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MULTIDIMENSIONAL DEGENERATE KELLER–SEGEL SYSTEM WITH CRITICAL DIFFUSION EXPONENT $2n/(n+2)$*

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Abstract. This paper deals with a degenerate diffusion Patlak–Keller–Segel system in $n \geq 3$ dimension. The main difference between the current work and many other recent studies on the same model is that we study the diffusion exponent $m = 2n/(n+2)$, which is smaller than the usual exponent $m^* = 2 - 2/n$ used in other studies. With the exponent $m = 2n/(n+2)$, the associated free energy is conformal invariant, and there is a family of stationary solutions $U_{\lambda,x_0}(x) = C(\frac{\lambda}{\lambda^2+|x-x_0|^2})^{\frac{n+2}{2}}$ $\forall \lambda > 0$, $x_0 \in \mathbb{R}^n$. For radially symmetric solutions, we prove that if the initial data are strictly below $U_{\lambda,0}(x)$ for some λ, then the solution vanishes in L^1_{loc} as $t \to \infty$; if the initial data are strictly above $U_{\lambda,0}(x)$ for some λ, then the solution either blows up at a finite time or has a mass concentration at $r = 0$ as time goes to infinity. For general initial data, we prove that there is a global weak solution provided that the L^m norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow-up of the solution if the L^m norm for initial data is larger than the L^m norm of $U_{\lambda,x_0}(x)$, which is constant independent of λ and x_0, and the free energy of initial data is smaller than that of $U_{\lambda,x_0}(x)$.

Key words. chemotaxis, critical diffusion exponent, nonlocal aggregation, critical stationary solution, global existence, mass concentration, radially symmetric solution

AMS subject classifications. 35K65, 35B45, 35J20

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1. Introduction and preliminaries. In this paper, we study the Patlak–Keller–Segel model in $n \geq 3$ dimension with homogeneous degenerate diffusion:

$$\begin{cases} \rho_t = \Delta\rho^m - \operatorname{div}(\rho\nabla c), & x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the diffusion exponent is taken to be $m = \frac{2n}{n+2} \in (1,2)$. This model is widely used to describe the collective motion of cells. Here $\rho(x,t)$ represents the bacteria density, and $c(x,t)$ represents the chemical substance concentration.

We assume the initial data $\rho_0(x) \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, where L^1_+ denotes nonnegative integrable functions. In the second equation of (1.1), $c(x,t)$ is given by the fundamental solution,

$$c(x,t) = \frac{1}{(n-2)n\alpha(n)} \int_{\mathbb{R}^n} \frac{\rho(y,t)}{|x-y|^{n-2}} dy, \quad \alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \tag{1.2}$$

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where $\alpha(n)$ is the volume of n-dimension unit ball.

The first equation in (1.1) can also be written as

$$\rho_t = \Delta\rho^m + \rho^2 - \nabla c \cdot \nabla\rho.$$

Thus the classical solution ρ of (1.1) preserves nonnegativity if it is initially so. Hence we are lead to study nonnegative solutions

$$\rho(x,t) \geq 0, \qquad x \in \mathbb{R}^n,\ t \geq 0.$$

There is a naturally associated free energy for (1.1) which is given by

$$\mathcal{F}(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^n} \rho(x,t)c(x,t)dx. \tag{1.3}$$

By using the fundamental solution representation in (1.2), we are able to rewrite the free energy as

$$\mathcal{F}(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{1}{2(n-2)n\alpha(n)}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy. \tag{1.4}$$

The different signs in the above free energy represent the competition between diffusion and nonlocal aggregation. This is the key feature of this system.

There is a natural variational structure for (1.1). The first order variation of $\mathcal{F}$ gives the chemical potential

$$\mu = \frac{\delta\mathcal{F}}{\delta\rho} = \frac{m}{m-1}\rho^{m-1} - c. \tag{1.5}$$

By defining the drift velocity $v = -\nabla\mu$, the first equation in (1.1) can be rewritten as a continuity equation:

$$\rho_t + \mathrm{div}(\rho v) = 0 \tag{1.6}$$

or

$$\rho_t = \mathrm{div}\left(\rho\nabla\left(\frac{m}{m-1}\rho^{m-1} - c\right)\right). \tag{1.7}$$

Moreover, by taking inner product of $\frac{\delta\mathcal{F}}{\delta\rho}$ with (1.6), we get the energy-dissipation relation

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho|\nabla\mu|^2 dx = 0$$

or

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1} - c\right)\right|^2 dx = 0, \tag{1.8}$$

which leads to the fact that $\mathcal{F}(\rho(\cdot,t))$ is a monotone nonincreasing function of t.

Another important group of quantities is useful in our analysis. They are the ith moments of ρ, $i = 0, 1, 2$, defined by

$$m_0(t) = \int_{\mathbb{R}^n} \rho(x,t)dx, \quad m_1(t) = \int_{\mathbb{R}^n} x\rho(x,t)dx, \quad m_2(t) = \int_{\mathbb{R}^n} |x|^2\rho(x,t)dx.$$

By a direct computation, we have the following conservation relations for these moments.

PROPOSITION 1.1. *The following equations for the* 0,1,2*th moments hold:*

$$m_0'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} \rho(x,t)dx = 0, \tag{1.9}$$

$$m_1'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} x\rho(x,t)dx = 0, \tag{1.10}$$

$$m_2'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} |x|^2\rho(x,t)dx = -4\int_{\mathbb{R}^n} \rho^m(x,t)dx + 2(n-2)\mathcal{F}(\rho(\cdot,t)). \tag{1.11}$$

The identity (1.11) will be used to show a finite time blow-up behavior when the L^m norm of the initial data is larger than a critical value in section 3.

The most commonly used version of the Keller–Segel model is a system with linear diffusion in density, i.e., $m = 1$. In the case of two dimensions, the nonlocal aggregation comes from the logarithmic potential, which is exactly the fundamental solution of the Laplacian. There are many papers in the literature that discuss global existence and blow-up criteria including the multidimensional case, and both for parabolic-elliptic and parabolic-parabolic systems. We will not give a detailed review in this direction, but we refer the reader to the review paper [19] or Chapter 5 in [28]. A sharp bound on the critical mass, $m_c = 8\pi$, was given by Dolbeault and Perthame in [15] by using the logarithmic Hardy–Littlewood–Sobolev inequality. Critical mass means that if the initial mass is less than m_c, the solution will exist globally; otherwise there must be mass concentration. There is an interesting connection between the two-dimensional (2D) Keller–Segel system and the Navier–Stokes equations in vorticity-stream function formulation, where the Navier–Stokes equations can be obtained by replacing the ∇c drift velocity with $\nabla^\perp c$. This connection was explored and an alternative proof of global weak solution was given in [10] based on Delort's theory on the 2D incompressible Euler equation [14]. There are in-depth analyses for the case of critical mass $m_c = 8\pi$ in [2, 4].

In space dimension $n \geq 3$, some authors worked with the classical Keller–Segel model, i.e., linear diffusion, in which case the typical space is proved to be $L^{\frac{n}{2}}$ [28]. Global existence, finite time blow-up, and large time asymptotic behavior were studied, for example, in [7, 28, 34]. In multiple dimensions, there are also several modifications of the Keller–Segel model. A simple and direct way is to use logarithmic interaction kernel instead of the $1/|x|^{n-2}$ kernel from the Laplacian [9].

A more physical modification in multiple dimensions is an introduction of degenerate slow diffusion to balance the nonlocal aggregation, as was suggested by Hillen and Painter in [17, 18], to describe volume filling and quorum sensing in models. Many works on the degenerate diffusion, or quasilinear parabolic type, Keller–Segel system can be found by different groups of mathematicians in the last several years [3, 5, 8, 12, 13, 17, 18, 20, 21, 23, 26, 27, 30, 31, 32, 33, 34]. The general model of nonlinear diffusion and nonlinear advection can be written as

$$\rho_t = \operatorname{div}\left(A(\rho)\nabla\rho - N(\rho)\nabla c\right). \tag{1.12}$$

In particular, many researchers were devoted to the following special form:

$$\rho_t = \Delta\rho^m - \operatorname{div}(\rho^{q-1}\nabla c). \tag{1.13}$$

The existence of solution, finite time blow-up, and large time asymptotic behavior were extensively studied, either in the whole space $\mathbb{R}^n$ or in a smooth bounded domain with homogeneous Neumann boundary conditions, either in a parabolic-parabolic form or in a parabolic-elliptic form. A critical diffusion exponent played a key role in their analysis: $m^* = q - 2/n$. Equation (1.13) can be recast as

$$\rho_t = \Delta\rho^m + \rho^q - (q-1)\rho^{q-2}\nabla\rho\cdot\nabla c. \tag{1.14}$$

For singularity of type $\rho(x) = |x|^{-\lambda}$, the last two terms in the above equation are of the same order $|x|^{-q\lambda}$. Hence $q = m+2/n$ is analogous to the Fujita exponent [16]. In the case $q = 2$, the exponent is $m^* = 2 - 2/n$. In 2005, Horstmann and Winkler [20] studied the case $m = 1$, and they found a critical exponent to be $m = 1 > q - 2/n$. When $q < 1 + 2/n$, they proved the existence of global solutions for large initial data. When $q > 1 + 2/n$, there are some cases such that the problem has an unbounded solution. In 2006, Sugiyama [31] and Sugiyama and Kunii [33] studied the cases $q = 2$ and $q \geq 2$. They proved that if $m > m^*$ (this case was referred to as subcritical), then diffusion dominates the system and there is a global solution for arbitrary $L^1 \cap L^\infty$ initial data; if $m < m^*$ (this case was referred to as supercritical), then aggregation dominates the system and there is a finite time blow-up of the solution for some large initial data; if $1 \leq m \leq q - 2/n$, then there is global existence of the solution with a decay property for small data. Luckhaus and Sugiyama [26] showed that with small initial data, the globally existing solution in the long term behaves like the Barenblatt solution of the porous media solution in the case $m < q - 2/n$. In 2006, Senba and Suzuki [30] proved that if the diffusion has a positive coefficient $A(\rho)$ which increases faster than ρ^{m-1} $(m \geq 2 - 2/n)$, then there is global existence of classical solution, with $C^{2,\alpha}$-type regularity. In 2008, Kowalczyk and Szymańska [23] obtained the global existence of the nonnegative weak solution in the case of $A(\rho) > C\rho^{m-1}$, $m > 3 - 4/n$ for $n \geq 2$. Note that $3 - 4/n \geq m^* = 2 - 2/n$. In 2009, Cieślak and Laurencot [12] showed that when $A(\rho) \geq C(1+\rho)^{m-1}$, $m > 2-2/n$, there is a global classic solution for any L^∞ initial data. In 2009, Blanchet, Carrillo, and Laurencot [3] studied the case $m = m^*$ and showed that there is a critical mass M_c such that if initial mass $m_0 < M_c$ and in addition $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$, then a global weak solution exists and satisfies an energy-dissipation inequality. They also proved that if $m_0 > M_c$, $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$, and the free energy is negative initially, then there is a finite time blow-up for the solution in $L^m(\mathbb{R}^n)$. For the critical mass $m_0 = M_c$ case, they discussed the large time behavior of the solution. In 2010, Sugiyama obtained some partial regularity results for the solution in the critical case $m = q-2/n$ in [32]. In 2012, Ishida and Yokota [21] established the global existence of the weak solution with large initial data when $m > q - 2/n$ for parabolic-parabolic type. Recently Bedrossian, Rodríguez, and Bertozzi [1] studied the corresponding critical exponent for general interaction potentials.

