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Global weak entropy solution to Doi–Saintillan–Shelley model for active and passive rod-like and ellipsoidal particle suspensions

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ABSTRACT

We prove the existence of the global weak entropy solution to the Doi–Saintillan–Shelley model for active and passive rod-like particle suspensions, which couples a Fokker–Planck equation with the incompressible Navier–Stokes or Stokes equation, under the no-flux boundary conditions, $L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ initial data, and finite initial entropy for the particle distribution function in two and three dimensions. Furthermore, for the model with the Stokes equation, we obtain the global $L^2(\Omega \times \mathbb{S}^{d-1})$ weak solution in two and three dimensions and the uniqueness in two dimension.

1. Introduction

Passive rod-like and ellipsoidal particle suspensions in fluid are common in nature, such as liquid crystal molecules moving in a solvent. The dilute suspensions of passive rod-like particles can be effectively modeled by a coupled microscopic Fokker–Planck equation and macroscopic (Navier–)Stokes equation, known as the Doi model (see Doi [12] and Doi and Edwards [13]). We refer to Hezel, Otto

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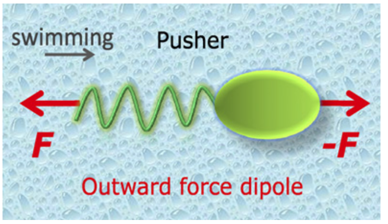

Fig. 1. Pusher.

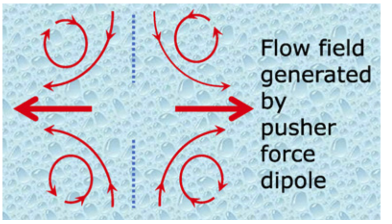

Fig. 2. Flow field generated by pusher.

and Tzavaras [15,25,16] for recent in-depth studies on the Doi model for suspensions of passive rod-like particles with and without considering the effects of gravity.

While the active rod-like and ellipsoidal particle suspensions are also very common in nature, such as in micro-organisms like bacteria locomoting, they also appear in technological applications such as the design of artificial swimmers. In 2008, Saintillan and Shelley [29,30] extended the Doi model for active rod-like and ellipsoidal particle suspensions. For completeness, we sketch below the derivation in Saintillan and Shelley [29,30] (we also follow some derivations in Doi and Edwards [13], Kim and Karrila [22] and Hezel, Otto and Tzavaras [16]).

1.1. Derivation of the model

Let $\Omega \subset \mathbb{R}^d$ be a macroscopic physical bounded domain with boundary $\partial\Omega$ of class C^1 and $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere. A system of identical active rod-like and ellipsoidal particles, described by $\{(\boldsymbol{x}_i, \boldsymbol{n}_i)\}_{i=1}^N$ where $\boldsymbol{x}_i \in \Omega$ is the position and $\boldsymbol{n}_i \in \mathbb{S}^{d-1}$ is the orientation, suspend in an incompressible fluid field with macroscopic velocity $\boldsymbol{u}(\boldsymbol{x}, t)$.

An active particle, labeled as $(\boldsymbol{x}_i, \boldsymbol{n}_i)$, moves along with the velocity $\boldsymbol{u}(\boldsymbol{x}_i, t)$, and with an active motion of self-propulsion in the direction of its orientation $\boldsymbol{n}_i$, where it experiences the least drag, at a constant speed U_0 (known as self-propelled speed or terminal speed). This dynamics is described by

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{u}(\boldsymbol{x}_i, t) + U_0 \boldsymbol{n}_i. \tag{1.1}$$

An active particle acts as a force dipole as we will explain below. The particle exerts a force of self-propulsion, denoted by $\boldsymbol{F}$, on the fluid. In the assumption of neglecting inertia, the particle also exerts an equal and opposite force against drag on the fluid, which equals $-\boldsymbol{F}$. According to the mechanism for swimming, a particle can be classified into either a pusher or a puller. A particle that swims by using its tail, is called a pusher. The tail of a pusher exerts a force $\boldsymbol{F}$ on the fluid in the opposite direction of swimming. Since at the same time the tail propels the head forward, the head also exerts a force $-\boldsymbol{F}$ on the fluid in the direction of swimming. Therefore, the pusher acts as an outward force dipole (see Fig. 1), and hence generates local flow field as shown in Fig. 2. A particle that swims by using its arms, is called a puller. In contrast with a pusher, a puller exerts an inward force dipole (see Fig. 3) on the fluid and also generates flow field (see Fig. 4) in an opposite direction of the flow field generated by a pusher.

A pusher $(\boldsymbol{x}_i, \boldsymbol{n}_i)$ force dipole (see Fig. 5) can be expressed as

$$\boldsymbol{F} = |\boldsymbol{F}|\boldsymbol{n}_i\left[\delta_{\mathbf{x}_i + \frac{\ell}{2}\mathbf{n}_i}(\boldsymbol{x}) - \delta_{\mathbf{x}_i - \frac{\ell}{2}\mathbf{n}_i}(\boldsymbol{x})\right] \tag{1.2}$$

where ℓ is the length of the particle. This force dipole can be uniquely decomposed by $\boldsymbol{F} = \nabla_{\mathbf{x}} \cdot \boldsymbol{S} + \nabla_{\mathbf{x}}\psi$ in $\mathscr{D}'$, where $\boldsymbol{S}$ is a symmetric traceless tensor with decay at infinity which is known as the stresslet, and ψ is a potential with decay at infinity. Let $\boldsymbol{u}$ be a fundamental solution to

$$\nabla_{\mathbf{x}} p = \mu \Delta_{\mathbf{x}} \boldsymbol{u} + F, \qquad \nabla_{\mathbf{x}} \cdot \boldsymbol{u} = 0.$$

Then $\boldsymbol{S} = -\mu(\nabla_{\mathbf{x}} \boldsymbol{u} + (\nabla_{\mathbf{x}} \boldsymbol{u})^\top)$.

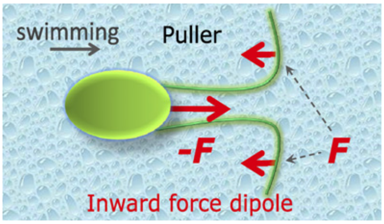

Fig. 3. Puller.

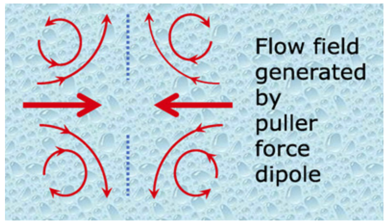

Fig. 4. Flow field generated by puller.

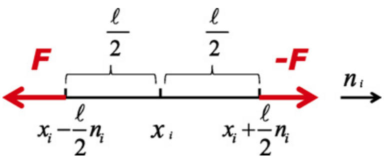

Fig. 5. Pusher outward force dipole.

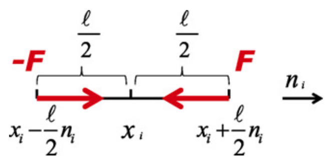

Fig. 6. Puller inward force dipole.

Using multipole approximation (see pp. 28–30 [22]), one has that the stresslet exerted by a pusher force dipole on fluid is approximated by

$$\boldsymbol{S} \approx -|\boldsymbol{F}|\ell\left(\boldsymbol{n}_i \otimes \boldsymbol{n}_i - \frac{1}{d}\mathrm{Id}\right)\delta_{\mathbf{x}_i}(\boldsymbol{x}). \tag{1.3}$$

Here we reformulate the multipole approximation in a weak form. Indeed, for any test function $\boldsymbol{v}(x)$ with $\nabla_{\mathbf{x}} \cdot \boldsymbol{v} = 0$, we have

$$\begin{aligned} \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v}\, d\boldsymbol{x} &= |\boldsymbol{F}|\boldsymbol{n}_i \cdot \left[\boldsymbol{v}\left(\boldsymbol{x}_i + \frac{\ell}{2}\boldsymbol{n}_i\right) - \boldsymbol{v}\left(\boldsymbol{x}_i - \frac{\ell}{2}\boldsymbol{n}_i\right)\right] \\ &\approx |\boldsymbol{F}|\ell\boldsymbol{n}_i \cdot \nabla_{\mathbf{x}}\boldsymbol{v}(\boldsymbol{x}_i)\boldsymbol{n}_i = |\boldsymbol{F}|\ell\left(\boldsymbol{n}_i \otimes \boldsymbol{n}_i - \frac{1}{d}\mathrm{Id}\right) : \nabla_{\mathbf{x}}\boldsymbol{v}(\boldsymbol{x}_i). \end{aligned} \tag{1.4}$$

It follows from integration-by-parts that

$$\int_\Omega \boldsymbol{F} \cdot \boldsymbol{v}\, d\boldsymbol{x} = \int_\Omega (\nabla_{\mathbf{x}} \cdot \boldsymbol{S}) \cdot \boldsymbol{v}\, d\boldsymbol{x} = -\int_\Omega \boldsymbol{S} : \nabla_{\mathbf{x}}\boldsymbol{v}\, d\boldsymbol{x}. \tag{1.5}$$

We deduce from (1.4) and (1.5) that (1.3) holds.

Similarly, the stresslet exerted by puller force dipole on fluid (see Fig. 6) is approximated by

$$\boldsymbol{S} \approx |\boldsymbol{F}|\ell\left(\boldsymbol{n}_i \otimes \boldsymbol{n}_i - \frac{1}{d}\mathrm{Id}\right)\delta_{\mathbf{x}_i}(\boldsymbol{x}). \tag{1.6}$$

Combining (1.3) and (1.6) together, we have that the stresslet exerted by the force dipole can be approximated by

$$\boldsymbol{S} \approx \sigma_0(d\boldsymbol{n}_i \otimes \boldsymbol{n}_i - \mathrm{Id})\delta_{\mathbf{x}_i}(\boldsymbol{x}) \tag{1.7}$$

where $\sigma_0 := \pm|\boldsymbol{F}|\ell/d$. For a pusher, $\sigma_0 < 0$; whereas, for a puller, $\sigma_0 > 0$.

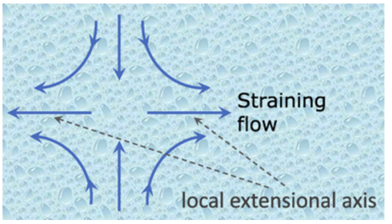

Fig. 7. Straining flow.

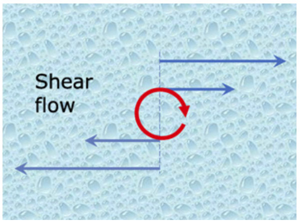

Fig. 8. Shear flow.

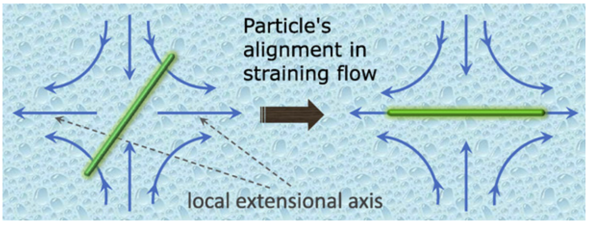

Fig. 9. Particle's alignment in straining flow.

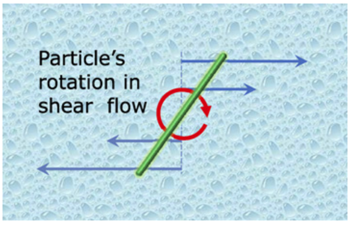

Fig. 10. Particle's rotation in shear flow.

The local linear flow $\nabla_{\mathbf{x}}\boldsymbol{u}$ is composed of two parts, the symmetric part $E = \frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u} + (\nabla_{\mathbf{x}}\boldsymbol{u})^\top)$, called straining flow (see Fig. 7) and the anti-symmetric part $W = \frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u} - (\nabla_{\mathbf{x}}\boldsymbol{u})^\top)$, called shear flow (see Fig. 8).

Under the straining flow, the particle aligns along the local extensional axis (see Fig. 9) and in the shear flow, the particle rotates (see Fig. 10) along the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Combining the two effects, the change of particle's direction arising from the local linear flow is described by the classical Jeffery's equation (p. 124, formula (5.33) [22]; also see [19])

$$\frac{d\boldsymbol{n}_i}{dt} = (\mathrm{Id} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i)\big(\gamma E(\boldsymbol{x}_i, t) + W(\boldsymbol{x}_i, t)\big)\boldsymbol{n}_i \tag{1.8}$$

where $\mathrm{Id} \in \mathbb{R}^{d\times d}$ denotes the unit matrix; $(\mathrm{Id} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i)(\gamma E(\boldsymbol{x}_i, t) + W(\boldsymbol{x}_i, t))\boldsymbol{n}_i$ is the projection of $(\gamma E(\boldsymbol{x}_i, t) + W(\boldsymbol{x}_i, t))\boldsymbol{n}_i$ on the tangential space at $\boldsymbol{n}_i$. For the general case, Eq. (1.8) is also known as Faxen's law [22]. Eq. (1.8) can be recast as

$$\frac{d\boldsymbol{n}_i}{dt} = -\nabla_{\mathbf{n}}\phi + \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{n} \tag{1.9}$$

where $\phi = -\gamma\frac{1}{2}\boldsymbol{n} \cdot E\boldsymbol{n}$ is a potential, which drives the particle from high potential to low potential and obtains the minimum at the eigenvector direction of the largest eigenvalue of E. This eigenvector direction is known as the local extensional axis.

Here $\gamma \in [-1, 1]$ is a shape parameter. For an ellipsoidal particle with aspect ratio A, $\gamma = (A^2 - 1)/(A^2 + 1)$. If $0 < \gamma < 1$ (i.e. $A > 1$), the particle is prolate spheroidal; in the limit $\gamma \to 1$, the prolate ellipsoid becomes a slender rod-like particle; if $0 > \gamma > -1$ (i.e. $A < 1$), the particle is oblate spheroidal; in the limit $\gamma \to -1$, the oblate spheroidal particle becomes a thin disk.

We next consider the effects of the flow field generated by the force dipole on the background flow. Since the particle most of time is aligned with the straining flow, local flow field generated by pusher force dipole is basically in the same direction as the strain flow. Consequently, the pusher force dipole increases the local background staining flow (see Fig. 11), and hence reduces the effective viscosity and enhances flow mixing, known as bio-mixing [34,21,29], which causes some kind of instability. Saintillan and Shelley [29] refer to this phenomenon as instability for pusher, by observing that low-wave number shear stress fluctuations will amplify exponentially in suspensions of pusher

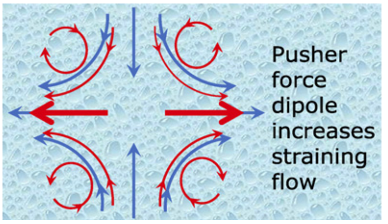

Fig. 11. Pusher force dipole increases straining flow.

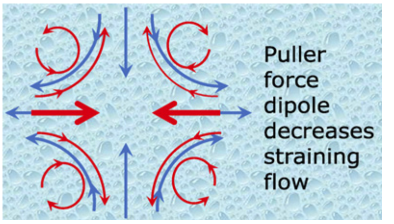

Fig. 12. Puller force dipole decreases straining flow.

(see also [31]). This instability can be explained by the fact that there is no entropy-dissipation relation for the pusher system. On the contrary, the puller force dipole decreases the local straining flow (see Fig. 12) and hence slows down the background flow which tends to be stable.

Taking account of the Brownian motions in (1.1) and (1.8), the particles satisfy the following system of coupled stochastic differential equations (which must be understood in the Stratonovich sense), for $1 \leqslant i \leqslant N$:

$$d\boldsymbol{x}_i = \boldsymbol{u}(\boldsymbol{x}_i, t)\,dt + U_0 \boldsymbol{n}_i\,dt + \sqrt{2D}\,dB^i, \tag{1.10}$$

$$d\boldsymbol{n}_i = (\mathrm{Id} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i)\big(\gamma E(\boldsymbol{x}_i, t) + W(\boldsymbol{x}_i, t)\big)\boldsymbol{n}_i\,dt + \sqrt{2D_r}(\mathrm{Id} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i) \circ dB^i, \tag{1.11}$$

where B^i is the translational Brownian motion which stands for the independent standard Brownian motions on $\mathbb{R}^d$ and $dB_r^i = (\mathrm{Id} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i) \circ dB^i$ is the rotational Brownian motion on $\mathbb{S}^{d-1}$ (see Hsu [18] for more details on how to define Brownian motion on a Riemannian manifold).

The translational (center-of-mass) diffusion coefficient D and rotational coefficient D_r in (1.10) and (1.11) are related to each other by the classic Taylor dispersion relation $D = U_0^2/6D_r$. This formula was first derived by Brenner [5,6] and was referred to as the generalized Taylor dispersion theory. There are many careful studies of this formula in experiments, simulations and theoretical derivations in literature. We refer to Saintillan and Shelley [28], Hohenegger and Shelley [17], Lauga [23] for recent discussions.

From the interacting particle dynamics system (1.10) and (1.11), we derive a mean-field limit as the number of particles N tends to infinity. We define the empirical distribution f^N by

$$f^N(\boldsymbol{x}, \boldsymbol{n}, t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{(\mathbf{x}_i, \mathbf{n}_i)}(\boldsymbol{x}, \boldsymbol{n}),$$

where the Dirac distribution is defined by $\langle \delta_{(\mathbf{x}_0, \mathbf{n}_0)}, \varphi \rangle_{\Omega \times \mathbb{S}^{d-1}} = \varphi(\boldsymbol{x}_0, \boldsymbol{n}_0)$ for any smooth function with compact support $\varphi \in C_0^\infty(\Omega \times \mathbb{S}^{d-1})$. For convenience, the total mass of the integration measure on the sphere $\mathbb{S}^{d-1}$ is supposed to be 1, so we have $\langle f^N, 1 \rangle_{\Omega \times \mathbb{S}^{d-1}} = 1$. If there is no noise (when $D = D_r = 0$), it is easy to see that f^N satisfies the following partial differential equation (in the sense of distributions)

$$\partial_t f^N + \nabla_{\mathbf{x}} \cdot \big((\boldsymbol{u} + U_0 \boldsymbol{n}) f^N\big) = -\nabla_{\mathbf{n}} \cdot \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E + W)\boldsymbol{n} f^N\big),$$

where $\nabla_{\mathbf{n}} \cdot$ denotes the tangential divergence operator on $\mathbb{S}^{d-1}$. When noise is added, the empirical distribution f^N tends to a probability density function f satisfying the following Smoluchowski equation

$$\partial_t f + \nabla_{\mathbf{x}} \cdot \big((\boldsymbol{u} + U_0 \boldsymbol{n}) f\big) = D \Delta_{\mathbf{x}} f + D_r \Delta_{\mathbf{n}} f - \nabla_{\mathbf{n}} \cdot \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E + W)\boldsymbol{n} f\big), \tag{1.12}$$

where $\Delta_{\mathbf{n}}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$.

We note that the interaction operator on the right side of (1.12) can be recast as

$$D_r \Delta_{\mathbf{n}} f - \nabla_{\mathbf{n}} \cdot \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E + W)\boldsymbol{n} f\big) = \nabla_{\mathbf{n}} \cdot \left(D_r \nabla_{\mathbf{n}} f + f\left(\nabla_{\mathbf{n}} \phi - \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{n}\right)\right). \quad (1.13)$$

If $f\boldsymbol{\omega} \times \boldsymbol{n}$ is absent, then (1.13) is called the Fokker–Planck operator and can be rewritten as

$$\nabla_{\mathbf{n}} \cdot (D_r \nabla_{\mathbf{n}} f + f \nabla_{\mathbf{n}} \phi) = D_r \nabla_{\mathbf{n}} \cdot \left(M \nabla_{\mathbf{n}} \frac{f}{M}\right), \quad M := \frac{e^{\phi/D_r}}{\int_{\mathbb{S}^{d-1}} e^{\phi/D_r}\, d\boldsymbol{n}}.$$

The major difficulty in the analysis of the Doi model is the presence of $\boldsymbol{\omega} \times \boldsymbol{n}$.

