Numerical Methods for Oscillatory Solutions to Hyperbolic Problems

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Abstract

Difference approximations of hyperbolic partial differential equations with highly oscillatory coefficients and initial values are studied. Analysis of strong and weak convergence is carried out in the practically interesting case when the discretization step sizes are essentially independent of the oscillatory wave lengths. ©1993 John Wiley & Sons, Inc.

1. Introduction

In order to guarantee a good numerical approximation to an oscillatory solution of a differential equation a fine computational grid is in general needed. There must often be a substantial number of grid points or elements per wave length in the oscillation; see [10].

There are many computational problems in physics and engineering with highly oscillatory solutions. They range from well understood linear problems in classical acoustics to the computation of turbulent flow. For many of these problems the requirement of several grid points per wave length is unrealistic in practical calculations.

One alternative is to replace the original problem with effective or homogenized equations; see, e.g., [4]. This is, however, not always possible or practical and the question naturally arises if the original formulation can be used, even if the oscillation in the continuous solution is not well resolved on the grid. There still may be some quantities in the solution which are well approximated. Modern shock capturing schemes are successful approximations of this type. The shock profile is not well approximated but the approximation may still converge in integral norms; see [5] and [15]. A similar study was initialed in [7] for problems with oscillatory solutions.

In the traditional definition of convergence, the error decreases for a fixed problem as the grid step size decreases. This concept is too weak and does not discriminate between solutions which are highly oscillatory and those which are not. A new concept of convergence was suggested in [7] to describe the convergence essentially independent of the wave length of the oscillation. With the expression essentially independent we mean that a set of arbitrary small measure in the ratio between the wave length and the grid size must be avoided (see the

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definition below). This notion of convergence is weak enough for a rigorous analysis but strong enough to be used as a convergence concept in practical cases. It is similar to the convergence for almost all sequences of random numbers in Monte Carlo methods; see [13]. Convergence fully independent of the oscillatory wave length is too strong a concept and will never be satisfied; see [7].

DEFINITION 1.1. Let u^ε represent a family of oscillatory solutions parameterized by ε (ε usually represents the oscillatory wave length), and let u_h^ε be a numerical approximation of u^ε with grid step size h. The approximation u_h^ε converges in norm $\|\cdot\|$ as $h \to 0$ essentially independent of ε to u^ε if for any $\delta > 0$, there exists $h_0 > 0$, independent of ε, and measurable sets $\mathscr{S}(\varepsilon, h_0) \subset (0, h_0)$ with Lebesgue measure $|\mathscr{S}(\varepsilon, h_0)| \geqq (1-\delta)h_0$ such that

$$\|u^\varepsilon - u_h^\varepsilon\| \leqq \delta \tag{1.1}$$

is valid for all $0 < \varepsilon < 1$ and $h \in \mathscr{S}(\varepsilon, h_0)$.

Numerical simulations and their rigorous justifications under the above definition of convergence have recently been successfully carried out in many important cases; see [2], [6]–[9], and [12]. Linear and semilinear hyperbolic equations with rapidly oscillating initial values, as well as elliptic equations with oscillatory coefficients have been considered. The purpose of this paper is to study some fundamental properties of oscillatory approximations. (Most results are new, a few of the results appeared in an unpublished report and they are referred to in the papers listed above.)

In this paper, we shall discuss numerical methods for two quite general classes of linear hyperbolic systems. In the first class, the initial values and forcing terms in the differential equations are rapidly oscillatory. As a result, the amplitude of the oscillations in the solutions does not vanish in the limit as the wave length converges to zero. It is referred to as singular homogenization problems (see [14]) or large amplitude oscillation problems. In the other class, the coefficients in the differential equations are rapidly oscillatory but the solutions converge strongly to the solutions of homogenized equations. It is often referred to as regular homogenization problems (see [14]) or small amplitude oscillation problems.

Although we can not expect to resolve the oscillatory solutions in detail without large amounts of computations, we can still "capture" their most important feature — the weak limit or local average with reasonable amounts of computations. We average the numerical solution by a convolution with a kernel function which has a support of range as small as square root of the computational grid size.

It is sometimes essential that not only the average of the approximation behaves well. The amplitude and the location of the oscillations should also be well approximated. This is a necessary step for the analysis of nonlinear problems. In the singular homogenization problems, numerical dissipation and numerical dispersive could cause $O(1)$ errors in the amplitude and the location of the oscillations. The oscillations in dissipative schemes are dampened out very fast.

Unitary schemes are more favorable but even they produce approximations with large errors in the location of the oscillations due to numerical dispersion and thus errors in the group velocities. These errors in the amplitude and the location of the oscillations are hidden by the averaging process in linear problems. Some oscillatory behavior in nonlinear dispersive schemes were discussed in [11].

The design of numerical schemes for linear convection problems in an oscillatory velocity field, an important physical problem which has been extensively studied recently (see [3]), needs particular care. Since the coefficients in the numerical schemes are highly oscillatory, we need some sampling conditions for the oscillatory coefficients instead of the consistency condition in Lax's equivalence theorem which is not valid in this case. Numerical schemes, which are stable for constant coefficients, sometimes become unstable when the coefficients are highly oscillatory.

The organization of this paper is as follows: in Section 2, we generalize the definition of the essential convergence in order to analyze the convergence rate. We shall give a sampling lemma for periodic oscillations with two scales. We also give some equivalent statements of the essential convergence.

In Section 3, the singular homogenization problems are considered. We shall prove essential convergence for consistent and stable difference approximation for linear systems with smooth coefficients and rapidly oscillatory initial values and forcing terms. We shall also discuss the damping of the oscillations in dissipative schemes and propagation of the oscillations in unitary schemes.

In Section 4, the regular homogenization problems are studied. We shall give convergence proof for finite difference methods and particle methods for one- and two-dimensional linear hyperbolic systems with oscillatory coefficients. For linear hyperbolic equations with both oscillatory coefficients and initial values, we remark on the weakly essential convergence for a special case by a probability method.

Finally, we shall present some numerical experiments in Section 5. The quantitative information in these experiments is consistent with our theoretical results.

2. Definition, Sampling Errors, and Averaging Operator

In order to analyze the convergence rates, we give a more precise definition of the essential convergence below.

DEFINITION 2.1. Let u^ε represent a family of oscillatory solutions parameterized by ε, and let u_h^ε be a numerical approximation of u^ε with grid step size h. The numerical approximations u_h^ε converges of order α in norm $\|\cdot\|$ as $h \to 0$ essentially independent of ε to u^ε, if there are an increasing continuous function $\nu(\tau)$ with $\nu(0) = 0$ and a constant C independent of ε and h, such that for all $\varepsilon > 0$ and $\tau > 0$

$$\mu_\tau\{h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq C h^\alpha\} \leqq \nu(\tau) \tag{2.1}$$

where μ_τ is the normalized Lebesgue measure on $(0,\tau)$ and defined by $\mu_\tau(A) = \frac{1}{\tau}\left|A \cap (0,\tau)\right|$.

Sampling lemma. Our main attention will be focused on periodic oscillations with two scales which appear in the coefficients, the initial values, and the forcing terms of the differential equations. We use functions of the form $a(x, x/\varepsilon)$ to represent the oscillations, where $x = (x^1, x^2, \dots, x^d)^\top \in R^d$ and $a(x,y)$ is unit periodic in all the components of $y = (y^1, y^2, \dots, y^d)^\top$. We denote $x_j = (j^1 h, j^2 h, \dots, j^d h)^\top$, the computation grid in the domain $Q = [0,1]^d$, where h is the grid size, $nh = 1$, and $j = (j^1, j^2, \dots, j^d)^\top \in Z^d$. We denote by C a generic constant which may not be the same in different formulas.

We first state the following sampling lemma. Some similar results can be found in [13], Section 5, [7], Theorem 1, and [2], Lemma 1, except for the order estimates and for allowing rational dependence between ε and h.

LEMMA 2.1. *Let $a(x,y) \in W^{1,\infty}(R^d_x, W^{3,\infty}(Q_y))$ with unit period in all the components of y. For any $\varepsilon > 0$ and*

$$h \in \mathscr{S}_\alpha(\varepsilon) = \left\{ h>0 : \left|\frac{h}{\varepsilon} - \frac{\ell}{k}\right| \geqq \frac{h^{1-\alpha}}{6k^{5/2}}, \quad \forall\ \ell, k \in Z,\ k \neq 0,\ |k| \leqq h^{-\alpha/4}, \right\}, \tag{2.2}$$

we have

$$\max_{0 \leqq J \leqq n} \left| \sum_{0 \leqq j \leqq J-1} a(x_j, x_j/\varepsilon)\, h^d - \int_{0 \leqq x \leqq x_J} \int_Q a(x,y)\, dy\, dx \right| \leqq C h^\alpha \left\|a\right\|_{W^{1,\infty}(R^d_x, W^{3,\infty}(Q_y))} \tag{2.3}$$

and

$$\mu_\tau(\mathscr{S}_\alpha(\varepsilon)) \geqq 1 - \tau^{1-\alpha} \tag{2.4}$$

where j and J are multi-index with each component of values $0, 1, \dots, n$.

Results of the type in Lemma 2.1 are standard in ergodic theory; see [1]. We sketch a few points here and refer to [2] and [12] for a more detailed discussion. Expanding $a(x,y)$ in a Fourier series and noting the rapid decrease of the Fourier coefficients, one has

$$\begin{aligned} &\left| \sum_{0 \leqq j \leqq J-1} a(x_j, x_j/\varepsilon)\, h^d - \int_{0 \leqq x \leqq x_J} \int_Q a(x,y)\, dy\, dx \right| \\ &\leqq C \sum_{\substack{m \neq 0 \\ |m| \leqq h^{-\alpha/4}}} \frac{h}{m^3 \left|1 - e^{2\pi i m h/\varepsilon}\right|} + O(h^\alpha) \\ &\leqq \frac{6h^\alpha}{\pi} \sum_{m \neq 0} \frac{1}{|m|^{3/2}} + O(h^\alpha) = O(h^\alpha) \end{aligned}$$

for any $\varepsilon > 0$ and $h \in \mathscr{S}_\alpha(\varepsilon)$. The definition of set $\mathscr{S}_\alpha(\varepsilon)$ implies

$$(0,\tau)\setminus\mathscr{S}_\alpha(\varepsilon) \subset \left\{ h \in (0,\tau) : \left| h - \frac{\ell\varepsilon}{k} \right| < \frac{\tau^{1-\alpha}\varepsilon}{6k^{5/2}}, \text{ for some } k \geqq 1, 0 \leqq \ell \leqq \frac{k\tau}{\varepsilon} \right\}$$
$$\subset \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{[k\tau/\varepsilon]} \left(\frac{\ell\varepsilon}{k} - \frac{\tau^{1-\alpha}\varepsilon}{6k^{5/2}}, \frac{\ell\varepsilon}{k} + \frac{\tau^{1-\alpha}\varepsilon}{6k^{5/2}} \right).$$

This fact leads to

$$\frac{1}{\tau}\left|(0,\tau)\setminus\mathscr{S}_\alpha(\varepsilon)\right| \leqq \frac{1}{\tau}\sum_{k=1}^{\infty} \frac{2\varepsilon\tau^{1-\alpha}}{6k^{5/2}}\frac{\tau k}{\varepsilon} = \frac{\tau^{1-\alpha}}{3}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \leqq \tau^{1-\alpha}.$$

Remark 2.1.

(a) The above lemma implies that the summation $\sum_{0\leqq j\leqq J-1} a(x_j, x_j/\varepsilon)\, h^d$ converges in L^∞ of order α, $0 < \alpha < 1$, essential independent of ε to the integration $\int_{0\leqq x\leqq x_J}\int_Q a(x,y)\,dy\,dx$.

(b) In the lemma above, we restrict h/ε away from rational numbers with the denominator smaller than $h^{-\alpha/4}$. This restriction can always be achieved in practical calculations.

(c) The estimate of (2.2) can be represented by the following Koksma-Hlawka inequality in Monte-Carlo methods; see [13].

$$\left| \frac{1}{N}\sum_{j=1}^{N} f(x_j/\varepsilon) - \int_0^1 f(x)\,dx \right| \leqq TV(f)D_N^*$$

where $TV(f)$ is the total variation of function f and D_N^* is the *discrepancy* of the set $\{y_j\}$, $y_j = x_j/\varepsilon \mod 1$, which is a key concept in Monte Carlo methods and is defined by

$$D_N^* = \sup_{0\leqq t\leqq 1} \left| \frac{1}{N}\sum_{j=1}^{N} \chi_{[0,t)}(y_j) - t \right|.$$

(d) The techniques used here are common in ergodic analysis and the method of good lattice points in Monte Carlo methods; see [13]. The key concept is the *Diophantine* numbers, or strong irrational number; see [1].