According to the above relatively complete discussions about the nonlinear diffusion in the Keller–Segel system, it seems that not much has been left for future study. The main reason to choose exponent $m^* = 2 - 2/n$ is that under the mass-invariant scaling $\rho_\lambda(x,t) = \lambda^n\rho(\lambda x, t)$ for the system (1.1), there is a balance between diffusion and potential drift. However, in our current work, we will try to understand more about the exponents in (1.1) from a different point of view. There are many reasons for us to take the diffusion exponent $m = \frac{2n}{n+2}$, which is smaller than $m^* = 2 - 2/n$ when $n \geq 3$, as we will discuss below.

We first show that there is a family of positive stationary solutions to (1.1). In fact, by taking $\rho = (\frac{m-1}{m}c)^{1/(m-1)}$ in (1.7) and plugging it into (1.1), we obtain the following equation:

$$-\Delta c = \left(\frac{m-1}{m}\right)^p c^p, \quad x \in \mathbb{R}^n, \quad p = \frac{1}{m-1}. \tag{1.15}$$

Solutions to the above equation are stationary solutions of (1.1). Indeed, from the energy-dissipation relation (1.8), c is given by (1.2), and ρ decay at infinity, and one knows that positive stationary solutions are given by the above equation. However, some compact support solutions do not satisfy this equation.

It is a well-known result [6, 11] that (1.15) has critical exponent $p_c = \frac{n+2}{n-2}$, or equivalently $m = \frac{2n}{n+2}$. Whenever $p < p_c$, or equivalently $m > \frac{2n}{n+2}$ (for example, $m^* = 2 - 2/n > \frac{2n}{n+2}$), the only nonnegative solution of (1.15) is 0. At $p = p_c$, all positive solutions of (1.15) must be of the form

$$C_{\lambda,x_0}(x) = \frac{2^{\frac{n+2}{4}} n^{\frac{n}{2}}}{n-2} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}} \qquad \text{for some } \lambda > 0,\ x_0 \in \mathbb{R}^n. \tag{1.16}$$

The corresponding stationary solutions $\rho(x)$ of (1.1) are given by

$$U_{\lambda,x_0}(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} C_{\lambda,x_0}^{\frac{1}{m-1}}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n+2}{2}}. \tag{1.17}$$

Remark 1.1. For 2D $n = 2$, one has the following (see [6, 11]).

1. $m = m^* = 1$: The system becomes the Keller–Segel system with linear diffusion.
2. The Lane–Emden equation (1.15) is replaced by

$$-\Delta c = e^c,$$

and the corresponding positive solutions (1.17) and (1.16) are replaced by

$$U_{\lambda,x_0}(x) = 8\left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^2, \quad C_{\lambda,x_0}(x) = \log U_{\lambda,x_0}(x), \quad \lambda > 0,\ x_0 \in \mathbb{R}^2.$$

3. The Hardy–Littlewood–Sobolev inequality used below will be replaced by the logarithmic Hardy–Littlewood–Sobolev inequality. The equality holds if and only if $\rho = AU_{\lambda,x_0}(x)$ for some positive constants A, $\lambda > 0$, and $x_0 \in \mathbb{R}^2$.

The function $U_{\lambda,x_0}(x)$ in (1.17) is known as the Lane–Emden function in astrophysics, which has infinite second moment for $n \geq 2$. The value $\|U_{\lambda,x_0}\|_{L^m}$ is independent of λ and x_0 and is given by (1.23). In particular, when $n = 2$, this value is 8π. Using the conservation laws (1.9)–(1.10), one can uniquely determine the parameters λ and x_0 in the stationary solution, and we state the result in the following proposition.

Proposition 1.2. *If $U_{\lambda,x_0}(x) = \lim_{t\to\infty} \rho(x,t)$, then the parameters $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are uniquely determined by m_0 and m_1 in the following relations:*

$$x_0 = m_1/m_0, \quad \lambda^{\frac{n-2}{2}} 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \alpha(n) = m_0.$$

Now we discuss connections among $U_{\lambda,x_0}(x)$, free energy, and the Hardy–Littlewood–Sobolev inequality. From (1.5) we have that $\frac{\delta \mathcal{F}}{\delta \rho}(U_{\lambda,x_0}(x)) = 0$. In other words, $U_{\lambda,x_0}(x)$ is also a family of critical points to $\mathcal{F}(\rho)$. Moreover, the stationary solutions

$U_{\lambda,x_0}(x)$ reach equality in the Hardy–Littlewood–Sobolev inequality. A special version of the Hardy–Littlewood–Sobolev inequality [25] is given by the following lemma.

LEMMA 1.1. *Let $\rho \in L^m(\mathbb{R}^n)$; then*

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy \le C(n)\|\rho\|^2_{L^m}, \tag{1.18}$$

where

$$C(n) = \pi^{(n-2)/2}\frac{1}{\Gamma(n/2+1)}\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-2/n}. \tag{1.19}$$

Moreover, the equality holds if and only if $\rho(x) = AU_{\lambda,x_0}(x)$ for some constant A and parameters $\lambda > 0$, $x_0 \in \mathbb{R}^n$.

Consequently, we have the following decomposition of the free energy:

$$\begin{aligned}\mathcal{F}(\rho) \;=\; &\frac{1}{m-1}\|\rho\|^m_{L^m}\left(1-\frac{(m-1)c_nC(n)}{2}\|\rho\|^{4/(n+2)}_{L^m}\right)\\ &+\frac{c_n}{2}\left(C(n)\|\rho\|^2_{L^m}-\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right)\\ =:\;& \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho),\end{aligned} \tag{1.20}$$

where $c_n = 1/(n(n-2)\alpha(n))$. Since $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$, it is also a critical point for $\mathcal{F}_1(\rho)$. Indeed, we will show that it is a maximum point for $\mathcal{F}_1(\rho)$. This property will be used in the proof of a finite time blow-up discussion in section 3.

All the facts we listed above are reflected by the following proposition; i.e., with diffusion exponent $m = \frac{2n}{n+2}$, the free energy $\mathcal{F}(\rho)$ is invariant under translations, similarities, orthogonal transformations, and inversions (Kelvin transformations).

PROPOSITION 1.3. *The following facts hold:*

(1) $\mathcal{F}(\rho_{\bar{x}}) = \mathcal{F}(\rho)$ *with* $\rho_{\bar{x}}(x) := \rho(x+\bar{x})$ $\forall \bar{x} \in \mathbb{R}^n$.

(2) $\mathcal{F}(\rho_\lambda) = \mathcal{F}(\rho)$ *with* $\rho_\lambda(x) := \lambda^{\frac{n+2}{2}}\rho(\lambda x)$ $\forall \lambda > 0$.

(3) $\mathcal{F}(\rho_\mathcal{R}) = \mathcal{F}(\rho)$ *with* $\rho_\mathcal{R}(x) := \rho(\mathcal{R}^{-1}x)$ $\forall$ $\mathcal{R}^*\mathcal{R} = I$.

(4) $\mathcal{F}(\rho_{\bar{x},\lambda}) = \mathcal{F}(\rho)$ *with* $\rho_{\bar{x},\lambda}(x) := (\frac{\lambda}{|x-\bar{x}|})^{n+2}\rho(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2})$ $\forall \bar{x} \in \mathbb{R}^n$, $\lambda > 0$.

We will give a proof of this proposition in the appendix.

Remark 1.2. By Liouville's theorem [24], any smooth conformal mapping on a domain of $\mathbb{R}^n$, $n > 2$, can be expressed as a composition of translations, similarities, orthogonal transformations, and Kelvin transformations. These transformations are all Möbius transformations.

Remark 1.3. The last transformation in Proposition 1.3 is motivated by the Kelvin transformation for c, i.e.,

$$c_{\bar{x},\lambda}(x) = \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n-2} c\left(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right).$$

Remark 1.4. Invariants of the translations and similarities in Proposition 1.3 allow that parameters x_0 and λ in $U_{\lambda,x_0}(x)$ be free. Invariants for free energy on the orthogonal and Kelvin transformations guarantee that the profile of the stationary solution is unique.

Moreover, using (1.3) with

$$C_{\lambda,x_0}(x) = \frac{m}{m-1}U^{m-1}_{\lambda,x_0}(x) = \frac{2n}{n-1}U^{m-1}_{\lambda,x_0}(x), \tag{1.21}$$

we get

$$\mathcal{F}(U_{\lambda,x_0}(x)) = \frac{2}{n-2}\|U_{\lambda,x_0}(x)\|^m_{L^m}, \tag{1.22}$$

while we can calculate the right-hand side explicitly,

$$\|U_{\lambda,x_0}(x)\|^m_{L^m} = n^n \pi^{\frac{n+1}{2}} 2^{1-\frac{n}{2}} \frac{1}{\Gamma(\frac{n+1}{2})}. \tag{1.23}$$

This is exactly the fact that the L^m norm of $U_{\lambda,x_0}(x)$ is a constant independent of λ and x_0. And this quantity will play an important role in our discussion of global existence and finite time blow-up in general initial data.

There is also an interesting connection between (1.1) and the Euler–Poisson equations for gaseous stars. In three dimensions, $m^* = 4/3$ and $m = 6/5$. Both exponents also appear to be some kind of critical adiabatic exponent in the Euler–Poisson equations in terms of stability and instability of spherically symmetric steady states [29, 22].

This paper is arranged as follows. In section 2, we prove that, for radially symmetric solutions, if the initial data are strictly below $U_{\lambda,0}(x)$ for some λ, then the solution vanishes in L^1_{loc} as $t \to \infty$; if the initial data are strictly above $U_{\lambda,0}(x)$ for some λ, then the solution either blows up at a finite time or has a mass concentration at $r = 0$ as $t \to \infty$.

In section 3, we prove that there is a global weak solution provided that the L^m norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow-up of the solution if the L^m norm for initial data is larger than that of $U_{\lambda,x_0}(x)$ in (1.23) and the free energy of initial data is smaller than that of $U_{\lambda,x_0}(x)$ in (1.23).

