bounded in M_+. Then (v_n^m) is bounded in $W^{1,q}(\Omega)$. Since $W^{1,q}(\Omega)$ embeddes compactly into $L^r(\Omega)$, up to a subsequence which is not relabeled, $v_n^m \to z$ in $L^r(\Omega)$ with $z \geq 0$. Again up to a subsequence, $v_n^m \to z$ a.e. and $v_n \to v := z^{1/m}$ a.e. Hence $v_n^m \to v^m$ in $L^r(\Omega)$ which yields

$$\lim_{n\to\infty} \|v_n\|_{L^{mr}(\Omega)} = \lim_{n\to\infty} \|v_n^m\|_{L^r(\Omega)}^{1/m} = \|v^m\|_{L^r(\Omega)}^{1/m} = \|v\|_{L^{mr}(\Omega)}.$$

Then it follows from Brezis-Lieb theorem (see [5, p. 298, 4.7.30] or [6]) that $v_n \to v$ in $L^{mr}(\Omega)$ (for a subsequence). This proves the claim. Next, let $w_n \to w$ in $L^{mr}(\Omega)$ and $w_n \to 0$ in $(H^\gamma(\Omega))'$. Since $L^{mr}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ and $(H^\gamma(\Omega))' \hookrightarrow \mathcal{D}'(\Omega)$, the convergences hold true in $\mathcal{D}'(\Omega)$ which gives $w = 0$. Furthermore, the following bound holds:

$$\|u_\tau\|_{L^{mp}(0,T;M_+)} = \|u_\tau^m\|_{L^p(0,T;W^{1,q}(\Omega))}^{1/m} \leq C.$$

By Theorem 2, (u_τ) is relatively compact in $L^{mp}(0,T;L^{mr}(\Omega))$.

(b) Note that the condition $\max\{0,(d-q)/(dq)\} < m < 1 + \min\{0,(d-q)/(dq)\}$ ensures that $s > 1$. By the first part of the proof, up to a subsequence, $u_\tau \to u$ a.e. It is shown in the proof of Proposition 2.1 in [13] that this convergence and (10) imply that $u_\tau \to u$ in $L^\infty(0,T;L^1(\Omega))$. We infer from the elementary inequality $|a-b|^{1/m} \leq |a^{1/m} - b^{1/m}|$ for all $a, b \geq 0$ that

$$\|u_\tau^m - u^m\|_{L^\infty(0,T;L^{1/m}(\Omega))} \leq \|u_\tau - u\|_{L^\infty(0,T;L^1(\Omega))} \to 0 \quad \text{as } \tau \to 0.$$

Then the Gagliardo-Nirenberg inequality gives

$$\begin{aligned}\|u_\tau^m - u^m\|_{L^{p/m}(0,T;L^{s/m}(\Omega))} &\leq C\|u_\tau^m - u^m\|_{L^p(0,T;W^{1,q}(\Omega))}^m \|u_\tau^m - u^m\|_{L^\infty(0,T;L^{1/m}(\Omega))}^{1-m} \\ &\leq C\|u_\tau^m - u^m\|_{L^\infty(0,T;L^{1/m}(\Omega))}^{1-m} \to 0.\end{aligned}$$

In particular, we infer that

$$\|u_\tau\|_{L^p(0,T;L^s(\Omega))}^m = \|u_\tau^m\|_{L^{p/m}(0,T;L^{s/m}(\Omega))} \leq C.$$

Furthermore, by the mean-value theorem, $|a-b| \leq \frac{1}{m}(a^{1-m} + b^{1-m})|a^m - b^m|$ for all a, $b \geq 0$, which yields, together with the Hölder inequality,

$$\begin{aligned}\|u_\tau - u\|_{L^p(0,T;L^s(\Omega))} &\leq C\big(\|u_\tau\|_{L^p(0,T;L^s(\Omega))}^{1-m} + \|u\|_{L^p(0,T;L^s(\Omega))}^{1-m}\big)\|u_\tau^m - u^m\|_{L^{p/m}(0,T;L^{s/m}(\Omega))} \\ &\leq C\|u_\tau^m - u^m\|_{L^{p/m}(0,T;L^{s/m}(\Omega))} \to 0.\end{aligned}$$

This proves the theorem.

3 Additional Results

Using Lemma 5, we can specify Maitre's nonlinear compactness result and Aubin-Lions lemma with an intermediate spaces assumption for piecewise constant functions in time.

Proposition 7 (Maitre nonlinear compactness) *Let either $1 \leq p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Let X, B be Banach spaces, and let $K : X \to B$ be a compact operator. Furthermore, let $(v_\tau) \subset L^1(0,T;X)$ be a sequence of functions, which are constant on each subinterval $((k-1)\tau, k\tau]$, $1 \leq k \leq N$, $T = N\tau$, and let $u_\tau = K(v_\tau) \in L^p(0,T;B)$. Assume that*

(i) *(v_τ) is bounded in $L^1(0,T;X)$, (u_τ) is bounded in $L^1(0,T;B)$.*
(ii) *There exists $C>0$ such that for all $\tau>0$, $\|\sigma_\tau u_\tau - u_\tau\|_{L^r(0,T-\tau;B)} \le C\tau$.*

Then, if $p<\infty$, (u_τ) is relatively compact in $L^p(0,T;B)$ and if $p=\infty$, there exists a subsequence of (u_τ) converging in $L^q(0,T;B)$ for all $1\le q<\infty$ to a limit function belonging to $C^0([0,T];B)$.