Averaging operator. We denote u_j^n as an approximate numerical solution of $u(x,t)$ at point (x_j, t_n) which may be highly oscillatory. In order get the "mean" value, one needs to define the following averaging operator:

$$\mathscr{A}_h^\beta u_j^n = \sum_k \theta_k^\beta u_{j+k}^n h^d \tag{2.5}$$

where $\theta_k^\beta = h^{-\beta d}\theta(h^{-\beta}x_k)$ and $\theta(x)$ is a kernel function with support in the unit disc and $\int_{R^d}\theta(x)dx = 1$.

LEMMA 2.2. *Suppose $v(x) \in H_0^1(R^d)$. Let $\mathscr{A}_h^\beta$ be the averaging operator defined by* (2.5). *Then*

$$\left\|\mathscr{A}_h^\beta v(x_j) - v(x_j)\right\|_{L^2} \leqq Ch^\beta \|v\|_{H^1} . \tag{2.6}$$

Proof: One can get directly from the definition of averaging operator (2.5) that

$$\mathscr{A}_h^\beta v(x_j) - v(x_j) = \sum_{|k| \leqq h^{\beta-1}} \theta(kh^{1-\beta})\big(v(x_{j+k}) - v(x_j)\big)h^{(1-\beta)d} .$$

Schwartz inequality implies

$$\begin{aligned}
&\left(\mathscr{A}_h^\beta v(x_j) - v(x_j)\right)^2 \\
&\leqq \sum_{|k| \leqq h^{\beta-1}} \theta^2(kh^{1-\beta})h^{(1-\beta)d} \sum_{|k| \leqq h^{\beta-1}} \big(v(x_{j+k}) - v(x_j)\big)^2 h^{(1-\beta)d} \\
&\leqq C \sum_{|x_k| \leqq h^\beta} \big(v(x_{j+k}) - v(x_j)\big)^2 h^{(1-\beta)d} \\
&\leqq Ch^{-\beta d} \sum_{\|x_k\| \leqq h^\beta} \|x_k\|^2 h^d \int_0^1 v_x^2(x_j + tx_k)\, dt .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathscr{A}_h^\beta v - v\|_{L^2}^2 &\leqq Ch^{-\beta d} \sum_{\|x_k\| \leqq h^\beta} \|x_k\|^2 h^d \int_0^1 \left(\sum_j v_x^2(x_j + tx_k)h^d\right) dt \\
&\leqq Ch^{-\beta d} h^{(d+2)\beta} \|v\|_{H^1}^2 = C h^{2\beta} \|v\|_{H^1}^2 .
\end{aligned}$$

This completes the proof of the lemma.

LEMMA 2.3. *Suppose $a(x,y) \in W^{1,\infty}\left(R_x^d, W^{3,\infty}(Q_y)\right)$ and is unit periodic in all the components of y. Let $\mathscr{S}_\alpha(\varepsilon)$ be the set defined by* (2.2), *$\mathscr{A}_h^\beta$ be the averaging operator defined by* (2.5), *and $\bar{a}(x) = \int_{Q_y} a(x,y)\, dy$. Then*

$$\max_j \left| \mathscr{A}_h^\beta a(x_j, x_j/\varepsilon) - \bar{a}(x_j) \right| \leqq C\left(h^{\alpha(1-\beta)} + h^\beta\right) \|a\|_{W^{1,\infty}\left(R_x^d, W^{3,\infty}(Q_y)\right)} \tag{2.7}$$

is valid for any $\varepsilon > 0$ and $h \in \mathscr{S}_\alpha(\varepsilon)$.

Proof: We decompose the error between the averaging operator and the mean value into the following

$$\begin{aligned}
\mathscr{A}_h^\beta a(x_j, x_j/\varepsilon) - \bar{a}(x_j) &= \sum_k \theta_k^\beta h^d \left(a(x_{j+k}, x_{j+k}/\varepsilon) - a(x_j, x_{j+k}/\varepsilon)\right) \\
&\quad + \sum_k \theta_k^\beta h^d \left(a(x_j, x_{j+k}/\varepsilon) - \bar{a}(x_j)\right) \equiv I_1 + I_2 .
\end{aligned}$$

Since the support of the kernel function θ^β is in $[-h^\beta, h^\beta]$, the first term can be estimated directly,

$$I_1 \leqq \sum_k \theta_k^\beta h^d \|x_k\| \|a\|_{W^{1,\infty}} \leqq h^\beta \|a\|_{W^{1,\infty}} .$$

Denoting $h_1 = h^{1-\beta}$ and $\varepsilon_1 = \varepsilon h^{-\beta}$, one can easily see that $h \in \mathscr{S}_\alpha(\varepsilon)$ always imply $h_1 \in \mathfrak{S}_\alpha(\varepsilon_1)$. Hence, applying Lemma 2.2 to function $a(x_j, y)$, one gets

$$\begin{aligned} I_2 &= \sum_k \theta\left(kh^{1-\beta}\right) a(x_j, (j+k)h/\varepsilon)\, h^{(1-\beta)d} - \bar{a}(x_j) \\ &= \sum_k \theta(kh_1) a(x_j, (j+k)h_1/\varepsilon_1) h_1^d - \bar{a}(x_j) \\ &\leqq C h_1^\alpha \|a(x_j, \cdot)\|_{W^{3,\infty}(Q_y)} \leqq C h^{\alpha(1-\beta)} \|a\|_{W^{1,\infty}\left(R_x^d, W^{3,\infty}(Q_y)\right)} . \end{aligned}$$

The lemma now follows directly from the above two estimates.

Some equivalent statements. The following three propositions discuss the relation between Definition 1.1, Definition 2.1, and some of their equivalent statements.

PROPOSITION 2.1. *Let u_h^ε represent the numerical approximations to the oscillatory solutions u^ε with grid step h, ε represent the wave length of oscillations in the solutions. Let μ_τ be the normalized Lebesgue measure on $(0, \tau)$. Then the following are equivalent.*

(1) *As $h \to 0$, the approximations u_h^ε converge to u^ε essentially independent of ε.*

(2) *For any $\delta > 0$, there exist $\tau_0 < 1$ independent of ε such that, for any $0 < \varepsilon < 1$,*

$$\mu_{\tau_0}\{ h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta\} \leqq \delta . \tag{2.8}$$

(3) *For any $\delta > 0$, there exist a sequence $\{\tau_n\}$, $\tau_n \to 0$, independent of ε such that, for any $0 < \varepsilon < 1$,*

$$\mu_{\tau_n}\{ h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta\} \leqq \delta . \tag{2.9}$$

Proof: Clearly, (2) is just a restatement of (1). (3) $\Longrightarrow$ (2) is obvious. We only need to show (2) $\Longrightarrow$ (3). If (3) is not true, then there are $\delta_0 > 0$, $\varepsilon_0 > 0$, and $\tau^* < 1$ such that

$$\mu_\tau\{ h > 0 \mid \|u^{\varepsilon_0} - u_h^{\varepsilon_0}\| \geqq \delta_0\} > \delta_0 , \qquad 0 < \tau \leqq \tau^* . \tag{2.10}$$

From (2), for $\delta = \tau^* \delta_0$ there is a $\tau_0 < 1$ such that

$$\mu_{\tau_0}\{ h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta\} \leqq \delta .$$

Let $\tau = \min(\tau_0, \tau^*)$. From (2.10) it follows that

$$\begin{aligned} \tau^* \delta_0 = \delta &\geqq \mu_{\tau_0}\{ h > 0 \mid \|u^{\varepsilon_0} - u_h^{\varepsilon_0}\| \geqq \delta\} \\ &\geqq \mu_{\tau_0}\{ h > 0 \mid \|u^{\varepsilon_0} - u_h^{\varepsilon_0}\| \geqq \delta_0 \} \\ &\geqq \frac{\tau}{\tau_0}\mu_\tau\{ h > 0 \mid \|u^{\varepsilon_0} - u_h^{\varepsilon_0}\| \geqq \delta_0 \} > \frac{\tau}{\tau_0}\delta_0 \geqq \tau^* \delta_0 . \end{aligned}$$

Above we have used the fact that $\delta \leqq \delta_0$ and $\tau_0 \leqq 1$, and we get a contradiction.

PROPOSITION 2.2. *Let u_h^ε represent the numerical approximations to the oscillatory solutions u^ε with grid step h, ε represent the wave length of oscillations in the solutions. Let μ_τ be the united Lebesgue measure on $(0,\tau)$. Then the following are equivalent.*

(1) *If there is a monotone continuous function $\nu(\tau)$ with $\nu(0) = 0$ and there are measurable sets $\mathscr{S}(\varepsilon) \subset (0,1)$ with measure $\mu_\tau \mathscr{S}(\varepsilon) \geqq 1 - \nu(\tau)$ for all τ, $\varepsilon \in (0,1)$ and if for any $\delta > 0$, there is a $\tau_0 > 0$ such that $\| u^\varepsilon - u_h^\varepsilon \| \leqq \delta$ for all $0 < \varepsilon < 1$ and $h \in \mathscr{S}(\varepsilon) \cap (0, \tau_0)$.*

(2) *For any $\delta > 0$, there exist $0 < \tau_0 < 1$ independent of ε such that*

$$\mu_\tau\{ h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta \} \leqq \delta , \qquad \forall\, \tau \leqq \tau_0 , \tag{2.11}$$

is valid for all $0 < \varepsilon < 1$.

Proof: Assume (1) holds. There is a monotonic continuous function $\nu(\tau)$ with $\nu(0) = 0$ and measurable sets $\mathscr{S}(\varepsilon)$ such that for any $\delta > 0$ there exist τ_0, $0 < \nu(\tau_0) < \delta$, $\| u^\varepsilon - u_h^\varepsilon \| \leqq \delta$ is valid for all $0 < \varepsilon < 1$, and $h \in \mathscr{S}(\varepsilon) \cap (0, \tau_0)$. This fact implies

$$\left\{ 0 < h < \tau \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta \right\} \subset (0,\tau) \setminus \mathscr{S}(\varepsilon) , \qquad \tau < \tau_0 .$$

Therefore,

$$\mu_\tau\{h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta\} \leqq \nu(\tau) \leqq \nu(\tau_0) \leqq \delta .$$

Thus, (2) holds. Now, we show (2) $\implies$ (1). From (2), we know that there is a monotonic continuous function $\gamma(\delta)$ with $\gamma(0) = 0$ such that, for any $0 < \varepsilon < 1$ and $0 < \delta < 1$,

$$\mu_\tau\{ h > 0 \mid \|u^\varepsilon - u_h^\varepsilon\| \geqq \delta \} \leqq \delta , \qquad \forall\, \tau \leqq \gamma(\delta) .$$

Denote $\nu(\tau)$ as the inverse function of $\gamma(\delta)$, which is also a monotone continuous function and $\nu(0) = 0$. Construct sets

$$\mathscr{S}_n(\varepsilon) = \left\{ \gamma(2^{-n}) \leqq h < \gamma(2^{-n+1}) \;\middle|\; \|u^\varepsilon - u_h^\varepsilon\| < 2^{-n-1} \right\}$$

and

$$\mathscr{S}(\varepsilon) = \bigcup_{n=0}^{\infty} \mathscr{S}_n(\varepsilon) .$$

It is easy to check that $\mathscr{S}(\varepsilon) \subset (0,1)$ and

$$| \mathscr{S}_n(\varepsilon) | \geqq \gamma(2^{-n+1}) - \gamma(2^{-n}) - 2^{-n-1}\gamma(2^{-n+1}) .$$

For any $0 < \tau < 1$, there is an integer n, $\gamma(2^{-n}) \leqq \tau < \gamma(2^{-n+1})$, such that

$$\mathscr{S}(\varepsilon) \cap (0,\tau) \supset \left(\bigcup_{k=n+1}^{\infty} \mathscr{S}_k(\varepsilon) \right) \bigcup \left(\mathscr{S}_n(\varepsilon) \cap (0,\tau) \right) .$$

Therefore,

$$\begin{aligned}
&| \mathscr{S}(\varepsilon) \cap (0,\tau) | \\
&\geqq \sum_{k=n+1}^{\infty} | \mathscr{S}_k(\varepsilon) | + | \mathscr{S}_n(\varepsilon) \cap (0,\tau) | \\
&\geqq \sum_{k=n+1}^{\infty} \Big(\gamma(2^{-k+1}) - \gamma(2^{-k}) - 2^{-k-1}\gamma(2^{-k+1}) \Big) + \tau - \gamma(2^{-n}) - 2^{-n-1}\tau \\
&\geqq \tau - \sum_{k=n+1}^{\infty} 2^{-k}\tau = \tau(1 - 2^{-n}) = \tau\big(1 - \nu(\gamma(2^{-n}))\big) \geqq \tau\big(1 - \nu(\tau)\big) .
\end{aligned}$$

For any $\delta > 0$, there is an integer n, $2^{-n} \leqq \delta < 2^{-n+1}$, such that

$$\| u^\varepsilon - u_h^\varepsilon \| \leqq 2^{-n} \leqq \delta$$

for $0 < \varepsilon < 1$, $h \in \mathscr{S}(\varepsilon) \cap (0, h_0)$, and $h_0 = \gamma(2^{-n})$. (1) now holds.