2. Decay and blow-up for radially symmetric solution. In this section, we will study large time behavior of the radially symmetric solution of (1.1). Radially symmetric solutions $(\rho(t,r),\ c(t,r))$ of the system (1.1) satisfy

$$\begin{cases} (r^{n-1}\rho)_t = \big(r^{n-1}(\rho^m)'\big)' - (r^{n-1}\rho c')', & r \in (0,\infty),\ t \geq 0, \\ -(r^{n-1}c')' = r^{n-1}\rho, & r \in (0,\infty),\ t \geq 0, \\ \rho'(t, r=0) = 0, & t \geq 0, \\ \rho(t=0,r) = \rho_0(r), & r \in (0,\infty), \end{cases} \tag{2.1}$$

where $'$ stands for the derivative with respect to r. We will show that the stationary solution $U_{\lambda,0}(x)$ ($x_0 = 0$ in (1.17)) is a critical profile in the following sense: if the initial datum ρ_0 is strictly below a stationary solution for some λ, then all radially symmetric solutions are vanishing in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$; if the initial datum ρ_0 is strictly above a stationary solution for some λ, then all radially symmetric solutions either have a finite time blow-up or have a mass concentration at the $x = 0$ point as $t \to \infty$. For simplicity, we will use the notation $U_\lambda(|x|) = U_{\lambda,0}(x)$ in this section. The following theorem is our main result in this section.

THEOREM 2.1. *Assume that the initial datum $\rho_0 \geq 0$ is radially symmetric.*

(1) *If* $\exists\ \lambda_0 > 0$ *s.t.*

$$\rho_0(r) < U_{\lambda_0}(r), \quad r > 0,$$

then any radially symmetric solution $\rho(r,t)$ *of* (1.1) *is vanishing in* $L^1_{loc}(\mathbb{R}^n)$ *as* $t \to \infty$.

(2) *If* $\exists\ \lambda_0 > 0$ *s.t.*

$$\rho_0(r) > U_{\lambda_0}(r), \quad r > 0,$$

then any radially symmetric solution $\rho(r,t)$ *of* (1.1) *must blow up at a finite time* t^* *or has a mass concentration at* $r = 0$ *as time goes to infinity in the sense that there are* $r(t) \to 0$ *as* $t \to \infty$ *and a positive constant* C *such that*

$$\int_{B(0,r(t))} \rho dx \geq C.$$

Inspired by a similar result in the 2D case [28], we work on the following weighted primitive variable (integral of density ρ in the ball with radius r and center at origin):

$$M(t,r) := n\alpha(n) \int_0^r \sigma^{n-1}\rho(t,\sigma)d\sigma. \tag{2.2}$$

By the second equation in (2.1), one has

$$M(t,r) = -n\alpha(n)r^{n-1}c'.$$

Thus (2.1) can be reduced to a single equation for $M(t,r)$. By integrating (2.1), we have

$$\begin{cases} M_t = n\alpha(n)r^{n-1}\left[\left(\frac{M'}{n\alpha(n)r^{n-1}}\right)^m\right]' + \frac{M'M}{n\alpha(n)r^{n-1}}, & r \in (0,\infty),\ t \geq 0, \\ M(t,0) = 0, M(t,\infty) = m_0, & t \geq 0, \\ M(0,r) = n\alpha(n)\int_0^r \sigma^{n-1}\rho_0(\sigma)d\sigma, & r \in (0,\infty). \end{cases} \tag{2.3}$$

From (2.2), we have

$$M' = n\alpha(n)r^{n-1}\rho \quad \forall r \in (0,\infty),\ t \geq 0.$$

Thus $M'(t,r) \geq 0$; i.e., $M(t,r)$ is an increasing function in r.

The main advantage of using (2.3) instead of (2.1) is that we can use the comparison principle by constructing a supersolution for decay estimates and constructing a subsolution for mass concentration estimates.

The stationary problem of (2.3) is reduced to

$$\begin{aligned} &n\alpha(n)r^{n-1}\left[\left(\frac{M'}{n\alpha(n)r^{n-1}}\right)^m\right]' + \frac{M'M}{n\alpha(n)r^{n-1}} = 0, \quad r \in (0,\infty), \\ &M(0) = 0, \quad M(\infty) = m_0. \end{aligned} \tag{2.4}$$

Recall (1.16) and (1.17); a family of stationary solutions to (1.1) is given by

$$(U_\lambda(r), C_\lambda(r)) = \left(2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{\frac{n+2}{2}},\ \ 2^{\frac{n+2}{4}} n^{\frac{n}{2}} (n-2)^{-1} \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{\frac{n-2}{2}}\right),$$

where $\lambda > 0$ is a free parameter. Hence, the corresponding family of explicit solutions to (2.4) is given by

$$\tilde{M}_\lambda(r) = n\alpha(n)\int_0^r \sigma^{n-1}U_\lambda(\sigma)d\sigma = K_\lambda(n)\frac{1}{(1+\lambda^2 r^{-2})^{\frac{n}{2}}} \tag{2.5}$$

with $K_\lambda(n) = \alpha(n)2^{\frac{n+2}{4}}n^{\frac{n+2}{2}}\lambda^{\frac{n-2}{2}}$.

Proof of Theorem 2.1. In the following two subsections we will prove (1) and (2) of the theorem in the form of Lemmas 2.1 and 2.2, respectively. ∎

2.1. Supersolution with subcritical initial data. In this subsection, we will show that the solutions of the problem (2.3) vanish as $t \to \infty$ for any finite space interval, when initial data are dominated by a stationary solution $\tilde{M}_{\lambda_0}(r)$ in (2.5) for some $\lambda_0 > 0$. More precisely, we have the following lemma.

LEMMA 2.1. *For $n \geq 3$, assume that*

$$m_0 = M(t,\infty) < K_{\lambda_0}(n), \quad M(0,r) < \tilde{M}_{\lambda_0}(r)\ \forall r > 0,$$

for some $\lambda_0 > 0$. Then the solutions of (2.3) *diminish in time in the following sense:*

$$M(t,r) \to 0 \ \text{ as } \ t\to\infty \ \text{ uniformly in any interval } 0 \leq r \leq R.$$

Thus $\rho(t,x)$ in (1.1) *vanishes in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$.*

Proof. Due to the facts that $M(0,r)$ and $\tilde{M}_{\lambda_0}(r)$ are bounded nondecreasing functions and $M(0,r) < \tilde{M}_{\lambda_0}(r)$, there exist a $\mu \in (0,1)$ s.t. $M(0,r) \leq \mu\tilde{M}_{\lambda_0}(r)$.

Noticing that $M(t,\infty) = m_0 < K_{\lambda_0}(n)$, without loss of generality, we can choose the same μ such that $M(t,\infty) = m_0 < \mu K_{\lambda_0}(n)$.

We construct a supersolution of (2.3) by modifying constant λ in the denominator of its stationary solution (2.5). By taking $\lambda = \lambda(t) = (A_1 t + \lambda_0^n)^{1/n}$, for some $A_1 > 0$ to be determined, and cutting $\mu\tilde{M}_\lambda(t)$ by a constant m_0 when $r \geq R(t)$, we can construct a supersolution $\bar{N}(t,r)$ in the following:

$$\bar{N}(t,r) := \min\left\{m_0, \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}\right\},$$

where the cut-off location $R(t)$ is given by

$$m_0 = \frac{\mu K_{\lambda_0}(n)}{\left(1+\lambda^2(t)R^{-2}(t)\right)^{n/2}}, \tag{2.6}$$

i.e.,

$$\left(1+\lambda^2(t)R^{-2}(t)\right)^{n/2} = \frac{\mu K_{\lambda_0}(n)}{m_0},$$

or

$$R(t) = \left(\frac{\lambda^2(t)}{\left(\frac{\mu K_{\lambda_0}(n)}{m_0}\right)^{2/n} - 1}\right)^{1/2}.$$

Hence $\bar{N}(t,r)$ is continuous and $\bar{N}(t,r) = m_0$ for $r > R(t)$, and $\bar{N}(t,r) = \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}$ for $r \leq R(t)$.

Now we prove that $\bar{N}(t,r)$ is a supersolution to (2.3). Obviously, the constant state m_0 is also a supersolution. Next we need only show that $\bar{N}(t,r) = \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}$ is a supersolution for $r \le R(t)$, i.e., prove that

$$LHS := \bar{N}_t - n\alpha(n)r^{n-1}\left[\left(\frac{\bar{N}'}{n\alpha(n)r^{n-1}}\right)^m\right]' - \frac{\bar{N}'\bar{N}}{n\alpha(n)r^{n-1}} \ge 0. \tag{2.7}$$

A direct calculation of (2.7) term by term gives that

$$\bar{N}_t = \bar{N}\frac{-n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\lambda'(t),$$
$$n\alpha(n)r^{n-1}\left[\left(\frac{\bar{N}'}{n\alpha(n)r^{n-1}}\right)^m\right]' = -2\bar{N}n^2\alpha(n)^{1-m}\lambda^{2m}(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}r^{-n-2}}{(1+\lambda^2(t)r^{-2})^{\frac{n}{2}+1}},$$
$$\frac{\bar{N}'\bar{N}}{n\alpha(n)r^{n-1}} = \mu K_{\lambda_0}(n)\bar{N}\frac{\lambda^2(t)r^{-n-2}}{\alpha(n)\left(1+\lambda^2(t)r^{-2}\right)^{\frac{n}{2}+1}}.$$

So, we have

$$\begin{aligned} LHS &= \bar{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Bigg(-\lambda'(t) + 2n\alpha(n)^{1-m}\lambda^{2m-1}(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}r^{-n}}{(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}} \\ &\qquad -\mu K_{\lambda_0}(n)\frac{\lambda(t)r^{-n}}{n\alpha(n)\left(1+\lambda^2(t)r^{-2}\right)^{\frac{n}{2}}}\Bigg) \\ &= \bar{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t) + A(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\right), \end{aligned} \tag{2.8}$$

where

$$A(t) := 2n^2\alpha(n)^{2-m}\lambda^{2m-2}(t) - (\mu K_{\lambda_0}(n))^{2-m}.$$

Moreover, (2.8) can be simplified as

$$LHS = \bar{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + A(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)(r^2+\lambda^2(t))^{\frac{n}{2}}}\right). \tag{2.9}$$

By using the expressions $m = \frac{2n}{n+2}$, $K_{\lambda_0}(n) = \alpha(n)2^{\frac{n+2}{4}}n^{\frac{n+2}{2}}\lambda_0^{\frac{n-2}{2}}$, and $\lambda(t) \ge \lambda_0$, we have

$$A(t) \ge 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1-\mu^{2-m}) =: A_0 > 0. \tag{2.10}$$

