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EXISTENCE OF WEAK SOLUTIONS TO p-NAVIER-STOKES EQUATIONS

Yuanyuan Feng[1], Lei Li[*2],
Jian-Guo Liu[3] and Xiaoqian Xu[4]

[1]School of Mathematical Sciences, Shanghai Key Laboratory of PMMP,
East China Normal University,
Shanghai, 200241, China

[2]School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSC,
Shanghai Jiao Tong University,
Shanghai, 200240, China

[3]Department of Mathematics, Department of Physics, Duke University,
Durham, NC 27708, USA

[4]Zu Chongzhi Center for Mathematics and Computational Sciences,
Duke Kunshan University,
Kunshan, 215316, China

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Abstract. We study the existence of weak solutions to the p-Navier-Stokes equations with a symmetric p-Laplacian on bounded domains. We construct a particular Schauder basis in $W_0^{1,p}(\Omega)$ with divergence free constraint and prove existence of weak solutions using the Galerkin approximation via this basis. Meanwhile, in the proof, we establish a chain rule for the L^p integral of the weak solutions, which fixes a gap in our previous work. The equality of energy dissipation is also established for the weak solutions considered.

1. **Introduction.** The system of Navier-Stokes equations is one of the most influential mathematical models in physical science and engineering fields [19]. The application of Navier-Stokes equations ranges from the design of a plane to weather forecasting. One of the Millennium Problems proposed by the Clay Mathematics Institute is about the global existence of smooth solutions to Navier-Stokes equations [1], which remains one of the most important open questions in the field of partial differential equations [14].

There are tons of models that are variants of the classical Navier-Stokes equations, typically for some Non-Newtonian fluids [8]. As an example, to study the shear thinning effect of the non-Newtonian flows, one could use the symmetric p-Laplacian term instead of Laplacian, and one may check [21, 5] for more discussion. In [13], the authors proposed the p-Euler equations as the Euler-Lagrange equations from Arnold's least action principle [2, 3], for which the action is represented by the Benamou-Brenier characterization of the Wasserstein-p distance between two

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*Corresponding author: Lei Li.

shapes with the incompressibility constraint. By adding p-Laplacian diffusion to the equation, the so-called p-Navier-Stokes equations were proposed:

$$\begin{aligned} &\partial_t v_p + v\cdot\nabla v_p = -\nabla\pi + \nu\,\Delta_p v,\\ &v_p = |v|^{p-2}v, \quad \nabla\cdot v = 0. \end{aligned} \tag{1}$$

Here, the p-Laplacian is given by $\Delta_p v = \nabla\cdot(|\nabla v|^{p-2}\nabla v)$, $|\nabla v| = \sqrt{\sum_{ij}(\partial_i v_j)^2}$. Mathematically, the p-Navier-Stokes equations are analogues of the classical Navier-Stokes equations and exhibit many similar properties. In particular, when $p=2$, such a system becomes the classical Navier-Stokes equations. The generalization to general p, on the other hand, has some particular difficulty and fine structures. Due to the lack of Hilbert structure in L^p space and the nonlinearity of all the terms in the differential equations, the analysis of such a system of differential equations is significantly more difficult than the classical problems.

In this paper, we are interested in the p-Navier-Stokes equations with the symmetric p-Laplacian arising in the models for shear thinning effect [21] on a bounded domain $\Omega\subset\mathbb{R}^d$ with C^∞ boundary $\partial\Omega$. In particular, we consider the initial-boundary value problem of the p-Navier-Stokes equations given by

$$\begin{cases} \partial_t v_p + v\cdot\nabla v_p = -\nabla\pi + \nu\,\mathcal{L}_p(v), & x\in\Omega,\ t\in(0,T)\,,\\ v_p = |v|^{p-2}v, \quad \nabla\cdot v = 0, & x\in\Omega,\ t\in(0,T)\,,\\ v(x,0) = v_0(x), & x\in\Omega\,,\\ v = 0, & x\in\partial\Omega\,. \end{cases} \tag{2}$$

Here, for the vector field v, the symmetric p-Laplacian $\mathcal{L}_p(v)$ is

$$\mathcal{L}_p(v) = \operatorname{div}(|\mathcal{D}(v)|^{p-2}\mathcal{D}(v)), \tag{3}$$

and

$$\mathcal{D}(v) = \frac{1}{2}(\nabla v + \nabla v^T). \tag{4}$$

We focus on the symmetric p-Laplacian because such a diffusion term appears in physical models [21]. We remark however that the analysis for the usual p-Laplacian diffusion $\Delta_p v$ would be similar (and in fact easier). When $1<p<2$, Δ_p corresponds to fast diffusion. When $p>2$, it is the case corresponding to slow diffusion. One can check this in standard textbooks or references on the p-Laplacian, for example [20]. One would expect the symmetric p-Laplacian term to exhibit similar diffusion effects.