PROPOSITION 2.3. *The statement* (2.11) *is stronger than the definition of essential convergence.*

Proof: We construct the following sequence of functions

$$u^h = \begin{cases} 1 , & a_{n+1} \leqq h < b_n , \\ 2^{-2n-1} , & b_n \leqq h < a_n , \end{cases}$$

where $a_n = 2^{-n^2}$ and $b_n = 2^{-n^2-2n}$. For any $\delta > 0$, there is a n, $2^{-2n} \leqq \delta < 2^{-2n-2}$, such that for $\tau_0 = a_n$

$$\mu_{\tau_0}\{h > 0 \mid |u^h| \geqq \delta\} = \frac{1}{a_n}\sum_{k=n}^{\infty}(b_k - a_{k+1}) = \frac{1}{2^{-n^2}}\sum_{k=n}^{\infty} 2^{-(k+1)^2} \leqq 2^{-2n} \leqq \delta .$$

Therefore, the statement (2.8) holds. While for $\delta = 1/2$ one has

$$\mu_{b_n}\{h > 0 \mid |u^h| \geqq \delta\} = \frac{1}{b_n}\sum_{k=n}^{\infty}(b_k - a_{k+1}) = \frac{1}{2^{-n^2-2n}}\sum_{k=n}^{\infty} 2^{-(k+1)^2} > \frac{1}{2} = \delta .$$

The statement (2.11) can not be true for all $0 < \tau_0 < 1$.

3. Hyperbolic Systems with Rapidly Oscillatory Data

In this section we shall discuss the singular homogenization problems or large amplitude oscillation problems. We shall first prove the weak convergence for consistent and stable difference schemes for the linear systems with oscillatory initial values and forcing terms. Then, we shall discuss the damping and propagation of the oscillations in dissipative and unitary schemes.

Convergence of difference schemes. For the linear hyperbolic system with oscillatory initial values and forcing terms,

$$(3.1) \qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \displaystyle\sum_{m=1}^{d} A_m(x,t)\frac{\partial u^\varepsilon}{\partial x_m} + B(x,t)u^\varepsilon = f(x,t,x/\varepsilon) , \\ u^\varepsilon(x,0) = u_0(x,x/\varepsilon) , \end{cases}$$

we use the difference scheme,

$$(3.2) \qquad \begin{cases} u_j^{n+1} = \displaystyle\sum_k \beta_k(x_j,t_n)u_{j+k}^n + f(x_j,t_n,x_j/\varepsilon)\Delta t , \\ u_j^0 = u_0(x_j,x_j/\varepsilon) , \end{cases}$$

as the approximation scheme, where $u_0(x,y)$ and $f(x,t,y)$ are smooth functions of compact support with respect to x and unit period in all components of y. It is known that the solution of (3.1) converges weakly in L^∞ to the following homogenized equation,

$$(3.3) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{m=1}^{d} A_m(x,t)\frac{\partial u}{\partial x_m} + B(x,t)u = \bar{f}(x,t) , \\ u(x,0) = \bar{u}_0(x) , \end{cases}$$

where $\bar{u}_0(x) = \int_Q u_0(x,y)\,dy$ and $\bar{f}(x,t) = \int_Q f(x,t,y)\,dy$. Denote the solution operators of (3.3) and (3.2) by $S(t,t_1)$ and S_n, respectively. The solutions of them can be represented by

$$u(x,t) = S(t,0)\,\bar{u}_0(x) + \int_0^t S(t-s,s)\bar{f}(x,s)\,ds$$

and

$$u^n = \prod_{k=0}^{n-1} S_k\, u_0^\varepsilon + \sum_{\ell=0}^{n-1}\prod_{k=\ell}^{n-1} S_k f_\ell^\varepsilon \Delta t\ . \tag{3.4}$$

We shall assume that the equation (3.3) is well posed in both L_2 and H^1,

$$\|u(\cdot,t)\|_{L^2} \leqq C\|u(\cdot,0)\|_{L^2} \quad \text{and} \quad \|u(\cdot,t)\|_{H^1} \leqq C\|u(\cdot,0)\|_{H^1}\ ,$$

and assume the difference scheme (3.2) is stable,

$$\prod_{k=n_1}^{n_2} \|S_k\|_{L^2} \leqq C\ , \tag{3.5}$$

and consistent to (3.3),

$$\left\|\left(S(\Delta t,t_n) - S_n\right)u(\cdot,t_n)\right\|_{L^2} \leqq C\,h^2\|u(\cdot,t_n)\|_{H^1}\ . \tag{3.6}$$

We also use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^{-1}}$ as discrete L_2-norm and H^{-1}-norm, respectively.

THEOREM 3.1. *Let u_j^n be the numerical solution of* (3.2), *$u(x,t)$ be the solution of the homogenized equation* (3.3), *$\mathcal{S}_\alpha(\varepsilon)$ be the set defined by* (2.2), *and $\mathcal{A}_h^\beta$ be the averaging operator defined by* (2.5). *Then, for any $0<\alpha<1$, $0<\beta<1$, $\varepsilon>0$, and $h \in \mathcal{S}_\alpha(\varepsilon)$, we have*

$$\left\|\mathcal{A}_h^\beta u_j^n - u(x_j,t_n)\right\|_{H^{-1}} \leqq C\left(h^{\alpha(1-\beta)} + h^\beta\right). \tag{3.7}$$

Furthermore, if the coefficient matrices A_m, $m = 1,2,\ldots,d$, and B in (3.1) *are constant matrices, then*

$$\left\|\mathcal{A}_h^\beta u_j^n - u(x_j,t_n)\right\|_{L^2} \leqq C\left(h^{\alpha(1-\beta)} + h^\beta\right) \tag{3.8}$$

where $C = C_1\left(\|u_0\|_{W^{1,\infty}(R_x^d,W^{3,\infty}(Q_y))} + \|f\|_{W^{1,\infty}(R_x^d\times R,W^{3,\infty}(Q_y))}\right)$ and C_1 is a constant independent of h and ε.

Proof: (1) For the constant coefficient case, the average operator and solution operator are commutate,

$$\mathcal{A}_h^\beta S_n u_j = \sum_k \theta_k^\beta \sum_\ell \beta_\ell u_{j+\ell+k} = \sum_\ell \beta_\ell \sum_k \theta_k^\beta u_{j+k+\ell} = S_n \mathcal{A}_h^\beta u_j\ .$$

Decompose the error between the average of the numerical solution and the homogenized solution into

$$
\begin{aligned}
\mathscr{A}_h^\beta u^n - u(\cdot, t_n) &= S_h^n \left(\mathscr{A}_h^\beta u_0^\varepsilon - \bar{u}_0\right) \\
&\quad + \sum_{\ell=0}^{n-1} S_h^{n-\ell-1} \left(\mathscr{A}_h^\beta f_\ell^\varepsilon - \bar{f}_\ell\right) \Delta t + \left(S_h^n - S(t_n)\right) \bar{u}_0 \\
&\quad + \sum_{\ell=0}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \left(S_h^{n-\ell-1} - S(t-s)\right) \bar{f}(x,s)\, ds \\
&\equiv I_1 + I_2 + I_3 + I_4 \,.
\end{aligned}
$$

Lemma 2.3 and the stability (3.5) tell us that

$$
I_1 \leqq \left\|\mathscr{A}_h^\beta u_0^\varepsilon - \bar{u}_0\right\|_{L^2} \leqq C\left(h^{\alpha(1-\beta)} + h^\beta\right) \|u_0\|_{W^{3,\infty}} \,.
$$

The estimate of I_3 follows from the consistent condition (3.6) and the well-posedness of (3.3) in H^1,

$$
I_3 = \left\|\left(S(t_n) - S_h^n\right)\bar{u}_0\right\|_{L^2} \leqq Ch \max_k \|S(t_k)\bar{u}_0\|_{H^1} \leqq Ch\|\bar{u}_0\|_{H^1} \,.
$$

One can similarly get estimates for I_2 and I_4 and thus (3.8) holds.

(2) For the smooth variable coefficient case, decompose the error terms into

$$
\begin{aligned}
&\mathscr{A}_h^\beta u^n - u(\cdot, t_n) \\
&\quad = \left(\mathscr{A}_h^\beta \prod_{k=1}^{n-1} S_k - \prod_{k=1}^{n-1} S_k \mathscr{A}_h^\beta\right) u_0^\varepsilon \\
&\quad\quad + \sum_{\ell=0}^{n-1}\left(\mathscr{A}_h^\beta \prod_{k=\ell}^{n-1} S_k - \prod_{k=\ell}^{n-1} S_k \mathscr{A}_h^\beta\right) f_\ell^\varepsilon \Delta t \\
&\quad\quad + \prod_{k=0}^{n-1} S_k \left(\mathscr{A}_h^\beta u_0^\varepsilon - \bar{u}_0\right) + \sum_{k=0}^{n-1} \prod_{k=\ell}^{n-1} S_k \left(\mathscr{A}_h^\beta f_\ell^\varepsilon - \bar{f}_\ell\right) \Delta t \\
&\quad\quad + \left(\prod_{k=0}^{n-1} S_k - S(t_n, 0)\right) \bar{u}_0 \\
&\quad\quad + \sum_{\ell=0}^{n-1} \int_{t_\ell}^{t_{\ell+1}} \left(\prod_{k=\ell}^{n-1} S_k - S(t-s, s)\right) \bar{f}(x,s)\, ds \\
&\quad \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \,.
\end{aligned} \tag{3.9}
$$

The estimates of the last four terms on the right-hand side of the above are the same as the estimates in the constant coefficient case, i.e.,

$$
|I_3 + I_4 + I_5 + I_6| \leqq C\left(h^{\alpha(1-\beta)} + h^\beta\right) \,. \tag{3.10}
$$

For any $v \in H^1$, duality gives

$$\langle v, I_1\rangle = \left\langle u_0^\varepsilon, \left(\prod_{k=0}^{n-1} S_{n-k}^* \mathscr{A}_h^\beta - \mathscr{A}_h^\beta \prod_{k=0}^{n-1} S_{n-k}^*\right) v\right\rangle$$
$$= \left\langle u_0^\varepsilon, \prod_{k=0}^{n-1} S_{n-k}^* \left(\mathscr{A}_h^\beta - I\right) v + \left(I - \mathscr{A}_h^\beta\right) S^*(t_n) v\right.$$
$$\left. + \left(I - \mathscr{A}_h^\beta\right)\left(\prod_{k=0}^{n-1} S_{n-k}^* - S^*(t_n)\right) v\right\rangle .$$

One has from the stability (3.5) that

$$|\langle v, I_1\rangle| \leqq C\|u_0^\varepsilon\|_{L^2}\left(\left\|\left(\mathscr{A}_h^\beta - I\right) v\right\|_{L^2} + \left\|\left(\mathscr{A}_h^\beta - I\right) S^*(t_n) v\right\|_{L^2}\right.$$
$$\left. + \left\|\left(S^*(t_n) - \prod_{k=0}^{n-1} S_{n-k}^*\right) v\right\|_{L^2}\right).$$

Similar to the constant coefficient case, one has estimates

$$\left\|\left(\mathscr{A}_h^\beta - I\right) v\right\|_{L^2} \leqq Ch^\beta \|v\|_{H^1} ,$$
$$\left\|\left(\mathscr{A}_h^\beta - I\right) S^*(t_n) v\right\|_{L^2} \leqq Ch^\beta \|S^*(t_n) v\|_{H^1} \leqq Ch^\beta \|v\|_{H^1} ,$$
$$\left\|\left(S^*(t_n) - \prod_{k=0}^{n-1} S_{n-k}^*\right) v\right\|_{L^2} \leqq Ch \max_k \|S^*(t_k) v\|_{H^1} \leqq Ch\|v\|_{H^1} .$$

Therefore

$$|\langle v, I_1\rangle| \leqq Ch^\beta \|u_0^\varepsilon\|_{L^2} \|v\|_{H^1}$$

or

$$\|I_1\|_{H^{-1}} \leqq Ch^\beta \|u_0^\varepsilon\|_{L^2} . \tag{3.11}$$

Similarly, one can get the following estimate for I_2,

$$\|I_2\|_{H^{-1}} \leqq C \sum_{\ell=0}^{n-1} h^\beta \|f_\ell^\varepsilon\|_{L^2} \leqq Ch^\beta \|f^\varepsilon\|_{L^\infty(0,T,L^2)} .$$

Now, (3.7) follows directly from (3.9), (3.10), (3.11) and above inequality. This completes the proof of the theorem.