Therefore, (2.9) and (2.10) together imply that

$$\begin{aligned} LHS &\ge \bar{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + A_0\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)(r^2+\lambda^2(t))^{\frac{n}{2}}}\right) \\ &\ge \bar{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + A_0\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)\left(R^2(t)+\lambda^2(t)\right)^{\frac{n}{2}}}\right) \\ &= \bar{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + A_1 n^{-1}\lambda^{-n+1}(t)\right), \end{aligned} \tag{2.11}$$

where we have used the notation

$$A_1 := A_0 \frac{(\mu K_{\lambda_0}(n))^{m-1} \lambda^n(t)}{\alpha(n)\left(R^2(t) + \lambda^2(t)\right)^{\frac{n}{2}}}.$$

From (2.6) and simple computation, one has that A_1 is a positive constant, i.e.,

$$A_1 = A_0(\mu K_{\lambda_0}(n))^{m-2} m_0 \alpha(n)^{-1} \left[\left(\frac{\mu K_{\lambda_0}(n)}{m_0} \right)^{\frac{2}{n}} - 1 \right]^{n/2} > 0.$$

Now we can choose $\lambda(t)$ such that the right-hand side of (2.11) is 0, and it follows that

$$LHS \geq 0.$$

The choice of $\lambda(t)$ must satisfy

$$\lambda'(t) = A_1 n^{-1} \lambda^{-n+1}(t) \text{ and } \lambda(0) = \lambda_0.$$

It is easy to see that $\lambda(t)$ is given by

$$\lambda(t) = (A_1 t + \lambda_0^n)^{1/n}, \quad t \geq 0. \tag{2.12}$$

Furthermore, $M(0, r) \leq \tilde{M}_{\lambda_0}(r)$, and

$$M(t, 0) \leq \lim_{r \to 0} \bar{N}(t, r) = \lim_{r \to 0} \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda^2(t) r^{-2})^{n/2}} = 0;$$

thus $\frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}$ is a supersolution of (2.3). The minimum of two supersolutions is also a supersolution; i.e., $\bar{N}(t, r)$ is a supersolution to (2.3).

By the comparison principle, we deduce that the solution of (2.3) satisfies $M(t, r) \leq \bar{N}(t, r)$ in $[0, \infty) \times [0, \infty)$. By (2.6) and (2.12), we have that

$$\lambda(t) \to \infty \text{ and } R(t) \to \infty \quad \text{ as } t \to \infty.$$

Therefore, for a given interval $r \in (0, R_0)$, it holds that

$$M(t, r) \leq \frac{\mu K_{\lambda_0}(n)}{(1 + \lambda^2(t) R_0^{-2})^{n/2}} \to 0 \quad \text{as} \ \ t \to \infty.$$

This completes the proof of Lemma 2.1. $\square$

2.2. Blow-up with supercritical initial data. In this subsection, we will prove that if the initial data are above a stationary solution $\tilde{M}_{\lambda_0}(r)$ in (2.5) for some $\lambda_0 > 0$ and there is no finite time blow-up in solution, then radially symmetric solutions must have mass concentration at $x = 0$ as time goes to infinity.

LEMMA 2.2. *For dimension $n \geq 3$, assume that*

$$m_0 = M(t, \infty) > K_{\lambda_0}(n), \qquad M(0, r) > \tilde{M}_{\lambda_0}(r) \quad \forall r > 0,$$

for some $\lambda_0 > 0$ and there is no finite time blow-up. Then there are a positive constant $C > 0$ and a function $r(t) \to 0$ as $t \to \infty$ such that all solutions $M(t, r)$ satisfy

$$M(t, r(t)) \geq C.$$

Or, equivalently, radially symmetric solutions ρ *to* (1.1) *have mass concentration at* $x = 0$ *as* $t \to \infty$, *i.e.,*

$$\int_{B(0,r(t))} \rho dx \geq C.$$

Proof. We assume there is no finite time blow-up. Hence the solution exists globally. We shall show that this global solution must have a mass concentration at $x = 0$ for large enough t. In other words, we will show that there exist a positive constant C and a radius function $r(t) > 0$ such that as $t \to \infty$, we have $r(t) \to 0$ and

$$M(t, r(t)) \geq C > 0, \tag{2.13}$$

i.e.,

$$\int_{B(0,r(t))} \rho dx \geq C > 0.$$

Similar to the discussion in the beginning of the proof of Lemma 2.1, we can choose $\mu_0 > 1$ such that $\mu_0 K_{\lambda_0}(n) < m_0 = M(t, \infty)$ and $M(0, r) > \mu_0 \tilde{M}_{\lambda_0}(r) \forall r > 0$. We construct a subsolution $\frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda(t)^2 r^{-2})^{n/2}}$ of (2.3) from the stationary solution $\tilde{M}_\lambda(r)$ in (2.5) by taking $\lambda = \lambda(t) = \lambda_0 e^{B_1 t}$ for some $B_1 < 0$. Similar to the construction of the supersolution in the previous subsection, we cut it off by $\frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}$ for $r \geq R(t)$ for some $R(t)$ which will be specified later. We take the subsolution in the following form:

$$\underline{N}(t, r) := \max \left\{ \frac{m_0}{(1 + \lambda_0^2 r^{-2})^{n/2}}, \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda(t)^2 r^{-2})^{n/2}} \right\}. \tag{2.14}$$

It can be shown that both terms on the right-hand side of (2.14) are subsolutions. For the first term, we have

$$\begin{aligned} &\underline{N}_t - n\alpha(n) r^{n-1} \left[\left(\frac{\underline{N}'}{n\alpha(n) r^{n-1}} \right)^m \right]' - \frac{\underline{N}' \underline{N}}{n\alpha(n) r^{n-1}} \\ &= \left[2\alpha(n)^{2-m} n^2 \lambda_0^{2m} - (m_0)^{2-m} \lambda_0^2 \right] \frac{n(m_0)^m r^{-n-2}}{n\alpha(n)(1 + \lambda_0^2 r^{-2})^{n+1}} \\ &\leq \left[2\alpha(n)^{2-m} n^2 \lambda_0^{2m} - K_{\lambda_0}(n)^{2-m} \lambda_0^2 \right] \frac{n(m_0)^m r^{-n-2}}{n\alpha(n)(1 + \lambda_0^2 r^{-2})^{n+1}} \\ &= 0. \end{aligned} \tag{2.15}$$

Together with the boundary conditions

$$\underline{N}(t, 0) = \lim_{r \to 0} \frac{m_0}{(1 + \lambda_0^2 r^{-2})^{n/2}} = 0,$$
$$\underline{N}(t, \infty) = \lim_{r \to \infty} \frac{m_0}{(1 + \lambda_0^2 r^{-2})^{n/2}} \leq m_0 = M(t, \infty)$$

and initial condition $\underline{N}(0, r) \leq M(0, r)$, we have that $\frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}$ is a subsolution.

Next we show that the second term $\frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t) r^{-2})^{n/2}}$ is also a subsolution in the interval $0 \leq r \leq R(t)$ where $\underline{N}$ achieves its maximum by $\frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t) r^{-2})^{n/2}}$. The radius $R(t)$ is defined by the following equality:

$$\frac{m_0}{(1 + \lambda_0^2 R^{-2}(t))^{n/2}} = \frac{\mu_0 K_{\lambda_0}(n)}{(1 + \lambda^2(t) R^{-2}(t))^{n/2}}. \tag{2.16}$$

Notice that there exists constant R_0 : $r \le R(t) \le R_0$ such that $\frac{m_0}{(1+\lambda_0^2 R_0^{-2})^{n/2}} = \mu_0 K_{\lambda_0}(n)$. Then a simple computation gives

$$LHS := \underline{N}_t - n\alpha(n)r^{n-1}\left[\left(\frac{\underline{N}'}{n\alpha(n)r^{n-1}}\right)^m\right]' - \frac{\underline{N}'\underline{N}}{n\alpha(n)r^{n-1}}$$

$$= \underline{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t) + B(t)\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\right), \tag{2.17}$$

where

$$B(t) := 2n^2\alpha(n)^{2-m}\lambda^{2m-2}(t) - (\mu_0 K_{\lambda_0}(n))^{2-m}$$

$$\le 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1-\mu_0^{2-m}) =: B_0 < 0. \tag{2.18}$$

By (2.17) and (2.18), we have

$$LHS \le \underline{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t) + B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\right)$$

$$\le \underline{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}\lambda(t)}{n\alpha(n)\left(R^2(t)+\lambda^2(t)\right)^{\frac{n}{2}}}\right)$$

$$= \underline{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + B_2(t)\lambda(t)\right), \tag{2.19}$$

where we have used the notation

$$B_2(t) := B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}}{n\alpha(n)\left(R^2(t)+\lambda^2(t)\right)^{\frac{n}{2}}}.$$

From (2.16) and simple computations, we have

$$B_2(t) \le B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}}{n\alpha(n)\left(R_0^2+\lambda_0^2\right)^{\frac{n}{2}}}$$

$$\le B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m}}{n\alpha(n)m_0 R_0^n}$$

$$= B_0(\mu_0 K_{\lambda_0}(n))^m(n\alpha(n)m_0)^{-1}R_0^{-n} < 0.$$

Denote the constant term above as B_1; (2.19) leads to

$$LHS \le \underline{N}\frac{n\lambda(t)}{r^2+\lambda^2(t)}\left(-\lambda'(t) + B_1\lambda(t)\right).$$

Since B_1 is a negative constant, we can set $\lambda'(t) = B_1\lambda(t)$. This gives the solution $\lambda(t) = \lambda_0 e^{B_1 t}$, and this is our choice of $\lambda(t)$. With this choice of $\lambda(t)$ we have shown that

$$LHS \le 0.$$

Now $\forall t > 0$, the comparison principle shows that

$$M(t,r) \ge \underline{N}(t,r) = \frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}} = \frac{K_{\lambda_0}(n)}{[1+(\lambda_0 e^{B_1 t})^2 r^{-2}]^{n/2}}.$$

We evaluate the above inequality at $r = \lambda_0 e^{B_1 t}$:

$$M(t,r(t)) \ge \underline{N}(t,r(t)) = \frac{K_{\lambda_0}(n)}{2^{n/2}} > 0.$$

This completes the proof of the lemma. ∎

3. Existence and blow-up with general initial data. We will discuss the existence and blow-up of the solution with more general initial data, not limited to the radially symmetric case. The main tools in this part are the entropy inequality and second moment estimates. For simplicity, we will use the notation L^p to represent $L^p(\mathbb{R}^n)$.

3.1. Global existence. In this subsection, we will prove the following theorem on global existence of weak solution of (1.1) if the initial data satisfy

$$\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s := \left(\frac{4m^2}{(2m-1)^2 C_{GNS}}\right)^{\frac{1}{2-m}}, \tag{3.1}$$

where C_{GNS} is the universal constant appearing in the Gagliardo–Nirenberg–Sobolev inequality (see (A.1) below).