The existence of weak solutions to the original p-Navier-Stokes equations (1) proposed in [13] has been explored in [13] and [15] by totally different methods. In [13], a regularized system was proposed for the approximation and for the existence of weak solutions. Meanwhile, in [15], the authors used the discrete time scheme to prove the existence of the weak solution. We remark that there are some minor gaps in the proof in [13]. For example, the well-posedness of the regularized system was taken for granted; second, the chain rule was not established rigorously as detailed in Section 5.1.

In this article, we focus on the equations (2) with symmetric p-Laplacian on a bounded domain and establish the existence of weak solutions rigorously using a totally different method, the Galerkin approximation. The reasons are as follows. First, the symmetric p-Laplacian is more frequently used for non-Newtonian fluids. Second, the well-posedness of the Galerkin system can be established rigorously.

Moreover, we also aim to fill the gaps for the existence in the previous work. The Galerkin method, or Galerkin approximation, is a very common method in numerical analysis as well as in applied analysis, especially for finding the local existence of the weak solutions to a particular differential equation. One can check more details in standard textbooks, for example [10]. In order to use the Galerkin approximation, one may need to choose a Hilbert space or a Banach space and find a Schauder basis. In this paper, by the natural structure of our differential equations (2), one has to use $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ spaces with divergence-free constraint as the reference spaces. The study of the existence of Schauder basis on such spaces can be tracked back to [11]. Later, in [4], by connecting with the Haar system in one-dimensional case, the authors constructed a special Schauder basis with orthogonality properties on $W_0^{1,p}(\Omega)$. As we shall see later, due to the boundary condition, the Leray projection cannot be used directly to obtain the basis for the subspaces with divergence-free constraint. For the self-consistency of this paper, we construct a Schauder basis based on the eigenfunctions of a compact operator. The significance of this work can be summarized as follows. First, the existence of weak solutions is established rigorously using the Galerkin approximation, fixing the previous gaps. In the proof, a chain rule for the L^p integral of the weak solution is proved using the finite difference approximation, and this is also used to show the energy dissipation equality. Note that this technique can also be used to fill in the gap in [13] for the original model in $\mathbb{R}^d$. Second, a Schauder basis is constructed explicitly for $W_0^{1,p}(\Omega)$ with divergence free constraint. This can be used further for other models in L^p type spaces.

The structure of this paper is as follows. In Section 2, we introduce the notations and definitions in this paper. In Section 3, we construct a Schauder basis in the Sobolev spaces with divergence-free constraints. The basis consists of eigenfunctions of the projected high order elliptic operator in the space with divergence free constraint. In Section 4, we use the Galerkin approximation and run the compactness argument. In Section 5, we finish the proof of the main theorem. Here, the chain rule is established using finite time differences.

2. Notations and definitions.

Fix $\Omega \subset \mathbb{R}^d$ simply connected, bounded with C^∞ boundary $\partial\Omega$. In the rest of this paper, we assume

$$p \geq d \geq 2. \tag{5}$$

To make this paper self-consistent, we recall some notations in tensor analysis. Let $a, b \in \mathbb{R}^d$ be vectors and A, B be matrices. We define $a \otimes b$ to be a matrix; called the tensor product of a and b:

$$(a \otimes b)_{ij} = a_i b_j. \tag{6}$$

We also define the dot product for vectors and matrices as

$$\begin{aligned} &(a \cdot A)_i = \sum_{j=1}^d a_j A_{ji}, \ (A \cdot a)_i = \sum_{i=1}^d A_{ij} a_j, \\ &(A \cdot B)_{ij} = \sum_k A_{ik} B_{kj}, \\ &A : B = \mathrm{tr}(A \cdot B^T) = \sum_{ij} A_{ij} B_{ij}. \end{aligned} \tag{7}$$

2.1. **The weak solutions.** To incorporate initial values in the definition of weak solution to (2), we introduce the following definition.