Integrating (1.12) over $\Omega \times \mathbb{S}^{d-1}$, we have that

$$\frac{d}{dt} \int_{\Omega \times \mathbb{S}^{d-1}} f\, d\boldsymbol{n}\, d\boldsymbol{x} = \int_{\partial\Omega \times \mathbb{S}^{d-1}} \big((\boldsymbol{u} + \alpha \boldsymbol{n}) f - D \nabla_{\mathbf{x}} f\big) \cdot \nu\, d\boldsymbol{n}\, dS,$$

where ν is the unit outer normal of Ω. To guarantee the conservation of $\int_{\Omega \times \mathbb{S}^{d-1}} f\, d\boldsymbol{n}\, d\boldsymbol{x}$, a natural condition is the no-flux boundary condition

$$(\boldsymbol{u} + \alpha \boldsymbol{n}) f - D \nabla_{\mathbf{x}} f \cdot \nu|_{\partial\Omega} = 0. \quad (1.14)$$

Based on Batchelor's slender-body theory [4,26], also known as Kirkwood theory (see [13]), in addition to the usual viscous stress, the stress exerted by the swimming of active particles is given by

$$\sigma = \sigma_0 \int_{\mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f\, d\boldsymbol{n}. \quad (1.15)$$

We recall that the sign of σ_0 depends on the swimming mechanism. For the pusher case, $\sigma_0 < 0$, whereas for the case, $\sigma_0 > 0$.

In fact, it follows from (1.7) and the definition of f^N that the average of the stresslets for all particles

$$\begin{aligned}
\sigma^N &\approx \sigma_0 \frac{1}{N} \sum_{i=1}^{N} (d\boldsymbol{n}_i \otimes \boldsymbol{n}_i - \mathrm{Id}) \delta_{\mathbf{x}_i}(\boldsymbol{x}) \\
&= \sigma_0 \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) \delta_{\mathbf{n}_i}(\boldsymbol{n}) \delta_{\mathbf{x}_i}(\boldsymbol{x})\, d\boldsymbol{n} \\
&= \sigma_0 \int_{\mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{(\mathbf{n}_i, \mathbf{x}_i)}(\boldsymbol{n}, \boldsymbol{x})\right) d\boldsymbol{n} \\
&= \sigma_0 \int_{\mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f^N\, d\boldsymbol{n}. \quad (1.16)
\end{aligned}$$

Then we get (1.15) by taking the mean-field limit as $N \to \infty$ in (1.16).

The velocity $\boldsymbol{u}$ of fluid is governed by the following incompressible Navier–Stokes equation with no slip boundary condition

$$\rho_f\big(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_{\mathbf{x}})\boldsymbol{u}\big) - \mu\Delta_{\mathbf{x}}\boldsymbol{u} + \nabla_{\mathbf{x}} p = \nabla_{\mathbf{x}}\cdot\sigma, \tag{1.17}$$

$$\nabla_{\mathbf{x}}\cdot\boldsymbol{u} = 0, \tag{1.18}$$

$$\boldsymbol{u}|_{\partial\Omega} = 0, \tag{1.19}$$

where ρ_f is the density of the fluid and is assumed to be constant, p is the pressure and μ denotes the dynamic viscosity coefficient of fluid.

In summary, combining (1.12)–(1.19) and using non-dimensionalization, we can deduce the following model:

$$\partial_t f + \nabla_{\mathbf{x}}\cdot\big((\boldsymbol{u}+\alpha\boldsymbol{n})f\big) - D\Delta_{\mathbf{x}} f = D_r\Delta_{\mathbf{n}} f - \nabla_{\mathbf{n}}\cdot\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W)\boldsymbol{n} f\big), \tag{1.20}$$

$$\sigma = \beta\int_{\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n} - \mathrm{Id}) f\, d\boldsymbol{n}, \tag{1.21}$$

$$\mathrm{Re}\big(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_{\mathbf{x}})\boldsymbol{u}\big) - \Delta_{\mathbf{x}}\boldsymbol{u} + \nabla_{\mathbf{x}} p = \nabla_{\mathbf{x}}\cdot\sigma, \tag{1.22}$$

$$\nabla_{\mathbf{x}}\cdot\boldsymbol{u} = 0, \tag{1.23}$$

with boundary conditions

$$(\alpha\boldsymbol{n} f - D\nabla_{\mathbf{x}} f)\cdot\nu|_{\partial\Omega} = 0, \tag{1.24}$$

$$\boldsymbol{u}|_{\partial\Omega} = 0, \tag{1.25}$$

where $E = \frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u} + (\nabla_{\mathbf{x}}\boldsymbol{u})^{\top})$ and $W = \frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u} - (\nabla_{\mathbf{x}}\boldsymbol{u})^{\top})$, $(t,\boldsymbol{x},\boldsymbol{n}) \in [0,T]\times\Omega\times\mathbb{S}^{d-1}$ with $T>0$. When $\alpha > 0$, the model is called active. Moreover, if we choose $\alpha = 0$ in (1.20), the model is the passive counterpart.

The coefficients $\mathrm{Re}\geqslant 0$, $\gamma\in[0,1]$ and $\beta\in\mathbb{R}$ are constants. If $\beta<0$, the model is called a pusher, whereas if $\beta>0$, it is called a puller. The case for $\beta<0$, i.e. the pusher case, is particularly interesting in its unstable dynamics (see Saintillan and Shelley [30]). Note that if $\gamma\in[-1,0]$, then the roles of pusher and puller are switched. This can be seen in the entropy estimates below.

If $\alpha=0$, $\beta>0$ and $\gamma=1$, the model is reduced to the standard Doi model for passive rod-like particle suspensions. Hence our results also include this case. If $\mathrm{Re}>0$, we call (1.20)–(1.23) the Navier–Stokes Doi–Saintillan–Shelley model and otherwise (i.e. $\mathrm{Re}=0$) the Stokes Doi–Saintillan–Shelley model. Likewise, we call the Doi model involving a (Navier–)Stokes equation the (Navier–)Stokes Doi model.

In this paper, we will investigate the Navier–Stokes Doi–Saintillan–Shelley model with boundary conditions (1.24)–(1.25) and initial condition

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_{in} \quad \text{in } \Omega, \tag{1.26}$$

$$f|_{t=0} = f_{in} \quad \text{in } \Omega\times\mathbb{S}^{d-1}; \tag{1.27}$$

and the Stokes Doi–Saintillan–Shelley model with the same boundary conditions and initial condition (1.27).

1.2. Basic entropy and energy estimates

We now show a formal entropy estimate below. The positivity of f follows directly from (1.20).

Multiplying (1.20) by $\ln f$ and integrating on $\Omega \times \mathbb{S}^{d-1}$, one could deduce that

$$\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}} \big(f(\ln f-1)+1\big)\,d\boldsymbol{n}\,d\boldsymbol{x}+4D\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+4D_r\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$=\alpha\int_{\Omega\times\mathbb{S}^{d-1}} \boldsymbol{n}\cdot\nabla_{\mathbf{x}} f\,d\boldsymbol{n}\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}} \big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E+W)\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.28}$$

The last term in (1.28) is a coupling term. We can gain one tangential gradient $\nabla_{\mathbf{n}}$ on $\mathbb{S}^{d-1}$ by using integration-by-parts as stated in the following lemma. This estimate will be used in entropy estimate (see also Section 3.2.1), global L^2 solution (Section 5.1) and uniqueness (Section 5.2).

Lemma 1.1. *Let $f \in W^{1,1}(\mathbb{S}^{d-1})$ and $X \in \mathbb{R}^{d\times d}$ be a constant matrix with $\mathrm{tr}(X)=0$. Then*

$$\int_{\mathbb{S}^{d-1}} \big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})X\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f\,d\boldsymbol{n}=\int_{\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f : X\,d\boldsymbol{n}.$$

Particularly, if $X=-X^\top$, then

$$\int_{\mathbb{S}^{d-1}} \big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})X\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f\,d\boldsymbol{n}=0.$$

Here and in the following, $X:Y$ denotes trace $(Y^\top X)$ for $X, Y \in \mathbb{R}^{d\times d}$. See Section 2 for the proof of Lemma 1.1.

Using Lemma 1.1, we have from (1.28) that

$$\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}} \big(f(\ln f-1)+1\big)\,d\boldsymbol{n}\,d\boldsymbol{x}+4D\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+4D_r\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$=\alpha\int_{\Omega\times\mathbb{S}^{d-1}} \boldsymbol{n}\cdot\nabla_{\mathbf{x}} f\,d\boldsymbol{n}\,d\boldsymbol{x}+\gamma\int_{\Omega\times\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f:\nabla_{\mathbf{x}}\boldsymbol{u}\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.29}$$

Multiplying (1.22) by $\boldsymbol{u}$ and integrating on Ω, we have that

$$\frac{\mathrm{Re}}{2}\frac{d}{dt}\int_{\Omega} |\boldsymbol{u}|^2\,d\boldsymbol{x}+\int_{\Omega} |\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}=-\beta\int_{\Omega\times\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f:\nabla_{\mathbf{x}}\boldsymbol{u}\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.30}$$

If $\beta\gamma>0$, by canceling the coupling terms in (1.29) and (1.30), we have that

$$\frac{d}{dt}\int_{\Omega}\bigg(\int_{\mathbb{S}^{d-1}} \big(f(\ln f-1)+1\big)\,d\boldsymbol{n}+\frac{\gamma\,\mathrm{Re}}{2\beta}|\boldsymbol{u}|^2\bigg)\,d\boldsymbol{x}+4D\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+4D_r\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{\gamma}{\beta}\int_{\Omega} |\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}$$
$$=\alpha\int_{\Omega\times\mathbb{S}^{d-1}} \boldsymbol{n}\cdot\nabla_{\mathbf{x}} f\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.31}$$

Furthermore, if in addition $\alpha = 0$ (i.e. $\alpha = 0$, $\beta\gamma > 0$, including the Doi model), one has

$$\frac{d}{dt}\int_{\Omega}\left(\int_{\mathbb{S}^{d-1}}\big(f(\ln f-1)+1\big)\,d\boldsymbol{n}+\frac{\gamma\,\mathrm{Re}}{2\beta}|\boldsymbol{u}|^2\right)d\boldsymbol{x}+4D\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+4D_r\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{\gamma}{\beta}\int_{\Omega}|\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}=0. \tag{1.32}$$

Then the total energy

$$E_0(\boldsymbol{u},f):=\int_{\Omega}\left(\int_{\mathbb{S}^{d-1}}\big(f(\ln f-1)+1\big)\,d\boldsymbol{n}+\frac{\gamma\,\mathrm{Re}}{2\beta}|\boldsymbol{u}|^2\right)d\boldsymbol{x}$$

is dissipated. Moreover, defining the concentration density $\rho := \int_{\mathbb{S}^{d-1}} f\,d\boldsymbol{n}$ and taking the integral over $\mathbb{S}^{d-1}$, when $\alpha = 0$ we find that (1.20) becomes the convection diffusion equation

$$\partial_t\rho+\boldsymbol{u}\cdot\nabla_{\mathbf{x}}\rho-D\Delta_{\mathbf{x}}\rho=0, \tag{1.33}$$

and the maximum principle holds, which makes the passive model easy to be tackled in the compactness argument.

However, for the active case, i.e. $\alpha \neq 0$, especially with $\beta\gamma < 0$ (for pushers $\beta < 0$, $\gamma > 0$), there exists no maximum principle for ρ and the total energy may increase due to the input of energy from the pushers. This gives some difficulties in analysis. The following is our strategies to handle these difficulties.

In fact, it follows from (1.29) and (1.30) that

$$\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}\big(f(\ln f-1)+1\big)\,d\boldsymbol{n}\,d\boldsymbol{x}+4D\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+4D_r\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{\mathrm{Re}}{2}\frac{d}{dt}\int_{\Omega}|\boldsymbol{u}|^2\,d\boldsymbol{x}+\int_{\Omega}|\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}$$
$$=\alpha\int_{\Omega\times\mathbb{S}^{d-1}}\boldsymbol{n}\cdot\nabla_{\mathbf{x}}f\,d\boldsymbol{n}\,d\boldsymbol{x}+(\gamma-\beta)\int_{\Omega\times\mathbb{S}^{d-1}}(d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f:\nabla_{\mathbf{x}}\boldsymbol{u}\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.34}$$

Applying Cauchy–Schwartz inequality, we can deduce that

$$\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}\big(f(\ln f-1)+1\big)\,d\boldsymbol{n}\,d\boldsymbol{x}+2D\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+4D_r\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{n}}\sqrt{f}|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+\frac{\mathrm{Re}}{2}\frac{d}{dt}\int_{\Omega}|\boldsymbol{u}|^2\,d\boldsymbol{x}+\frac{1}{2}\int_{\Omega}|\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}$$
$$\leqslant C\big(\|\rho\|^2_{L^2(\Omega)}+1\big). \tag{1.35}$$

Now, the key idea is to estimate $\|\rho\|_{L^2(\Omega)}$. Indeed, integrating (1.20) over $\mathbb{S}^{d-1}$, we deduce that

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla_{\mathbf{x}} \rho + \nabla_{\mathbf{x}} \cdot \left(\int\limits_{\mathbb{S}^{d-1}} (\alpha \boldsymbol{n} f) \, d\boldsymbol{n} - D \nabla_{\mathbf{x}} \rho \right) = 0. \tag{1.36}$$

The no-flux boundary condition (1.24) implies

$$\left(\int\limits_{\mathbb{S}^{d-1}} (\alpha \boldsymbol{n} f) \, d\boldsymbol{n} - D \nabla_{\mathbf{x}} \rho \right) \cdot \nu|_{\partial\Omega} = 0. \tag{1.37}$$

Multiplying (1.36) by ρ and integrating on Ω, we have

$$\frac{d}{dt} \frac{1}{2} \int\limits_{\Omega} |\rho|^2 \, d\boldsymbol{x} + D \int\limits_{\Omega} |\nabla_{\mathbf{x}} \rho|^2 \, d\boldsymbol{x} = \alpha \int\limits_{\Omega} \left(\int\limits_{\mathbb{S}^{d-1}} \boldsymbol{n} f \, d\boldsymbol{n} \right) \cdot \nabla_{\mathbf{x}} \rho \, d\boldsymbol{x}. \tag{1.38}$$

One has from Cauchy–Schwartz inequality that

$$\frac{d}{dt} \frac{1}{2} \int\limits_{\Omega} |\rho|^2 \, d\boldsymbol{x} + D \int\limits_{\Omega} |\nabla_{\mathbf{x}} \rho|^2 \, d\boldsymbol{x} \leqslant C \|\rho\|^2_{L^2(\Omega)}, \tag{1.39}$$

and hence from Gronwall's inequality that

$$\|\rho\|_{L^2(\Omega)} \leqslant C e^{Ct}. \tag{1.40}$$

Therefore (1.35) yields

$$\begin{aligned} E_1(f, \boldsymbol{u})(t) &:= \int\limits_{\Omega} \left(\int\limits_{\mathbb{S}^{d-1}} \big(f(\ln f - 1) + 1 \big) \, d\boldsymbol{n} + \frac{\mathrm{Re}}{2} \int\limits_{\Omega} |\boldsymbol{u}|^2 \right) d\boldsymbol{x} \\ &\leqslant C e^{Ct} + E_1(f, \boldsymbol{u})(0). \end{aligned} \tag{1.41}$$

The self-propelled motion and the pusher continuously pump energy into the system and results in the increasing of total energy $E_1(f, \boldsymbol{u})$ in (1.41). This increasing of energy is consistent with a linear stability analysis by Saintillan and Shelley [29] shown as before.

For the Stokes Doi–Saintillan–Shelley model ($\mathrm{Re} = 0$), we also have L^2-weak solutions in two and three dimensions and uniqueness in two dimension.

In fact, (1.30) with $\mathrm{Re} = 0$ implies $\|\nabla_{\mathbf{x}} \boldsymbol{u}\|_{L^2(\Omega)} \leqslant C \|\rho\|_{L^2(\Omega)}$, and hence from (1.40) that

$$\|\nabla_{\mathbf{x}} \boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C. \tag{1.42}$$

Inner producting (1.20) with f and using Lemma 1.1, we deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int\limits_{\Omega \times \mathbb{S}^{d-1}} |f|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + D \int\limits_{\Omega \times \mathbb{S}^{d-1}} |\nabla_{\mathbf{x}} f|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + D_r \int\limits_{\Omega \times \mathbb{S}^{d-1}} |\nabla_{\mathbf{n}} f|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} \\ &\quad = \alpha \int\limits_{\Omega \times \mathbb{S}^{d-1}} \boldsymbol{n} f \cdot \nabla_{\mathbf{x}} f \, d\boldsymbol{n} \, d\boldsymbol{x} + \frac{\gamma}{2} \int\limits_{\Omega \times \mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f^2 : \nabla_{\mathbf{x}} \boldsymbol{u} \, d\boldsymbol{n} \, d\boldsymbol{x}. \end{aligned} \tag{1.43}$$

By (1.42) and Hölder inequality, this yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}|f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{D}{2}\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+D_r\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{n}}f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant C\int_{\Omega\times\mathbb{S}^{d-1}}|f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\|f\|^2_{L^4(\Omega;L^2(\mathbb{S}^{d-1}))}. \tag{1.44}$$

Applying Gagliardo–Nirenberg inequality and Young inequality (see (5.8) and (5.9)), we have from (1.44) that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}|f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{D}{4}\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+D_r\int_{\Omega\times\mathbb{S}^{d-1}}|\nabla_{\mathbf{n}}f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant C\int_{\Omega\times\mathbb{S}^{d-1}}|f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{1.45}$$

Gronwall's inequality implies the formal L^2-estimate for Stokes Doi–Saintillan–Shelley model.

We follow the usual procedure in proving the existence of a global weak entropy solution. First, we use a semi-implicit scheme to construct a sequence of approximate solutions. In this construction, we apply the Leray–Schauder fixed-point theorem and cut-off techniques to prove the existence of a solution to the discrete problem. Here we are motivated by Barrett and Süli's [2,3] idea with cut-off in the study of FENE-type and Hookean-type bead-spring chains model. Then, we use compactness to show that these constructed solutions converge to a weak solution.

In the previous literature about the analysis of the Doi model for passive particle suspensions, the energy is dissipated; the density ρ satisfies a transport equation (1.33) with and without diffusion and hence the maximum principle holds. These are common foundations of their proofs. However, their methods cannot be adapted to the analysis of active suspensions which is the main objective of this paper. We shall also point out that there exist no discussions on the Doi model with no-flux conditions in the literature.

More precisely, as for the Stokes Doi model, with a novel estimate for the Smoluchowski equation, Otto and Tzavaras [25] obtained the stationary solution and showed that discontinuities in the velocity gradient cannot occur in finite time. For the case without the center-of-mass diffusion (i.e. $\Delta_{\mathbf{x}}f$), Constantin [8] established the global smooth solution in $\mathbb{T}^3$.

As for the Navier–Stokes Doi model without the center-of-mass diffusion, a series of papers [9–11] by Constantin and his coauthors proved the global smooth solutions of coupled Navier–Stokes and Fokker–Planck equations, which covered the Doi case, in $\mathbb{R}^2$ and $\mathbb{T}^2$. Sun and Zhang [32] investigated the two-dimensional case in a bounded domain. Using the propagation of the compactness, Lions and Masmoudi [24] established the global weak solution for this Doi model in $\mathbb{T}^d$ $(d=2,3)$. Recently, based on a quasi-compressible approximation of the pressure, Bae and Trivisa [1] investigated the three-dimensional case with Dirichlet boundary condition and obtained the existence of a global weak solution.

The paper is organized as follows. Section 2 collects some preliminary notions, definitions and lemmas which will be pertinent to our study. In Section 3, we establish the global weak entropy solution for the two- and three-dimensional Navier–Stokes Doi–Saintillan–Shelley models, where a semi-implicit scheme is used to construct the approximate problem and compactness was shown. Section 4 provides similar results for Stokes Doi–Saintillan–Shelley model. Then in Section 5, we prove the existence of a global $L^2(\Omega\times\mathbb{S}^{d-1})$ weak solution to the Stokes Doi–Saintillan–Shelley model with $d=2,3$ and its uniqueness with $d=2$.

For conciseness in presentation, we set $D=D_r=1$; and $\mathrm{Re}=1$ in the Navier–Stokes Doi–Saintillan–Shelley model in the rest of this paper.

2. Preliminaries

The following notations will be used in this paper.

$$\boldsymbol{L}^p(\Omega) = L^p(\Omega, \mathbb{R}^d), \qquad \boldsymbol{H}^m(\Omega) = H^m(\Omega, \mathbb{R}^d), \qquad \boldsymbol{C}_0^\infty(\Omega) = C_0^\infty(\Omega, \mathbb{R}^d),$$
$$\mathscr{V} = \{\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega)\colon \nabla_{\mathbf{x}} \cdot \boldsymbol{u} = 0\}, \qquad \boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\colon \nabla_{\mathbf{x}} \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0\},$$
$$\boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)\colon \nabla_{\mathbf{x}} \cdot \boldsymbol{u} = 0\}, \qquad \boldsymbol{V}^m = \boldsymbol{V} \cap \boldsymbol{H}^m(\Omega),$$

where $\mathscr{V}$ is dense in $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{V}^m$; $A \hookrightarrow B$ (or $A \hookrightarrow\hookrightarrow B$) denotes A is continuously (or compactly) embedded in B; $f_\tau \to$ ($\rightharpoonup$ or $\overset{*}{\rightharpoonup}$) f in A denotes a sequence $\{f_\tau\}_{\tau>0} \subset A$ converges strongly (weakly or weakly star) to f in A as $\tau \to 0$; $C(a, b, \ldots)$ denotes a constant only dependent on $a, b, \ldots$; C denotes a constant independent of L and N.