THEOREM 3.1. *For initial data $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^n)$ and $\|\rho_0\|_{L^m(\mathbb{R}^n)} < C_s$, there is a global weak solution (ρ, c) to (1.1) with regularity*

$$\begin{aligned} &\rho \in L^\infty\left(0,T;\, L^1_+ \cap L^m(\mathbb{R}^n)\right) \cap L^{m+1}\left(0,T;\, L^{m+1}(\mathbb{R}^n)\right), \\ &\nabla\rho \in L^r\left(0,T;\, L^r(\mathbb{R}^n)\right), \quad r = \min\left(2, \frac{3n+2}{n+4}\right), \\ &c \in L^\infty\left(0,T;\, L^s(\mathbb{R}^n)\right), \quad \frac{n}{n-2} < s \le \frac{2n}{n-2}, \\ &\nabla c \in L^\infty\left(0,T;\, L^2(\mathbb{R}^n)\right). \end{aligned} \tag{3.2}$$

Moreover, $\|\rho(\cdot,t)\|_{L^m(\mathbb{R}^n)}$ decays algebraically in time,

$$\|\rho(\cdot,t)\|_{L^m(\mathbb{R}^n)} \le C t^{-\frac{1}{m(\beta-1)}} \qquad \text{for large } t, \tag{3.3}$$

where $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Proof. We split the proof into six steps.

Step 1. We start with the regularized problem for $\varepsilon > 0$,

$$\begin{cases} \partial_t \rho_\varepsilon = \Delta\rho_\varepsilon^m + \varepsilon\Delta\rho_\varepsilon - \operatorname{div}(\rho_\varepsilon \nabla c_\varepsilon), & x \in \mathbb{R}^n,\ t \ge 0, \\ -\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon, & x \in \mathbb{R}^n,\ t \ge 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.4}$$

where J_ε is a mollifier with radius ε and satisfies $\int_{\mathbb{R}^n} J_\varepsilon dx = 1$. We know from parabolic theory that the above regularized problem has a global smooth nonnegative solution ρ_ε for $t > 0$ if the initial data are nonnegative.

Step 2. In this step, we will prove the following basic energy estimates:

$$\|\rho_\varepsilon\|_{L^\infty(\mathbb{R}_+;L^m)} + \|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2(\mathbb{R}_+;L^2)} + \|\varepsilon^{\frac{1}{2}}\nabla\rho_\varepsilon^{\frac{m}{2}}\|_{L^2(\mathbb{R}_+;L^2)} \le C, \tag{3.5}$$

$$\|\rho_\varepsilon\|_{L^{m+1}(\mathbb{R}_+;L^{m+1})} \le C, \tag{3.6}$$

$$\|\nabla c_\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2)} + \|c_\varepsilon\|_{L^\infty(\mathbb{R}_+;\,L^s)} \le C, \quad \frac{n}{n-2} < s \le \frac{2n}{n-2}, \tag{3.7}$$

where C stands for positive constants depending only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^m}$, and n.

Taking $m\rho_\varepsilon^{m-1}$ as a test function in (1.1), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^m dx + \frac{4m^2(m-1)}{(2m-1)^2}\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dx + \varepsilon\frac{4(m-1)}{m}\int_{\mathbb{R}^n}\left|\nabla\rho_\varepsilon^{\frac{m}{2}}\right|^2 dx \\ &\quad = (m-1)\int_{\mathbb{R}^n} \rho_\varepsilon^m J_\varepsilon * \rho_\varepsilon dx. \end{aligned} \tag{3.8}$$

The right-hand side of the above equation can be estimated by the Gagliardo–Nirenberg–Sobolev inequality

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_\varepsilon^m J_\varepsilon * \rho_\varepsilon dx &\le \|\rho_\varepsilon^m\|_{L^{\frac{m+1}{m}}} \|J_\varepsilon * \rho_\varepsilon\|_{L^{m+1}} \le \|\rho_\varepsilon\|_{L^{m+1}}^m \|\rho_\varepsilon\|_{L^{m+1}} \\ &= \left\|\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^{\frac{m+1}{m-\frac{1}{2}}}}^{\frac{m+1}{m-\frac{1}{2}}} \le C_{GNS} \left\|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^2}^2 \left\|\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^{\frac{m}{m-\frac{1}{2}}}}^{\frac{2(2-m)}{2m-1}} \\ &= C_{GNS} \left\|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^2}^2 \|\rho_\varepsilon\|_{L^m}^{2-m}.\end{aligned} \tag{3.9}$$

If the last term of (3.8) can be strictly dominated by the second term of (3.8), which can be realized by taking initial data satisfying (3.1), then we can close the estimate. In other words, if we choose ρ_0 such that

$$(m-1)\left(\frac{4m^2}{(2m-1)^2} - C_{GNS}\|\rho_0\|_{L^m}^{2-m}\right) =: \delta > 0,$$

then we can obtain the estimate

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^m dx + \delta\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dx + \varepsilon\frac{4(m-1)}{m}\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{\frac{m}{2}}\right|^2 dx \le 0. \tag{3.10}$$

Thus one has

$$\|\rho_\varepsilon\|_{L^m} \le \|\rho_0\|_{L^m} < C_s, \tag{3.11}$$

and

$$\delta\int_0^\infty\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dxdt + \varepsilon\frac{4(m-1)}{m}\int_0^\infty\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{\frac{m}{2}}\right|^2 dxdt < C_s. \tag{3.12}$$

Combining the last inequality of (3.9) with (3.11) and (3.12), we obtain that

$$\int_0^\infty \|\rho(\cdot,t)\|_{L^{m+1}}^{m+1} dt \le C_{GNS} C_s^{2-m} \int_0^\infty\int_{\mathbb{R}^n} \left|\nabla\rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dxdt \le \frac{C_{GNS} C_s^{3-m}}{\delta}.$$

Therefore, (3.6) follows. Applying the weak Young inequality [25, p. 107, formula (9)] to

$$\nabla c_\varepsilon = -\frac{1}{n\alpha(n)}\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}(J_\varepsilon * \rho_\varepsilon)(y)dy,$$

one has

$$\|\nabla c_\varepsilon\|_{L^2} \le C\Big\|\frac{x-y}{|x-y|^n}\Big\|_{L^q_w} \|J_\varepsilon * \rho_\varepsilon\|_{L^m} \le C\|\rho_\varepsilon\|_{L^m} \le C, \tag{3.13}$$

where $q = \frac{n}{n-1}$ satisfies $1 + \frac{1}{2} = \frac{1}{q} + \frac{1}{m}$. L^q_w stands for the weak L^q-space and $|x|^{1-n} \in L_w^{n/(n-1)}$; see [25, p. 106]. This fact was used in the second inequality above.

By the second equation of (3.4), we know that

$$c_\varepsilon = \frac{1}{(n-2)n\alpha(n)}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}(J_\varepsilon * \rho_\varepsilon)(y)dy.$$

Using the weak Young inequality again, $\rho_\varepsilon \in L^\infty(\mathbb{R}^+, L^1 \cap L^m)$, we can easily get

$$\|c_\varepsilon\|_{L^\infty(\mathbb{R}^+,\, L^s)} \le C, \tag{3.14}$$

where $\frac{n}{n-2} < s \le \frac{2n}{n-2}$. Inequalities (3.13) and (3.14) imply that (3.7) is true.

Step 3. We will show the following time regularity:

$$\|\partial_t \rho_\varepsilon\|_{L^2(0,T;\; W^{-1,p}(U))} \le C(U,T), \tag{3.15}$$

where $p = \min\{\frac{2m}{m+1}, \frac{2(m+1)}{m+3}, m\} = \frac{2(m+1)}{m+3} > 1$ and U is any bounded open subset of $\mathbb{R}^n$. $C(U,T)$ stands for a positive constant dependent only on U, T, $\|\rho_0\|_{L^m}$, and n. For simplicity in presentation, we drop the dependence of $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^m}$, and n in the constant notation $C(U,T)$.

Inequality (3.15) follows by directly using the approximate equation (3.4) and the following estimates (details will be given below):

$$\|\nabla \rho_\varepsilon^m\|_{L^2\left(\mathbb{R}_+; L^{\frac{2m}{m+1}}\right)} \le C, \tag{3.16}$$

$$\|\rho_\varepsilon \nabla c_\varepsilon\|_{L^{m+1}\left(\mathbb{R}_+; L^{\frac{2(m+1)}{m+3}}\right)} \le C, \tag{3.17}$$

and

$$\|\varepsilon^{1/2} \nabla \rho_\varepsilon\|_{L^2(\mathbb{R}_+; L^m)} \le C. \tag{3.18}$$

By the weak formulation of (3.4), for any test function $\phi(x) \in C_0^\infty(U)$, we deduce that for any t,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \partial_t \rho_\varepsilon \phi \, dx \right| \\
&\quad = \left| - \int_U \nabla \rho_\varepsilon^m \cdot \nabla \phi \, dx - \varepsilon \int_U \nabla \rho_\varepsilon \cdot \nabla \phi \, dx + \int_U \rho_\varepsilon \nabla c_\varepsilon \cdot \nabla \phi \, dx \right| \\
&\quad \le \|\nabla \rho_\varepsilon^m\|_{L^{\frac{2m}{m+1}}} \|\nabla \phi\|_{L^{\frac{2m}{m-1}}} + \|\rho_\varepsilon \nabla c_\varepsilon\|_{L^{\frac{2(m+1)}{m+3}}} \|\nabla \phi\|_{L^{\frac{2(m+1)}{m-1}}} \\
&\qquad\quad + \|\varepsilon^{1/2} \nabla \rho_\varepsilon\|_{L^m} \varepsilon^{\frac{1}{2}} \|\nabla \phi\|_{L^{\frac{m}{m-1}}} \\
&\quad \le C(U) \left(\|\nabla \rho_\varepsilon^m\|_{L^{\frac{2m}{m+1}}} + \|\rho_\varepsilon \nabla c_\varepsilon\|_{L^{\frac{2(m+1)}{m+3}}} + \|\varepsilon^{1/2} \nabla \rho_\varepsilon\|_{L^m} \right) \|\nabla \phi\|_{L^{\frac{2(m+1)}{m-1}}}.
\end{aligned}$$

Hence,

$$\|\partial_t \rho_\varepsilon\|_{W^{-1,p}(U)} \le C(U) \left(\|\nabla \rho_\varepsilon^m\|_{L^{\frac{2m}{m+1}}} + \|\rho_\varepsilon \nabla c_\varepsilon\|_{L^{\frac{2(m+1)}{m+3}}} + \|\varepsilon^{1/2} \nabla \rho_\varepsilon\|_{L^m} \right).$$

Furthermore, integrating it into $t \in [0,T]$, we have

$$\begin{aligned}
&\int_0^T \|\partial_t \rho_\varepsilon\|_{W^{-1,p}(U)}^2 dt \\
&\le C(U) \left(\int_0^T \|\nabla \rho_\varepsilon^m\|^2_{L^{\frac{2m}{m+1}}} dt + \int_0^T \|\rho_\varepsilon \nabla c_\varepsilon\|^2_{L^{\frac{2(m+1)}{m+3}}} dt + \int_0^T \|\varepsilon^{1/2} \nabla \rho_\varepsilon\|^2_{L^m} dt \right) \\
&\le C(U,T) \left(\int_0^T \|\nabla \rho_\varepsilon^m\|^2_{L^{\frac{2m}{m+1}}} dt + \int_0^T \|\rho_\varepsilon \nabla c_\varepsilon\|^{m+1}_{L^{\frac{2(m+1)}{m+3}}} dt + \int_0^T \|\varepsilon^{1/2} \nabla \rho_\varepsilon\|^2_{L^m} dt \right) \\
&\le C(U,T).
\end{aligned}$$

Thus, (3.15) is true.