Definition 2.1. A function $f \in L^1[0,T]$ is said to have the weak time derivative $w \in (C_c^\infty[0,T))'$ with initial value f_0 if

$$\int_0^T \phi w \, dt = -\int_0^T \phi' f \, dt - \phi(0) f_0, \ \forall \phi \in C_c^\infty([0,T)).$$

A function $f \in L^1_{\text{loc}}(\Omega \times [0,T])$ is said to have weak time derivative $w \in (C_c^\infty(\Omega \times [0,T)))'$ and initial data $f_0(x) \in L^1_{\text{loc}}(\Omega)$ if

$$\int_0^T \int_\Omega \phi w \, dx dt = -\int_0^T \int_\Omega \partial_t \phi f \, dx dt - \int_\Omega \phi(x,0) f_0(x) \, dx, \ \forall \phi \in C_c^\infty(\Omega \times [0,T)).$$

We define the bounded trace operator $\mathcal{T}_r : W^{1,p}(\Omega; \mathbb{R}^d) \to L^p(\partial\Omega; \mathbb{R}^d)$, such that $\mathcal{T}_r(u) = u|_{\partial\Omega}$, for any $u \in C^\infty(\Omega; \mathbb{R}^d)$. Such operator is unique, and one can check this fact from standard PDE textbooks, for instance [10]. We then denote by $W_0^{1,p}(\Omega; \mathbb{R}^d)$ all the functions u in $W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\mathcal{T}_r(u) = 0$. One could verify that the space $W_0^{1,p}(\Omega; \mathbb{R}^d)$ is the completion of $C_0^\infty(\Omega; \mathbb{R}^d)$ under the $W^{1,p}(\Omega; \mathbb{R}^d)$ norm. We further use $W^{-1,q}(\Omega; \mathbb{R}^d)$ to denote the the dual space of $W_0^{1,p}(\Omega; \mathbb{R}^d)$, where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Following from the Poincaré inequality, $\|v\|_{L^p} \le c\|\nabla v\|_{L^p}$, for $v \in W_0^{1,p}(\Omega; \mathbb{R}^d)$. As a consequence, the norm in $W_0^{1,p}(\Omega; \mathbb{R}^d)$ is equivalent to $\|\nabla v\|_{L^p} + \|v\|_{L^p}$. Such norm is also equivalent to $\|\mathcal{D}(v)\|_{L^p} + \|v\|_{L^p}$, for which the proof can be found in [16]. Here, we state the fact as the following lemma and prove it in the Appendix (Section A) for completeness.

Lemma 2.2. *There exist two positive constants C_1, C_2; such that for any $v \in W_0^{1,p}(\Omega; \mathbb{R}^d)$,*

$$\begin{aligned} &C_1(\|v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}) \\ \le &\|v\|_{L^p(\Omega)} + \|\mathcal{D}(v)\|_{L^p(\Omega)} \\ \le &C_2(\|v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}). \end{aligned} \tag{8}$$

The action of symmetric p-Laplacian can be considered as an operator $\mathcal{L}_p$: $W_0^{1,p}(\Omega; \mathbb{R}^d) \to W^{-1,q}(\Omega; \mathbb{R}^d)$. For $u, v \in W_0^{1,p}(\Omega; \mathbb{R}^d)$, define

$$\langle \mathcal{L}_p(u), v \rangle := -\int_\Omega |\mathcal{D}(u)|^{p-2} \mathcal{D}(u) : \nabla v dx \, . \tag{9}$$

In addition, by Hölder's inequality, one has

$$|\langle \mathcal{L}_p(u), v \rangle| = \left| -\int_\Omega |\mathcal{D}(u)|^{p-2} \mathcal{D}(u) : \nabla v dx \right| \le \|\mathcal{D}(u)\|_p^{p-1} \|\nabla v\|_p,$$

for $u, v \in W_0^{1,p}(\Omega; \mathbb{R}^d)$. Hence,

$$\int_0^T \|\mathcal{L}_p(u)\|^q_{W^{-1,q}(\Omega;\mathbb{R}^d)} dt \le \int_0^T (\|\mathcal{D}(u)\|_p^{p-1})^q dt = \int_0^T \|\mathcal{D}(u)\|_p^p dt$$

for $u \in L^p(0,T; W_0^{1,p}(\Omega; \mathbb{R}^d))$. This means that $\mathcal{L}_p$ maps bounded sets in $L^p(0,T; W_0^{1,p}(\Omega; \mathbb{R}^d))$ to bounded sets in $L^q(0,T; W^{-1,q}(\Omega; \mathbb{R}^d))$.