Damping and propagation of oscillations. In the analysis of the previous subsection we showed that the presence of oscillations in the general cases does not distort the convergence of the averaged approximation to the ε-weak limit as $h \to 0$. But this does not mean that we resolved the oscillations. The group velocity for the differential equation and the corresponding discretization may be quite different; see Figure 1(c). We shall in this subsection briefly consider another measure of the shape of the solutions. We want a measure that gives more information on the size and the location of the oscillations than the local average. This is needed for an understanding of approximations to nonlinear equations. It is natural to study the powers or the moments of the solution first. We shall give two simple results regarding the weak limit of the squares of the solutions. The purpose is to point out the effect of errors in the amplitude or in the location of the oscillations due to numerical dissipation and numerical dispersive. We shall show that the oscillations in dissipative schemes are almost vanished after a few time steps and the oscillations in the dispersive schemes have wrong group velocities.

For the constant coefficient case, the coefficients β_i in (3.2) become constants. We denote

$$\hat{S}_h(\omega) = \sum_k \beta_k \exp\{ik\omega h\} \equiv \rho(\omega)\exp\{i\Omega(\omega)\Delta t\}$$

as the symbol for the difference operator. We say the difference scheme (3.2) is dissipative if

$$\left|\hat{S}_h(\omega)\right| \leqq 1 - \delta|\omega h|^{2r}, \qquad \forall\ |\omega h| \leqq \pi, \tag{3.12}$$

for some constants $\delta > 0$ and some positive integers r. We say the difference scheme (3.2) is unitary if

$$\left|\hat{S}_h(\omega)\right| \equiv 1. \tag{3.13}$$

We apply the difference scheme (3.2) to the following scalar equation in one space dimension,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \\ u(x,0) = \sqrt{2}\sin(x)\sin(x/\varepsilon). \end{cases} \tag{3.14}$$

Denoting $h/\varepsilon = \omega h \mod 2\pi$, we have

$$\begin{aligned} S_h \exp\{ix/\varepsilon\} &= \sum_k \beta_k \exp\{i(x+kh)/\varepsilon\} \\ &= \sum_k \beta_k \exp\{ik\omega h\}\exp\{ix/\varepsilon\} = \hat{S}_h(\omega)\exp\{ix/\varepsilon\}. \end{aligned}$$

Simple calculations lead to

$$\begin{aligned} &S_h^n \sqrt{2}\sin(x)\sin(x/\varepsilon) \\ &= \rho^n(\omega)\sqrt{2}\sin\left(x - \frac{\Omega(\omega_+) - \Omega(\omega_-)}{2}t_n\right)\sin\left(\frac{x}{\varepsilon} - \frac{\Omega(\omega_+) + \Omega(\omega_-)}{2}t_n\right) \end{aligned}$$

where $h/\varepsilon \pm h = \omega_\pm h \mod 2\pi$.

The most information on the size and the location of the oscillations is contained by the weak limit of $(u^n)^2$ which is given by

$$\mathscr{A}_h^\beta \left(S_h^n \sqrt{2} \sin(x) \sin(x/\varepsilon)\right)^2 \to \rho^{2n}(\omega) \sin^2 \left(x - \frac{\Omega(\omega_+) + \Omega(\omega_-)}{2} t_n\right) \tag{3.15}$$

as $h \to 0$. The error in the size and the location of the oscillation can be described by

$$E(x_j, t_n) = \mathscr{A}_h^\beta \left(u(x_j, t_n)^2 - (u_j^n)^2\right).$$

From (3.15) it follows that,

$$E(x_j, t_n) \to \sin^2(x - t_n) - \rho^{2n}(\omega) \sin^2 \left(x - \frac{\Omega(\omega_+) - \Omega(\omega_-)}{2} t_n\right) \tag{3.16}$$

as $h \to 0$. For the dissipative scheme, if h and ε satisfy $|1 - \exp\{ih/\varepsilon\}| \geqq c_0$, for some $c_0 > 0$, then $|\omega h| \geqq \arcsin(c_0)$. One can get from (3.12) that

$$\rho(\omega) \leqq 1 - \delta \arcsin^{2r}(c_0) \equiv \delta_0 < 1 .$$

Therefore,

$$\|E(\cdot, t_n)\|_{L^\infty} \geqq 1 - \delta_0^{2n} .$$

For the unitary scheme, where $\rho(\omega) = 1$, if the order accuracy of the difference approximation is finite, then there exists an interval I such that

$$\left(\Omega(\omega_+) - \Omega(\omega_-)\right)/2 \neq 1$$

for $h/\varepsilon \in I$. From (3.16), we know there exists a constant $\delta_0 > 0$ independent of ε and h such that

$$|E(x_j, t_n)| \geqq \delta_0 \, t_n$$

for $h/\varepsilon \in I$, $t_n \leqq T$. Thus, we have the following theorem.

THEOREM 3.2. *Let $E(x_j, t_n)$ be the error function defined by* (3.15) *for the scheme* (3.2) *approximating* (3.14).

(1) *If* (3.2) *is a dissipative scheme and h and ε satisfy $|1 - \exp\{ih/\varepsilon\}| \geqq c_0$, for some $c_0 > 0$, then there exists a constant $\delta_0 < 1$ independent of ε and h such that*

$$|E(x_j, t_n)| \geqq 1 - \delta_0^{2n} .$$

(2) *If* (3.2) *is a unitary scheme, then*

$$|E(x_j, t_n)| \leqq C t_n$$

for some C independent of h and ε. Furthermore, if the order of accuracy of (3.2) *is finite, then there exist constants $\delta_0 > 0$, $T > 0$ and an interval I such that*

$$|E(x_j, t_n)| \geqq \delta_0 \, t_n$$

for $h/\varepsilon \in I$, $t_n \leqq T$.

4. Hyperbolic Equations with Rapidly Oscillatory Coefficients

In this section, we shall discuss the regular homogenization problems or small amplitude oscillation problems. We shall consider difference methods and particle methods for linear hyperbolic equations with oscillatory coefficients. First, we shall prove essential convergence of the upwind scheme for the linear systems. Then we discuss stability and convergence for more general schemes in the one-dimensional scalar case. Finally, we shall consider the particle methods for multidimensional scalar problems.

Consider the following linear hyperbolic system with micro inhomogeneous coefficients

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + A^\varepsilon(x,t)\dfrac{\partial u^\varepsilon}{\partial x} + B(x,t)\,u^\varepsilon = f(x,t)\,, \\ u^\varepsilon(x,0) = u_0(x)\,. \end{cases} \tag{4.1}$$

In many cases, the solution of (4.1) converges to that of a homogenized equation (see [5]),

$$\begin{cases} \dfrac{\partial u}{\partial t} + A^*(x,t)\dfrac{\partial u}{\partial x} + B(x,t)u = f(x,t)\,, \\ u(x,0) = u_0(x)\,. \end{cases} \tag{4.2}$$

In the periodic oscillatory case, e.g., $A^\varepsilon(x,t) = A(x,t,x/\varepsilon)$ and $A(x,t,y)$ is a smooth function with unit period in y, we know the corresponding homogenized coefficient is

$$A^*(x,t) = \left(\int_0^1 A^{-1}(x,t,y)\,dy\right)^{-1} \tag{4.3}$$

and the solutions of (4.1) converge strongly in L^∞ to the solution of (4.2), (4.3); see [12]. We approximate to (4.1) with the following difference method,

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{A^\varepsilon(x_j,t_n)}{h} \displaystyle\sum_{|k|\leqq k_0} (\beta_k^1 u_{j+k}^{n+1} + \beta_k^0 u_{j+k}^n) \\ \qquad\qquad + B(x_j,t_n)u_j^n = f(x_j,t_n)\,, \\ u_j^0 = u_0(x_j)\,, \end{cases} \tag{4.4}$$

where $x_j = jh$ and $t_n = n\Delta t$. For the upwind scheme, assuming A^ε is positive for simplicity, we take

$$\beta_0^0 = 1\,, \qquad \beta_{-1}^0 = -1\,; \tag{4.5}$$

for the Crank-Nicholson scheme, we take

$$\beta_1^0 = \beta_1^1 = 1/4\,, \qquad \beta_{-1}^0 = \beta_{-1}^1 = -1/4\,; \tag{4.6}$$

for the backward Euler scheme, we take

$$\beta_1^1 = 1/2\,, \qquad \beta_{-1}^1 = -1/2\,. \tag{4.7}$$

In order to guarantee that the difference scheme (4.4) converges to the homogenized equation (4.2), we need a stability condition for (4.4) and a good sample of the oscillatory coefficients of the numerical schemes.

Convergence of the upwind scheme. We denote the sampling error as

$$\begin{aligned} S(\varepsilon,h) &= \max_{j,n}\left(|S^\varepsilon(x_j,t_n)| + \frac{|S^\varepsilon(x_j,t_{n+1}) - S^\varepsilon(x_j,t_n)|}{\Delta t} \right)\,, \\ S^\varepsilon(x_j,t_n) &= \sum_{k=1}^{j}\left((A^\varepsilon)^{-1}(x_k,t_n) - (A^*)^{-1}(x_k,t_n)\right) h\,. \end{aligned} \tag{4.8}$$

In the periodic oscillatory case, e.g., $A^\varepsilon(x,t) = A(x,t,x/\varepsilon)$, one has from Lemma 2.1 that

$$\|S(\varepsilon,h)\| \leqq C h^\alpha$$

for all $\varepsilon > 0$ and $h \in \mathscr{S}_\alpha(\varepsilon)$. We assume the following CFL condition is always held:

$$\frac{\Delta t}{h}\max_{x,t}\|A^\varepsilon(x,t)\| < 1\,. \tag{4.9}$$

THEOREM 4.1. *Assume an upwind scheme* (4.4), (4.5) *approximating* (4.1) *satisfies the CFL condition* (4.9). *Let* u_j^n *be the approximated numerical solution,* $u(x,t)$ *be the solution of the corresponding homogenized equation* (4.2), *and* $S(\varepsilon,h)$ *be the sampling error* (4.8). *Then we have*

$$\sup_{j,n} |\, u_j^n - u(x_j,t_n)\,| \leqq C\,(S(\varepsilon,h) + h)\,(\|\,u_0\,\|_{W^{2,\infty}} + \|\,f\,\|_{W^{1,\infty}}) \tag{4.10}$$

where C is a constant independent of ε *and h.*

Proof: For simplicity, we assume $\Delta t = h$. Let

$$w_j^n = U_j^n - u_j^n + S_j^n A_{n,j}^* \frac{U_j^n - U_{j-1}^n}{h} \tag{4.11}$$

where $S_j^n = S^\varepsilon(x_j,t_n)$, $A_{n,j}^* = A^*(t_n,x_j)$, and U_j^n is the solution of the following upwind scheme for the homogenized equation

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + A_{n,j}^* \dfrac{U_j^n - U_{j-1}^n}{h} + B_j^n U_j^n = f_j^n\,, \\ U_j^0 = u_0(x_j)\,. \end{cases} \tag{4.12}$$