To prove Lemma 1.1, we need the following basic result.

Lemma 2.1. *Let $\boldsymbol{a} : \mathbb{S}^{d-1} \to \mathbb{R}^d$ be a vector-valued function and f, g be scalar-valued functions on $\mathbb{S}^{d-1}$. Then*

$$\int_{\mathbb{S}^{d-1}} (\nabla_{\mathbf{n}} \cdot \boldsymbol{a}) f \, d\boldsymbol{n} = -\int_{\mathbb{S}^{d-1}} \boldsymbol{a} \cdot \nabla_{\mathbf{n}} f \, d\boldsymbol{n} + (d-1) \int_{\mathbb{S}^{d-1}} (\boldsymbol{a} \cdot \boldsymbol{n}) f \, d\boldsymbol{n}, \tag{2.1}$$

$$\int_{\mathbb{S}^{d-1}} (\Delta_{\mathbf{n}} g) f \, d\boldsymbol{n} = -\int_{\mathbb{S}^{d-1}} \nabla_{\mathbf{n}} g \cdot \nabla_{\mathbf{n}} f \, d\boldsymbol{n}. \tag{2.2}$$

Proof. Otto and Tzavaras [25, Appendix II], have proved formulas (2.1) and (2.2) for $d = 3$ with spherical coordinates. For $d = 2$, the proof is similar with coordinate transformation in $\mathbb{R}^2$. □

Proof of Lemma 1.1. Since $C^\infty(\mathbb{S}^{d-1})$ is dense in $W^{1,1}(\mathbb{S}^{d-1})$, we assume $f \in C^\infty(\mathbb{S}^{d-1})$. By noting that $((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})X\boldsymbol{n}) \perp \boldsymbol{n}$, we deduce from (2.1) of Lemma 2.1 that

$$\int_{\mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})X\boldsymbol{n}\big) \cdot \nabla_{\mathbf{n}} f \, d\boldsymbol{n} = -\int_{\mathbb{S}^{d-1}} \big(\nabla_{\mathbf{n}} \cdot \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})X\boldsymbol{n}\big)\big) f \, d\boldsymbol{n}.$$

It follows from $\mathrm{tr}(X) = 0$ that

$$-\nabla_{\mathbf{n}} \cdot \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})X\boldsymbol{n}\big) = d(\boldsymbol{n} \cdot X\boldsymbol{n}) \quad \text{and} \quad (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f : X = d(\boldsymbol{n} \cdot X\boldsymbol{n}) f.$$

This finishes the proof of Lemma 1.1. □

Denote

$$\digamma(s) := s(\ln s - 1) + 1, \quad s \in [0, \infty)$$

and define some cut-off functions which will be used in the approximate problem, entropy estimate and L^2 estimate.

Definition 2.2. Let $L > 1$. Define

$$Q^L(s) := \begin{cases} 0, & \text{if } s \leqslant 0, \\ s, & \text{if } 0 \leqslant s \leqslant L, \\ L, & \text{if } s \geqslant L; \end{cases}$$

$$\digamma^L(s) := \begin{cases} s(\ln s - 1) + 1, & 0 \leqslant s \leqslant L, \\ \frac{s^2 - L^2}{2L} + s(\ln L - 1) + 1, & s \geqslant L; \end{cases}$$

$$G^L(s) := \begin{cases} \frac{s^2}{2}, & s \leqslant L, \\ \frac{L^2}{2} + L(s - L), & s \geqslant L. \end{cases}$$

With elementary computation, one could verify the following properties (also see Barrett and Süli [2,3] for some of them).

Lemma 2.3. *Let $L > 1$. Then*

$$Q^L \in C^{0,1}(\mathbb{R}); \qquad G^L \in C^{1,1}(\mathbb{R}); \qquad \digamma^L \in C^{2,1}\big(\mathbb{R}^+\big) \cap C\big([0, \infty)\big), \tag{2.3}$$

$$\big(G^L\big)'(s) = Q^L(s), \qquad G^L(s) \leqslant \frac{s^2}{2}, \quad \forall s \in [0, \infty), \tag{2.4}$$

$$\digamma^L(s) \geqslant \digamma(s), \quad \forall s \in [0, \infty), \tag{2.5}$$

$$\big(\digamma^L\big)''(s) = \big(Q^L(s)\big)^{-1} \geqslant s^{-1}, \quad \forall s \in \mathbb{R}^+, \tag{2.6}$$

$$\big(\digamma^L\big)''(s + \delta) \leqslant \frac{1}{\delta}, \quad \forall \delta \in (0, 1),\ \forall s \in [0, \infty), \tag{2.7}$$

$$\forall s \in [0, \infty), \quad \lim_{L \to \infty} Q^L(s) = s, \tag{2.8}$$

$$\digamma^L\big(Q^L(s) + \delta\big) \leqslant \delta + \frac{\delta^2}{2} + \digamma(s + \delta), \quad \forall \delta \in (0, 1),\ \forall s \in [0, \infty). \tag{2.9}$$

The global weak solutions with finite entropy for Navier–Stokes Doi–Saintillan–Shelley model and Stokes Doi–Saintillan–Shelley model are defined as follows.

Definition 2.4. Let $d = 2, 3$. Suppose $\boldsymbol{u}_{in} \in \boldsymbol{H}$ and $f_{in} \in L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ such that

$$f_{in} \geqslant 0 \quad \text{a.e. in } \Omega \times \mathbb{S}^{d-1}, \qquad \int_{\Omega \times \mathbb{S}^{d-1}} \digamma(f_{in})\, d\boldsymbol{n}\, d\boldsymbol{x} < \infty. \tag{2.10}$$

A pair of measurable functions $(\boldsymbol{u}, f)$ is called a global weak entropy solution of Navier–Stokes Doi–Saintillan–Shelley model with boundary conditions (1.24)–(1.25) and initial conditions (1.26)–(1.27) if

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V}), \qquad \boldsymbol{u} \in H^1\big(0, T; \big(\boldsymbol{V}^2\big)'\big), \tag{2.11}$$

$$f \geqslant 0 \quad \text{a.e. in } [0, T] \times \Omega \times \mathbb{S}^{d-1}, \qquad \int_{\Omega \times \mathbb{S}^{d-1}} \digamma\big(f(t)\big)\, d\boldsymbol{n}\, d\boldsymbol{x} < \infty \quad \text{a.e. in } [0, T], \tag{2.12}$$

$$f \in L^\infty\big(0, T; L^2\big(\Omega; L^1\big(\mathbb{S}^{d-1}\big)\big)\big), \qquad \sqrt{f} \in L^2\big(0, T; H^1\big(\Omega \times \mathbb{S}^{d-1}\big)\big), \tag{2.13}$$

$$f \in H^1\big(0, T; \big(H^4\big(\Omega \times \mathbb{S}^{d-1}\big)\big)'\big); \tag{2.14}$$

and for any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty([0,T) \times \Omega)$ with $\nabla_{\mathbf{x}} \cdot \boldsymbol{v} = 0$,

$$-\int_0^T \int_\Omega \boldsymbol{u} \cdot \partial_t \boldsymbol{v}\, d\boldsymbol{x}\, dt + \int_0^T \int_\Omega \nabla_{\mathbf{x}} \boldsymbol{u} : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{x}\, dt + \int_0^T \int_\Omega (\boldsymbol{u} \cdot \nabla_{\mathbf{x}}) \boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x}\, dt$$
$$= -\beta \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt + \int_\Omega \boldsymbol{u}_{in}(\boldsymbol{x}) \cdot \boldsymbol{v}(0,\boldsymbol{x})\, d\boldsymbol{x}; \tag{2.15}$$

for any $\varphi \in C^\infty([0,T] \times \overline{\Omega} \times \mathbb{S}^{d-1})$ with $\varphi(T) = 0$,

$$-\int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} f \partial_t \varphi\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} (\boldsymbol{u} \cdot \nabla_{\mathbf{x}} f) \varphi\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt - \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} (\alpha \boldsymbol{n} f) \cdot \nabla_{\mathbf{x}} \varphi\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$+ \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} (\nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{x}} \varphi + \nabla_{\mathbf{n}} f \cdot \nabla_{\mathbf{n}} \varphi)\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$= \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E + W) \boldsymbol{n} f\big) \cdot \nabla_{\mathbf{n}} \varphi\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$+ \int_{\Omega \times \mathbb{S}^{d-1}} f_{in}(\boldsymbol{x},\boldsymbol{n}) \varphi(0,\boldsymbol{x},\boldsymbol{n})\, d\boldsymbol{n}\, d\boldsymbol{x}. \tag{2.16}$$

Definition 2.5. Let $d = 2, 3$. Suppose $f_{in} \in L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ satisfying (2.10). A pair of measurable functions $(\boldsymbol{u}, f)$ is called a global weak entropy solution of Stokes Doi–Saintillan–Shelley model with boundary conditions (1.24)–(1.25) and initial conditions (1.27) if $\boldsymbol{u} \in L^\infty(0,T; \boldsymbol{V})$, f satisfies (2.12)–(2.14) and for any $\boldsymbol{v} \in L^2(0,T; \boldsymbol{V})$,

$$\int_0^T \int_\Omega \nabla_{\mathbf{x}} \boldsymbol{u} : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{x}\, dt = -\beta \int_0^T \int_{\Omega \times \mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{n}\, d\boldsymbol{x}\, dt; \tag{2.17}$$

for any $\varphi \in C^\infty([0,T] \times \overline{\Omega} \times \mathbb{S}^{d-1})$ with $\varphi(T) = 0$, (2.16) holds.

3. Global weak entropy solutions to Navier–Stokes Doi–Saintillan–Shelley model

In this section, we prove the following theorem.

Theorem 3.1. *Let $d = 2, 3$. Suppose $\boldsymbol{u}_{in} \in \boldsymbol{H}$ and $f_{in} \in L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ such that $f_{in} \geqslant 0$ a.e. on $\Omega \times \mathbb{S}^{d-1}$, $\int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}(f_{in})\, d\boldsymbol{n}\, d\boldsymbol{x} < \infty$. Then for any coefficients $\gamma \in [-1,1]$, $\beta \in \mathbb{R}$ and $\alpha \in [0,\infty)$, there exists a global weak entropy solution $(\boldsymbol{u}, f)$ to Navier–Stokes Doi–Saintillan–Shelley model with boundary conditions (1.24)–(1.25) and initial conditions (1.26)–(1.27) which satisfies the following energy inequalities for a.e. $t \in [0,T]$,*

$$\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + 2 \int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}\big(f(t)\big)\, d\boldsymbol{n}\, d\boldsymbol{x} + \int_0^t \|\nabla_{\mathbf{x}} u(s)\|_{L^2(\Omega)}^2\, ds$$

$$+4\int_0^t \left(\left\|\nabla_{\mathbf{x}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}+\left\|\nabla_{\mathbf{n}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right)ds$$
$$\leqslant \|\boldsymbol{u}_{in}\|^2_{L^2(\Omega)}+2\int_{\Omega\times\mathbb{S}^{d-1}} \digamma(f_{in})\,d\boldsymbol{n}\,d\boldsymbol{x}+C(\alpha,\beta,\gamma,T)\left(\|f_{in}\|^2_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))}+1\right). \tag{3.1}$$

Moreover, if $\alpha=0$, $\beta\gamma>0$ (*including the Navier–Stokes Doi model*), *then*

$$\frac{\gamma}{\beta}\left\|\boldsymbol{u}(t)\right\|^2_{L^2(\Omega)}+2\int_{\Omega\times\mathbb{S}^{d-1}} \digamma\left(f(t)\right)d\boldsymbol{n}\,d\boldsymbol{x}+\frac{2\gamma}{\beta}\int_0^t \left\|\nabla_{\mathbf{x}}u(s)\right\|^2_{L^2(\Omega)}\,ds$$
$$+4\int_0^t \left(\left\|\nabla_{\mathbf{x}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}+\left\|\nabla_{\mathbf{n}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right)ds$$
$$\leqslant \frac{\gamma}{\beta}\|\boldsymbol{u}_{in}\|^2_{L^2(\Omega)}+2\int_{\Omega\times\mathbb{S}^{d-1}} \digamma(f_{in})\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{3.2}$$

3.1. Approximate problem

In this subsection, we will use a semi-implicit time scheme to construct the approximate problem with cut-off from the top by L and below by 0 and then apply Leray–Schauder Fixed-point theorem to solve it. In the proof, the cut-off is the key to proving some boundedness of the linear functional defined by the discrete Fokker–Planck equation and the boundedness for fixed-points. Using this effective cut-off, we obtain the $\boldsymbol{V}\times H^1$ weak solution for approximate problem, and then by applying the standard method for elliptic equation we get the positivity.

Let $N\in\mathbb{N}$ and set $\tau=T/N$, $k=0,1,\dots,N$. We can divide the time interval $(0,T]$ into $\bigcup_{k=1}^N((k-1)\tau,k\tau]$. For any $k=1,2,\dots,N$, given $(\boldsymbol{u}^L_{k-1},f^L_{k-1})$, the approximate problem with cut-off reads

$$\int_\Omega \frac{\boldsymbol{u}^L_k-\boldsymbol{u}^L_{k-1}}{\tau}\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_\Omega \nabla_{\mathbf{x}}\boldsymbol{u}^L_k:\nabla_{\mathbf{x}}\boldsymbol{v}\,d\boldsymbol{x}+\int_\Omega \left(\boldsymbol{u}^L_{k-1}\cdot\nabla_{\mathbf{x}}\right)\boldsymbol{u}^L_k\cdot\boldsymbol{v}\,d\boldsymbol{x}$$
$$=-\beta\int_{\Omega\times\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f^L_k:\nabla_{\mathbf{x}}\boldsymbol{v}\,d\boldsymbol{n}\,d\boldsymbol{x},\quad \forall\boldsymbol{v}\in\boldsymbol{V}; \tag{3.3}$$

$$\int_{\Omega\times\mathbb{S}^{d-1}} \frac{f^L_k-f^L_{k-1}}{\tau}\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}} \left(\boldsymbol{u}^L_{k-1}\cdot\nabla_{\mathbf{x}}f^L_k\right)\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$-\int_{\Omega\times\mathbb{S}^{d-1}} \left(\alpha\boldsymbol{n}Q^L\left(f^L_k\right)\right)\cdot\nabla_{\mathbf{x}}\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}} \left(\nabla_{\mathbf{x}}f^L_k\cdot\nabla_{\mathbf{x}}\varphi+\nabla_{\mathbf{n}}f^L_k\cdot\nabla_{\mathbf{n}}\varphi\right)d\boldsymbol{n}\,d\boldsymbol{x}$$
$$=\int_{\Omega\times\mathbb{S}^{d-1}} \left((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})\left(\gamma E^L_k+W^L_k\right)\boldsymbol{n}\right)Q^L\left(f^L_k\right)\cdot\nabla_{\mathbf{n}}\varphi\,d\boldsymbol{n}\,d\boldsymbol{x},\quad \forall\varphi\in H^1\left(\Omega\times\mathbb{S}^{d-1}\right), \tag{3.4}$$

where $E^L_k=\frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u}^L_k+(\nabla_{\mathbf{x}}\boldsymbol{u}^L_k)^\top)$, $W^L_k=\frac{1}{2}(\nabla_{\mathbf{x}}\boldsymbol{u}^L_k-(\nabla_{\mathbf{x}}\boldsymbol{u}^L_k)^\top)$.

Remark 3.2. We note that (3.4) implies a weak formulation of the discrete (1.36), saying for any $\psi \in H^1(\Omega)$,

$$\int_\Omega \frac{\rho_k^L - \rho_{k-1}^L}{\tau}\psi\, d\boldsymbol{x} + \int_\Omega \big(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}}\rho_k^L\big)\psi\, d\boldsymbol{x} + \int_\Omega \nabla_{\mathbf{x}}\rho_k^L \cdot \nabla_{\mathbf{x}}\psi\, d\boldsymbol{x}$$
$$= \int_\Omega \bigg(\int_{\mathbb{S}^{d-1}} \alpha \boldsymbol{n} Q^L\big(f_k^L\big)\, d\boldsymbol{n}\bigg) \cdot \nabla_{\mathbf{x}}\psi\, d\boldsymbol{x}, \tag{3.5}$$

where $\rho_k^L = \int_{\mathbb{S}^{d-1}} f_k^L\, d\boldsymbol{n}$, $0 \leqslant k \leqslant N$. This plays a crucial role in the proof of the uniform estimate (see Section 3.2.1) and strong convergence for f_k^L (see Section 3.3.1).

Define

$$Z := \big\{ f \in L^2\big(\Omega \times \mathbb{S}^{d-1}\big) \colon\ f \geqslant 0 \text{ a.e. on } \Omega \times \mathbb{S}^{d-1} \big\}. \tag{3.6}$$

Proposition 3.3. *Let* $(\boldsymbol{u}_{k-1}^L, f_{k-1}^L) \in \boldsymbol{V} \times Z$. *Then there exists* $(\boldsymbol{u}_k^L, f_k^L) \in \boldsymbol{V} \times (Z \cap H^1(\Omega \times \mathbb{S}^{d-1}))$ *which solves* (3.3)–(3.4).

Proof. *Step* 1. Let $\bar{f} \in L^2(\Omega \times \mathbb{S}^{d-1})$. We claim that there exists a unique element $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = A(\bar{f})(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.7}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \tau \int_\Omega \nabla_{\mathbf{x}}\boldsymbol{u} : \nabla_{\mathbf{x}}\boldsymbol{v}\, d\boldsymbol{x} + \tau \int_\Omega \big(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}}\big)\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V};$$
$$A(\bar{f})(\boldsymbol{v}) = \int_\Omega \boldsymbol{u}_{k-1}^L \cdot \boldsymbol{v}\, d\boldsymbol{x} - \tau\beta \int_{\Omega \times \mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id})\bar{f} : \nabla_{\mathbf{x}}\boldsymbol{v}\, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

In fact, noting that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\nabla_{\mathbf{x}} \cdot \boldsymbol{u}_{k-1}^L = 0$, we have

$$\bigg|\int_\Omega \big(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}}\big)\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x}\bigg| \leqslant \big\|\boldsymbol{u}_{k-1}^L\big\|_{L^4(\Omega)} \|\nabla_{\mathbf{x}}\boldsymbol{u}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^4(\Omega)} \leqslant C\|\boldsymbol{u}\|_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)}$$

and $\int_\Omega (\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}})\boldsymbol{u} \cdot \boldsymbol{u}\, d\boldsymbol{x} = 0$. Then thanks to

$$\bigg\| \int_{\mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id})\bar{f}\, d\boldsymbol{n} \bigg\|_{L^2(\Omega)} \leqslant C \bigg\| \int_{\mathbb{S}^{d-1}} |\bar{f}|\, d\boldsymbol{n} \bigg\|_{L^2(\Omega)} \leqslant C\|\bar{f}\|_{L^2(\Omega \times \mathbb{S}^{d-1})},$$

we have that $a(\cdot,\cdot)$ is a bounded, coercive bilinear functional on $\boldsymbol{V} \times \boldsymbol{V}$ and $A(\bar{f}) \in \boldsymbol{V}'$. Hence by the Lax–Milgram theorem, we finish the proof of Step 1.