Now we return to the proof of (3.16)–(3.18). Hölder's inequality gives that

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla \rho_\varepsilon^m|^{\frac{2m}{m+1}} dx &= C \int_{\mathbb{R}^n} |\rho_\varepsilon^{\frac{1}{2}} \nabla \rho_\varepsilon^{m-\frac{1}{2}}|^{\frac{2m}{m+1}} dx \\ &\le C \left(\int_{\mathbb{R}^n} \rho_\varepsilon^m dx \right)^{1/(m+1)} \left(\int_{\mathbb{R}^n} |\nabla \rho_\varepsilon^{m-\frac{1}{2}}|^2 dx \right)^{m/(m+1)} \\ &= C \|\rho_\varepsilon\|_{L^m}^{m/(m+1)} \|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2}^{2m/(m+1)}.\end{aligned}$$

Estimates in (3.12) and the above discussion imply that (3.16) is true. More precisely, by using $r = \frac{m+1}{m}$ we have

$$\begin{aligned}\|\nabla \rho_\varepsilon^m\|^2_{L^2\left(\mathbb{R}_+; L^{\frac{2m}{m+1}}\right)} &= \int_0^\infty \|\nabla \rho_\varepsilon^m\|_{L^{\frac{2m}{m+1}}}^{\frac{2m}{m+1} r} dt \\ \le C \int_0^\infty \|\rho_\varepsilon\|_{L^m}^{rm/(m+1)} \|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2}^{2rm/(m+1)} dt & \\ \le C \int_0^\infty \|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2}^{2rm/(m+1)} dt = C \int_0^\infty \|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\|_{L^2}^2 dt \le C. &\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\rho_\varepsilon \nabla c_\varepsilon|^{\frac{2(m+1)}{m+3}} dx &\le \left(\int_{\mathbb{R}^n} \rho_\varepsilon^{m+1} dx \right)^{2/(m+3)} \left(\int_{\mathbb{R}^n} |\nabla c_\varepsilon|^2 dx \right)^{(m+1)/(m+3)} \\ &= \|\rho_\varepsilon\|_{L^{m+1}}^{2(m+1)/(m+3)} \|\nabla c_\varepsilon\|_{L^2}^{2(m+1)/(m+3)}.\end{aligned}$$

Moreover, by (3.6) and (3.7) we can get that (3.17) is true in the following estimates with $r_1 = \frac{m+3}{2}$:

$$\begin{aligned}\|\rho_\varepsilon \nabla c_\varepsilon\|^{m+1}_{L^{m+1}\left(\mathbb{R}_+; L^{\frac{2(m+1)}{m+3}}\right)} &= \int_0^\infty \|\rho_\varepsilon \nabla c_\varepsilon\|_{L^{\frac{2(m+1)}{m+3}}}^{\frac{2(m+1)}{m+3} r_1} dt \\ \le \int_0^\infty \|\rho_\varepsilon\|_{L^{m+1}}^{2r_1(m+1)/(m+3)} \|\nabla c_\varepsilon\|_{L^2}^{2r_1(m+1)/(m+3)} dt & \\ \le C \int_0^\infty \|\rho_\varepsilon\|_{L^{m+1}}^{2r_1(m+1)/(m+3)} dt = C \int_0^\infty \|\rho_\varepsilon\|_{L^{m+1}}^{m+1} dt \le C. &\end{aligned}$$

Since we can write

$$\varepsilon^{1/2} \nabla \rho_\varepsilon = \frac{2}{m} \varepsilon^{1/2} \rho_\varepsilon^{1-\frac{m}{2}} \nabla \rho_\varepsilon^{\frac{m}{2}},$$

the term for parabolic regularization is easily done with Hölder's inequality and (3.5):

$$\|\varepsilon^{1/2} \nabla \rho_\varepsilon\|_{L^2(\mathbb{R}_+; L^m)} \le C.$$

Step 4. In this step we show the estimates for $\nabla \rho_\varepsilon$:

$$\|\nabla \rho_\varepsilon\|_{L^r(0,T;\, L^r(\mathbb{R}^n))} \le C, \qquad \text{with } r = \min\left(2, \frac{3n+2}{n+4}\right). \tag{3.19}$$

The estimate will be divided into two cases: $n < 6$ and $n \ge 6$. In the case of $n < 6$, which is equivalently $m - \frac{1}{2} < 1$, we can use the above estimate (3.12) to get the estimate for $\nabla\rho_\varepsilon$, while in the case of $n \geq 6$, (3.12) is not useful. We will use ρ_ε^{2-m} as a test function and get the estimate for $\nabla\rho_\varepsilon$ directly from the diffusion term.

In the case of $n < 6$, we recast $\nabla\rho_\varepsilon$ as

$$\nabla\rho_\varepsilon = \frac{1}{m-1/2}\rho_\varepsilon^{3/2-m}\nabla\rho_\varepsilon^{m-1/2}.$$

From the estimates we obtained before, $\rho_\varepsilon \in L^{m+1}\left(\mathbb{R}_+; L^{m+1}\right)$, and $\nabla\rho_\varepsilon^{m-1/2} \in L^2\left(\mathbb{R}_+; L^2\right)$, by Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla\rho_\varepsilon|^{\frac{2(m+1)}{4-m}} dx &= C\int_{\mathbb{R}^n} |\rho_\varepsilon^{3/2-m}|^{\frac{2(m+1)}{4-m}} |\nabla\rho_\varepsilon^{m-1/2}|^{\frac{2(m+1)}{4-m}} dx \\ &\leq C\left(\int_{\mathbb{R}^n} |v|^{\frac{2(m+1)}{4-m}p} dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |\nabla\rho_\varepsilon^{m-1/2}|^{\frac{2(m+1)}{4-m}q} dx\right)^{1/q},\end{aligned}$$

where $v := \rho_\varepsilon^{3/2-m} \in L^{\frac{m+1}{3/2-m}}(\mathbb{R}_+; L^{\frac{m+1}{3/2-m}})$, with $\frac{m+1}{3/2-m} > 2$. We choose $p = \frac{4-m}{3-2m} > 1$ and $q = \frac{4-m}{m+1} > 1$ such that $\frac{2(m+1)}{4-m}q = 2$. Hence we have

$$\|\nabla\rho_\varepsilon\|_{L^{\frac{2(m+1)}{4-m}}} \leq C\|v\|_{L^{\frac{m+1}{3/2-m}}} \|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}.$$

Furthermore, by using Hölder's inequality in the time integral, we have that

$$\begin{aligned}\int_0^\infty \|\nabla\rho_\varepsilon\|_{L^{\frac{2(m+1)}{4-m}}}^{\frac{2(m+1)}{4-m}} dt &\leq C\left(\int_0^\infty \|v\|_{L^{\frac{m+1}{3/2-m}}}^{\frac{2(m+1)}{4-m}p} dt\right)^{1/p} \left(\int_0^\infty \|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}^{\frac{2(m+1)}{4-m}q} dt\right)^{1/q} \\ &= C\left(\int_0^\infty \|v\|_{L^{\frac{m+1}{3/2-m}}}^{\frac{m+1}{3/2-m}} dt\right)^{1/p} \left(\int_0^\infty \|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}^2 dt\right)^{1/q} \\ &\leq C,\end{aligned}$$

where p and q are the same as before. Thus,

$$\nabla\rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}\left(\mathbb{R}_+; L^{\frac{2(m+1)}{4-m}}\right) \qquad \text{bounded uniformly in } \varepsilon.$$

Combining the above estimate with the fact that $\rho_\varepsilon \in L^\infty(\mathbb{R}_+; L^1 \cap L^m) \cap L^{m+1}(\mathbb{R}_+; L^{m+1})$, we deduce that

$$\rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}\left(\mathbb{R}_+; W^{1,\frac{2(m+1)}{4-m}}\right) \qquad \text{bounded uniformly in } \varepsilon. \tag{3.20}$$

In the case of $n \geq 6$, by using ρ_ε^{2-m} as a test function in the approximate problem (3.4), one has

$$\begin{aligned}&\frac{1}{3-m}\frac{d}{dt}\int_{\mathbb{R}^n} \rho_\varepsilon^{3-m} dx + m(2-m)\int_{\mathbb{R}^n} |\nabla\rho_\varepsilon|^2 dx + \varepsilon(2-m)\int_{\mathbb{R}^n} \rho_\varepsilon^{1-m} |\nabla\rho_\varepsilon|^2 dx \\ &= \int_{\mathbb{R}^n} \rho_\varepsilon \nabla c_\varepsilon \cdot \nabla\rho_\varepsilon^{2-m} dx = \frac{2-m}{3-m}\int_{\mathbb{R}^n} \nabla c_\varepsilon \cdot \nabla\rho_\varepsilon^{3-m} dx \leq C\int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx.\end{aligned}$$

Now we need only estimate $\int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx$. Letting $u := \rho_\varepsilon^{m-1/2}$, we will use $u \in L^\infty\left(\mathbb{R}_+; L^{\frac{m}{m-1/2}}\right)$, which is exactly $\rho_\varepsilon \in L^\infty(\mathbb{R}_+; L^m)$. From (3.12), we have $\nabla u \in$

$L^2(\mathbb{R}_+;L^2)$. By the Gagliardo–Nirenberg–Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dx &= \int_{\mathbb{R}^n} u^{\frac{4-m}{m-1/2}} dx \le C\|\nabla u\|_{L^2}^{\theta\frac{4-m}{m-1/2}} \|u\|_{L^{\frac{m}{m-1/2}}}^{(1-\theta)\frac{4-m}{m-1/2}} \\ &= C\|\nabla u\|_{L^2}^{\frac{16}{n+2}} \|u\|_{L^{\frac{m}{m-1/2}}}^{(1-\theta)\frac{4-m}{m-1/2}},\end{aligned}$$

where $0 < \theta = \frac{4(3n-2)}{(n+2)(n+4)} < 1$. Hence for any fixed $T > 0$, it holds that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{4-m} dxdt &\le C\int_0^T \|\nabla u\|_{L^2}^{\frac{16}{n+2}} \|u\|_{L^{\frac{m}{m-1/2}}}^{(1-\theta)\frac{4-m}{m-1/2}} dt \\ &\le C\left(\|u\|_{L^\infty(\mathbb{R}_+;L^{\frac{m}{m-1/2}})}, \|\nabla u\|_{L^2(\mathbb{R}_+;L^2)}, T\right),\end{aligned}$$

where we have used $\frac{16}{n+2} \le 2$ for $n \ge 6$. Consequently, for any fixed $T > 0$,

$$\begin{aligned}&\frac{1}{(3-m)}\int_{\mathbb{R}^n} \rho_\varepsilon^{3-m} dx + m(2-m)\int_0^T \int_{\mathbb{R}^n} |\nabla\rho_\varepsilon|^2 dxdt \\ &\quad + \varepsilon(2-m)\int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{1-m} |\nabla\rho_\varepsilon|^2 dxdt \\ &\le \frac{1}{(3-m)}\|\rho_0\|_{L^{3-m}}^{3-m} + C \le C\left(\|\rho_0\|_{L^m}, \|\rho_0\|_{L^1}\right) + C.\end{aligned}$$

In the above discussion we have used the fact that $3-m \le m$ in the case of $n \ge 6$.