Manipulating from (4.4), (4.5), (4.11), and (4.12), one gets

$$\tag{4.13} \begin{cases} \dfrac{w_j^{n+1} - w_j^n}{\Delta t} + A_{n,j}^{\varepsilon} \dfrac{w_j^n - w_{j-1}^n}{h} + B_j^n w_j^n = R_j^n \,, \\ w_j^0 = S_j^n A_{n,j}^* \dfrac{u_0(x_j) - u_0(x_{j-1})}{h} \,, \end{cases}$$

where

$$\begin{aligned} R_j^n = {} & \left(S_j^{n+1} A_{n+1,j}^* - A_{n,j}^{\varepsilon} S_{j-1}^n\right) \frac{U_j^{n+1} - U_j^n - U_{j-1}^{n+1} + U_{j-1}^n}{h\Delta t} \\ & + \left(B_j^n S_j^n A_{n,j}^* - A_{n,j}^{\varepsilon} S_{j-1}^n B_j^n + \frac{S_j^{n+1} A_{n+1,j}^* - S_j^n A_{n,j}^*}{\Delta t}\right) \frac{U_j^n - U_{j-1}^n}{h} \\ & + A_{n,j}^{\varepsilon} S_{j-1}^n \frac{f_j^n - f_{j-1}^n}{h} - A_{n,j}^{\varepsilon} S_{j-1}^n \frac{B_j^n - B_{j-1}^n}{h} U_{j-1}^n \,. \end{aligned}$$

From the smoothness of $A^*(x,t)$ and $B(x,t)$, one has estimate

$$\begin{aligned} |R_j^n| \leqq {} & C\left(|S_j^n| + |S_{j-1}^n| + |S_j^{n+1}| + \frac{|S_j^{n+1} - S_j^n|}{\Delta t}\right) \\ & \times \left(\frac{|U_j^{n+1} - U_j^n - U_{j-1}^{n+1} + U_{j-1}^n|}{h\Delta t} + \frac{|U_j^n - U_{j-1}^n|}{h} \right. \\ & \left. + \frac{|f_j^n - f_{j-1}^n|}{h} + |U_{j-1}^n|\right) . \end{aligned}$$

Since (4.12) is the scheme for the smooth equation (without small parameter ε in it), it is easy to get estimate

$$\begin{aligned} & |U_{j-1}^n| + \frac{|U_j^n - U_{j-1}^n|}{\Delta t} + \frac{|U_j^{n+1} - U_j^n - U_{j-1}^{n+1} + U_{j-1}^n|}{h\Delta t} \\ & \leqq C\left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right) . \end{aligned}$$

Therefore,

$$|R_j^n| \leqq C \max\left(|S_j^n| + \frac{|S_j^{n+1} - S_j^n|}{\Delta t}\right) \left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right) .$$

Clearly,

$$|w_j^0| \leqq C\,|S_j^n|\, \|u_0\|_{W^{1,\infty}} .$$

Together with the stability of scheme (4.13), one has

$$\begin{aligned} \max \|w_j^n\| & \leqq C \max\left(|R_j^n| + |w_j^0|\right) \\ & \leqq C \max\left(|S_j^n| + \frac{|S_j^{n+1} - S_j^n|}{\Delta t}\right) \left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right) \\ & \leqq C\, S(\varepsilon, h)\left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right) . \end{aligned}$$

From (4.11) and the above inequality, it follows that

$$\begin{aligned}\max |u_j^n - U_j^n| &\leqq \max\left(|w_j^n| + C|S_j^n|\frac{|U_j^n - U_{j-1}^n|}{h}\right)\\ &\leqq C\max\left(|w_j^n| + |S_j^n|\right)\left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right)\\ &\leqq C\left(S(\varepsilon,h)+h\right)\left(\|u_0\|_{W^{2,\infty}} + \|f\|_{W^{1,\infty}}\right) .\end{aligned}$$

This completes the proof of the theorem.

COROLLARY 4.1. *Assume the CFL condition is held and the coefficient matrix of the equation* (4.1) *has the form* $A^\varepsilon = A(x,t,x/\varepsilon)$, *where* $A(x,t,y)$ *is a smooth positive matrix with unit period in* y. *Then the upwind scheme* (4.4), (4.5) *is essential convergence of order* α *for all* $\alpha < 1$.

Remark 4.1. For the linear hyperbolic equation with coefficients oscillating at time variables,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + A(x,\,t,\,t/\varepsilon)\dfrac{\partial u^\varepsilon}{\partial x} + b(x,t)u^\varepsilon = f(x,t)\,, \\ u^\varepsilon(x,0) = u_0(x)\,, \end{cases}$$

where $A(x,t,\tau) > 0$ is a smooth matrix with unit period in τ, we can similarly prove the essential convergence for its standard upwind schemes. In this case, the corresponding homogenized coefficient matrix becomes

$$A^*(x,t) = \int_0^1 A(x,t,\tau)\,d\tau .$$

Stability analysis. We shall now discuss stability and convergence for the more general scheme (4.4). There are large classes of algorithms with stability even for the oscillatory coefficients. Monotone schemes, such as the upwind scheme and the Lax-Friedrichs scheme, satisfy the maximum principle and are stable even for oscillatory coefficients. For the L^2 stable schemes, such as the Crank-Nicholson scheme and the backward Euler scheme, the stability may be lost if the eigenvalues of the oscillatory coefficient matrix $A^\varepsilon(x_j,t_n)$ has both signs. For the Crank-Nicholson approximation with initial values and the function $A^\varepsilon(x)$ given below

$$u_j^\varepsilon = \begin{cases} \vdots & \vdots \\ 0 & j=-1 \\ -\alpha_0 & j=0 \\ \alpha_0 & j=1 \\ 0 & j=2 \\ \vdots & \vdots \end{cases} \qquad A_j^\varepsilon = \begin{cases} \vdots & \vdots \\ 0 & j=-1 \\ -1 & j=0 \\ 1 & j=1 \\ 0 & j=2 \\ \vdots & \vdots \end{cases},$$

the algorithm,

$$u_j^{n+1} + \frac{\Delta t A_j^\varepsilon}{4h}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n - \frac{\Delta t A_j^\varepsilon}{4h}\left(u_{j+1}^n - u_{j-1}^n\right),$$

has a solution u_j^n that blows up for $j = 0, 1$ independently of the values of u and A for $j < -1$ and $j > 2$,

$$u_j^n = \frac{\alpha^n}{\alpha_0} u_j^\varepsilon, \quad j = -1, 0, 1, 2, \quad \alpha = \frac{1 + \Delta t/(4h)}{1 - \Delta t/(4h)}.$$

Local blowups of this type are difficult to control with sampling properties. But if we modify the space differing

$$A\partial_x \to \frac{1}{2}\left(A_j D_0 + D_0 A_j\right)\left(u_j^{n+1} + u_j^n\right),$$

then the Crank-Nicholson scheme is stable in L^2.

Convergence for general schemes. The original consistency condition for the scheme (4.4) with smooth coefficients is given by

$$\sum_{|k|\leqq k_0} \beta_k^1 = \sum_{|k|\leqq k_0} \beta_k^0 = 0, \qquad \sum_{|k|\leqq k_0} \left(\beta_k^1 + \beta_k^0\right) k = 2.$$

Clearly, the difference scheme (4.4) is not consistent with the homogenized equation. In the case where the oscillatory coefficient $A^\varepsilon = A^\varepsilon(x)$ depends only on the space variables, we can show that some schemes, such as the upwind scheme, the Crank-Nicholson scheme, and the backward Euler scheme, are consistent with the homogenized equation after redefining the computational grid. More precisely, we need the following equivalence conditions: there is a set of grid points $\{y_j\}$ and numbers $S(\varepsilon, h)$, $S(\varepsilon, h)$ essentially converge to zero, such that

$$a^\varepsilon(x_j) \sum_{|k|\leqq k_0} \left(\beta_k^1 + \beta_k^0\right) y_{j+k} = a^*(x_j) h \tag{4.14}$$

and

$$|y_j - x_j| \leqq S(\varepsilon, h). \tag{4.15}$$

THEOREM 4.2. *Let $u(x,t)$ be the solution of* (4.2) *and u_j^n be the solution of scheme* (4.4). *Assume the coefficient $a^\varepsilon = a^\varepsilon(x)$ only dependent on x and the numerical scheme* (4.4) *is stable and satisfies equivalent condition* (4.14) (4.15). *Then, we have*

$$\| u^n - u(\cdot, t_n) \| \leqq C\left(S(\varepsilon, h) + h\right), \qquad \textit{for } t_n \leqq T \tag{4.16}$$

where C is a constant independent of ε and h.

Proof: The equivalent condition (4.14) gives

$$\frac{a^\varepsilon(x_j)}{h} = \frac{a^*(x_j)}{\sum_{|k| \leqq k_0} (\beta_k^1 + \beta_k^0) y_{j+k}} .$$

Plug the above identity into (4.4) to get

$$(4.17) \qquad \begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a^*(x_j) \dfrac{\sum_{|k| \leqq k_0} (\beta_k^1 u_{j+k}^{n+1} + \beta_k^0 u_{j+k}^n)}{\sum_{|k| \leqq k_0} (\beta_k^1 + \beta_k^0) y_{j+k}} \\ \qquad + b(x_j, t_n) u_j^n = f(x_j, t_n) , \\ u_j^0 = u_0(x_j) . \end{cases}$$

Taking y_j as computation grids, we construct the following scheme for the homogenized equation (4.2)

$$(4.18) \qquad \begin{cases} \dfrac{v_j^{n+1} - v_j^n}{\Delta t} + a^*(y_j) \dfrac{\sum_{|k| \leqq k_0} (\beta_k^1 v_{j+k}^{n+1} + \beta_k^0 v_{j+k}^n)}{\sum_{|k| \leqq k_0} (\beta_k^1 + \beta_k^0) y_{j+k}} \\ \qquad + b(y_j, t_n) v_j^n = f(y_j, t_n) , \\ v_j^0 = u_0(y_j) . \end{cases}$$

Since (4.2) is a smooth equations, i.e., does not involve small parameter ε, and scheme (4.18) are stable and consistent to (4.2), one has the following error estimate

$$(4.19) \qquad \| v_j^n - u(y_j, t_n) \| \leqq C \max |y_j - y_{j-1}| \leqq C \left(S(\varepsilon, h) + h \right) .$$

Now, we estimate the error between (4.17) and (4.18). Denote $w_j^n = u_j^n - v_j^n$. Subtracting (4.18) from (4.17), one gets

$$(4.20) \qquad \begin{cases} \dfrac{w_j^{n+1} - w_j^n}{\Delta t} + a^*(x_j) \dfrac{\sum_{|k| \leqq k_0} (\beta_k^1 w_{j+k}^{n+1} + \beta_k^0 w_{j+k}^n)}{\sum_{|k| \leqq k_0} (\beta_k^1 + \beta_k^0) y_{j+k}} \\ \qquad + b(x_j, t_n) w_j^n = \theta_j^n , \\ w_j^0 = u_0(x_j) - u_0(y_j) , \end{cases}$$

where

$$\theta_j^n = f(x_j, t_n) - f(y_j, t_n) - \left(b(x_j, t_n) - b(y_j, t_n) \right) v_j^n \\ - \left(a^*(x_j) - a^*(y_j) \right) \frac{\sum_{|k| \leqq k_0} (\beta_k^1 v_{j+k}^{n+1} + \beta_k^0 v_{j+k}^n)}{\sum_{|k| \leqq k_0} (\beta_k^1 + \beta_k^0) y_{j+k}} .$$

From the Lipschitz continuity of functions a^*, b and f, one knows

$$|\theta_j^n| \leqq C\,|x_j - y_j| \left(\|f\|_{W^{1,\infty}} + |v_j^n| + \left| \frac{\sum_{|k|\leqq k_0} (\beta_k^1 v_{j+k}^{n+1} + \beta_k^0 v_{j+k}^n)}{\sum_{|k|\leqq k_0} (\beta_k^1 + \beta_k^0)\, y_{j+k}} \right| \right).$$

The stability of scheme (4.18) implies

$$\|\theta^n\| \leqq C\,|x_j - y_j| \leqq C\,S(\varepsilon, h)\,.$$

Clearly,

$$|w_j^0| = \left|u_0(x_j) - u_0(y_j)\right| \leqq C\,S(\varepsilon, h)\,.$$

One has from the stability of (4.20) that

$$\|w_j^n\| \leqq C\,S(\varepsilon, h)\,.$$

Therefore, it follows from (4.15), (4.19), and above inequality that

$$\begin{aligned} \|u_j^n - u(x_j, t_n)\| &\leqq \|w_j^n\| + \|v_j^n - u(y_j, t_n)\| + \|u(y_j, t_n) - u(x_j, t_n)\| \\ &\leqq C\,\left(S(\varepsilon, h) + h\right)\,. \end{aligned}$$

This completes the proof of the theorem.

COROLLARY 4.2. *Assume CFL condition* (4.9) *is valid and the coefficient in the equation* (4.1) *has the form* $a^\varepsilon = a(x, x/\varepsilon)$, $a(x, y)$ *is a smooth positive function with unit period in* y. *Then the upwind scheme* (4.4), (4.5), *the backward scheme* (4.4), (4.6), *and the Crank-Nicholson scheme* (4.4), (4.7) *are essentially convergence of order* α, *for all* $\alpha < 1$.