Step 2. We prove that for such $\bar{f} \in L^2(\Omega \times \mathbb{S}^{d-1})$ and solution $\boldsymbol{u} \in \boldsymbol{V}$ in (3.7), there exists a unique element $f \in H^1(\Omega \times \mathbb{S}^{d-1})$ such that

$$b(f,\varphi) = B(\bar f, \boldsymbol{u})(\varphi), \quad \forall \varphi \in H^1\big(\Omega \times \mathbb{S}^{d-1}\big), \tag{3.8}$$

where

$$\begin{aligned} b(f,\varphi) = {} & \int_{\Omega\times\mathbb{S}^{d-1}} f\varphi \, d\boldsymbol{n}\, d\boldsymbol{x} + \tau \int_{\Omega\times\mathbb{S}^{d-1}} (\nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{x}}\varphi + \nabla_{\mathbf{n}} f \cdot \nabla_{\mathbf{n}}\varphi)\, d\boldsymbol{n}\, d\boldsymbol{x} \\ & + \tau \int_{\Omega\times\mathbb{S}^{d-1}} \big(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}} f\big)\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall f, \varphi \in H^1\big(\Omega \times \mathbb{S}^{d-1}\big); \\ B(\bar f, \boldsymbol{u})(\varphi) = {} & \int_{\Omega\times\mathbb{S}^{d-1}} f_{k-1}^L \varphi \, d\boldsymbol{n}\, d\boldsymbol{x} + \tau \int_{\Omega\times\mathbb{S}^{d-1}} \big(\alpha \boldsymbol{n} Q^L(\bar f)\big) \cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x} \\ & + \tau \int_{\Omega\times\mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W)\boldsymbol{n}\big) Q^L(\bar f) \cdot \nabla_{\mathbf{n}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall \varphi \in H^1\big(\Omega \times \mathbb{S}^{d-1}\big). \end{aligned}$$

Indeed, noting $H^1(\Omega\times\mathbb{S}^{d-1}) \hookrightarrow L^3(\Omega\times\mathbb{S}^{d-1})$ and $\nabla_{\mathbf{x}} \cdot \boldsymbol{u}_{k-1}^L = 0$, we have

$$\begin{aligned} \left| \int_{\Omega\times\mathbb{S}^{d-1}} \big(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}} f\big)\varphi \, d\boldsymbol{n}\, d\boldsymbol{x} \right| & \leqslant \big\|\boldsymbol{u}_{k-1}^L\big\|_{L^6(\Omega)} \|\nabla_{\mathbf{x}} f\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \|\varphi\|_{L^3(\Omega\times\mathbb{S}^{d-1})} \\ & \leqslant C\|f\|_{H^1(\Omega\times\mathbb{S}^{d-1})} \|\varphi\|_{H^1(\Omega\times\mathbb{S}^{d-1})} \end{aligned}$$

and $\int_{\Omega\times\mathbb{S}^{d-1}} (\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}} f) f \, d\boldsymbol{n}\, d\boldsymbol{x} = 0$. Therefore $b(\cdot,\cdot)$ is a bounded and coercive bilinear functional on $H^1(\Omega\times\mathbb{S}^{d-1})$. It follows from $|Q^L(s)| \leqslant L$ $(\forall s \in \mathbb{R})$ that $B(\bar f, \boldsymbol{u}) \in (H^1(\Omega\times\mathbb{S}^{d-1}))'$. We thus finish the proof of Step 2 by Lax–Milgram theorem.

Step 3. Define the mapping $\Phi : L^2(\Omega\times\mathbb{S}^{d-1}) \to L^2(\Omega\times\mathbb{S}^{d-1})$ by $\Phi(\bar f) = f \in H^1(\Omega\times\mathbb{S}^{d-1})$ via the procedure (3.7) and (3.8). By the Leray–Schauder fixed-point theorem (see [14, p. 280, Theorem 11.3]), we obtain a solution f to $\Phi(f) = f$, and hence a solution $(\boldsymbol{u}, f) \in V \times H^1(\Omega\times\mathbb{S}^{d-1})$ to (3.3) and (3.4). For explicitness, we relabel $(\boldsymbol{u}, f)$ as $(\boldsymbol{u}_k^L, f_k^L)$.

To prove this, we only need to show the following three claims to apply Leray–Schauder fixed-point theorem.

Claim 1. $\Phi : L^2(\Omega\times\mathbb{S}^{d-1}) \to L^2(\Omega\times\mathbb{S}^{d-1})$ *is continuous.*

Claim 2. Φ *is compact.*

Claim 3. $\Lambda := \{f \in L^2(\Omega\times\mathbb{S}^{d-1}):\ f = \sigma\Phi(f)$ *for some* $\sigma \in (0,1]\}$ *is bounded in* $L^2(\Omega\times\mathbb{S}^{d-1})$.

Proof of Claim 1. Set $f := \Phi(\bar f)$ and $f_i := \Phi(\bar f_i)$, $i \in \mathbb{N}$. If

$$\bar f_i \to \bar f \quad \text{in } L^2\big(\Omega\times\mathbb{S}^{d-1}\big) \text{ as } i \to \infty, \tag{3.9}$$

we need to show

$$f_i \to f \quad \text{in } L^2\big(\Omega\times\mathbb{S}^{d-1}\big) \text{ as } i \to \infty. \tag{3.10}$$

Indeed, for $\bar f$ and $\bar f_i$, in view of the definition of Φ, there exist unique $\boldsymbol{u} \in \boldsymbol{V}$ and $\boldsymbol{u}_i \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = A(\bar f)(\boldsymbol{v}), \qquad a(\boldsymbol{u}_i,\boldsymbol{v}) = A(\bar f_i)(\boldsymbol{v}), \quad \forall \boldsymbol{v}\in \boldsymbol{V}, \tag{3.11}$$

$$b(f,\varphi) = B(\bar f,\boldsymbol{u})(\varphi), \qquad b(f_i,\varphi) = B(\bar f_i,\boldsymbol{u}_i)(\varphi), \quad \forall \varphi \in H^1\big(\Omega\times\mathbb{S}^{d-1}\big). \tag{3.12}$$

In (3.11), subtracting $a(\boldsymbol{u}_i,\boldsymbol{v}) - a(\boldsymbol{u},\boldsymbol{v}) = A(\bar f_i)(\boldsymbol{v}) - A(\bar f)(\boldsymbol{v})$ and taking $\boldsymbol{v} = \boldsymbol{u}_i - \boldsymbol{u}$, we have by noting $\int_\Omega (\boldsymbol{u}^L_{k-1}\cdot\nabla_{\mathbf{x}})(\boldsymbol{u}_i-\boldsymbol{u})\cdot(\boldsymbol{u}_i-\boldsymbol{u})\,d\boldsymbol{x} = 0$ that

$$\int_\Omega |\boldsymbol{u}_i-\boldsymbol{u}|^2\,d\boldsymbol{x} + \tau\int_\Omega |\nabla_{\mathbf{x}}\boldsymbol{u}_i - \nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x} \leqslant C\tau \int_{\Omega\times\mathbb{S}^{d-1}} |\bar f_i - \bar f||\nabla_{\mathbf{x}}\boldsymbol{u}_i - \nabla_{\mathbf{x}}\boldsymbol{u}|\,d\boldsymbol{n}\,d\boldsymbol{x}$$

and from Cauchy–Schwartz inequality that $\|\boldsymbol{u}_i-\boldsymbol{u}\|^2_{H^1(\Omega)} \leqslant C\|\bar f_i - \bar f\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}$. Thus (3.9) yields

$$\boldsymbol{u}_i \to \boldsymbol{u} \quad \text{in } \boldsymbol{H}^1(\Omega) \text{ as } i\to\infty. \tag{3.13}$$

In (3.12), using the same procedure and noting $\int_{\Omega\times\mathbb{S}^{d-1}}(\boldsymbol{u}^L_{k-1}\cdot\nabla_{\mathbf{x}}(f_i-f))(f_i-f)\,d\boldsymbol{n}\,d\boldsymbol{x} = 0$, one has

$$\begin{aligned}\|f_i - f\|^2_{H^1(\Omega\times\mathbb{S}^{d-1})} &\leqslant C\Big(\big\|Q^L(\bar f_i) - Q^L(\bar f)\big\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \big\|\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E_i + W_i)\boldsymbol{n}\big)Q^L(\bar f_i) \\ &\quad - \big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W)\boldsymbol{n}\big)Q^L(\bar f)\big\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\Big) \\ &=: C\big(I_1^2 + I_2^2\big).\end{aligned}$$

Clearly, we have from $Q^L \in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1 that $I_1 \leqslant \|\bar f_i - \bar f\|_{L^2(\Omega\times\mathbb{S}^{d-1})}$. Now we estimate I_2

$$\begin{aligned}I_2 &\leqslant \big\|\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E_i + W_i - \gamma E - W)\boldsymbol{n}\big)Q^L(\bar f_i)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \\ &\quad + \big\|\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W)\boldsymbol{n}\big)\big(Q^L(\bar f_i) - Q^L(\bar f)\big)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \\ &=: I_{2,1} + I_{2,2}.\end{aligned}$$

Clearly (3.13) yields $I_{2,1} \leqslant CL\|\nabla_{\mathbf{x}}(\boldsymbol{u}_i - \boldsymbol{u})\|_{L^2(\Omega)} \to 0$ as $i\to\infty$. We only need to deal with $I_{2,2}$. In fact, since $|\gamma E + W| \leqslant C|\nabla_{\mathbf{x}} u| \in L^2(\Omega)$ and $C^\infty(\Omega)$ is dense in $L^2(\Omega)$, we deduce that

$$\forall \varepsilon > 0,\ \exists X \in C^\infty(\Omega) \text{ such that } \quad \big\|(\gamma E + W) - X\big\|_{L^2(\Omega)} < \frac{\varepsilon}{L}.$$

Moreover, we have from $Q^L \in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1 and (3.9) that

$$\exists K\in\mathbb{N},\ \forall i > K, \quad \big\|X\big(Q^L(\bar f_i) - Q^L(\bar f)\big)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \leqslant \|X\|_{L^\infty(\Omega)}\|\bar f_i - \bar f\|_{L^2(\Omega\times\mathbb{S}^{d-1})} < \varepsilon.$$

Therefore

$$\begin{aligned}I_{2,2} &\leqslant \big\|\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W - X)\boldsymbol{n}\big)\big(Q^L(\bar f_i) - Q^L(\bar f)\big)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \\ &\quad + \big\|\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})X\boldsymbol{n}\big)\big(Q^L(\bar f_i) - Q^L(\bar f)\big)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \\ &\leqslant CL\big\|(\gamma E + W) - X\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} + C\big\|X\big(Q^L(\bar f_i) - Q^L(\bar f)\big)\big\|_{L^2(\Omega\times\mathbb{S}^{d-1})} < C\varepsilon.\end{aligned}$$

Consequently $f_i \to f$ in $H^1(\Omega\times\mathbb{S}^{d-1})$ and hence (3.10) holds. This ends the proof of Claim 1. □

Proof of Claim 2. It is quite easy to deduce that

$$\exists C(\tau, L) > 0,\ \forall \bar{f} \in L^2\big(\Omega \times \mathbb{S}^{d-1}\big), \quad \big\|\Phi(\bar{f})\big\|_{H^1(\Omega\times\mathbb{S}^{d-1})} \leqslant C(\tau, L)\big(1 + \|\bar{f}\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\big).$$

This and $H^1(\Omega \times \mathbb{S}^{d-1}) \hookrightarrow\hookrightarrow L^2(\Omega \times \mathbb{S}^{d-1})$ establish Claim 2. □

Proof of Claim 3. For any $f \in \Lambda$, there exists a unique $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = A(f)(v), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.14}$$

$$b(f, \varphi) = \sigma B(f, \boldsymbol{u})(\varphi), \quad \forall \varphi \in H^1\big(\Omega \times \mathbb{S}^{d-1}\big). \tag{3.15}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ in (3.14) and using the identity

$$2(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 + |\boldsymbol{a} - \boldsymbol{b}|^2 - |\boldsymbol{b}|^2, \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d, \tag{3.16}$$

one has from $\int_\Omega (\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}})\boldsymbol{u} \cdot \boldsymbol{u}\, d\boldsymbol{x} = 0$ and Cauchy–Schwartz inequality that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\boldsymbol{u}|^2\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega \big|\boldsymbol{u} - \boldsymbol{u}_{k-1}^L\big|^2\, d\boldsymbol{x} + \tau \int_\Omega |\nabla_{\mathbf{x}}\boldsymbol{u}|^2\, d\boldsymbol{x} \\
&\quad \leqslant \frac{1}{2}\int_\Omega \big|\boldsymbol{u}_{k-1}^L\big|^2\, d\boldsymbol{x} + C(\beta)\tau \int_{\Omega\times\mathbb{S}^{d-1}} |f||\nabla_{\mathbf{x}}\boldsymbol{u}|\, d\boldsymbol{n}\, d\boldsymbol{x} \\
&\quad \leqslant \frac{1}{2}\int_\Omega \big|\boldsymbol{u}_{k-1}^L\big|^2\, d\boldsymbol{x} + \frac{\tau}{2}\int_\Omega |\nabla_{\mathbf{x}}\boldsymbol{u}|^2\, d\boldsymbol{x} + C(\beta)\tau \|f\|^2_{L^2(\Omega; L^1(\mathbb{S}^{d-1}))}.
\end{aligned} \tag{3.17}$$

Therefore

$$\tau \|\nabla_{\mathbf{x}}\boldsymbol{u}\|^2_{L^2(\Omega)} \leqslant C(k-1) + C(\beta)\tau \|f\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}. \tag{3.18}$$

Taking $\varphi = f$ in (3.15), we deduce from (3.16), $\int_{\Omega\times\mathbb{S}^{d-1}} (\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}} f) f\, d\boldsymbol{n}\, d\boldsymbol{x} = 0$ and Cauchy–Schwartz inequality that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\times\mathbb{S}^{d-1}} |f|^2\, d\boldsymbol{n}\, d\boldsymbol{x} + \frac{1}{2}\int_{\Omega\times\mathbb{S}^{d-1}} \big|f - \sigma f_{k-1}^L\big|^2\, d\boldsymbol{n}\, d\boldsymbol{x} + \tau \int_{\Omega\times\mathbb{S}^{d-1}} \big(|\nabla_{\mathbf{x}} f|^2 + |\nabla_{\mathbf{n}} f|^2\big)\, d\boldsymbol{n}\, d\boldsymbol{x} \\
&= \frac{1}{2}\int_{\Omega\times\mathbb{S}^{d-1}} \big|\sigma f_{k-1}^L\big|^2\, d\boldsymbol{n}\, d\boldsymbol{x} + \sigma\tau \int_{\Omega\times\mathbb{S}^{d-1}} \big(\alpha \boldsymbol{n} Q^L(f)\big) \cdot \nabla_{\mathbf{x}} f\, d\boldsymbol{n}\, d\boldsymbol{x} \\
&\quad + \sigma\tau \int_{\Omega\times\mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n})(\gamma E + W)\boldsymbol{n}\big) Q^L(f) \cdot \nabla_{\mathbf{n}} f\, d\boldsymbol{n}\, d\boldsymbol{x} \\
&\leqslant \frac{1}{2}\int_{\Omega\times\mathbb{S}^{d-1}} \big|f_{k-1}^L\big|^2\, d\boldsymbol{n}\, d\boldsymbol{x} + C(\alpha)\tau \int_{\Omega\times\mathbb{S}^{d-1}} |f||\nabla_{\mathbf{x}} f|\, d\boldsymbol{n}\, d\boldsymbol{x} + C(\gamma)L\tau \int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}}\boldsymbol{u}||\nabla_{\mathbf{n}} f|\, d\boldsymbol{n}\, d\boldsymbol{x}
\end{aligned}$$

$$\leqslant C(k-1)+\frac{\tau}{2}\int_{\Omega\times\mathbb{S}^{d-1}}\left(|\nabla_{\mathbf{x}} f|^2+|\nabla_{\mathbf{n}} f|^2\right)d\boldsymbol{n}\,d\boldsymbol{x}$$

$$+C(\alpha)\tau\int_{\Omega\times\mathbb{S}^{d-1}}|f|^2\,d\boldsymbol{n}\,d\boldsymbol{x}+C(\gamma)L^2\tau\int_{\Omega}|\nabla_{\mathbf{x}}\boldsymbol{u}|^2\,d\boldsymbol{x}. \tag{3.19}$$

Thus (3.18) and (3.19) yield $\|f\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\leqslant C(k-1,\gamma,L)+C(\alpha,\beta,\gamma)L^2\tau\|f\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}$. Note that

$$\forall\tau<\frac{1}{2C(\alpha,\beta,\gamma)L^2}\ \text{which is independent of}\ k-1, \tag{3.20}$$

we obtain $\|f\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\leqslant C(k-1,\gamma,L)$ and then establish Claim 3. □

Step 4. We prove the positivity.

In fact, set $[f_k^L]^-:=\min\{f_k^L,0\}$. Then $[f_k^L]^-\in H^1(\Omega\times\mathbb{S}^{d-1})$. Choosing $\varphi=[f_k^L]^-$ in (3.4) and noting that $Q^L(f_k^L)\nabla_{\mathbf{x}}[f_k^L]^-=Q^L(f_k^L)\nabla_{\mathbf{n}}[f_k^L]^-=0$, we deduce that

$$\int_{\Omega\times\mathbb{S}^{d-1}}\left|\left[f_k^L\right]^-\right|^2 d\boldsymbol{n}\,d\boldsymbol{x}+\tau\int_{\Omega\times\mathbb{S}^{d-1}}\left(\left|\nabla_{\mathbf{x}}\left[f_k^L\right]^-\right|^2+\left|\nabla_{\mathbf{n}}\left[f_k^L\right]^-\right|^2\right)d\boldsymbol{n}\,d\boldsymbol{x}$$

$$=\int_{\Omega\times\mathbb{S}^{d-1}}f_{k-1}^L\left[f_k^L\right]^-\,d\boldsymbol{n}\,d\boldsymbol{x}\leqslant 0. \tag{3.21}$$

Therefore $[f_k^L]^-=0$ a.e. on $\Omega\times\mathbb{S}^{d-1}$ and hence $f_k^L\geqslant 0$ a.e. on $\Omega\times\mathbb{S}^{d-1}$. Thus $f_k^L\in Z$. This finishes the proof of Proposition 3.3. □

3.2. *Uniform estimates in L and N*

Suppose $\boldsymbol{u}_{in}\in\boldsymbol{H}$ and $f_{in}\in L^2(\Omega;L^1(\mathbb{S}^{d-1}))$ satisfying (2.10). We regularize $\boldsymbol{u}_{in}$ by $\boldsymbol{u}_{in}^L$ which is the weak solution of $\boldsymbol{u}_{in}^L-\frac{1}{L}\Delta\boldsymbol{u}_{in}^L=\boldsymbol{u}_{in}$ $(L>1)$. Therefore

$$\left\|\boldsymbol{u}_{in}^L\right\|^2_{L^2(\Omega)}+\frac{1}{L}\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_{in}^L\right\|^2_{L^2(\Omega)}\leqslant\|\boldsymbol{u}_{in}\|^2_{L^2(\Omega)} \tag{3.22}$$

and $\boldsymbol{u}_{in}^L\rightharpoonup\boldsymbol{u}_{in}$ in $\boldsymbol{H}$ as $L\to\infty$. Furthermore, let $\boldsymbol{u}_0^L=\boldsymbol{u}_{in}^L$, $f_0^L=Q^L(f_{in})$. Then $(\boldsymbol{u}_0^L,f_0^L)\in\boldsymbol{V}\times Z$. Using Proposition 3.3 iteratively, we obtain a sequence of approximate solutions

$$\left(\boldsymbol{u}_k^L,f_k^L\right)\in\boldsymbol{V}\times\left(Z\cap H^1\left(\Omega\times\mathbb{S}^{d-1}\right)\right),\quad\text{for}\ k=1,2,\ldots,N \tag{3.23}$$

to (3.3)–(3.4). We will establish the uniform estimates in L and N. Define $\rho_{in}:=\int_{\mathbb{S}^{d-1}}f_{in}\,d\boldsymbol{n}$ and recall that $\rho_k^L:=\int_{\mathbb{S}^{d-1}}f_k^L\,d\boldsymbol{n}$ $(0\leqslant k\leqslant N)$.