This result extends Theorem 1 in [8], which was proven for $r=p$ only, for piecewise constant functions in time. In fact, Lemma 5 shows that condition (ii) implies a bound on $\sigma_h u_\tau - u_\tau$ in $L^p(0,T-h;B)$, and Theorem 1 in [8] applies for $p<\infty$. The case $p=\infty$ is treated as in the proof of Theorem 2.

Proposition 8 (Aubin-Lions compactness) *Let X, B, Y be Banach spaces and $1\le p<\infty$. Assume that $X\hookrightarrow Y$ compactly, $X\hookrightarrow B\hookrightarrow Y$ continuously and there exist $\theta\in(0,1)$, $C_\theta>0$ such that for any $u\in X$, $\|u\|_B\le C_\theta\|u\|_X^{1-\theta}\|u\|_Y^\theta$. Furthermore, let (u_τ) be a sequence of functions, which are constant on each subinterval $((k-1)\tau,k\tau]$, $1\le k\le N$, $T=N\tau$. If*

(i) *(u_τ) is bounded in $L^p(0,T;X)$.*
(ii) *There exists $C>0$ such that for all $\tau>0$, $\|\sigma_\tau u_\tau - u_\tau\|_{L^1(0,T-\tau;Y)}\le C\tau$.*

Then (u_τ) is relatively compact in $L^q(0,T;B)$ for all $p\le q<p/(1-\theta)$.

Proof Let $p\le q<p/(1-\theta)$ and set $\ell=\theta/(1/q-(1-\theta)/p)$. Then $\ell\in[1,\infty)$ and $1/q=(1-\theta)/p+\theta/\ell$. Hence it follows from Lemma 5 that $\|\sigma_h u_\tau - u_\tau\|_{L^\ell(0,T-h;Y)}\le Ch^{1/\ell}$ for all $0<h<T$. This and Theorem 7 of [22] prove the result. □

This result improves Theorem 1 in [10] for the case $p<\infty$. For piecewise constant functions, Lemma 5 can be applied to Theorem 1.1 of [2] which yields another compactness result.

In finite-element or finite-volume approximation, $u_n\in Y_n$ may be the solution of a discretized evolution equation, where (Y_n) is a sequence of (finite-dimensional) Banach spaces which "approximates" the (infinite-dimensional) Banach space Y. Since the spaces Y_n depend on the index n, the classical Aubin-Lions lemma generally does not apply. Gallouët and Latché [12] have proved a discrete version of this lemma. We generalize their result for seminormed cones M_n and allow for the case $p=\infty$.

Proposition 9 (Discrete Aubin-Lions-Dubinskiĭ) *Let B, Y_n be Banach spaces ($n\in\mathbb{N}$) and let M_n be seminormed nonnegative cones in B with "seminorms" $[\cdot]_n$. Let $1\le p\le\infty$. Assume that*

(i) *$(u_n)\subset L^p(0,T;M_n\cap Y_n)$ and there exists $C>0$ such that $\|u_n\|_{L^p(0,T;M_n)}\le C$.*
(ii) *$\|\sigma_h u_n - u_n\|_{L^p(0,T-h;Y_n)}\to 0$ as $h\to 0$, uniformly in $n\in\mathbb{N}$.*

Then (u_n) is relatively compact in $L^p(0,T;B)$ (and in $C^0([0,T];B)$ if $p=\infty$).

Proof The proof uses the same techniques as in Sect. 2, therefore we give only a sketch. Similarly as in Lemma 4, a Ehrling-type inequality holds: Let $u_n\in M_n$ ($n\in\mathbb{N}$). Assume that (i) if $[u_n]_n\le C$ for all $n\in\mathbb{N}$, for some $C>0$, then (u_n) is relatively compact in B; (ii) if $u_n\to u$ in B as $n\to\infty$ and $\lim_{n\to\infty}\|u_n\|_{Y_n}=0$ then $u=0$. Then for all $\varepsilon>0$, there exists $C_\varepsilon>0$ such that for all $n\in\mathbb{N}$, $u,v\in M_n\cap Y_n$,

$$\|u-v\|_B\le\varepsilon([u]_n+[v]_n)+C_\varepsilon\|u-v\|_{Y_n}.$$

We infer as in the proof of Theorem 1 that conditions (i) and (ii) imply that

$$\|\sigma_h u_n - u_n\|_{L^p(0,T-h;B)} \to 0 \quad \text{as } h \to 0, \quad \text{uniformly for } n \in \mathbb{N}.$$

Finally, as in the proof of Lemma 6 in [8], the relative compactness of (u_n) in $L^p(0,T;B)$ (and in $C^0([0,T];B)$ if $p=\infty$) follows. □

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