Proof: The convergence of the upwind scheme is covered by Corollary 4.1. For the backward Euler scheme and the Crank-Nicholson scheme, we take

$$y_{2j} = x_0 + 2h \sum_{k=1}^{j} \frac{a^*(x_{2k-1})}{a^\varepsilon(x_{2k-1})}\,, \qquad y_{2j+1} = x_1 + 2h \sum_{k=1}^{j} \frac{a^*(x_{2k})}{a^\varepsilon(x_{2k})}\,.$$

It is easy to check that

$$\frac{1}{2} a^\varepsilon(x_j)(y_{j+1} - y_{j-1}) = h a^*(x_j)$$

and

$$|y_j - x_j| \leqq C h^\alpha\,, \qquad \forall\ \varepsilon > 0, h \in \mathcal{S}_\alpha(\varepsilon)\,.$$

The equivalence condition (4.14) and (4.15) is satisfied. The corollary now follows directly from Theorem 4.2.

Remark 4.1. If both of the coefficients and initial values in the linear hyperbolic equations are highly oscillatory, then the behavior of solutions is very complicate (see [12]). In here, we consider the difference methods for the following model equations which share some similarity with the general linear equations with highly oscillatory coefficients, initial values, and forcing terms

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \dfrac{\partial}{\partial x}\left(a(x/\varepsilon)u^\varepsilon \right) = f(x,t) \,, \\ u^\varepsilon(x,0) = u_0(x) \,. \end{cases} \tag{4.21}$$

Letting $v^\varepsilon = a(x/\varepsilon)u^\varepsilon$, then (4.21) becomes

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} + a(x/\varepsilon)\dfrac{\partial v^\varepsilon}{\partial x} = a(x/\varepsilon)f(x,t) \,, \\ v^\varepsilon(x,0) = a(x/\varepsilon)u_0(x) \,. \end{cases}$$

The solution of (4.21) is highly oscillatory and it converges weakly to the following homogenized equation (see [12])

$$\begin{cases} \dfrac{\partial u}{\partial t} + a^* \dfrac{\partial u}{\partial x} = f(x,t) \,, \\ u(x,0) = u_0(x) \,, \end{cases} \tag{4.22}$$

where a^* is the harmonic average of $a(x)$. We approximate (4.21) with the following upwind scheme,

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{a(x_j/\varepsilon)u_j^n - a(x_{j-1}/\varepsilon)u_{j-1}^n}{h} = f_j^n \,, \\ u_j^0 = u_0(x_j) \,. \end{cases} \tag{4.23}$$

Assume $\Delta t = h$, $h/\varepsilon = \gamma$ is held fixed and γ is any irrational number. The above equation can be rewritten as

$$u_j^{n+1} = a((j-1)\gamma)\, u_{j-1}^n + \left(1 - a(j\gamma)\right) u_j^n + h f_j^n \,.$$

We can view the above equation as a renewal equation describing the distribution of the particles: initially, they have distribution $u_0(x_j)$ at position j; at each time step, particles have probability $a(j\gamma)$ of jumping one unit to the right and probability $1 - a(j\gamma)$ of staying fixed if the particles local at position j. We can show, by a stochastic analysis, that the upwind scheme (4.23) converges weakly of order $1/2$ essential independent of ε to the homogenized equation (4.22). We refer to [12] for a detailed discussion.

Particle methods. Particle methods or the method of characteristic have been successfully used in numerical simulation of the semi-linear hyperbolic systems with oscillatory initial values; see [7]–[10]. In this subsection, we apply the particle method to the following linear convection equation,

$$\text{(4.24)}\qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + a_\varepsilon(x)\cdot\nabla u^\varepsilon = f(x,t)\,, \\ u^\varepsilon(x,0) = u_0(x)\,. \end{cases}$$

We suppose the velocity field $a_\varepsilon(x)$ is periodic oscillatory at one direction c and divergence free. Consequently, there is a smooth unit periodic function $a(y)$ with $c\cdot a(y)\equiv c_0$, such that $a_\varepsilon = a(c\cdot x/\varepsilon)$. Equations of the above form are typical models for miscible displacement problems in the oil reservoir simulation. The unknown u^ε corresponds to the concentration of the invading fluid.

Denote $X^\varepsilon(x,t)$ as the solution of the following equation

$$\text{(4.25)}\qquad \frac{d}{dt}X^\varepsilon(x,t) = a(c\cdot x/\varepsilon)\,,\qquad X^\varepsilon(0,x) = x\,.$$

Taking the scalar product with c to get $d(c\cdot X^\varepsilon)/dt = c_0$. Therefore,

$$\frac{d}{dt}X^\varepsilon(x,t) = a\big((c\cdot x + c_0 t)/\varepsilon\big)\,.$$

This leads to the following homogenized behavior: if $c_0 \neq 0$, then

$$\text{(4.26)}\qquad X^\varepsilon(x,t) \to x + t\bar{a}\,;$$

if $c_0 = 0$, then

$$\text{(4.27)}\qquad X^\varepsilon(x,t) = x + ta(c\cdot x/\varepsilon)\,.$$

THEOREM 4.3. *The particle method for* (4.24) *is essential convergence of order* $\alpha < 1$.

The proof follows from the convergence of the following Euler method for (4.25)

$$\text{(4.28)}\qquad x^{n+1} = x^n + a(c\cdot x^n/\varepsilon)\Delta t\,.$$

Taking the scalar product with c to get

$$c\cdot x^n = c\cdot x^0 + c_0 n\Delta t\,.$$

Therefore,

$$\begin{aligned} x^n &= x^{n-1} + a\big((c\cdot x^0 + c_0(n-1)\Delta t)/\varepsilon\big)\Delta t \\ &= x^0 + \frac{1}{n}\sum_{j=0}^{n-1} a\big((c\cdot x^0 + c_0 j\Delta t)/\varepsilon\big)t_n\,. \end{aligned}$$

If $c_0 = 0$, then

$$x^n = x^0 + a(c \cdot x^0/\varepsilon)\, t_n \; ; \tag{4.29}$$

if $c_0 \neq 0$, Lemma 2.1 implies that

$$\| x^n - x^0 - \bar{a} t_n \| \leqq \Delta t^\alpha \|a\|_{W^{3,1}} \tag{4.30}$$

for all $0 < \varepsilon < 1$ and $c_0 \Delta t \in \mathscr{S}_\alpha(\varepsilon)$. The above theorem follows from (4.26), (4.27), (4.29), and (4.30).

5. Numerical Experiments

Numerical experiments are carried out for the algorithms discussed in Section 3 and Section 4 for linear hyperbolic equations with oscillatory coefficients and oscillatory initial values. In the numerical experiments, we take the time step size Δt equal to the space grid size h (the CFL condition is satisfied) and always compare the numerical results with the "homogenized" solution which is obtained by using the same numerical schemes for the corresponding homogenized differential equations.

Experiment 5.1 (unitary and dissipative schemes): For the linear convection equation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \, ,$$

with initial oscillatory pulse,

$$u(x, 0) = \begin{cases} \frac{1}{2}\cos(\pi x)(1 + \cos(\pi x/\varepsilon)) \, , & \text{for } -0.5 < x < 0.5 \, , \\ 0 \, , & \text{otherwise} \, , \end{cases}$$

we use the leapfrog scheme (a unitary scheme),

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0 \, , \qquad u_j^0 = u_0(x_j, x_j/\varepsilon) \, ,$$

and the Lax-Friedrichs scheme (a dissipative scheme),

$$u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + \frac{\Delta t}{2h}(u_{j+1}^n - u_{j-1}^n) = 0 \, , \qquad u_j^0 = u_0(x_j, x_j/\varepsilon) \, .$$

We choose grid size $h = 0.02$ and $\varepsilon = 0.0141423$, and plot the initial oscillation pulse in Figure 1(a). The exact solution at time $t = 2$ is plotted in Figure 1(b), the numerical solution for the leapfrog scheme at time $t = 2$ in Figure 1(c), and the numerical solution for the Lax-Friedrichs scheme at time $t = 2$ in Figure 1(d).

We can see that both the unitary scheme and the dissipative scheme give good approximations in the weak sense (Theorem 3.2). We can also see that the oscillation pulse in the dissipative scheme is dampen out very fast; see Figure 1(d). For the unitary scheme, the oscillation pulse moves at its own numerical group velocity instead of the convection velocity; see Figure 1(c). Even the oscillations move to the wrong location instead of being with the average, the oscillatory pulse has a zero local average so that the averaged solution is well approximated (Theorem 3.1). When the oscillatory plus is propagated in a nonlinear equation, it then also contributes to the smooth part of the solution too.

Experiment 5.2 (upwind schemes): For the linear hyperbolic equation with oscillatory coefficient,

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{1}{3.75 + 2.5\sin(2\pi x/\varepsilon)} \frac{\partial u^\varepsilon}{\partial x} = 0\,, \tag{5.1}$$

and initial condition,

$$u_0(x) = \begin{cases} \sin(2\pi x)\,, & \text{for } 0 < x < 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

we use the upwind scheme,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a^\varepsilon(x_j)\frac{u_j^n - u_{j-1}^n}{h} = 0\,.$$

We choose $\varepsilon = 0.0014142$, grid size $h = 0.1$, 0.05, 0.025, and 0.0125 and plot the numerical results at time $t = 1.0$ in Figure 2(a)-(d), respectively. The grid size $h \gg \varepsilon$ and the numerical solutions converge essentially to the "homogenized" solution. The rate of convergence is basically of order $O(h)$ as showed by Corollary 3.1.

In Figure 3(a)-(c), we also plot the approximation errors and the sampling errors at different grid sizes in L^1-norm, L^2-norm, and L^∞-norm, respectively. The approximation errors are linearly proportional to the sampling errors as proved in Theorem 3.1. In Figure 3(d), we plot the approximation errors in L^∞-norm for $h = 0.00625$ in all time steps up to $t = 1$. The approximation errors are not increasing as the time increases.

Figure 4 shows the sampling effect in the upwind schemes for the linear hyperbolic equation,

$$\frac{\partial u^\varepsilon}{\partial t} + \big(0.42 + 0.4\sin(2\pi x/\varepsilon)\big)\frac{\partial u^\varepsilon}{\partial x} = 0\,.$$

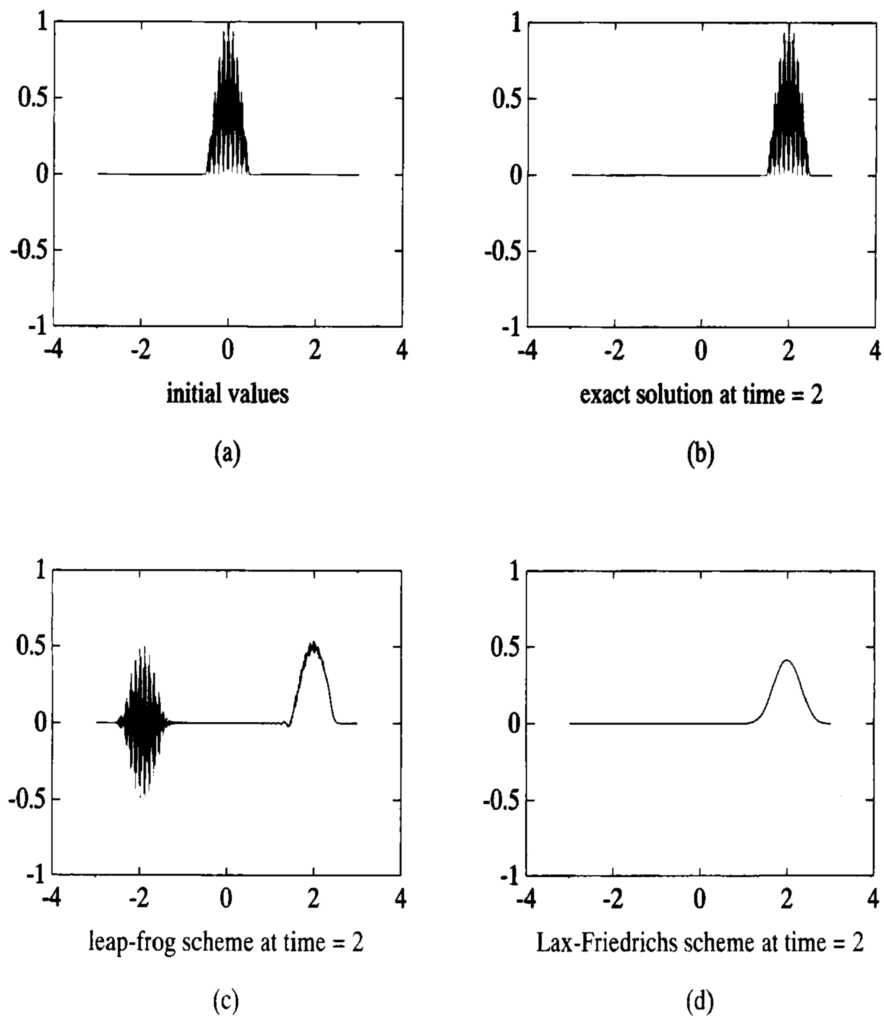

Figure 1. Leap-frog scheme and Lax-Friedrichs scheme for the equation $u_t + u_x = 0$ with initial data $u_0(x) = \frac{1}{2}\cos(x)(1 + \cos(x/\varepsilon))$, for $-\pi < x < \pi$ and equal to zero else. (a) Initial data. (b) Exact solution at time $t = 2$. (c) Numerical solution for leapfrog scheme at time $t = 2$ with grid size $h = 0.02$ and $\varepsilon = 0.0141423$. (d) Numerical solution for Lax-Friedrichs scheme at time $t = 2$ with grid size $h = 0.02$ and $\varepsilon = 0.0141423$.