3.2.1. $\|\rho_k^L\|_{L^2(\Omega)}$ *estimate*

Lemma 3.4.

$$\sup_{1\leqslant k\leqslant N}\left\|\rho_k^L\right\|^2_{L^2(\Omega)}+\sum_{k=1}^N\left\|\rho_k^L-\rho_{k-1}^L\right\|^2_{L^2(\Omega)}+\tau\sum_{k=1}^N\left\|\nabla_{\mathbf{x}}\rho_k^L\right\|^2_{L^2(\Omega)}\leqslant C(\alpha,T)\|\rho_{in}\|^2_{L^2(\Omega)}. \tag{3.24}$$

Proof. Clearly $\rho_k^L \in H^1(\Omega)$. Taking $\psi = \rho_k^L$ in (3.5), we have

$$\int_\Omega \frac{\rho_k^L - \rho_{k-1}^L}{\tau} \rho_k^L \, d\boldsymbol{x} + \int_\Omega \left(\boldsymbol{u}_{k-1}^L \cdot \nabla_{\mathbf{x}} \rho_k^L\right) \rho_k^L \, d\boldsymbol{x} + \int_\Omega \left|\nabla_{\mathbf{x}} \rho_k^L\right|^2 d\boldsymbol{x}$$
$$= \int_\Omega \left(\int_{\mathbb{S}^{d-1}} \alpha \boldsymbol{n} Q^L\left(f_k^L\right) d\boldsymbol{n} \right) \cdot \nabla_{\mathbf{x}} \rho_k^L \, d\boldsymbol{x}. \tag{3.25}$$

Integrating by parts and noting $\rho_k^L = \|f_k^L\|_{L^1(\mathbb{S}^{d-1})}$, $\nabla_{\mathbf{x}} \cdot \boldsymbol{u}_k^L = 0$ and (3.16), one has from Cauchy–Schwartz inequality that

$$\frac{1}{2} \int_\Omega \left|\rho_k^L\right|^2 d\boldsymbol{x} + \frac{1}{2} \int_\Omega \left|\rho_k^L - \rho_{k-1}^L\right|^2 d\boldsymbol{x} + \tau \int_\Omega \left|\nabla_{\mathbf{x}} \rho_k^L\right|^2 d\boldsymbol{x}$$
$$\leqslant \frac{1}{2} \int_\Omega \left|\rho_{k-1}^L\right|^2 d\boldsymbol{x} + \frac{\tau}{2} \int_\Omega \left|\nabla_{\mathbf{x}} \rho_k^L\right|^2 d\boldsymbol{x} + \frac{|\alpha|^2 \tau}{2} \int_\Omega \left|\rho_k^L\right|^2 d\boldsymbol{x}. \tag{3.26}$$

Summing up (3.26) and then for $\alpha \neq 0$, letting $\tau < \frac{1}{2|\alpha|^2}$, one has that

$$\frac{1}{4} \int_\Omega \left|\rho_k^L\right|^2 d\boldsymbol{x} + \frac{\tau}{2} \sum_{i=1}^{k} \int_\Omega \left|\nabla_{\mathbf{x}} \rho_i^L\right|^2 d\boldsymbol{x} \leqslant \frac{1}{2} \int_\Omega \left|\rho_0^L\right|^2 d\boldsymbol{x} + \frac{|\alpha|^2 \tau}{2} \sum_{i=1}^{k-1} \int_\Omega \left|\rho_i^L\right|^2 d\boldsymbol{x}.$$

Then it follows from the discrete Gronwall's inequality that

$$\frac{1}{2} \left\|\rho_k^L\right\|_{L^2(\Omega)}^2 \leqslant \left\|\rho_0^L\right\|_{L^2(\Omega)}^2 e^{2|\alpha|^2 T} \leqslant \|\rho_{in}\|_{L^2(\Omega)}^2 e^{2|\alpha|^2 T}. \tag{3.27}$$

Clearly, (3.27) is also true for $\alpha = 0$. Moreover (3.26) and (3.27) imply (3.24). This ends the proof of Lemma 3.4. □

3.2.2. Entropy estimate

The entropy estimate is the key to the construction of a global entropy weak solution. We use $(\mathcal{F}^L)'(f_k^L + \delta)$ as a test function and then let $\delta \to 0$, to deal with the singularity when $f_k(x, \boldsymbol{n}) = 0$ on some subset of $\Omega \times \mathbb{S}^{d-1}$. Another problem is tackling the term $\int_{\Omega \times \mathbb{S}^{d-1}} ((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E_k^L + W_k^L)\boldsymbol{n}) \cdot \nabla_{\mathbf{n}} f_k^L \, d\boldsymbol{n} \, d\boldsymbol{x}$ in the proof. By gaining one tangential gradient on $\mathbb{S}^{d-1}$, we apply Lemma 1.1 to solve this problem.

Lemma 3.5. *For any* $1 \leqslant k \leqslant N$,

$$\frac{1}{2} \left\|\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}^2 + \int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}\left(f_k^L\right) d\boldsymbol{n} \, d\boldsymbol{x} + \frac{1}{2} \sum_{i=1}^{k} \left\|\boldsymbol{u}_i^L - \boldsymbol{u}_{i-1}^L\right\|_{L^2(\Omega)}^2$$
$$+ \frac{1}{2} \tau \sum_{i=1}^{k} \left\|\nabla_{\mathbf{x}} \boldsymbol{u}_i^L\right\|_{L^2(\Omega)}^2 + 2\tau \sum_{i=1}^{k} \left(\left\|\nabla_{\mathbf{x}} \sqrt{f_i^L}\right\|_{L^2(\Omega \times \mathbb{S}^{d-1})}^2 + \left\|\nabla_{\mathbf{n}} \sqrt{f_i^L}\right\|_{L^2(\Omega \times \mathbb{S}^{d-1})}^2 \right)$$
$$\leqslant \frac{1}{2} \|\boldsymbol{u}_{in}\|_{L^2(\Omega)}^2 + \int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}(f_{in}) \, d\boldsymbol{n} \, d\boldsymbol{x} + C(\alpha, \beta, \gamma, T)\left(\|\rho_{in}\|_{L^2(\Omega)}^2 + 1\right). \tag{3.28}$$

Moreover, if $\alpha = 0$, $\beta\gamma > 0$, *then*

$$\frac{\gamma}{2\beta}\|\boldsymbol{u}_k^L\|_{L^2(\Omega)}^2 + \int\limits_{\Omega\times\mathbb{S}^{d-1}} F\big(f_k^L\big)\,d\boldsymbol{n}\,d\boldsymbol{x} + \frac{\gamma}{2\beta}\sum_{i=1}^k \|\boldsymbol{u}_i^L - \boldsymbol{u}_{i-1}^L\|_{L^2(\Omega)}^2$$
$$+ \frac{\gamma}{\beta}\tau\sum_{i=1}^k \|\nabla_{\mathbf{x}}\boldsymbol{u}_i^L\|_{L^2(\Omega)}^2 + 2\tau\sum_{i=1}^k\big(\|\nabla_{\mathbf{x}}\sqrt{f_i^L}\|_{L^2(\Omega\times\mathbb{S}^{d-1})}^2 + \|\nabla_{\mathbf{n}}\sqrt{f_i^L}\|_{L^2(\Omega\times\mathbb{S}^{d-1})}^2\big)$$
$$\leqslant \frac{\gamma}{2\beta}\|\boldsymbol{u}_{in}\|_{L^2(\Omega)}^2 + \int\limits_{\Omega\times\mathbb{S}^{d-1}} F(f_{in})\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{3.29}$$

Proof. Let $\delta \in (0,1)$. Taking $\varphi = (F^L)'(f_k^L+\delta) \in H^1(\Omega\times\mathbb{S}^{d-1})$ in (3.4) and noting

$$\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(\boldsymbol{u}_{k-1}^L\cdot\nabla_{\mathbf{x}} f_k^L\big)\big(F^L\big)'\big(f_k^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x} = \int\limits_{\Omega\times\mathbb{S}^{d-1}} \boldsymbol{u}_{k-1}^L\cdot\nabla_{\mathbf{x}}\big(F^L\big(f_k^L+\delta\big)\big)\,d\boldsymbol{n}\,d\boldsymbol{x} = 0,$$

we have from the convexity of F^L that

$$\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(F^L\big(f_k^L+\delta\big) - F^L\big(f_{k-1}^L+\delta\big)\big)\,d\boldsymbol{n}\,d\boldsymbol{x} + \tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(\big|\nabla_{\mathbf{x}} f_k^L\big|^2 + \big|\nabla_{\mathbf{n}} f_k^L\big|^2\big)\big(F^L\big)''\big(f_k^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant \tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(Q^L\big(f_k^L\big)\big(F^L\big)''\big(f_k^L+\delta\big)\big)\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})\big(\gamma E_k^L + W_k^L\big)\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+ \alpha\tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(Q^L\big(f_k^L\big)\big(F^L\big)''\big(f_k^L+\delta\big)\big)\boldsymbol{n}\cdot\nabla_{\mathbf{x}} f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x} =: J_1 + J_2. \tag{3.30}$$

For J_1, one divides it into two parts as below

$$J_1 = \tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big(Q^L\big(f_k^L\big)\big(F^L\big)''\big(f_k^L+\delta\big) - 1\big)\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})\big(\gamma E_k^L + W_k^L\big)\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+ \tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})\big(\gamma E_k^L + W_k^L\big)\boldsymbol{n}\big)\cdot\nabla_{\mathbf{n}} f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x} =: J_{1,1} + J_{1,2}. \tag{3.31}$$

The Cauchy–Schwartz inequality, $Q^L \in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1, (2.6) and (2.7) imply

$$J_{1,1} \leqslant C\tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|\big|\nabla_{\mathbf{n}} f_k^L\big|\big|\big(F^L\big)''\big(f_k^L+\delta\big)\big|\big|Q^L\big(f_k^L+\delta\big) - Q^L\big(f_k^L\big)\big|\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant C\sqrt{\delta}\tau\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|\big|\nabla_{\mathbf{n}} f_k^L\big|\sqrt{\big(F^L\big)''\big(f_k^L+\delta\big)}\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant C\delta\tau\int\limits_{\Omega} \big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|^2\,d\boldsymbol{x} + \frac{\tau}{2}\int\limits_{\Omega\times\mathbb{S}^{d-1}} \big|\nabla_{\mathbf{n}} f_k^L\big|^2\big(F^L\big)''\big(f_k^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{3.32}$$

It follows from Lemma 1.1 that

$$J_{1,2}=\gamma\tau\int_{\Omega\times\mathbb{S}^{d-1}}(d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f_k^L:\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}. \tag{3.33}$$

Now we estimate J_2. Since

$$\begin{aligned}J_2&=\alpha\tau\int_{\Omega\times\mathbb{S}^{d-1}}\big(Q^L\big(f_k^L\big)\big(F^L\big)''\big(f_k^L+\delta\big)-1\big)\boldsymbol{n}\cdot\nabla_{\mathbf{x}}f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}+\alpha\tau\int_{\Omega\times\mathbb{S}^{d-1}}\boldsymbol{n}\cdot\nabla_{\mathbf{x}}f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}\\&=:J_{2,1}+J_{2,2},\end{aligned}\tag{3.34}$$

similarly as (3.32), we have

$$J_{2,1}\leqslant C\delta\tau+\frac{\tau}{4}\int_{\Omega\times\mathbb{S}^{d-1}}\big|\nabla_{\mathbf{x}}f_k^L\big|^2\big(F^L\big)''\big(f_k^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x}; \tag{3.35}$$

$$\begin{aligned}J_{2,2}&\leqslant C\tau\int_{\Omega\times\mathbb{S}^{d-1}}\big|\nabla_{\mathbf{x}}f_k^L\big|\,d\boldsymbol{n}\,d\boldsymbol{x}=C\tau\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\nabla_{\mathbf{x}}f_k^L|}{\sqrt{f_k^L+\delta}}\sqrt{f_k^L+\delta}\,d\boldsymbol{n}\,d\boldsymbol{x}\\&\leqslant C\delta\tau+C\tau\big\|f_k^L\big\|_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))}+\frac{\tau}{4}\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\nabla_{\mathbf{x}}f_k^L|^2}{f_k^L+\delta}\,d\boldsymbol{n}\,d\boldsymbol{x}.\end{aligned}\tag{3.36}$$

Taking $\boldsymbol{v}=\boldsymbol{u}_k^L$ in (3.3), one has from (3.16) that

$$\begin{aligned}&\frac{1}{2}\int_\Omega\big|\boldsymbol{u}_k^L\big|^2\,d\boldsymbol{x}+\frac{1}{2}\int_\Omega\big|\boldsymbol{u}_k^L-\boldsymbol{u}_{k-1}^L\big|^2\,d\boldsymbol{x}+\tau\int_\Omega\big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|^2\,d\boldsymbol{x}\\&\quad=\frac{1}{2}\int_\Omega\big|\boldsymbol{u}_{k-1}^L\big|^2\,d\boldsymbol{x}-\beta\tau\int_{\Omega\times\mathbb{S}^{d-1}}(d\boldsymbol{n}\otimes\boldsymbol{n}-\mathrm{Id})f_k^L:\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\,d\boldsymbol{n}\,d\boldsymbol{x}\\&\quad=:\frac{1}{2}\int_\Omega\big|\boldsymbol{u}_{k-1}^L\big|^2\,d\boldsymbol{x}-K.\end{aligned}\tag{3.37}$$

Since

$$J_{1,2}-K\leqslant C\tau\int_{\Omega\times\mathbb{S}^{d-1}}\big|f_k^L\big|\big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|\,d\boldsymbol{n}\,d\boldsymbol{x}\leqslant\frac{\tau}{2}\int_\Omega\big|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\big|^2\,d\boldsymbol{x}+C\tau\big\|\rho_k^L\big\|_{L^2(\Omega)}^2, \tag{3.38}$$

combining (3.30)–(3.38) and summing up, we have by noting $f_0^L=Q^L(f_{in})$ that

$$\begin{aligned}&\frac{1}{2}\int_\Omega\big|\boldsymbol{u}_k^L\big|^2\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}}F^L\big(f_k^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x}+\frac{1}{2}\sum_{i=1}^k\int_\Omega\big|\boldsymbol{u}_i^L-\boldsymbol{u}_{i-1}^L\big|^2\,d\boldsymbol{x}\\&\quad+\frac{\tau}{2}(1-C\delta)\sum_{i=1}^k\int_\Omega\big|\nabla_{\mathbf{x}}\boldsymbol{u}_i^L\big|^2\,d\boldsymbol{x}+\frac{\tau}{2}\sum_{i=1}^k\int_{\Omega\times\mathbb{S}^{d-1}}\big(\big|\nabla_{\mathbf{x}}f_i^L\big|^2+\big|\nabla_{\mathbf{n}}f_i^L\big|^2\big)\big(F^L\big)''\big(f_i^L+\delta\big)\,d\boldsymbol{n}\,d\boldsymbol{x}\end{aligned}$$

$$+\frac{\tau}{4}\sum_{i=1}^{k}\int_{\Omega\times\mathbb{S}^{d-1}}\left|\nabla_{\mathbf{x}} f_i^L\right|^2\left(\left(F^L\right)''\left(f_i^L+\delta\right)-\frac{1}{f_i^L+\delta}\right)d\boldsymbol{n}\,d\boldsymbol{x}$$

$$\leqslant \frac{1}{2}\int_\Omega |\boldsymbol{u}_{in}|^2\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}} F^L\left(Q^L(f_{in})+\delta\right)d\boldsymbol{n}\,d\boldsymbol{x}+C\tau\sum_{i=1}^{k}\left(\left\|\rho_i^L\right\|_{L^2(\Omega)}^2+1\right)+CT\delta. \tag{3.39}$$

Thus it follows from (2.5), (2.6), (2.9) and (3.24) that

$$\frac{1}{2}\int_\Omega \left|\boldsymbol{u}_k^L\right|^2 d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}} F\left(f_k^L+\delta\right)d\boldsymbol{n}\,d\boldsymbol{x}+\frac{1}{2}\sum_{i=1}^{k}\int_\Omega\left|\boldsymbol{u}_i^L-\boldsymbol{u}_{i-1}^L\right|^2 d\boldsymbol{x}$$
$$+\frac{\tau}{2}(1-C\delta)\sum_{i=1}^{k}\int_\Omega\left|\nabla_{\mathbf{x}}\boldsymbol{u}_i^L\right|^2 d\boldsymbol{x}+\frac{\tau}{2}\sum_{i=1}^{k}\int_{\Omega\times\mathbb{S}^{d-1}}\left(\frac{|\nabla_{\mathbf{x}} f_i^L|^2}{f_i^L+\delta}+\frac{|\nabla_{\mathbf{n}} f_i^L|^2}{f_i^L+\delta}\right)d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant \frac{1}{2}\int_\Omega |\boldsymbol{u}_{in}|^2\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}}\left(\delta+\frac{\delta^2}{2}+F(f_{in}+\delta)\right)d\boldsymbol{n}\,d\boldsymbol{x}+CT\left(\|\rho_{in}\|_{L^2(\Omega)}^2+1\right)+CT\delta.$$

Choosing a sufficiently small $\delta>0$ and then performing $\delta\to 0$, one finishes the proof by applying Lebesgue's dominated convergence theorem and Fatou's lemma. With more concern on constants in the proof, we find that $C=C(\alpha,\beta,\gamma,T)$ in (3.28).

If $\alpha=0$, $\beta\gamma>0$, then $J_{1,2}-\frac{\gamma}{\beta}K=0$. Therefore, with a similar discussion, one could deduce (3.29). This finishes the proof of Lemma 3.5. □

3.2.3. Time regularity estimate

Lemma 3.6.

$$\tau\sum_{k=1}^{N}\left\|\frac{\boldsymbol{u}_k^L-\boldsymbol{u}_{k-1}^L}{\tau}\right\|_{(\mathbf{V}^2)'}^2+\tau\sum_{k=1}^{N}\left\|\frac{f_k^L-f_{k-1}^L}{\tau}\right\|_{(H^4(\Omega\times\mathbb{S}^{d-1}))'}^2\leqslant C. \tag{3.40}$$

Proof. It follows from (3.3) that for any $\boldsymbol{v}\in\boldsymbol{V}^2\hookrightarrow L^\infty(\Omega)$,

$$\left|\int_\Omega \frac{\boldsymbol{u}_k^L-\boldsymbol{u}_{k-1}^L}{\tau}\cdot\boldsymbol{v}\,d\boldsymbol{x}\right|\leqslant\left(\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}+\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}\left\|\boldsymbol{u}_{k-1}^L\right\|_{L^2(\Omega)}+C\left\|\rho_k^L\right\|_{L^2(\Omega)}\right)\|\boldsymbol{v}\|_{\mathbf{V}^2}$$

and hence from (3.24) and (3.28) that

$$\tau\sum_{k=1}^{N}\left\|\frac{\boldsymbol{u}_k^L-\boldsymbol{u}_{k-1}^L}{\tau}\right\|_{(\mathbf{V}^2)'}^2\leqslant C\left(\tau\sum_{k=1}^{N}\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}^2+\tau\sum_{k=1}^{N}\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}^2\cdot\sup_{1\leqslant k\leqslant N}\left\|\boldsymbol{u}_{k-1}^L\right\|_{L^2(\Omega)}^2+T\right)$$
$$\leqslant C.$$

For any $\varphi\in H^4(\Omega\times\mathbb{S}^{d-1})\hookrightarrow W^{1,\infty}(\Omega\times\mathbb{S}^{d-1})$ and noting

$$\int_{\Omega\times\mathbb{S}^{d-1}}\left(\boldsymbol{u}_{k-1}^L\cdot\nabla_{\mathbf{x}} f_k^L\right)\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}=-\int_{\Omega\times\mathbb{S}^{d-1}}\left(\boldsymbol{u}_{k-1}^L\cdot\nabla_{\mathbf{x}}\varphi\right)f_k^L\,d\boldsymbol{n}\,d\boldsymbol{x},$$

we deduce from (3.4) that

$$\left|\int_\Omega \frac{f_k^L - f_{k-1}^L}{\tau}\cdot\varphi\,d\boldsymbol{x}\right| \leqslant \int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}} f_k^L\right||\nabla_{\mathbf{x}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x} + \int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{n}} f_k^L\right||\nabla_{\mathbf{n}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+ \int_{\Omega\times\mathbb{S}^{d-1}} \left|f_k^L\right|\left|\boldsymbol{u}_{k-1}^L\right||\nabla_{\mathbf{x}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x} + C\int_{\Omega\times\mathbb{S}^{d-1}} \left|f_k^L\right||\nabla_{\mathbf{x}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$+ C\int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right|\left|f_k^L\right||\nabla_{\mathbf{n}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x}.$$

Now it follows from (3.24) and Hölder inequality that

$$\int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}} f_k^L\right||\nabla_{\mathbf{x}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x} = 2\int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}}\sqrt{f_k^L}\right|\left|\sqrt{f_k^L}\right||\nabla_{\mathbf{x}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x}$$
$$\leqslant 2\left\|\nabla_{\mathbf{x}}\sqrt{f_k^L}\right\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\left\|\sqrt{f_k^L}\right\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\|\nabla_{\mathbf{x}}\varphi\|_{L^\infty(\Omega\times\mathbb{S}^{d-1})}$$
$$\leqslant C\left\|\nabla_{\mathbf{x}}\sqrt{f_k^L}\right\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\|\varphi\|_{H^4(\Omega\times\mathbb{S}^{d-1})};$$
$$\int_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right|\left|f_k^L\right||\nabla_{\mathbf{n}}\varphi|\,d\boldsymbol{n}\,d\boldsymbol{x} \leqslant \left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}\left\|f_k^L\right\|_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))}\|\nabla_{\mathbf{n}}\varphi\|_{L^\infty(\Omega\times\mathbb{S}^{d-1})}$$
$$\leqslant C\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}\|\varphi\|_{H^4(\Omega\times\mathbb{S}^{d-1})}.$$

Similarly, we deal with the other three parts. Therefore

$$\left|\int_\Omega \frac{f_k^L - f_{k-1}^L}{\tau}\cdot\varphi\,d\boldsymbol{x}\right| \leqslant C\Big(\left\|\nabla_{\mathbf{x}}\sqrt{f_k^L}\right\|_{L^2(\Omega\times\mathbb{S}^{d-1})} + \left\|\nabla_{\mathbf{n}}\sqrt{f_k^L}\right\|_{L^2(\Omega\times\mathbb{S}^{d-1})} + \left\|\boldsymbol{u}_{k-1}^L\right\|_{L^2(\Omega)}$$
$$+ 1 + \left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|_{L^2(\Omega)}\Big)\|\varphi\|_{H^4(\Omega\times\mathbb{S}^{d-1})}.$$

This and (3.28) imply that

$$\tau\sum_{k=1}^N\left\|\frac{f_k^L - f_{k-1}^L}{\tau}\right\|^2_{(H^4(\Omega\times\mathbb{S}^{d-1}))'} \leqslant C\Bigg(\tau\sum_{k=1}^N\left\|\nabla_{\mathbf{x}}\sqrt{f_k^L}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \tau\sum_{k=1}^N\left\|\nabla_{\mathbf{n}}\sqrt{f_k^L}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}$$
$$+ T\sup_{1\leqslant k\leqslant N}\left\|\boldsymbol{u}_{k-1}^L\right\|^2_{L^2(\Omega)} + T + \tau\sum_{k=1}^N\left\|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right\|^2_{L^2(\Omega)}\Bigg) \leqslant C.$$

This finishes the proof of Lemma 3.6. □

3.3. *Convergence and proof of Theorem 3.1*

Definition 3.7. Define the piecewise function in t by

$$\boldsymbol{u}_\tau^L(t,\cdot) := \boldsymbol{u}_k^L(\cdot), \qquad \pi_\tau \boldsymbol{u}_\tau^L(t,\cdot) := \boldsymbol{u}_{k-1}^L(\cdot), \quad t \in \big((k-1)\tau, k\tau\big],\ k = 1, 2, \ldots, N$$

and the difference quotient of size τ by

$$\partial_t^\tau \boldsymbol{u}_\tau^L(t,\cdot) := \frac{\boldsymbol{u}_k^L(\cdot) - \boldsymbol{u}_{k-1}^L(\cdot)}{\tau}, \quad t \in \big((k-1)\tau, k\tau\big],\ k = 1, 2, \ldots, N.$$

Likewise, define ρ_τ^L, f_τ^L and $\partial_t^\tau f_\tau^L$. To pass the limit with $\tau \to 0$ and $L \to \infty$ simultaneously, we choose $\tau = o(L^{-2})$ in view of (3.20).