To motivate the definition of weak solutions, let us perform some formal estimate (*a priori* estimate). Multiplying v on both sides of the first equation in (2) and integrating over space and time, one has

$$\int_\Omega |v|^p(x,T)dx - \int_\Omega |v|^p(x,0)dx = -q\nu \int_0^T \int_\Omega |\mathcal{D}(v)|^p dxdt. \tag{10}$$

Hence, if $v_0 \in L^p(\Omega;\mathbb{R}^d)$, one is expected to have

$$v \in L^\infty(0,T,L^p(\Omega;\mathbb{R}^d)) \cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^d)). \tag{11}$$

We remark that the energy dissipation equality (10) often reduces to inequality for weak solutions (we will show this equality holds later for our weak solutions). Nevertheless, the regularity of the v with this *a priori* estimate is expected to hold. Let $\hat{v}$ be the unit vector with the same direction as v. Based on the observation $\nabla v_p = |v|^{p-2}\nabla v \cdot (I + (p-2)\hat{v} \otimes \hat{v})$, we immediately obtain that in the case $p > 2$

$$v_p \in L^\infty(0,T;L^q(\Omega;\mathbb{R}^d)) \cap L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^d)). \tag{12}$$

By Definition 2.1, with these a *priori* estimates, it is natural for us to define the weak solutions to the initial-boundary p-Navier-Stokes problems as follows.

Definition 2.3. Given $v_0 \in L^p(\Omega;\mathbb{R}^d)$ with $\int_\Omega \nabla\psi \cdot v_0\, dx = 0$ for all $\psi \in C^\infty(\overline{\Omega};\mathbb{R})$, we say $v \in L^\infty(0,T,L^p(\Omega;\mathbb{R}^d) \cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^d))$ is a weak solution of the p-Navier-Stokes problem (Equation (2)) with initial value v_0; if

$$\lim_{h\to 0^+} \int_0^{T-h} \|v(t+h) - v(t)\|^p_{L^p(\Omega;\mathbb{R}^d)} dt = 0, \tag{13}$$

and for any $\varphi \in C_c^\infty(\Omega \times [0,T);\mathbb{R}^d)$, $\nabla\cdot\varphi = 0$, $\psi \in C_c^\infty(\overline{\Omega} \times [0,T);\mathbb{R})$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega v_p \cdot \partial_t\varphi\, dxdt + \int_0^T \int_\Omega \nabla\varphi : (v \otimes v_p)dxdt - \nu \int_0^T \int_\Omega \nabla\varphi : \mathcal{D}(v)|\mathcal{D}(v)|^{p-2}dxdt \\
&+ \int_\Omega |v_0|^{p-2}v_0 \cdot \varphi(x,0)dx = 0, \\
&\int_0^T \int_\Omega \nabla\psi \cdot v\, dxdt = 0.
\end{aligned} \tag{14}$$

If $v \in L^\infty_{\mathrm{loc}}(0,\infty;L^p(\Omega;\mathbb{R}^d)) \cap L^p_{\mathrm{loc}}(0,\infty;W_0^{1,p}(\Omega;\mathbb{R}^d))$, and (14) holds with ∞ instead of T for all $\varphi \in C_c^\infty(\Omega \times [0,\infty);\mathbb{R}^d)$ with $\nabla\cdot\varphi = 0$ and $\psi \in C_c^\infty(\overline{\Omega} \times [0,\infty);\mathbb{R})$, we say v is a global solution.