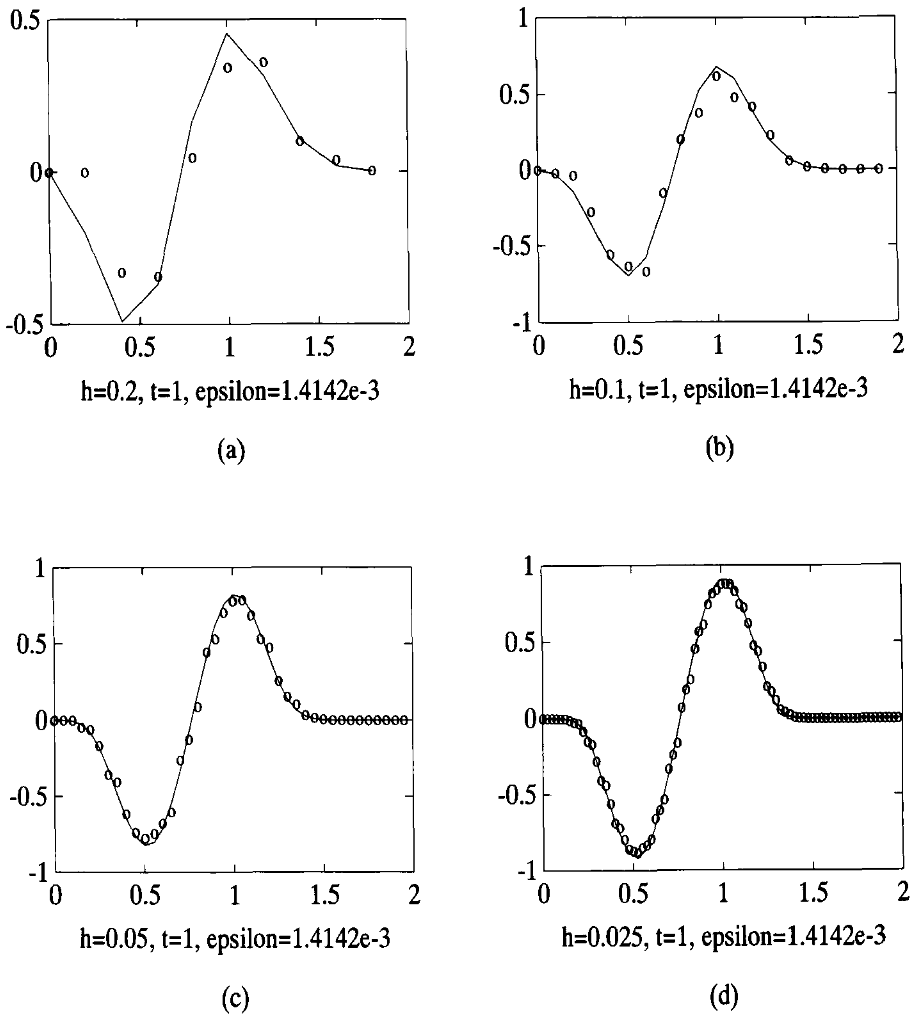

Figure 2. Upwind scheme for the linear hyperbolic equation $u_t + 1/(3.75 + 2.5\sin(2\pi x/\varepsilon))u_x = 0$. The solid lines are direct approximation of the 'homogenized' solution. (a) $h = 0.2$. (b) $h = 0.1$. (c) $h = 0.05$. (d) $h = 0.025$ for time $= 1$ and $\varepsilon \approx 1.4142e - 3$ (about 0.14 points per wavelength).

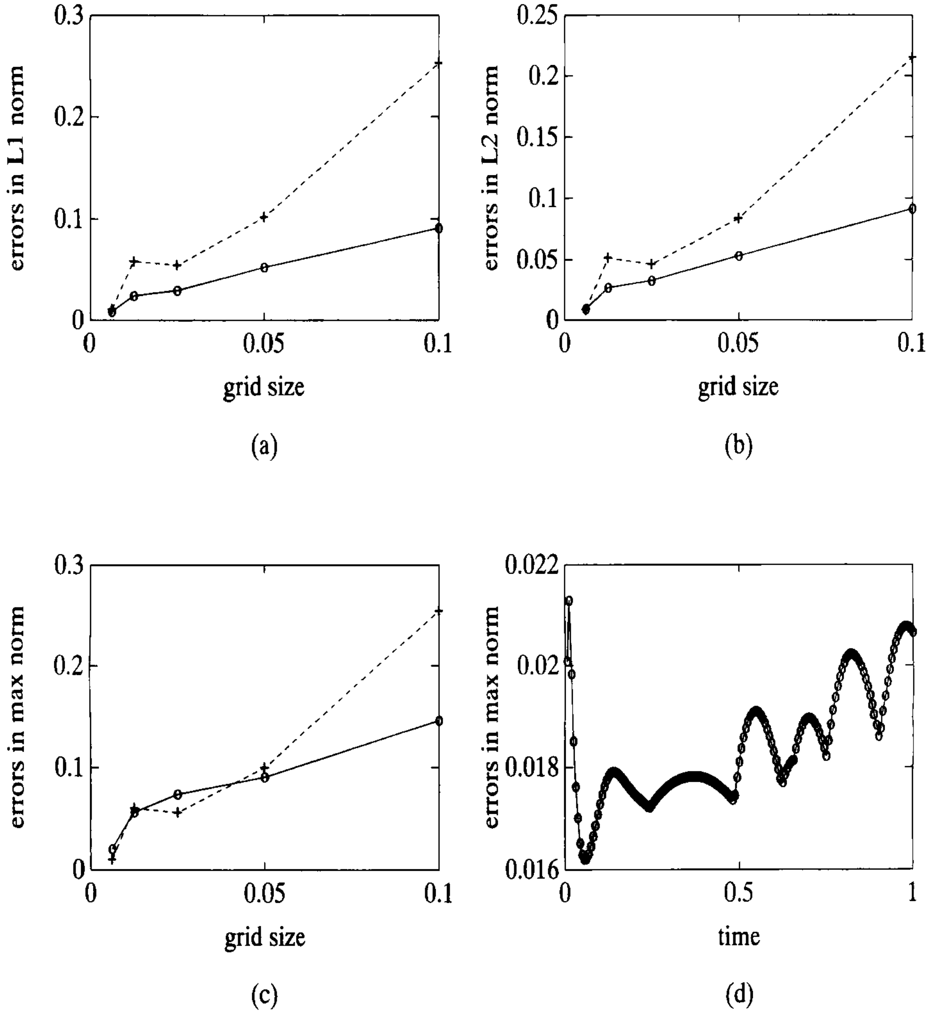

Figure 3. Upwind scheme for the linear hyperbolic equation $u_t + 1/(3.75 + 2.5\sin(2\pi x/\varepsilon))u_x = 0$. The solid lines are the approximation errors and the dish lines are the sample errors defined by (4.8). (a) L^1-norm. (b) L^2-norm. (c) L^∞-norm, for a sequence of grid sizes, time = 1 and $\varepsilon \approx 1.4142e-3$. (d) L^∞-norm, for all the time steps up to $t = 1$, $h = 0.00625$ and $\varepsilon \approx 1.4142e-3$.

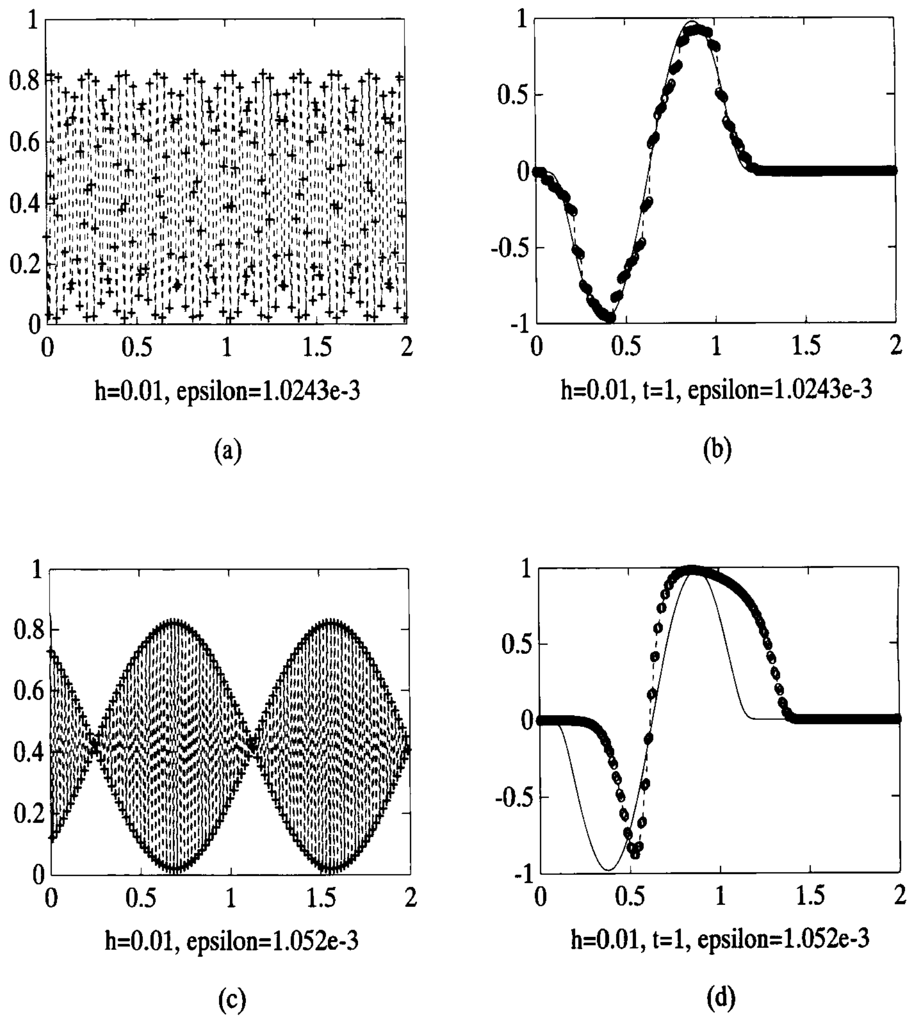

Figure 4. Upwind scheme for the linear hyperbolic equation $u_t + (0.42 + 0.4\sin(2\pi x/\varepsilon))u_x = 0$. (a) $h = 0.01$ and $\varepsilon = 1.0243e-3$ give a good sample of oscillatory coefficients. (b) $h = 0.01$ and $\varepsilon = 1.0243e-3$ give a good approximation of solutions. (c) $h = 0.01$ and $\varepsilon = 1.052e-3$ give a bad sample of oscillatory coefficients. (d) $h = 0.01$ and $\varepsilon = 1.052e-3$ give a bad approximation of solutions.