3.3.1. *Convergence*

The compactness discussion is crucial in obtaining strong convergence. Using Aubin–Lions–Simon lemma with hypothesis on derivatives, the traditional Rothe method in evolution PEDs (see [27] and [20]) requires the construction of linear interpolation functions (also known as Rothe functions). However, the dealing with Rothe functions is fairly indirect and tedious, where more estimates and sometimes even more regularity discussion of initial data are needed. Here, we shall apply a simple version of Aubin–Lions–Simon lemma with hypothesis on time translation (see Dreher and Jüngel [7, Theorem 1]) directly to avoid using Rothe functions and making the discussion more clean.

Proposition 3.8. *As $\tau = o(L^{-2}) \to 0$, there exist a subsequence of $\{(\boldsymbol{u}_\tau^L, f_\tau^L)\}_{L>1}$, not relabeled, and a pair of function $(\boldsymbol{u}, f)$ satisfying $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$, (2.12)–(2.13) and (3.1) such that*

$$\boldsymbol{u}_\tau^L \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^\infty(0,T;\boldsymbol{H}), \tag{3.41}$$

$$\boldsymbol{u}_\tau^L \rightharpoonup \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{V}), \tag{3.42}$$

$$\boldsymbol{u}_\tau^L \to \boldsymbol{u} \quad \text{in } L^2\big(0,T;L^p(\Omega)\big)\ (\forall 2<p<6), \tag{3.43}$$

$$\pi_\tau \boldsymbol{u}_\tau^L \to \boldsymbol{u} \quad \text{in } L^2\big((0,T)\times\Omega\big), \tag{3.44}$$

$$f_\tau^L \to f \quad \text{in } L^2\big((0,T)\times\Omega; L^1\big(\mathbb{S}^{d-1}\big)\big), \tag{3.45}$$

$$\sqrt{f_\tau^L} \to \sqrt{f} \quad \text{in } L^4\big((0,T)\times\Omega; L^2\big(\mathbb{S}^{d-1}\big)\big), \tag{3.46}$$

$$\sqrt{f_\tau^L} \rightharpoonup \sqrt{f} \quad \text{in } L^2\big(0,T; H^1\big(\Omega\times\mathbb{S}^{d-1}\big)\big), \tag{3.47}$$

$$Q^L\big(f_\tau^L\big) \to f \quad \text{in } L^2\big((0,T)\times\Omega; L^1\big(\mathbb{S}^{d-1}\big)\big). \tag{3.48}$$

Proof. Applying (3.28), we deduce that there exists a subsequence of $\{\boldsymbol{u}_\tau^L\}_{L>1}$, not relabeled, and $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$ such that (3.41)–(3.42) hold. For any $\tau \in (0,1)$, we have from (3.40) that

$$\big\|\tau_\tau \boldsymbol{u}_\tau^L - \boldsymbol{u}_\tau^L\big\|_{L^2(0,T-\tau;(\mathbf{V}^2)')}^2 = \tau \sum_{k=1}^{N-1} \big\|\boldsymbol{u}_{k+1}^L - \boldsymbol{u}_k^L\big\|_{(\mathbf{V}^2)'}^2 \leqslant C\tau^2, \tag{3.49}$$

where $\tau_\tau \boldsymbol{u}_\tau^L(t) := \boldsymbol{u}_\tau^L(t+\tau)$. Since (3.28) implies $\|\boldsymbol{u}_\tau^L\|_{L^2(0,T;\mathbf{V})} \leqslant C$, we obtain (3.43) from (3.49) and $\boldsymbol{V} \hookrightarrow\hookrightarrow L^p(\Omega) \cap \boldsymbol{H} \hookrightarrow (\mathbf{V}^2)'$ $(\forall 2 < p < 6)$ by employing Dreher and Jüngel [7, Theorem 1]. It follows from (3.28) that

$$\left\|\pi_\tau \boldsymbol{u}_\tau^L - \boldsymbol{u}_\tau^L\right\|^2_{L^2((0,T)\times\Omega)} = \tau \sum_{k=1}^{N} \left\|\boldsymbol{u}_k^L - \boldsymbol{u}_{k-1}^L\right\|^2_{L^2(\Omega)} \leqslant C\tau.$$

This and (3.43) yield (3.44).

It follows from (3.40) that

$$\left\|\tau_\tau f_\tau^L - f_\tau^L\right\|^2_{L^2(0,T-\tau;(H^4(\Omega\times\mathbb{S}^{d-1}))')} = \tau \sum_{k=1}^{N-1} \left\|f_{k+1}^L - f_k^L\right\|^2_{(H^4(\Omega\times\mathbb{S}^{d-1}))'} \leqslant C\tau^2. \tag{3.50}$$

Since (3.24) and (3.28) imply

$$\begin{aligned}\left\|\nabla_{\mathbf{x}} f_\tau^L; \nabla_{\mathbf{n}} f_\tau^L\right\|_{L^2(0,T;L^1(\Omega\times\mathbb{S}^{d-1}))} &\leqslant 2\left\|\nabla_{\mathbf{x}}\sqrt{f_\tau^L}; \nabla_{\mathbf{n}}\sqrt{f_\tau^L}\right\|_{L^2(0,T;L^2(\Omega\times\mathbb{S}^{d-1}))}\left\|\sqrt{f_\tau^L}\right\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{S}^{d-1}))} \\ &\leqslant C,\end{aligned}$$

we have $\|f_\tau^L\|_{L^2(0,T;W^{1,1}(\Omega\times\mathbb{S}^{d-1}))} \leqslant C$. This and (3.50), together with the embedding

$$W^{1,1}\left(\Omega\times\mathbb{S}^{d-1}\right) \hookrightarrow\hookrightarrow L^p\left(\Omega\times\mathbb{S}^{d-1}\right) \hookrightarrow \left(H^4\left(\Omega\times\mathbb{S}^{d-1}\right)\right)' \quad \left(\forall 1 < p < \frac{2d-1}{2d-2}\right)$$

yield

$$f_\tau^L \to f \quad \text{in } L^2\left(0,T; L^p\left(\Omega\times\mathbb{S}^{d-1}\right)\right) \left(\forall 1 < p < \frac{2d-1}{2d-2}\right) \tag{3.51}$$

by applying Dreher and Jüngel [7, Theorem 1]. Also employing Dreher and Jüngel [7, Theorem 1], with the same discussion as (3.43), we deduce from (3.24) that

$$\rho_\tau^L \to \rho \quad \text{in } L^2\left(0,T; L^p(\Omega)\right) \ (\forall 2 < p < 6). \tag{3.52}$$

By noting $\rho_\tau^L = \int_{\mathbb{S}^{d-1}} f_\tau^L \, d\boldsymbol{n}$, one has from (3.51) and (3.52) that

$$\rho = \int_{\mathbb{S}^{d-1}} f \, d\boldsymbol{n} \quad \text{and} \quad \int_{\mathbb{S}^{d-1}} f_\tau^L \, d\boldsymbol{n} \to \int_{\mathbb{S}^{d-1}} f \, d\boldsymbol{n} \quad \text{in } L^2\left(0,T; L^p(\Omega)\right) \ (\forall 2 < p < 6). \tag{3.53}$$

It follows from Gagliardo–Nirenberg inequality that

$$\left\|\rho_\tau^L\right\|_{L^4(\Omega)} \leqslant \left\|\rho_\tau^L\right\|^{d/4}_{H^1(\Omega)}\left\|\rho_\tau^L\right\|^{1-d/4}_{L^2(\Omega)}$$

and then from (3.24) and Hölder inequality that

$$\left\|f_\tau^L\right\|_{L^{8/d}(0,T;L^4(\Omega;L^1(\mathbb{S}^{d-1})))} = \left\|\rho_\tau^L\right\|_{L^{8/d}(0,T;L^4(\Omega))} \leqslant \left\|\rho_\tau^L\right\|^{d/4}_{L^2(0,T;H^1(\Omega))}\left\|\rho_\tau^L\right\|^{1-d/4}_{L^\infty(0,T;L^2(\Omega))} \leqslant C.$$

Moreover, (3.24) also yields

$$\rho_\tau^L \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big), \tag{3.54}$$

$$\rho_\tau^L \rightharpoonup \rho \quad \text{in } L^2\big(0,T;H^1(\Omega)\big). \tag{3.55}$$

Therefore $f \in L^{8/d}(0,T;L^4(\Omega;L^1(\mathbb{S}^{d-1})))$. Consequently, by the interpolation inequality for L^p-norms, we have

$$\begin{aligned}\|f_\tau^L - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} &\leqslant \|f_\tau^L - f\|_{L^{8/d}((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))}^{4/(8-d)} \|f_\tau^L - f\|_{L^1((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))}^{(4-d)/(8-d)} \\ &\leqslant C\|f_\tau^L - f\|_{L^1((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))}^{(4-d)/(8-d)}.\end{aligned}$$

This and (3.51) imply (3.45) and hence $f \geqslant 0$ a.e. on $[0,T]\times\Omega\times\mathbb{S}^{d-1}$. Therefore we have (3.46) by considering the inequality $|\sqrt{a}-\sqrt{b}|^2 \leqslant |a-b|$ $(a,b\geqslant 0)$ and (3.47) in view of (3.28). In addition, one has from (3.53) and (3.54) that $f \in L^\infty(0,T;L^2(\Omega;L^1(\mathbb{S}^{d-1})))$. In light of the weakly lower semi-continuity of norm, we obtain the energy inequalities (3.1) directly from (3.28) and the convergent results (3.41)–(3.43) and (3.45)–(3.47).

At last, we prove (3.48). Indeed, we have from $Q^L \in C^{0,1}(\mathbb{R})$ with Lipschitz coefficient 1 that

$$\begin{aligned}&\|Q^L(f_\tau^L) - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} \\ &\quad\leqslant \|Q^L(f_\tau^L) - Q^L(f)\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} + \|Q^L(f) - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} \\ &\quad\leqslant \|f_\tau^L - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} + \|Q^L(f) - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))}.\end{aligned} \tag{3.56}$$

Moreover, employing Lebesgue' dominated convergence theorem, one deduces from (2.8) and $|Q^L(f)| \leqslant f$ that

$$\|Q^L(f) - f\|_{L^2((0,T)\times\Omega;L^1(\mathbb{S}^{d-1}))} \to 0 \quad \text{as } \tau = o\big(L^{-2}\big) \to 0. \tag{3.57}$$

This, (3.45) and (3.56) imply (3.48) and end the proof of Proposition 3.8. □

3.3.2. Proof of Theorem 3.1

Next we shall prove Theorem 3.1. The key point is to establish the convergence of discrete derivatives $\partial_t^\tau \boldsymbol{u}_\tau^L$ and $\partial_t^\tau f_\tau^L$ as well as their weak integrals. These follow from the time regularity estimate (Lemma 3.6) of $\partial_t^\tau \boldsymbol{u}_\tau^L$, $\partial_t^\tau f_\tau^L$ and their convergence to $\partial_t \boldsymbol{u}$, $\partial_t f$ in the sense of distribution.

Proof of Theorem 3.1. In view of Definition 3.7, the weak approximation form of (3.3)–(3.4) reads

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^\tau \boldsymbol{u}_\tau^L\cdot\boldsymbol{v}\,d\boldsymbol{x}\,dt + \int_0^T\int_\Omega \nabla_{\mathbf{x}}\boldsymbol{u}_\tau^L : \nabla_{\mathbf{x}}\boldsymbol{v}\,d\boldsymbol{x}\,dt + \int_0^T\int_\Omega \big(\pi_\tau\boldsymbol{u}_\tau^L\cdot\nabla_{\mathbf{x}}\big)\boldsymbol{u}_\tau^L\cdot\boldsymbol{v}\,d\boldsymbol{x}\,dt \\ &\quad = -\beta\int_0^T\int_{\Omega\times\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n} - \mathrm{Id})f_\tau^L : \nabla_{\mathbf{x}}\boldsymbol{v}\,d\boldsymbol{n}\,d\boldsymbol{x}\,dt, \\ &\quad \forall\boldsymbol{v}\in\boldsymbol{C}_0^\infty\big([0,T)\times\Omega\big) \text{ with } \nabla_{\mathbf{x}}\cdot\boldsymbol{v} = 0,\end{aligned} \tag{3.58}$$

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \partial_t^\tau f_\tau^L \varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\pi_\tau \boldsymbol{u}_\tau^L \cdot \nabla_{\mathbf{x}} f_\tau^L\right)\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$- \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\alpha \boldsymbol{n} Q^L\left(f_k^L\right)\right)\cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt + \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\nabla_{\mathbf{x}} f_\tau^L \cdot \nabla_{\mathbf{x}}\varphi + \nabla_{\mathbf{n}} f_\tau^L \cdot \nabla_{\mathbf{n}}\varphi\right) d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$= \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left((\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n})\left(\gamma E_\tau^L + W_\tau^L\right)\boldsymbol{n}\right) Q^L\left(f_\tau^L\right)\cdot \nabla_{\mathbf{n}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt,$$
$$\forall \varphi \in C^\infty\left([0,T]\times\overline{\Omega}\times\mathbb{S}^{d-1}\right) \text{ with } \varphi(T) = 0. \tag{3.59}$$

We first claim that as $\tau = o(L^{-2}) \to 0$,

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \partial_t^\tau f_\tau^L \varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$\to - \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} f\, \partial_t \varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt - \int_{\Omega\times\mathbb{S}^{d-1}} f_{in}(\boldsymbol{x},\boldsymbol{n})\varphi(0,\boldsymbol{x},\boldsymbol{n})\, d\boldsymbol{n}\, d\boldsymbol{x}, \tag{3.60}$$
$$\partial_t^\tau f_\tau^L \rightharpoonup \partial_t f \quad \text{in } L^2\left(0,T;\left(H^4\left(\Omega\times\mathbb{S}^{d-1}\right)\right)'\right). \tag{3.61}$$

Indeed,

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \partial_t^\tau f_\tau^L \varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$= \int_\tau^T \int_{\Omega\times\mathbb{S}^{d-1}} \frac{f_\tau^L(t) - f_\tau^L(t-\tau)}{\tau}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt + \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} \frac{f_\tau^L(t) - f_{\mathrm{in}}^L}{\tau}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$= \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \frac{f_\tau^L(t)}{\tau}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt - \int_0^{T-\tau} \int_{\Omega\times\mathbb{S}^{d-1}} \frac{f_\tau^L(t)}{\tau}\varphi(t+\tau) \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$- \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} \frac{Q^L(f_{\mathrm{in}})}{\tau}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$= \int_{T-\tau}^T \int_{\Omega\times\mathbb{S}^{d-1}} f_\tau^L(t)\frac{\varphi}{\tau} \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt - \int_0^{T-\tau} \int_{\Omega\times\mathbb{S}^{d-1}} f_\tau^L(t)\frac{\varphi(t+\tau) - \varphi(t)}{\tau} \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt$$
$$- \int_0^\tau \int_{\Omega\times\mathbb{S}^{d-1}} Q^L(f_{\mathrm{in}})\frac{\varphi}{\tau} \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt.$$

Then

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega\times\mathbb{S}^{d-1}}\partial_t^\tau f_\tau^L\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}\,dt+\int_0^T\int_{\Omega\times\mathbb{S}^{d-1}}f\,\partial_t\varphi\,d\boldsymbol{n}\,d\boldsymbol{x}\,dt+\int_{\Omega\times\mathbb{S}^{d-1}}f_{in}(\boldsymbol{x},\boldsymbol{n})\varphi(0,\boldsymbol{x},\boldsymbol{n})\,d\boldsymbol{n}\,d\boldsymbol{x}\right|\\
&\quad\leqslant\left|\int_{T-\tau}^T\int_{\Omega\times\mathbb{S}^{d-1}}\left(f_\tau^L\frac{\varphi}{\tau}+f\,\partial_t\varphi\right)d\boldsymbol{n}\,d\boldsymbol{x}\,dt\right|+\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{S}^{d-1}}\left(f\,\partial_t\varphi-f_\tau^L\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\right)d\boldsymbol{n}\,d\boldsymbol{x}\,dt\right|\\
&\qquad+\left|\int_{\Omega\times\mathbb{S}^{d-1}}\left(f_{in}\varphi(0)-Q^L(f_{in})\int_0^\tau\frac{\varphi}{\tau}dt\right)d\boldsymbol{n}\,d\boldsymbol{x}\right|\\
&\quad=:P_1+P_2+P_3.
\end{aligned}$$

Thanks to $\varphi(T)=0$, we have from the mean value theorem of differentials that

$$\begin{aligned}
P_1&\leqslant\tau\|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{S}^{d-1})}\left(\left\|f_\tau^L\right\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{S}^{d-1}))}+\|f\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{S}^{d-1}))}\right)\leqslant C\tau,\\
P_2&\leqslant\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{S}^{d-1}}f\left(\partial_t\varphi-\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\right)d\boldsymbol{n}\,d\boldsymbol{x}\right|+\left|\int_0^{T-\tau}\int_{\Omega\times\mathbb{S}^{d-1}}\left(f-f_\tau^L\right)\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\,d\boldsymbol{n}\,d\boldsymbol{x}\right|\\
&\leqslant\tau\|\partial_{tt}\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{S}^{d-1})}\|f\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{S}^{d-1}))}\\
&\quad+\left\|f-f_\tau^L\right\|_{L^1((0,T)\times\Omega\times\mathbb{S}^{d-1})}\|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega\times\mathbb{S}^{d-1})}\\
&\leqslant C\left(\tau+\left\|f-f_\tau^L\right\|_{L^1((0,T)\times\Omega\times\mathbb{S}^{d-1})}\right).
\end{aligned}$$

It follows from the proof of (3.57) and the mean value theorem that,

$$P_3\leqslant\int_{\Omega\times\mathbb{S}^{d-1}}\left|f_{in}-Q^L(f_{in})\right|\left|\varphi(0)\right|d\boldsymbol{n}\,d\boldsymbol{x}+\int_{\Omega\times\mathbb{S}^{d-1}}Q^L(f_{in})\left|\varphi(0)-\frac{1}{\tau}\int_0^\tau\varphi(t)\,dt\right|d\boldsymbol{n}\,d\boldsymbol{x}\to 0.$$