Above, following (6) and (7), the double dots are interpreted as

$$\begin{aligned}
\nabla\varphi : (v \otimes v_p) &= \sum_{ij} \partial_i\varphi_j v_i (v_p)_j, \\
\nabla\varphi : \mathcal{D}(v) &= \sum_{ij} \partial_i\varphi_j (\mathcal{D}(v))_{ij}.
\end{aligned} \tag{15}$$

2.2. **The working subspaces.** For the convenience of the discussion, we aim to incorporate the divergence-free constraint into the working spaces. In particular, we need to seek a solution v in the subspaces of $L^p(\Omega;\mathbb{R}^d)$ and $W_0^{1,p}(\Omega)$ with certain divergence-free constraints. Define

$$\mathcal{U} := \{\phi \in C_c^\infty(\Omega;\mathbb{R}^d) : \mathrm{div}\phi = 0\}. \tag{16}$$

For the space $L^p(\Omega;\mathbb{R}^d)$, we recall the Helmohotlz-Weyl decomposition [12]. Denote by $U_p(\Omega)$ the L^p-completion of the space $\mathcal{U}$, which is given by

$$U_p(\Omega)=\left\{w\in L^p(\Omega;\mathbb{R}^d):\int_\Omega w\cdot\nabla\varphi\,dx=0,\forall\varphi\in C^1(\overline{\Omega})\right\}. \tag{17}$$

This is the weak form of $\{w\in L^p(\Omega;\mathbb{R}^d):\nabla\cdot w=0 \text{ in } \Omega,\, w\cdot n=0 \text{ on } \partial\Omega\}$, where n represents a normal vector field on the boundary. Let $G_p(\Omega)=\{w\in L^p(\Omega,\mathbb{R}^d):\exists\varphi\in W^{1,p}_{\mathrm{loc}}(\Omega), w=\nabla\varphi\}$. Theorem III.1.2 and relevant results in [12] can be summarized as the following lemma:

Lemma 2.4. *Let $\Omega\subset\mathbb{R}^d$, $d\geq 2$ be either a domain of class C^2 or the whole space or a half space, and then the Helmholtz-Weyl decomposition holds,*

$$L^p(\Omega;\mathbb{R}^d)=U_p(\Omega)\oplus G_p(\Omega)\,, \tag{18}$$

where $\oplus$ denotes direct sum. This defines the Leray projection operator $\mathcal{P}:L^p(\Omega;\mathbb{R}^d)\to U_p(\Omega)$. There is a constant $C(p,\Omega)$ such that for any $w\in L^p(\Omega;\mathbb{R}^d)$,

$$\|\mathcal{P}w\|_p\leq C(p,\Omega)\|w\|_p. \tag{19}$$

This says that any $w\in L^p(\Omega;\mathbb{R}^d)$ can be uniquely decomposed as

$$w=w_1+w_2\,,$$

with $w_1\in U_p(\Omega)$; and $w_2\in G_p(\Omega)$, and thus $w_1=\mathcal{P}w$. The decomposition here is the so-called Helmholtz-Weyl decomposition.

Remark 2.5. The boundary condition matters. For example, $\phi=(-y,x)$ is divergence free in $\Omega=\{(x,y):2x^2+y^2<1\}$, but $\phi\cdot n\neq 0$ on $\partial\Omega$. Then, $\mathcal{P}\phi\neq\phi$ since $\int_\Omega\nabla\varphi\cdot\phi\,dx\neq 0$ for some φ.

For $W^{1,p}_0$, since the weak derivatives are well-defined, we can introduce directly

$$W:=\left\{v\in W^{1,p}_0(\Omega;\mathbb{R}^d):\nabla\cdot v=0\right\}. \tag{20}$$

Here, $\nabla\cdot$ means divergence in the weak sense. Moreover, we will also use

$$V:=L^p(0,T;W), \tag{21}$$

equipped with the $L^p(0,T;W^{1,p})$ norm.

Remark 2.6. As in Lemma 2.4, for any $\phi\in W^{1,p}_0$, the Helmholtz-Weyl decomposition of ϕ is given by

$$\phi=\mathcal{P}\phi+\nabla\varphi,$$

where $\mathcal{P}\phi\in U_p(\Omega)$, $\nabla\varphi\in G_p(\Omega)$, and φ is unique up to a constant. Then, φ can be determined by the following Poisson equation:

$$\begin{cases}\Delta\varphi=\nabla\cdot\phi \text{ in } \Omega,\\ \dfrac{\partial\varphi}{\partial n}=0 \text{ on } \partial\Omega.\end{cases}$$

Clearly, one has $\mathcal{P}\phi\in W^{1,p}$ by the elliptic regularity. Unfortunately, the boundary value of $\mathcal{P}$ is not necessarily zero (only the normal component is zero). Hence, the projection of $v\in W^{1,p}_0$ onto W *cannot* be simply obtained using the Leray projection.