Thus we have

$$\int_0^T \int_{\mathbb{R}^n} |\nabla\rho_\varepsilon|^2 dxdt \le C \text{ for any fixed } T > 0.$$

Combining the above estimate with the fact that $\rho_\varepsilon \in L^\infty(\mathbb{R}_+;L^1\cap L^m)\cap L^{m+1}(\mathbb{R}_+;L^{m+1})$, we deduce that

$$\rho_\varepsilon \in L^2(0,T;W^{1,2}) \qquad \text{bounded uniformly in } \varepsilon. \tag{3.21}$$

From (3.20), (3.21), and the fact that $\frac{2(m+1)}{4-m} = \frac{3n+2}{n+4}$, we have proved (3.19).

Step 5. From bounds (3.5), (3.6), (3.15), and (3.19) there exist subsequences of ρ_ε and c_ε. Without relabeling for convenience, we have the following weak convergence:

$$\begin{aligned}&\rho_\varepsilon \rightharpoonup \rho \text{ in } L^{m+1}\left(0,T;L^{m+1}(\mathbb{R}^n)\right), \\ &\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty\left(0,T;L^1\cap L^m(\mathbb{R}^n)\right), \\ &c_\varepsilon \overset{*}{\rightharpoonup} c \text{ in } L^\infty(0,T;L^s(\mathbb{R}^n)), \quad \frac{n}{n-2} < s \le \frac{2n}{n-2}, \\ &\nabla\rho_\varepsilon \rightharpoonup \nabla\rho \text{ in } L^r\left(0,T;\, L^r(\mathbb{R}^n)\right), \quad r = \min\left(2, \frac{3n+2}{n+4}\right), \\ &\nabla c_\varepsilon \overset{*}{\rightharpoonup} \nabla c \text{ in } L^\infty\left(0,T;\, L^2(\mathbb{R}^n)\right).\end{aligned} \tag{3.22}$$

From (3.15), (3.17), and the Lions–Aubin lemma, for any bounded domain $U \subset \mathbb{R}^n$, there exists a subsequence of ρ_ε, not relabeled, such that

$$\rho_\varepsilon \to \rho \qquad \text{in } L^r\left(0,T;L^{\bar p}(U)\right),$$

where $r = \min(2, \frac{3n+2}{n+4})$ and $\bar{p} = \min\{\frac{(3n+2)n}{n^2+n-2}, \frac{2n}{n-2}\} > 2$. Let $\{B_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ be a sequence of balls centered at 0 with radius R_k, $R_k \to \infty$. By a standard diagonal argument, we can find a subsequence of ρ_ε. Without relabeling, we have the following uniform strong convergence:

$$\rho_\varepsilon \to \rho \qquad \text{in } L^r(0,T;L^{\bar{p}}(B_k)) \qquad \forall k.$$

This leads to the existence of a global weak solution. The regularity (3.2) comes directly from (3.22).

Step 6. In this step, we prove that the global weak solution obtained in Step 5 decays to zero as $t \to \infty$.

By the Gagliardo–Nirenberg–Sobolev inequality,

$$\begin{aligned}(3.23)\qquad \int_{\mathbb{R}^n} \rho^{m+1}dx &= \left\|\rho^{m-\frac{1}{2}}\right\|_{L^{\frac{m+1}{m-\frac{1}{2}}}}^{\frac{m+1}{m-\frac{1}{2}}} \\ &\le C_{GNS}\left\|\nabla\rho^{m-\frac{1}{2}}\right\|_{L^2}^2 \left\|\rho^{m-\frac{1}{2}}\right\|_{L^{\frac{m}{m-\frac{1}{2}}}}^{\frac{2(2-m)}{2m-1}} = C_{GNS}\left\|\nabla\rho^{m-\frac{1}{2}}\right\|_{L^2}^2 \|\rho\|_{L^m}^{2-m}.\end{aligned}$$

Or equivalently,

$$\left\|\nabla\rho^{m-\frac{1}{2}}\right\|_{L^2}^2 \ge \frac{1}{C_{GNS}\|\rho\|_{L^m}^{2-m}}\int_{\mathbb{R}^n}\rho^{m+1}dx.$$

We have the following inequality for weak solution:

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho^m dx \le -\delta\int_{\mathbb{R}^n}|\nabla\rho^{m-\frac{1}{2}}|^2dx \le -\frac{\delta}{C_{GNS}\|\rho\|_{L^m}^{2-m}}\int_{\mathbb{R}^n}\rho^{m+1}dx.$$

On the other hand, we have

$$\|\rho\|_{L^m} \le \|\rho\|_{L^{m+1}}^{\theta}\|\rho\|_{L^1}^{1-\theta}, \qquad \theta = \frac{m^2-1}{m^2}.$$

Combining the above estimate with the previous inequality, we have an inequality for $\|\rho\|_{L^m}$,

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho^m dx \le -\frac{\delta}{C_{GNS}}\|\rho\|_{L^m}^{m-2}\|\rho\|_{L^m}^{\frac{m^2}{m-1}}\|\rho_0\|_{L^1}^{\frac{1}{1-m}} = -C_n\left(\int_{\mathbb{R}^n}\rho^m dx\right)^{\beta},$$

where $C_n = \frac{\delta}{C_{GNS}}\|\rho_0\|_{L^1}^{\frac{1}{1-m}}$, $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Then by solving this ordinary differential inequality, we have

$$\|\rho(\cdot,t)\|_{L^m} \le \left(\frac{1}{(\beta-1)C_n t + \|\rho_0\|_{L^m}^{m(1-\beta)}}\right)^{\frac{1}{m(\beta-1)}},$$

which implies that the solution decays to zero in the L^m norm as $t \to \infty$:

$$\|\rho(\cdot,t)\|_{L^m} \le Ct^{-\frac{1}{m(\beta-1)}} \qquad \text{for large } t.$$

This completes the proof of the theorem. $\square$

3.2. Blow-up of the general solution. In this subsection, we will discuss the blow-up of the solution when $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$.

Recall that (1.20) gives a decomposition of the free energy

$$\begin{aligned}\mathcal{F}(\rho) &= \frac{1}{m-1}\|\rho\|_{L^m}^m\left(1-\frac{(m-1)c_nC(n)}{2}\|\rho\|_{L^m}^{4/(n+2)}\right)\\ &\quad+\frac{c_n}{2}\left(C(n)\|\rho\|_{L^m}^2-\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right)\\ &=: \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho),\end{aligned}$$

where $c_n = 1/(n(n-2)\alpha(n))$ and $\mathcal{F}_2(\rho) \geq 0$ from the Hardy–Littlewood–Sobolev inequality and

$$\mathcal{F}_1(\rho) = f(\|\rho\|_{L^m}^m), \qquad f(s) = \frac{1}{m-1}s - \frac{c_n}{2}C(n)s^{\frac{2}{m}}.$$

As we have already mentioned in the introduction, $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$. Hence it is also a critical point for $\mathcal{F}_1(\rho)$. In the following lemma, we show that $\|U_{\lambda,x_0}\|_{L^m}^m$ is indeed a maximum point for $f(s)$.

LEMMA 3.1. *$f(s)$ is a strict concave function, and it reaches the unique maximum point at*

$$s^* := \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n}{2}} = \|U_{\lambda,x_0}\|_{L^m}^m,$$

where U_{λ,x_0} is any stationary solution of (1.1), *and constants $\alpha(n)$ and $C(n)$ are given by* (1.2) *and* (1.19), *respectively.*

Proof. Taking the first and second order derivatives for $f(s)$, one has $f'(s) = \frac{1}{m-1} - \frac{c_n}{m}C(n)s^{\frac{2-m}{m}}$ and $f''(s) = -\frac{c_n(2-m)}{m^2}C(n)s^{\frac{2-m}{m}-1} < 0 \ \forall s > 0$. Thus $f(s)$ attains its maximum at

$$s^* = \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n}{2}}.$$

Now we show that $s^* = \|U_{\lambda,x_0}\|_{L^m}^m$. By the formula of free energy with $\rho = U_{\lambda,x_0}$

$$\mathcal{F}(U_{\lambda,x_0}) = \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2}\int_{\mathbb{R}^n} U_{\lambda,x_0}C_{\lambda,x_0}dx,$$

and by the critical case of the Hardy–Littlewood–Sobolev inequality (1.18)

$$\begin{aligned}\mathcal{F}(U_{\lambda,x_0}) &= \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{U_{\lambda,x_0}(x)U_{\lambda,x_0}(y)}{|x-y|^{n-2}}dxdy\\ &= \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2(n-2)n\alpha(n)}C(n)\|U_{\lambda,x_0}\|_{L^m}^2,\end{aligned}$$

we have

$$\int_{\mathbb{R}^n} U_{\lambda,x_0}C_{\lambda,x_0}dx = \frac{1}{(n-2)n\alpha(n)}C(n)\|U_{\lambda,x_0}\|_{L^m}^2.$$

Noticing that $C_{\lambda,x_0} = \frac{2n}{n-2}U_{\lambda,x_0}^{m-1}$ as in (1.21), we have

$$2n^2\alpha(n)\|U_{\lambda,x_0}\|_{L^m}^m = C(n)\|U_{\lambda,x_0}\|_{L^m}^2,$$

from which we have

$$\|U_{\lambda,x_0}\|_{L^m}^m = \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n}{2}} = s^*.$$

Another way to prove $s^* = \|U_{\lambda,x_0}\|_{L^m}^m$ is direct verification of (1.23). ∎

Before we state and prove our main theorem in this section, let us first prove the following technical lemma.