Therefore (3.60) is proved. Moreover, by taking $\varphi\in C_0^\infty((0,T)\times\overline{\Omega}\times\mathbb{S}^{d-1})$, (3.60) implies

$$\partial_t^\tau f_\tau^L\rightharpoonup\partial_t f\quad\text{in }\mathscr{D}'\left((0,T);\left(C^4\left(\overline{\Omega}\times\mathbb{S}^{d-1}\right)\right)'\right).\tag{3.62}$$

We have from (3.40) that $\|\partial_t^\tau f_\tau^L\|_{L^2(0,T;(H^4(\Omega\times\mathbb{S}^{d-1}))')}\leqslant C$. This and (3.62) yield (3.61). Likewise, we deduce from (3.40) that

$$\int_0^T\int_\Omega\partial_t^\tau\boldsymbol{u}_\tau^L\cdot\boldsymbol{v}\,d\boldsymbol{x}\,dt\to-\int_0^T\int_\Omega\boldsymbol{u}\partial_t\boldsymbol{v}\,d\boldsymbol{n}\,d\boldsymbol{x}\,dt-\int_\Omega\boldsymbol{u}_{in}(\boldsymbol{x})\boldsymbol{v}(0,\boldsymbol{x})\,d\boldsymbol{x},\tag{3.63}$$

$$\partial_t^\tau\boldsymbol{u}_\tau^L\rightharpoonup\partial_t\boldsymbol{u}\quad\text{in }L^2\left(0,T;\left(\boldsymbol{V}^2\right)'\right).\tag{3.64}$$

Next, we prove

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \nabla_{\mathbf{x}} f_\tau^L \cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt \to \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt, \tag{3.65}$$

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \nabla_{\mathbf{n}} f_\tau^L \cdot \nabla_{\mathbf{n}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt \to \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \nabla_{\mathbf{n}} f \cdot \nabla_{\mathbf{n}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt, \tag{3.66}$$

$$\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\pi_\tau \boldsymbol{u}_\tau^L \cdot \nabla_{\mathbf{x}} f_\tau^L\right)\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt \to \int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} (\boldsymbol{u}\cdot \nabla_{\mathbf{x}} f)\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt. \tag{3.67}$$

In fact, it follows from (3.46) and (3.47) that

$$\begin{aligned}
&\left|\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\nabla_{\mathbf{x}} f_\tau^L - \nabla_{\mathbf{x}} f\right)\cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt\right| \\
&\quad \leqslant 2\left|\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \left(\sqrt{f_\tau^L} - \sqrt{f}\right)\nabla_{\mathbf{x}}\sqrt{f_\tau^L}\cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt\right| \\
&\qquad + 2\left|\int_0^T \int_{\Omega\times\mathbb{S}^{d-1}} \sqrt{f}\left(\nabla_{\mathbf{x}}\sqrt{f_\tau^L} - \nabla_{\mathbf{x}}\sqrt{f}\right)\cdot \nabla_{\mathbf{x}}\varphi \, d\boldsymbol{n}\, d\boldsymbol{x}\, dt\right| \to 0.
\end{aligned}$$

Similarly, one has (3.66). Noting $\pi_\tau \boldsymbol{u}_\tau^L$, $\boldsymbol{u}\in\boldsymbol{V}$ and integrating by parts, we then establish (3.67) directly from (3.41) and (3.45).

The convergence of two terms with cut-off $Q^L(f_k^L)$ in (3.59) could be deduced directly by (3.42) and (3.48). The convergence of the remaining terms in (3.58) is clearly implied by (3.42), (3.44) and (3.45). This and Proposition 3.8 finish the proof of Theorem 3.1. □

4. Global weak entropy solution to Stokes Doi–Saintillan–Shelley model

In this section, we prove the following theorem.

Theorem 4.1. *Let $d = 2, 3$. Suppose $f_{in}\in L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ satisfy (2.10). Then for any coefficients $\gamma\in[-1,1]$, $\beta\in\mathbb{R}$ and $\alpha\in[0,\infty)$, there exists a global weak entropy solution $(\boldsymbol{u}, f)$ to Stokes Doi–Saintillan–Shelley model with boundary conditions (1.24)–(1.25) and initial condition (1.27) which satisfies the following energy inequalities for a.e. $t\in[0,T]$,*

$$\begin{aligned}
&\int_{\Omega\times\mathbb{S}^{d-1}} F\left(f(t)\right) d\boldsymbol{n}\, d\boldsymbol{x} + 2\int_0^t \left(\left\|\nabla_{\mathbf{x}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \left\|\nabla_{\mathbf{n}}\sqrt{f(s)}\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right) ds \\
&\quad \leqslant \int_{\Omega\times\mathbb{S}^{d-1}} F(f_{in})\, d\boldsymbol{n}\, d\boldsymbol{x} + C(\alpha,\beta,\gamma,T)\left(\|f_{in}\|^2_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))} + 1\right).
\end{aligned}$$

Moreover, if $\alpha = 0$, $\beta\gamma > 0$ (including the Stokes Doi model), then $C(\alpha,\beta,\gamma,T) = 0$.

Let $f_{in} \in L^2(\Omega; L^1(\mathbb{S}^{d-1}))$ satisfying (2.10) and $f_0^L = Q^L(f_{in})$. Then $f_0^L \in Z$. With a similar discussion as Section 3.1, we have that, for any $k = 1, 2, \ldots, N$, if given $f_{k-1}^L \in Z$, then there exists $(\boldsymbol{u}_k^L, f_k^L) \in \boldsymbol{V} \times (Z \cap H^1(\Omega \times \mathbb{S}^{d-1}))$ which solves the approximate problem with cut-off

$$\int_{\Omega} \nabla_{\mathbf{x}} \boldsymbol{u}_k^L : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{x} = -\beta \int_{\Omega \times \mathbb{S}^{d-1}} (d\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id}) f_k^L : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}; \tag{4.1}$$

$$\begin{aligned} &\int_{\Omega \times \mathbb{S}^{d-1}} \frac{f_k^L - f_{k-1}^L}{\tau} \varphi\, d\boldsymbol{n}\, d\boldsymbol{x} + \int_{\Omega \times \mathbb{S}^{d-1}} \big(\boldsymbol{u}_k^L \cdot \nabla_{\mathbf{x}} f_k^L\big) \varphi\, d\boldsymbol{n}\, d\boldsymbol{x} \\ &\quad - \int_{\Omega \times \mathbb{S}^{d-1}} \big(\alpha \boldsymbol{n} Q^L\big(f_k^L\big)\big) \cdot \nabla_{\mathbf{x}} \varphi\, d\boldsymbol{n}\, d\boldsymbol{x} + \int_{\Omega \times \mathbb{S}^{d-1}} \big(\nabla_{\mathbf{x}} f_k^L \cdot \nabla_{\mathbf{x}} \varphi + \nabla_{\mathbf{n}} f_k^L \cdot \nabla_{\mathbf{n}} \varphi\big)\, d\boldsymbol{n}\, d\boldsymbol{x} \\ &= \int_{\Omega \times \mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})\big(\gamma E_k^L + W_k^L\big)\boldsymbol{n}\big) Q^L\big(f_k^L\big) \cdot \nabla_{\mathbf{n}} \varphi\, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall \varphi \in H^1\big(\Omega \times \mathbb{S}^{d-1}\big), \end{aligned} \tag{4.2}$$

where $E_k^L = \frac{1}{2}(\nabla_{\mathbf{x}} \boldsymbol{u}_k^L + (\nabla_{\mathbf{x}} \boldsymbol{u}_k^L)^\top)$, $W_k^L = \frac{1}{2}(\nabla_{\mathbf{x}} \boldsymbol{u}_k^L - (\nabla_{\mathbf{x}} \boldsymbol{u}_k^L)^\top)$. Then similarly as Section 3.2, we deduce the following lemmas.

Lemma 4.2.

$$\sup_{1 \leqslant k \leqslant N} \big\| \rho_k^L \big\|_{L^2(\Omega)}^2 + \sum_{k=1}^N \big\| \rho_k^L - \rho_{k-1}^L \big\|_{L^2(\Omega)}^2 + 2\tau \sum_{k=1}^N \big\| \nabla_{\mathbf{x}} \rho_k^L \big\|_{L^2(\Omega)}^2 \leqslant C(\alpha, T) \|\rho_{in}\|_{L^2(\Omega)}^2.$$

Lemma 4.3.

$$\sup_{1 \leqslant k \leqslant N} \big\| \nabla_{\mathbf{x}} \boldsymbol{u}_k^L \big\|_{L^2(\Omega)}^2 \leqslant C(\alpha, \beta, T) \|\rho_{in}\|_{L^2(\Omega)}^2.$$

Lemma 4.4 *(Entropy estimate). For any $1 \leqslant k \leqslant N$,*

$$\begin{aligned} &\int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}\big(f_k^L\big)\, d\boldsymbol{n}\, d\boldsymbol{x} + 2\tau \sum_{i=1}^k \Big(\Big\| \nabla_{\mathbf{x}} \sqrt{f_i^L} \Big\|_{L^2(\Omega \times \mathbb{S}^{d-1})}^2 + \Big\| \nabla_{\mathbf{n}} \sqrt{f_i^L} \Big\|_{L^2(\Omega \times \mathbb{S}^{d-1})}^2 \Big) \\ &\quad \leqslant \int_{\Omega \times \mathbb{S}^{d-1}} \mathcal{F}(f_{in})\, d\boldsymbol{n}\, d\boldsymbol{x} + C(\alpha, \beta, \gamma, T)\big(\|\rho_{in}\|_{L^2(\Omega)}^2 + 1\big). \end{aligned}$$

Moreover, if $\alpha = 0$, $\beta\gamma > 0$, then $C(\alpha, \beta, \gamma, T) = 0$.

Lemma 4.5 *(Time regularity estimate).*

$$\tau \sum_{k=1}^N \left\| \frac{f_k^L - f_{k-1}^L}{\tau} \right\|_{(H^4(\Omega \times \mathbb{S}^{d-1}))'}^2 \leqslant C.$$

Proof of Theorem 4.1. Using these uniform estimates, it is sufficient for us to prove Theorem 4.1 with a similar discussion as Section 3.3. □

5. Global L^2 solution to Stokes Doi–Saintillan–Shelley model and uniqueness

The main difficulty in obtaining a global L^2 solution and uniqueness is how to deal with the discrete weak form of $\nabla_{\mathbf{n}} \cdot ((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E + W)\boldsymbol{n} f)$. By gaining one tangential gradient on $\mathbb{S}^{d-1}$, we apply Lemma 1.1 to solve this problem.

Theorem 5.1. *Let $d = 2, 3$. Suppose $f_{in} \in L^2(\Omega \times \mathbb{S}^{d-1})$ satisfying (2.10). Then the solution in Theorem 4.1 has more regularity*

$$\boldsymbol{u} \in L^2\big(0, T; \boldsymbol{H}^2(\Omega)\big), \qquad f \in L^\infty\big(0, T; L^2\big(\Omega \times \mathbb{S}^{d-1}\big)\big) \cap L^2\big(0, T; H^1\big(\Omega \times \mathbb{S}^{d-1}\big)\big), \tag{5.1}$$

$$f \in W^{1,4/d}\big(0, T; \big(H^1\big(\Omega \times \mathbb{S}^{d-1}\big)\big)'\big) \tag{5.2}$$

and moreover

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))} + \|f\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{S}^{d-1})) \cap L^2(0,T;H^1(\Omega \times \mathbb{S}^{d-1}))} \leqslant C. \tag{5.3}$$

Theorem 5.2. *Let $d = 2$. Suppose $f_{in} \in L^2(\Omega \times \mathbb{S}^{d-1})$ satisfying (2.10). Then the solution in Theorem 4.1 is unique.*

5.1. Global L^2 solution

Let $d = 2, 3$ and $f_{in} \in L^2(\Omega \times \mathbb{S}^{d-1})$ satisfying (2.10). To prove Theorem 5.1, we need more estimates of the approximation solution $(\boldsymbol{u}_k^L, f_k^L)$ uniformly in L and N.

Lemma 5.3 *(L^2-estimate). For any $1 \leqslant k \leqslant N$,*

$$\begin{aligned} &\|f_k^L\|^2_{L^2(\Omega \times \mathbb{S}^{d-1})} + \sum_{i=1}^{k} \|f_k^L - f_{k-1}^L\|^2_{L^2(\Omega \times \mathbb{S}^{d-1})} \\ &\quad + \tau \sum_{i=1}^{k} \big(\|\nabla_{\mathbf{x}} f_i^L\|^2_{L^2(\Omega \times \mathbb{S}^{d-1})} + \|\nabla_{\mathbf{n}} f_i^L\|^2_{L^2(\Omega \times \mathbb{S}^{d-1})}\big) \leqslant C. \end{aligned} \tag{5.4}$$

Proof. Taking $\varphi = f_k^L$ in (4.2) and performing a similar procedure as (3.19), we have

$$\begin{aligned} &\frac{1}{2} \int\limits_{\Omega \times \mathbb{S}^{d-1}} |f_k^L|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + \frac{1}{2} \int\limits_{\Omega \times \mathbb{S}^{d-1}} |f_k^L - f_{k-1}^L|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + \tau \int\limits_{\Omega \times \mathbb{S}^{d-1}} \big(|\nabla_{\mathbf{x}} f_k^L|^2 + |\nabla_{\mathbf{n}} f_k^L|^2\big) \, d\boldsymbol{n} \, d\boldsymbol{x} \\ &\quad = \frac{1}{2} \int\limits_{\Omega \times \mathbb{S}^{d-1}} |f_{k-1}^L|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + \tau \int\limits_{\Omega \times \mathbb{S}^{d-1}} \big(\alpha \boldsymbol{n} Q^L\big(f_k^L\big)\big) \cdot \nabla_{\mathbf{x}} f_k^L \, d\boldsymbol{n} \, d\boldsymbol{x} \\ &\qquad + \tau \int\limits_{\Omega \times \mathbb{S}^{d-1}} \big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})\big(\gamma E_k^L + W_k^L\big)\boldsymbol{n}\big) Q^L\big(f_k^L\big) \cdot \nabla_{\mathbf{n}} f_k^L \, d\boldsymbol{n} \, d\boldsymbol{x} \\ &\quad =: \frac{1}{2} \int\limits_{\Omega \times \mathbb{S}^{d-1}} |f_{k-1}^L|^2 \, d\boldsymbol{n} \, d\boldsymbol{x} + O_1 + O_2 \end{aligned} \tag{5.5}$$

and

$$O_1 \leqslant C\tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left|f_k^L\right|\left|\nabla_{\mathbf{x}} f_k^L\right| d\boldsymbol{n}\,d\boldsymbol{x} \leqslant \frac{\tau}{4} \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left|\nabla_{\mathbf{x}} f_k^L\right|^2 d\boldsymbol{n}\,d\boldsymbol{x} + C\tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left|f_k^L\right|^2 d\boldsymbol{n}\,d\boldsymbol{x}. \tag{5.6}$$

It follows from Lemma 1.1, (2.4), Lemma 4.3 and Hölder inequality that

$$\begin{aligned} O_2 &= \tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left((\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n})\left(\gamma E_k^L + W_k^L\right)\boldsymbol{n}\right)\cdot\nabla_{\mathbf{n}} G^L\left(f_k^L\right) d\boldsymbol{n}\,d\boldsymbol{x} \\ &= \gamma\tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} (d\boldsymbol{n}\otimes\boldsymbol{n} - \mathrm{Id})G^L\left(f_k^L\right) : \nabla_{\mathbf{x}}\boldsymbol{u}_k^L\,d\boldsymbol{n}\,d\boldsymbol{x} \leqslant C\tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left|G^L\left(f_k^L\right)\right|\left|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right| d\boldsymbol{n}\,d\boldsymbol{x} \\ &\leqslant C\tau \int\limits_{\Omega\times\mathbb{S}^{d-1}} \left|f_k^L\right|^2\left|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L\right| d\boldsymbol{n}\,d\boldsymbol{x} \leqslant C\tau \left\|f_k^L\right\|^2_{L^4(\Omega;L^2(\mathbb{S}^{d-1}))}. \end{aligned} \tag{5.7}$$

Applying Gagliardo–Nirenberg inequality, we have that

$$\begin{aligned} \left\|f_k^L\right\|^2_{L^4(\Omega;L^2(\mathbb{S}^{d-1}))} &\leqslant C\left\|f_k^L\right\|^{d/2}_{H^1(\Omega;L^2(\mathbb{S}^{d-1}))}\left\|f_k^L\right\|^{2-d/2}_{L^2(\Omega;L^2(\mathbb{S}^{d-1}))} \\ &\leqslant C\left(\left\|\nabla_{\mathbf{x}} f_k^L\right\|^{d/2}_{L^2(\Omega\times\mathbb{S}^{d-1})}\left\|f_k^L\right\|^{2-d/2}_{L^2(\Omega\times\mathbb{S}^{d-1})} + \left\|f_k^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right). \end{aligned} \tag{5.8}$$

Therefore by Young inequality, one has

$$O_2 \leqslant \frac{\tau}{4}\left\|\nabla_{\mathbf{x}} f_k^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + C\tau\left\|f_k^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}. \tag{5.9}$$

Combining (5.5), (5.6) and (5.9), then summing up, we deduce that

$$\begin{aligned} &\left\|f_k^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \sum_{i=1}^k \left\|f_i^L - f_{i-1}^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \tau\sum_{i=1}^k\left(\left\|\nabla_{\mathbf{x}} f_i^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + \left\|\nabla_{\mathbf{n}} f_i^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right) \\ &\quad\leqslant \left\|f_{in}^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})} + C\tau\sum_{i=1}^k\left\|f_i^L\right\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}. \end{aligned} \tag{5.10}$$

Letting $\tau < \frac{1}{2C}$, this implies (5.4) by employing discrete Gronwall's inequality. □

Lemma 5.4. *For any* $1 \leqslant k \leqslant N$,

$$\left\|\boldsymbol{u}_k^L\right\|_{H^2(\Omega)} \leqslant C\left\|\nabla_{\mathbf{x}} f_k^L\right\|_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))}. \tag{5.11}$$

Proof. It follows from the regularity of Stokes equation (see [33, p. 35, Proposition 2.3]) that

$$\left\|\boldsymbol{u}_k^L\right\|_{H^2(\Omega)} \leqslant C\left\|\nabla_{\mathbf{x}}\cdot\beta\int\limits_{\mathbb{S}^{d-1}}(d\boldsymbol{n}\otimes\boldsymbol{n} - \mathrm{Id})f_k^L\,d\boldsymbol{n}\right\|_{L^2(\Omega)} \leqslant C\left\|\nabla_{\mathbf{x}} f_k^L\right\|_{L^2(\Omega;L^1(\mathbb{S}^{d-1}))}.$$

This ends the proof. □

Lemma 5.5.

$$\|\boldsymbol{u}_\tau^L\|_{L^\infty(0,T;\mathbf{V})\cap L^2(0,T;H^2(\Omega))} \leqslant C, \tag{5.12}$$

$$\|f_\tau^L\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{S}^{d-1}))\cap L^2(0,T;H^1(\Omega\times\mathbb{S}^{d-1}))} \leqslant C. \tag{5.13}$$

Proof. We deduce (5.12)–(5.13) directly from Lemma 4.3 and Lemmas 5.3–5.4. □

Lemma 5.6 *(Time regularity estimate).*

$$\|\partial_t^\tau f_\tau^L\|_{L^{4/d}(0,T;(H^1(\Omega\times\mathbb{S}^{d-1}))')} \leqslant C. \tag{5.14}$$

Proof. We deduce from (4.2) that for any $\varphi \in H^1(\Omega\times\mathbb{S}^{d-1})$,

$$\begin{aligned}\left|\int_\Omega \frac{f_k^L - f_{k-1}^L}{\tau}\varphi\, d\boldsymbol{x}\right| \leqslant &\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}} f_k^L||\nabla_{\mathbf{x}}\varphi|\, d\boldsymbol{n}\, d\boldsymbol{x} + \int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{n}} f_k^L||\nabla_{\mathbf{n}}\varphi|\, d\boldsymbol{n}\, d\boldsymbol{x} \\ &+ \int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}} f_k^L||\boldsymbol{u}_k^L||\varphi|\, d\boldsymbol{n}\, d\boldsymbol{x} + C\int_{\Omega\times\mathbb{S}^{d-1}} |f_k^L||\nabla_{\mathbf{x}}\varphi|\, d\boldsymbol{n}\, d\boldsymbol{x} \\ &+ C\int_{\Omega\times\mathbb{S}^{d-1}} |\nabla_{\mathbf{x}}\boldsymbol{u}_k^L||f_k^L||\nabla_{\mathbf{n}}\varphi|\, d\boldsymbol{n}\, d\boldsymbol{x}.\end{aligned}$$