With the spaces in hand, clearly, we will then seek solutions in $L^\infty(0,T;U_p(\Omega))\cap V$. For this purpose, we need a Schauder basis for $U_p(\Omega)$ and W.

3. **A Schauder basis.** The existence of the Schauder basis of $W_0^{1,p}$ is well known (see, for example, [4]). However, as commented in the last section, one cannot simply use the Leray projection $\mathcal{P}$ to obtain a basis for W. In this section, we will show that the eigenfunctions in certain spaces of $\mathcal{P}\Delta^m$, if m is large enough, will form a Schauder basis for both $U_p(\Omega)$ and W.

We first of all consider the following elliptic problem:

$$\begin{aligned}&(-1)^m\mathcal{P}\Delta^m u = f,\\ &\Delta^s u|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial n}\Delta^\ell u|_{\partial\Omega} = 0, \quad \forall s \in S,\ \forall \ell \in L.\end{aligned} \tag{22}$$

Here, $f \in U_2(\Omega)$, the space which is an L^2-completion of the divergence-free smooth functions as defined in (17). If $m = 2k$, then $S = \{s \in \mathbb{Z} : 0 \le s \le k-1\}$, and $L = \{\ell \in \mathbb{Z} : 0 \le \ell \le k-1\}$. If $m = 2k+1$, then $S = \{s \in \mathbb{Z} : 0 \le s \le k\}$, $L = \{\ell \in \mathbb{Z} : 0 \le \ell \le k-1\}$. We use $U_2(\Omega)$ here to make use of its Hilbert structure. The domain of the operator $\mathcal{A} = (-1)^m\mathcal{P}\Delta^m : U_2(\Omega) \to U_2(\Omega)$ is given by

$$\mathcal{D}(\mathcal{A}) = H^{2m} \cap \tilde{H}^m,$$

where

$$\tilde{H}^m = \left\{u \in H^m : \operatorname{div}(u) = 0,\ \Delta^s u|_{\partial\Omega} = 0,\ \frac{\partial}{\partial n}\Delta^\ell u|_{\partial\Omega} = 0,\ \forall s \in S,\ \forall \ell \in L\right\}.$$

We remark that $\mathcal{U}$ is not dense in Hilbert space $\tilde{H}^m$ because the completion of $\mathcal{U}$ is $\{u \in H_0^m : \operatorname{div}(u) = 0\}$.

Now, consider the weak solution to the problem (22). The associated bilinear form $B : \tilde{H}^m \times \tilde{H}^m \to \mathbb{R}$ is given by

$$B[u,v] = \begin{cases}\int \Delta^k u \Delta^k v\, dx, & m = 2k,\\ \int \nabla\Delta^k u \cdot \nabla\Delta^k v\, dx, & m = 2k+1.\end{cases} \tag{23}$$

A weak solution $u \in \tilde{H}^m$ is the one for which

$$B[u,v] = \int_\Omega f v\, dx, \quad \forall v \in \tilde{H}^m.$$

We remark that this definition of a weak solution is consistent with problem (22). In fact, a weak solution is called a strong solution if the left hand side is a locally integrable function, and (22) holds for a.e. x. If a weak solution $u \in H^{2m}$, then integration by parts gives

$$\int_\Omega (-1)^m \Delta^m u v\, dx = \int_\Omega (-1)^m \mathcal{P}\Delta^m u v\, dx = \int_\Omega f v\, dx,$$

for all $v \in \tilde{H}^m$. Hence, u is a strong solution of (22).

By the Lax-Milgram theorem (see [10, Chapter 6]), the existence and uniqueness of the weak solution hold. Hence, the solution map

$$\mathcal{S} := ((-1)^m\mathcal{P}\Delta^m)^{-1} : U_2(\Omega) \to \tilde{H}^m \subset U_2(\Omega)$$

is well-defined.

Remark 3.1. The bilinear form for the equation $(-1)^m\Delta^m u = f$ with the same boundary conditions has the same expression, but the domain is $\hat{H}^m \times \hat{H}^m$ with $\hat{H}^m = \{u \in H^m,\ \Delta^s u|_{\partial\Omega} = 0,\ \frac{\partial}{\partial n}\Delta^\ell u|_{\partial\Omega} = 0,\ \forall s \in S,\ \forall \ell \in L\}$. Note that though $(-1)^m\mathcal{P}\Delta^m$ and $(-1)^m\Delta^m$ agree on $\mathcal{U}$, they do not agree as maps $\tilde{H}^m \to (\hat{H}^m)'$ (they are identical as $\tilde{H}^m \to (\tilde{H}^m)'$ though). Here, prime means the dual space.