LEMMA 3.2. *Assume $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$, and ρ is a solution of* (1.1). *Then there is $\mu > 1$ such that ρ satisfies*

$$\|\rho\|_{L^m}^m > \mu\|U_{\lambda,x_0}\|_{L^m}^m.$$

Proof. Since $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, we can choose $\delta : 0 < \delta < 1$ such that $\mathcal{F}(\rho_0) < \delta\mathcal{F}(U_{\lambda,x_0})$. By (1.20) with the Hardy–Littlewood–Sobolev inequality (1.18) and the fact that $\mathcal{F}(\rho(\cdot,t))$ is nonincreasing in t, we have

$$f(\|\rho\|_{L^m}^m) = \mathcal{F}_1(\rho) \le \mathcal{F}(\rho) \le \mathcal{F}(\rho_0) < \delta\mathcal{F}(U_{\lambda,x_0}) = \delta f(s^*).$$

Then for any $s > \|U_{\lambda,x_0}\|_{L^m}^m$, $f(s)$ is a strictly decreasing function. So it has a strictly decreasing inverse function f^{-1}. Hence if $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$, we have for some $\mu > 1$

$$\|\rho\|_{L^m}^m > \mu\|U_{\lambda,x_0}\|_{L^m}^m. \qquad ∎$$

THEOREM 3.2. *Assume $m_2(0) = \int_{\mathbb{R}^n} |x|^2\rho_0(x)dx < \infty$, $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, and $\|\rho_0\|_{L^m(\mathbb{R}^n)} > \|U_{\lambda,x_0}\|_{L^m(\mathbb{R}^n)}$. Then there is a finite time $t^* > 0$ such that*

$$\|\rho(\cdot,t)\|_{L^m(\mathbb{R}^n)} \to \infty \quad \text{as } t \to t^*.$$

Proof. Here we use the formula

$$\nabla c = -\frac{1}{n\alpha(n)}\frac{x}{|x|^n} * \rho(x).$$

By Lemma 3.2, (1.11), and the monotonicity of free energy, we deduce that

$$\begin{aligned}\frac{d}{dt}m_2(t) &\le -4\mu\|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(\rho_0)\\ &\le -4\mu\|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(U_{\lambda,x_0})\\ &= -4(\mu-1)\|U_{\lambda,x_0}\|_{L^m}^m < 0,\end{aligned}$$

where we have used $\mathcal{F}(U_{\lambda,x_0}) = \frac{2}{n-2}\|U_{\lambda,x_0}\|_{L^m}^m$ in the third equality; see (1.22). This means that there is a $t^* > 0$ such that $\lim_{t\to t^*} m_2(t) = 0$.

On the other hand, $\forall R > 0$, by using the Hölder inequality, we have

$$\int_{\mathbb{R}^n}\rho(x)dx \le \int_{B_R}\rho(x)dx + \int_{B_R^c}\rho(x)dx \le CR^{(n-2)/2}\|\rho\|_{L^m} + \frac{1}{R^2}m_2(t).$$

Now by choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^m}})^{2/(n+2)}$, we have

$$\|\rho\|_{L^1} \le C\|\rho\|_{L^m}^{\frac{4}{n+2}} m_2(t)^{\frac{n-2}{n+2}}.$$

So, $\lim_{t\to t^*}\|\rho\|_{L^m}^{\frac{4}{n+2}} \ge \lim_{t\to t^*}\frac{\|\rho\|_{L^1}}{\bar{C}(n)m_2(t)^{\frac{n-2}{n+2}}} = \infty.$ ∎

Remark 3.1. There is a gap between C_s and $\|U_{\lambda,x_0}\|_{L^m}$. So there is still a space, in the case that $C_s \le \|\rho_0\|_{L^m} \le \|U_{\lambda,x_0}\|_{L^m}$ where we do not know whether the solution exists or blows up. See section A.1 for the comparison between C_s and $\|U_{\lambda,x_0}\|_{L^m}$.

Appendix. Proof of Proposition 1.3.

The invariance of the free energy $F(\rho)$ is obvious in the translation transformation $\rho_{\bar{x}}(x) = \rho(x+\bar{x})$ and the orthogonal transformation $\mathcal{R}\rho(x) = \rho(\mathcal{R}^{-1}x)$, since the determinant to the Jacobian matrix is 1 for these two transformations.

By the scaling transformation $\rho_\lambda(x) = \lambda^{\frac{n+2}{2}}\rho(\lambda x)$ and direct computation, we have

$$\begin{aligned}
\mathcal{F}(\rho_\lambda) &= \int_{\mathbb{R}^n} \frac{\lambda^{\frac{(n+2)m}{2}}\rho^m(\lambda x)}{m-1}dx - \frac{c_n}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\lambda^{n+2}\rho(\lambda x)\rho(\lambda y)}{|x-y|^{n-2}}dxdy \\
&= \int_{\mathbb{R}^n} \frac{\lambda^n\rho^m(\lambda x)}{m-1}dx - \frac{c_n}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(\lambda x)\rho(\lambda y)}{|x-y|^{n-2}\lambda^{n-2}}\lambda^{n-2}\lambda^{n+2}dxdy \\
&= \mathcal{F}(\rho).
\end{aligned}$$

Notice that the Kelvin transformation of c is

$$c_{\bar{x},\lambda}(x) = \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n-2} c\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right);$$

we thus have the related transformation for ρ in the following:

$$\rho_{\bar{x},\lambda}(x) = \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n+2} \rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right).$$

Therefore, the free energy for $\rho_{\bar{x},\lambda}$ is

$$\begin{aligned}
\mathcal{F}(\rho_{\bar{x},\lambda}) = &\int_{\mathbb{R}^n} \frac{1}{m-1}\left(\frac{\lambda}{|x-\bar{x}|}\right)^{(n+2)m} \left[\rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right)\right]^m dx \\
&- \frac{c_n}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right) \\
&\rho\left(\bar{x} + \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}\right)\left(\frac{\lambda^2}{|x-\bar{x}||y-\bar{x}|}\right)^{n+2} dxdy \\
=: &\, I_1 - \frac{c_n}{2}I_2.
\end{aligned}$$

Here

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^n} \frac{1}{m-1}\left(\frac{\lambda}{|x-\bar{x}|}\right)^{2n} \left[\rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right)\right]^m dx \\
&= \int_{\mathbb{R}^n} \frac{1}{m-1}\left[\rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right)\right]^m d\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right), \\
I_2 &:= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left[\frac{\lambda^2|x-y|}{|x-\bar{x}|\cdot|y-\bar{x}|}\right]^{2-n} \rho(z)\rho(w)dzdw \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(z)\rho(w)}{|z-w|^{n-2}}dzdw,
\end{aligned}$$

where $z = \bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}$, $w = \bar{x} + \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}$, and we have used

$$|z-w| = \left|\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2} - \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}\right| = \frac{\lambda^2|x-y|}{|x-\bar{x}|\cdot|y-\bar{x}|}.$$

Hence we have

$$\mathcal{F}(\rho_{\bar{x},\lambda}) = I_1 - \frac{c_n}{2} I_2 = \int_{\mathbb{R}^n} \frac{\rho^m(z)}{m-1} dz - \frac{c_n}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(z)\rho(w)}{|z-w|^{n-2}} dz dw = \mathcal{F}(\rho).$$

A.1. Gap between C_s and $\|U_{\lambda,x_0}\|_{L^m}$. Since the L^m norm of U_{λ,x_0} does not depend on λ or x_0, we will use $\|U\|_{L^m} = \|U_{\lambda,x_0}\|_{L^m}$ in the following. Among the estimates in the proof of global existence of the weak solution, Theorem 3.1, we used only an important Gagliardo–Nirenberg–Sobolev inequality, which is

$$\|v\|_{L^r} \le \|v\|_{L^{2^*}}^{\theta} \|v\|_{L^q}^{1-\theta} \le \bar{C}_{GNS} \|\nabla v\|_{L^2}^{\theta} \|v\|_{L^q}^{1-\theta},$$

where, in our case, $r = \frac{2(m+1)}{2m-1}$, $\theta = \frac{2m-1}{m+1}$, and $C_{GNS} = (\bar{C}_{GNS})^{\frac{m+1}{m-\frac{1}{2}}}$.

From [25, p. 202], we know the best constant for Sobolev embedding,

$$\|\nabla f\|_{L^2}^2 \ge S_n \|f\|_{L^{2^*}}^2, \qquad S_n = \frac{n(n-2)}{4} 2^{\frac{2}{n}} \pi^{1+\frac{1}{n}} \Gamma\left(\frac{n+1}{2}\right)^{-\frac{2}{n}},$$

which gives

$$C_{GNS} = \left(S_n^{-\frac{\theta}{2}}\right)^{\frac{m+1}{m-\frac{1}{2}}}. \tag{A.1}$$

We can calculate that C_s is strictly less than $\|U\|_{L^m}$. In fact, from (3.1) and (1.23),

$$\begin{aligned}
&C_s - \|U\|_{L^m} \\
&= \left(\frac{4m^2}{(2m-1)^2 C_{GNS}}\right)^{\frac{1}{2-m}} - \left(n^n \pi^{\frac{n+1}{2}} 2^{1-\frac{n}{2}} \Gamma^{-1}\left(\frac{n+1}{2}\right)\right)^{\frac{1}{m}} \\
&= \left(\frac{4m^2}{(2m-1)^2 \frac{4}{n(n-2)} 2^{-\frac{2}{n}} \pi^{-(1+\frac{1}{n})} \Gamma^{\frac{2}{n}}\left(\frac{n+1}{2}\right)}\right)^{\frac{n+2}{4}} - \left(n^n \pi^{\frac{n+1}{2}} 2^{1-\frac{n}{2}} \Gamma^{-1}\left(\frac{n+1}{2}\right)\right)^{\frac{n+2}{2n}} \\
&= \left[\left(\frac{m^2(n-2)}{(2m-1)^2}\right)^{\frac{n+2}{4}} - n^{\frac{n+2}{4}} 2^{-\frac{n+2}{4}}\right] 2^{\frac{n+2}{2n}} \pi^{\frac{(n+1)(n+2)}{4n}} \Gamma^{-\frac{n+2}{2n}} \left(\frac{n+1}{2}\right) n^{\frac{n+2}{4}} < 0,
\end{aligned}$$

due to $\frac{m^2(n-2)}{(2m-1)^2} < n/2 \, \forall \, n \ge 3$.

Remark A.1. Although there is a big gap between these two constants, we can see that our globally existing weak solution also has algebraic decay in time when $\|\rho_0\|_{L^m} < C_s$, while for initial data $C_s \le \|\rho_0\|_{L^m} < \|U\|_{L^m}$, we do not expect decay in time. Our conjecture is to have global existence in this case. We leave this topic to our future work.

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