Therefore in light of $H^1(\Omega\times\mathbb{S}^{d-1}) \hookrightarrow L^4(\Omega; L^2(\mathbb{S}^{d-1}))$ that

$$\begin{aligned}&\left\|\frac{f_k^L - f_{k-1}^L}{\tau}\right\|_{(H^1(\Omega\times\mathbb{S}^{d-1}))'} \\ &\quad\leqslant C\big(\|f_k^L\|_{H^1(\Omega\times\mathbb{S}^{d-1})} + \|\nabla_{\mathbf{x}} f_k^L\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\|\boldsymbol{u}_k^L\|_{L^4(\Omega)} + \|\nabla_{\mathbf{x}}\boldsymbol{u}_k^L f_k^L\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\big).\end{aligned} \tag{5.15}$$

Hence

$$\begin{aligned}&\|\partial_t^\tau f_\tau^L\|_{L^{4/d}(0,T;(H^1(\Omega\times\mathbb{S}^{d-1}))')} \\ &\quad= \left(\tau\sum_{k=1}^N \left\|\frac{f_k^L - f_{k-1}^L}{\tau}\right\|_{(H^1(\Omega\times\mathbb{S}^{d-1}))'}^{4/d}\right)^{d/4} \\ &\quad\leqslant C\left(\sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \big(\|f_\tau^L\|_{H^1(\Omega\times\mathbb{S}^{d-1})} + \|\nabla_{\mathbf{x}} f_\tau^L\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\|\boldsymbol{u}_\tau^L\|_{L^4(\Omega)}\right. \\ &\qquad\left. + \|\nabla_{\mathbf{x}}\boldsymbol{u}_\tau^L f_\tau^L\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\big)^{4/d}\right)^{d/4} \\ &\quad\leqslant C\big(\|f_\tau^L\|_{L^2(0,T;H^1(\Omega\times\mathbb{S}^{d-1}))} + \|\nabla_{\mathbf{x}} f_\tau^L\|_{L^2((0,T)\times\Omega\times\mathbb{S}^{d-1})}\|\boldsymbol{u}_\tau^L\|_{L^\infty(0,T;L^4(\Omega))} \\ &\qquad + \|f_\tau^L\|_{L^{4/(d-1)}(0,T;L^{2d}(\Omega;L^2(\mathbb{S}^{d-1})))}\|\nabla_{\mathbf{x}}\boldsymbol{u}_\tau^L\|_{L^4(0,T;L^{2d/(d-1)}(\Omega))}\big).\end{aligned} \tag{5.16}$$

It follows from Gagliardo–Nirenberg inequality that

$$\left\| \nabla_{\mathbf{x}} \boldsymbol{u}_\tau^L \right\|_{L^{2d/(d-1)}(\Omega)} \leqslant C \left\| \boldsymbol{u}_\tau^L \right\|_{H^2(\Omega)}^{1/2} \left\| \nabla_{\mathbf{x}} \boldsymbol{u}_\tau^L \right\|_{L^2(\Omega)}^{1/2}$$

and hence by Hölder inequality that

$$\left\| \nabla_{\mathbf{x}} \boldsymbol{u}_\tau^L \right\|_{L^4(0,T;L^{2d/(d-1)}(\Omega))} \leqslant C \left\| \boldsymbol{u}_\tau^L \right\|_{L^2(0,T;H^2(\Omega))}^{1/2} \left\| \nabla_{\mathbf{x}} \boldsymbol{u}_\tau^L \right\|_{L^\infty(0,T;L^2(\Omega))}^{1/2}. \tag{5.17}$$

Now we estimate $\|f_\tau^L\|_{L^{4/(d-1)}(0,T;L^{2d}(\Omega;L^2(\mathbb{S}^{d-1})))}$.

Case 1. $d=2$. Similar as the estimate of (5.17), we deduce from Gagliardo–Nirenberg inequality and Hölder inequality that

$$\left\| f_\tau^L \right\|_{L^4((0,T)\times\Omega;L^2(\mathbb{S}^{d-1}))} \leqslant \left\| f_\tau^L \right\|_{L^2(0,T;H^1(\Omega\times\mathbb{S}^{d-1}))}^{1/2} \left\| f_\tau^L \right\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{S}^{d-1}))}^{1/2}. \tag{5.18}$$

Case 2. $d=3$. The embedding $H^1(\Omega\times\mathbb{S}^{d-1}) \hookrightarrow L^6(\Omega; L^2(\mathbb{S}^{d-1}))$ implies

$$\left\| f_\tau^L \right\|_{L^2(0,T;L^6(\Omega;L^2(\mathbb{S}^{d-1})))} \leqslant C \left\| f_\tau^L \right\|_{L^2(0,T;H^1(\Omega\times\mathbb{S}^{d-1}))}. \tag{5.19}$$

Clearly, $H^1(\Omega) \hookrightarrow L^4(\Omega)$ implies $\|\boldsymbol{u}_\tau^L\|_{L^\infty(0,T;L^4(\Omega))} \leqslant C\|\boldsymbol{u}_\tau^L\|_{L^\infty(0,T;H^1(\Omega))}$. Combining this and (5.16)–(5.19), we get (5.14) from Lemma 5.5. This finishes the proof of the lemma. □

Proof of Theorem 5.1. We could deduce the same result as (3.62). Then it follows from (5.14) that (5.2) holds. Clearly, Lemma 5.5 implies (5.1) and (5.3). This ends the proof. □

5.2. *Uniqueness for L^2 solution ($d=2$)*

Proof of Theorem 5.2. Since $f \in L^2(0,T;H^1(\Omega\times\mathbb{S}^1)) \cap H^1(0,T;(H^1(\Omega\times\mathbb{S}^1))')$, we have $f \in C([0,T];L^2(\Omega\times\mathbb{S}^1))$ and

$$\frac{d}{dt}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}^2 = 2\langle \partial_t f, f\rangle, \tag{5.20}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product between $H^1(\Omega\times\mathbb{S}^1)$ and its dual $(H^1(\Omega\times\mathbb{S}^1))'$ (see [33, p. 260, Lemma 1.2]). Suppose $(\boldsymbol{u}_1, f_1)$ and $(\boldsymbol{u}_2, f_2)$ are both solutions of (2.16)–(2.17). That is, for a.e. $t\in[0,T]$ and $i=1,2$, they satisfy

$$\int_\Omega \nabla_{\mathbf{x}} \boldsymbol{u}_i : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{x} = -\beta \int_\Omega (2\boldsymbol{n}\otimes\boldsymbol{n} - \mathrm{Id}) f_i : \nabla_{\mathbf{x}} \boldsymbol{v}\, d\boldsymbol{n}\, d\boldsymbol{x}, \quad \forall v \in \boldsymbol{V}; \tag{5.21}$$

$$\begin{aligned}
&(\partial_t f_i, \varphi) + (\boldsymbol{u}_i\cdot\nabla_{\mathbf{x}} f_i, \varphi) + (\nabla_{\mathbf{x}} f_i - \alpha\boldsymbol{n} f_i, \nabla_{\mathbf{x}}\varphi) + (\nabla_{\mathbf{n}} f_i, \nabla_{\mathbf{n}}\varphi) \\
&\quad = \big((\mathrm{Id} - \boldsymbol{n}\otimes\boldsymbol{n})(\gamma E_i + W_i)\boldsymbol{n} f_i, \nabla_{\mathbf{n}}\varphi\big), \quad \forall \varphi \in H^1\big(\Omega\times\mathbb{S}^1\big);
\end{aligned} \tag{5.22}$$

$$f_i|_{t=0} = f_{in} \quad \text{a.e. on } \Omega\times\mathbb{S}^1, \tag{5.23}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega\times\mathbb{S}^1)$. By subtracting and then setting $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ in (5.21), we have by Cauchy–Schwartz inequality and the regularity of the weak solution to Stokes equation (see [33, p. 35, Proposition 2.3]) that for a.e. $t\in[0,T]$,

$$\|\nabla_{\mathbf{x}}\boldsymbol{u}_1-\nabla_{\mathbf{x}}\boldsymbol{u}_2\|_{L^2(\Omega)}\leqslant C\|f_1-f_2\|_{L^2(\Omega\times\mathbb{S}^1)},\tag{5.24}$$

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{H^2(\Omega)}\leqslant C\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|_{L^2(\Omega\times\mathbb{S}^1)}.\tag{5.25}$$

By subtracting and then setting $\varphi=f_1-f_2$ in (5.22), we have by noting (5.20) that for a.e. $t\in[0,T]$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|f_1-f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\|\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\\
&=-\big((\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla_{\mathbf{x}}f_1,f_1-f_2\big)-\big(\boldsymbol{u}_2\cdot\nabla_{\mathbf{x}}(f_1-f_2),f_1-f_2\big)+\big(\alpha\boldsymbol{n}(f_1-f_2),\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\big)\\
&\quad+\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})\big((\gamma E_1+W_1)-(\gamma E_2+W_2)\big)\boldsymbol{n}f_1,\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\big)\\
&\quad+\big((\mathrm{Id}-\boldsymbol{n}\otimes\boldsymbol{n})(\gamma E_2+W_2)\boldsymbol{n}(f_1-f_2),\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\big)\\
&=:S_1+S_2+S_3+S_4+S_5.
\end{aligned}\tag{5.26}$$

We deduce from (5.24), Hölder inequality, Gagliardo–Nirenberg inequality, Poincaré inequality and Young inequality that for a.e. $t\in[0,T]$,

$$\begin{aligned}
S_1&\leqslant\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^4(\Omega)}\|\nabla_{\mathbf{x}}f_1\|_{L^2(\Omega;L^2(\mathbb{S}^1))}\|f_1-f_2\|_{L^4(\Omega;L^2(\mathbb{S}^1))}\\
&\leqslant C\|\nabla_{\mathbf{x}}\boldsymbol{u}_1-\nabla_{\mathbf{x}}\boldsymbol{u}_2\|^{1/2}_{L^2(\Omega)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|^{1/2}_{L^2(\Omega)}\\
&\quad\times\|f_1-f_2\|^{1/2}_{H^1(\Omega;L^2(\mathbb{S}^1))}\|f_1-f_2\|^{1/2}_{L^2(\Omega\times\mathbb{S}^1)}\|\nabla_{\mathbf{x}}f_1\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leqslant C\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^{1/2}_{L^2(\Omega\times\mathbb{S}^1)}\|f_1-f_2\|^{3/2}_{L^2(\Omega\times\mathbb{S}^1)}\|\nabla_{\mathbf{x}}f_1\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\quad+\|f_1-f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\|\nabla_{\mathbf{x}}f_1\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leqslant\frac{1}{8}\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}+C\|f_1-f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\\
&\quad\times\big(\|\nabla_{\mathbf{x}}f_1\|^{4/3}_{L^2(\Omega\times\mathbb{S}^1)}+\|\nabla_{\mathbf{x}}f_1\|_{L^2(\Omega\times\mathbb{S}^1)}\big).
\end{aligned}\tag{5.27}$$

Clearly

$$S_2=-\frac{1}{2}\big(\boldsymbol{u}_2,\nabla_{\mathbf{x}}(f_1-f_2)^2\big)=0;\tag{5.28}$$

$$S_3\leqslant\frac{1}{8}\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}+C\|f_1-f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}.\tag{5.29}$$

Similar as (5.27), in light of (5.24)–(5.25) and (5.3) we have that for a.e. $t\in[0,T]$,

$$\begin{aligned}
S_4&\leqslant C\|\nabla_{\mathbf{x}}\boldsymbol{u}_1-\nabla_{\mathbf{x}}\boldsymbol{u}_2\|_{L^4(\Omega)}\|f_1\|_{L^4(\Omega;L^2(\mathbb{S}^1))}\|\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leqslant C\|\boldsymbol{u}_1-\boldsymbol{u}_2\|^{1/2}_{H^2(\Omega)}\|\nabla_{\mathbf{x}}\boldsymbol{u}_1-\nabla_{\mathbf{x}}\boldsymbol{u}_2\|^{1/2}_{L^2(\Omega)}\|f_1\|^{1/2}_{H^1(\Omega;L^2(\mathbb{S}^1))}\|f_1\|^{1/2}_{L^2(\Omega\times\mathbb{S}^1)}\|\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leqslant C\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^{1/2}_{L^2(\Omega\times\mathbb{S}^1)}\|f_1-f_2\|^{1/2}_{L^2(\Omega\times\mathbb{S}^1)}\|f_1\|^{1/2}_{H^1(\Omega\times\mathbb{S}^1)}\|\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leqslant\frac{1}{2}\|\nabla_{\mathbf{n}}f_1-\nabla_{\mathbf{n}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{8}\|\nabla_{\mathbf{x}}f_1-\nabla_{\mathbf{x}}f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\\
&\quad+C\|f_1-f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\|f_1\|^2_{H^1(\Omega\times\mathbb{S}^1)}.
\end{aligned}\tag{5.30}$$

It follows from Lemma 1.1, Gagliardo–Nirenberg inequality and (5.3) that for a.e. $t \in [0, T]$,

$$\begin{aligned} S_5 &= \frac{1}{2}\big((\mathrm{Id} - \boldsymbol{n} \otimes \boldsymbol{n})(\gamma E_2 + W_2)\boldsymbol{n}, \nabla_{\mathbf{n}}(f_1 - f_2)^2\big) \\ &= \frac{\gamma}{2}\big((2\boldsymbol{n} \otimes \boldsymbol{n} - \mathrm{Id})(f_1 - f_2)^2 : \nabla_{\mathbf{x}}\boldsymbol{u}_2\big) \\ &\leqslant C\|f_1 - f_2\|^2_{L^4(\Omega;L^2(\mathbb{S}^1))}\|\nabla_{\mathbf{x}}\boldsymbol{u}_2\|_{L^2(\Omega)} \\ &\leqslant C\big(\|\nabla_{\mathbf{x}} f_1 - \nabla_{\mathbf{x}} f_2\|_{L^2(\Omega\times\mathbb{S}^1)}\|f_1 - f_2\|_{L^2(\Omega\times\mathbb{S}^1)} + \|f_1 - f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}\big) \\ &\leqslant \frac{1}{8}\|\nabla_{\mathbf{x}} f_1 - \nabla_{\mathbf{x}} f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)} + C\|f_1 - f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}. \end{aligned} \tag{5.31}$$

Combining (5.26)–(5.31), we have for a.e. $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt}\|f_1 - f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)} \leqslant C\big(1 + \|f_1\|_{H^1(\Omega\times\mathbb{S}^1)} + \|f_1\|^{4/3}_{H^1(\Omega\times\mathbb{S}^1)} + \|f_1\|^2_{H^1(\Omega\times\mathbb{S}^1)}\big) \\ \times \|f_1 - f_2\|^2_{L^2(\Omega\times\mathbb{S}^1)}. \end{aligned} \tag{5.32}$$

Since (5.3) implies $\|f_1\|_{L^2(0,T;H^1(\Omega\times\mathbb{S}^1))} \leqslant C$ and (5.23) yields $(f_1 - f_2)|_{t=0} = 0$, it follows from Gronwall's inequality that $f_1 \equiv f_2$ a.e. on $[0, T] \times \Omega \times \mathbb{S}^1$ and hence from (5.24) and Poincaré inequality that $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$ on a.e. $[0, T] \times \Omega \times \mathbb{S}^1$. This ends the proof of uniqueness. □

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References

[1] H. Bae, K. Trivisa, On the Doi model for the suspensions of rod-like molecules: global-in-time existence, Commun. Math. Sci. (2012), in press.
[2] J.W. Barrett, E. Süli, Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains, Math. Models Methods Appl. Sci. 21 (6) (2011) 1211–1289.
[3] J.W. Barrett, E. Süli, Existence and equilibration of global weak solutions to kinetic models for dilute polymers II: Hookean-type bead-spring chains, Math. Models Methods Appl. Sci. 22 (5) (2012) 1150024.
[4] G.K. Batchelor, Slender-body theory for particles of arbitrary cross-section in Stokes flow, J. Fluid Mech. 44 (1970) 419–440.
[5] H. Brenner, Taylor dispersion in systems of sedimenting nonspherical Brownian particles, I: Homogeneous, centrosymmetric, axisymmetric particles, J. Colloid Interface Sci. 71 (1979) 189.
[6] H. Brenner, A general theory of Taylor dispersion phenomena, Phys. Chem. Hydrodynam. (PCH) 1 (1980) 91.
[7] M. Dreher, A. Jüngel, Compact families of piecewise constant functions in $L^p(0, T; B)$, Nonlinear Anal. 75 (2012) 3072–3077.
[8] P. Constantin, Nonlinear Fokker–Planck Navier–Stokes Systems, Commun. Math. Sci. 3 (2005) 531–544.
[9] P. Constantin, C. Fefferman, E.S. Titi, A. Zarnescu, Regularity of coupled two-dimensional nonlinear Fokker–Planck and Navier–Stokes systems, Comm. Math. Phys. 270 (2007) 789–811.
[10] P. Constantin, N. Masmoudi, Global well-posedness for a Smoluchowski equation coupled with Navier–Stokes equations in 2D, Comm. Math. Phys. 278 (2008) 179–191.
[11] P. Constantin, G. Seregin, Global Regularity of solutions of coupled Navier–Stokes equations and nonlinear Fokker–Planck equations, Discrete Contin. Dyn. Syst. 26 (2010) 1185–1196.
[12] M. Doi, Molecular-dynamics and rheological properties of concentrated solutions of rodlike polymers in isotropic and liquid-crystalline phases, J. Polym. Sci. Polym. Phys. Ed. 19 (1981) 229–243.
[13] M. Doi, S.F. Edwards, The Theory of Polymer Dynamics, Oxford Univ. Press, Oxford, 1986.

[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 2001.
[15] C. Helzel, F. Otto, Multiscale simulations for suspensions of rod-like molecules, J. Comput. Phys. 216 (1) (2006) 52–75.
[16] C. Helzel, F. Otto, A.E. Tzavaras, A kinetic model for the sedimentation of rod-like particles, preprint, 2011.
[17] C. Hohenegger, M.J. Shelley, Stability of active suspensions, Phys. Rev. E. 81 (2010) 046311.
[18] E.P. Hsu, Stochastic Analysis on Manifolds, Amer. Math. Soc., Providence, RI, 2002.
[19] G.B. Jeffery, The motion of ellipsoidal particles immersed in a viscous fluid, Proc. R. Soc. London, Ser. A. 102 (2010) 161.
[20] J. Kačur, Method of Rothe in evolution equations, Teubner-Texte Math., vol. 80, B.G. Teubner, Leipzig, 1985.
[21] M.J. Kim, K.S. Breuera, Enhanced diffusion due to motile bacteria, Phys. Fluids 16 (2004) L78–L81.
[22] S. Kim, S.J. Karrila, Microhydrodynamics: Principles and Selected Applications, Butterworth–Heinemann, Boston, 1991.
[23] E. Lauga, Enhanced diffusion by reciprocal swimming, Phys. Rev. Lett. 106 (2011) 178101.
[24] P.L. Lions, N. Masmoudi, Global solutions of weak solutions to some micro-macro models, C. R. Math. Acad. Sci. Paris 345 (2007) 15–20.
[25] F. Otto, A.E. Tzavaras, Continuity of velocity gradient in suspensions of rod-like molecules, Comm. Math. Phys. 277 (2008) 729–758.
[26] T.J. Pedley, J.O. Kessler, Hydrodynamic phenomena in suspensions of swimming micro-organisms, Annu. Rev. Fluid Mech. 24 (1992) 313–358.
[27] E. Rothe, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, Math. Ann. 102 (1930) 650–670.
[28] D. Saintillan, M.J. Shelley, Orientational order and instabilities in suspensions of self-locomoting rods, Phys. Rev. Lett. 99 (2007) 058102.
[29] D. Saintillan, M.J. Shelley, Instabilities, pattern formation, and mixing in active suspensions, Phys. Fluids 20 (2008) 123304.
[30] D. Saintillan, M.J. Shelley, Instabilities and pattern formation in active particle suspensions: Kinetic theory and continuum simulations, Phys. Rev. Lett. 100 (2008) 178103.
[31] R.A. Simha, S. Ramaswamy, Hydrodynamic fluctuations and instabilities in ordered suspensions of self-propelled particles, Phys. Rev. Lett. 89 (2002) 058101.
[32] Y.Z. Sun, Z.F. Zhang, Global well-posedness for the 2D micro-macro models in the bounded domain, Comm. Math. Phys. 303 (2011) 361–383.
[33] R. Teman, Navier–Stokes Equations. Theory and Numerical Analysis, third ed., North-Holland, Amsterdam, 1984.
[34] X.-L. Wu, A. Libchaber, Particle diffusion in a quasi-two-dimensional bacterial bath, Phys. Rev. Lett. 84 (2000) 3017–3020.