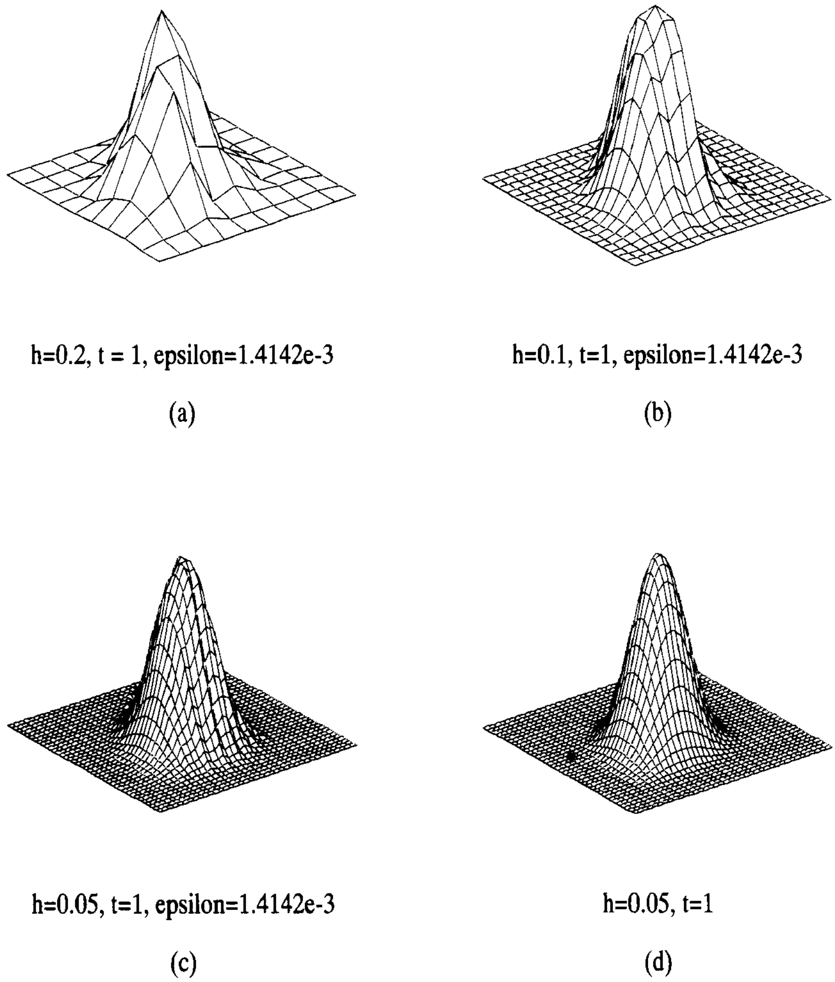

Figure 5. Upwind scheme for the 2-D linear hyperbolic equation $u_t + (0.42 + 0.4\sin(2\pi y/\varepsilon))u_x + 0.5u_y = 0$. (a) $h = 0.2$. (b) $h = 0.1$. (c) $h = 0.05$, for $t = 1.0$ and $\varepsilon \approx 1.4142e-3$. (d) The corresponding 'homogenized' solution.

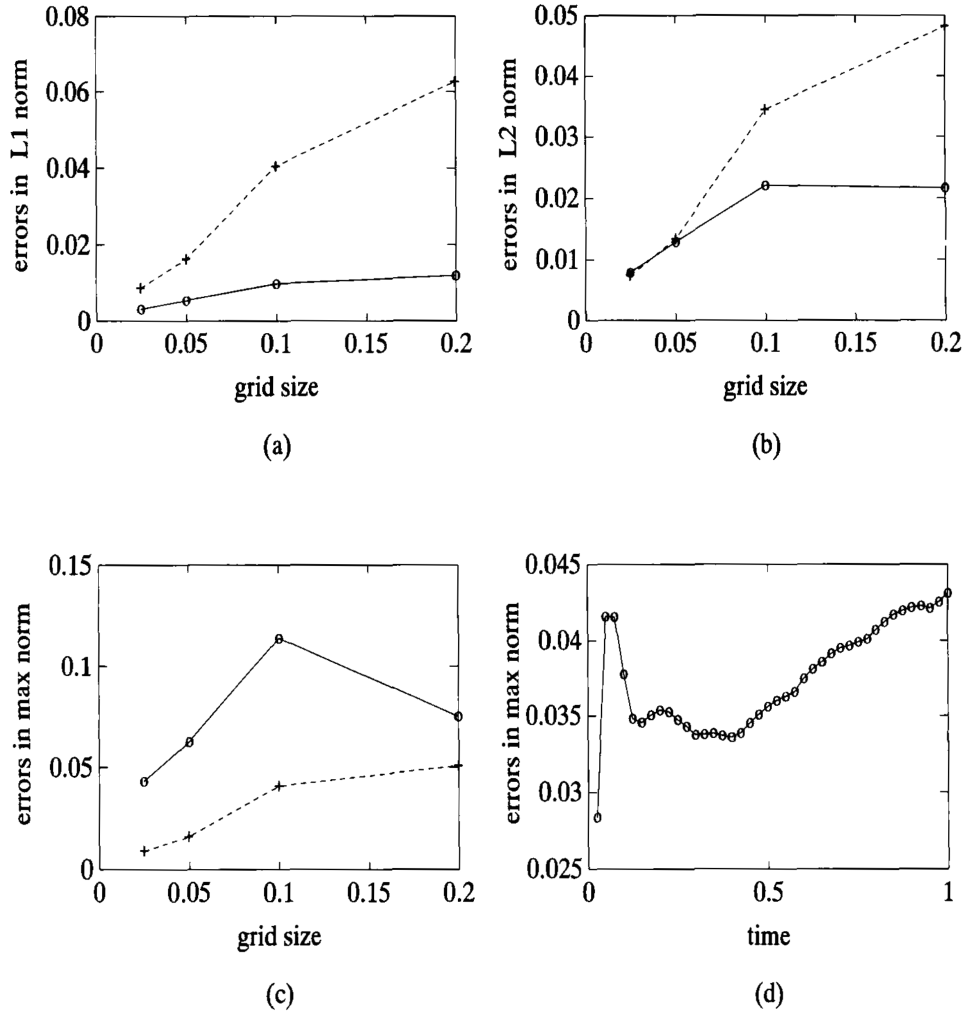

Figure 6. Upwind scheme for the 2-D linear hyperbolic equation $u_t + (0.42 + 0.4\sin(2\pi y/\varepsilon))u_x + 0.5u_y = 0$. The solid lines are the approximation errors and the dish lines are the sample errors. (a) L^1-norm. (b) L^2-norm. (c) L^∞-norm, for a sequence of grid sizes, time = 1 and $\varepsilon \approx 1.4142e-3$. (d) L^∞-norm, for all the time steps up to $t = 1$, $h = 0.0125$ and $\varepsilon \approx 1.4142e-3$.

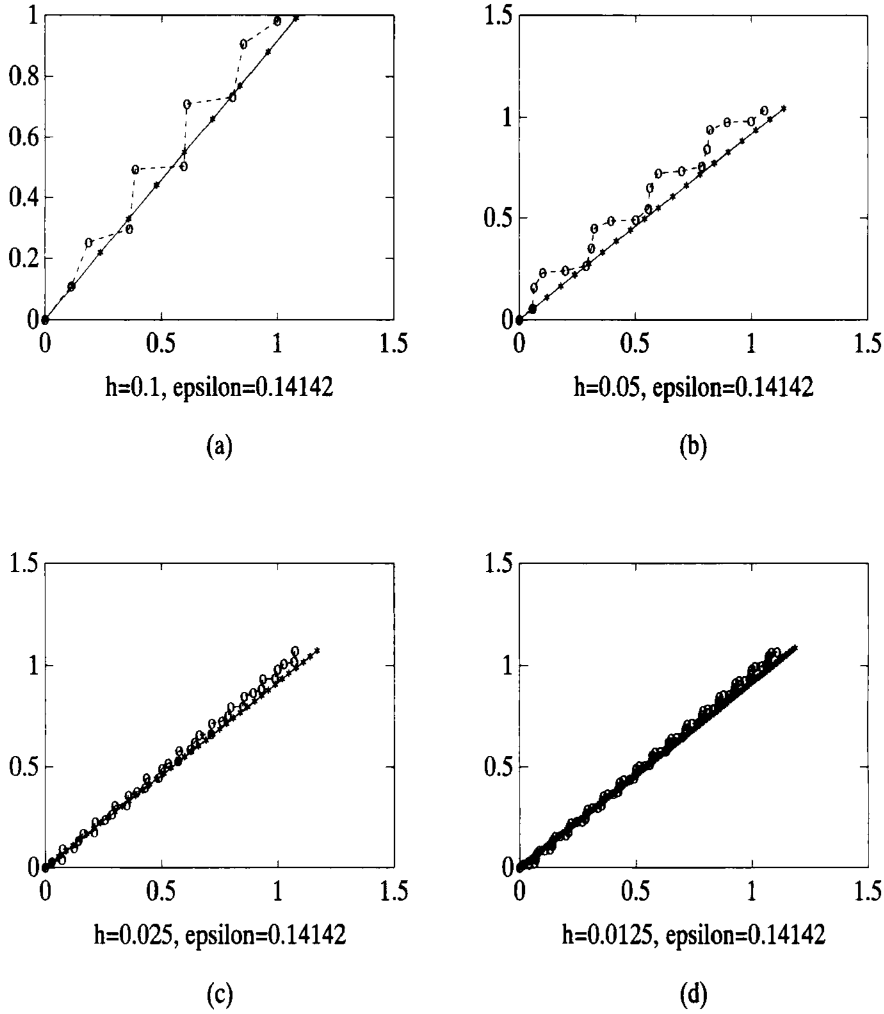

Figure 7. Particle method for the equation $u_t + (1.2 + \sin(2\pi(x+y)/\varepsilon))u_x + (1.1 - \sin(2\pi(x+y)/\varepsilon))u_y = 0$. The cycles are numerical particle paths up to time $t = 1.0$ with starting point (0,0) and $\varepsilon \approx 0.14142$. The solid lines are the corresponding 'homogenized' particle paths (characteristic). (a) $h = 0.1$. (b) $h = 0.05$. (c) $h = 0.025$. (d) $h = 0.0125$.

We take $h = 0.01$ and $\varepsilon = 0.0010243$ and $\varepsilon = 0.001052$, respectively, plot the sample of the oscillatory coefficients in Figures 4(a) and 3(c) and plot the approximation in the solutions in Figures 4(b) and 3(d). We can see that a good sampling of the oscillatory coefficients leads to a good approximation of the solutions.

For the following 2-D linear hyperbolic equation with oscillatory coefficient,

$$\frac{\partial u^\varepsilon}{\partial t} + (0.42 + 0.4\sin(2\pi y/\varepsilon))\frac{\partial u^\varepsilon}{\partial x} + 0.5\frac{\partial u^\varepsilon}{\partial y} = 0 ,$$

and initial condition,

$$u_0(x, y) = \begin{cases} \sin(\pi x)^2 \sin(\pi x)^2 , & \text{for } 0 < x, y < 1 , \\ 0 , & \text{otherwise} , \end{cases}$$

we use the upwind scheme,

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + a(y_j/\varepsilon)\frac{u_{i,j}^n - u_{i-1,j}^n}{h} + b\frac{u_{i,j}^n - u_{i,j-1}^n}{h} = 0 .$$

We choose $\varepsilon = 0.0014142$, grid size $h = 0.2$, $h = 0.1$, and $h = 0.05$ and plot the numerical results at time $t = 1.0$ in Figure 5(a)-(c), respectively. Comparing this with the corresponding "homogenized" solutions in Figure 5(d). We can see that the numerical solutions agree well with the "homogenized" solution.

In Figure 6(a)-(d), we plot the approximation errors via sampling errors at different grid sizes, in L^1-norm, L^2-norm, and L^∞-norm, with $\varepsilon = 0.0014142$. We can see the rate of convergence is linear and the errors do not propagate with time.

Experiment 5.3 (particle methods): Consider the two-dimensional linear convection equation,

$$\frac{\partial u^\varepsilon}{\partial t} + (1.2 + \sin(2\pi(x + y)/\varepsilon))\frac{\partial u^\varepsilon}{\partial x} + (1.1 - \sin(2\pi(x + y)/\varepsilon))\frac{\partial u^\varepsilon}{\partial y} = 0 .$$

We approximate it with the following particle method

$$\begin{cases} x^{n+1} = x^n + \Delta t(1.2 + \sin(2\pi(x^n + y^n)/\varepsilon) , \\ y^{n+1} = y^n + \Delta t(1.1 - \sin(2\pi(x^n + y^n)/\varepsilon) . \end{cases}$$

We hold $\varepsilon = 0.14142$ fixed, choose grid sizes $h = 0.1, 0.05, 0.025$, and 0.0125, and plot the numerical results up to time $t = 1.0$ in Figure 7(a)-(d), respectively. We can see that the numerical solutions converge to the "homogenized" solution. The rate of convergence is basically of order $O(h)$.

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