This suggests that $((-1)^m\mathcal{P}\Delta^m)^{-1}f$ is different from $((-1)^m\Delta^m)^{-1}f$ as elements in $\hat{H}^m$ when $f \in U_2(\Omega)$. In particular, $((-1)^m\Delta^m)^{-1}f$ may not be divergence free even if f is.

Proposition 3.2. *The eigenfunctions of $\mathcal{S}$ in $U_2(\Omega)$ form a Schauder basis for both $U_p\Omega)$ and W if m is sufficiently large.*

Proof. The operator $\mathcal{S}$ is self-adjoint and compact as a map from $U_2(\Omega)$ to $U_2(\Omega)$. Then, it has a complete set of eigenfunctions in $U_2(\Omega)$. Denote the set of eigenfunctions as $\{\phi_k\}_{k=1}^{\infty}$ and the corresponding eigenvalues as $\{\lambda_k\}_{k=1}^{\infty}$. They are orthogonal in L^2, and thus they form a Schauder basis for $U_2(\Omega)$.

First, we show that the eigenfunctions form a Schauder basis for $\tilde{H}^m$. This is done by the same argument as in the proof of [10, Section 6.5, Theorem 2]. In fact, by the elliptic regularity, one can show that $\phi_k \in H^{2m}$. Hence, ϕ_k are also the eigenfunctions of $(-1)^m\mathcal{P}\Delta^m$. For any $u \in \tilde{H}^m$, by definition of B in (23) and integration by parts, one has

$$B[u,\phi_k] = \int_\Omega u(-1)^m\Delta^m\phi_k = \int_\Omega u(-1)^m\mathcal{P}\Delta^m\phi_k = \lambda_k^{-1}\int_\Omega u\phi_k\,dx.$$

Since $\{\phi_k\}$ form a basis for $U_2(\Omega)$, then $B[u,\phi_k] = 0$ for all $k \geq 1$ implies that $u = 0$ in $U_2(\Omega)$ and thus in $\tilde{H}^m$. Hence, $\{\phi_k\}$ is also complete in $\tilde{H}^m$. Moreover, it is orthogonal as well in $\tilde{H}^m$; hence, it is a Schauder basis for $\tilde{H}^m$. The convergence in $\tilde{H}^m$ clearly implies the convergence in $U_2(\Omega)$. Hence, the expansion coefficient is the same in the two spaces.

Second, choose m sufficiently large such that $\tilde{H}^m \subset W \subset U_2(\Omega)$, and the embeddings are continuous due to the Sobolev inequalities.

It is clear that $\mathcal{U} \subset \mathcal{S}(U_2(\Omega)) \subset \tilde{H}^m \subset W$ where all the embeddings are continuous. Hence, $\mathcal{S}(U_2(\Omega))$ is dense both in $U_p(\Omega)$ and W. Let $\mathcal{X}$ be $U_p(\Omega)$ or W, and $\|\cdot\|$ is the corresponding norm.

For every $u \in \mathcal{S}(U_2(\Omega)) \subset W$, one has the expansion in $\tilde{H}^m$ and thus

$$u = \sum_{k=1}^{\infty} c_k\phi_k, \quad \text{in} \quad \mathcal{X}.$$

Consider the projection operator on $\mathcal{S}(U_2(\Omega))$

$$P_{m,m'}u := \sum_{k=m}^{m'} c_k\phi_k.$$

One has

$$\|P_{m,m'}u\| \leq C(m,m')\sqrt{\sum_{k=m}^{m'} c_k^2} \leq C(m,m')\|u\|_{L^2} \leq \tilde{C}(m,m',\Omega,p)\|u\|.$$

The first inequality is by the equivlence of norms for finite dimensional space, while others are trivial. Hence, $P_{m,m'}$ can be extended to the whole $\mathcal{X}$. Since

$$\|P_{m,m'}u\| \leq \left\|\sum_{k=1}^{m'} c_k\phi_k\right\| + \left\|\sum_{k=1}^{m} c_k\phi_k\right\|,$$

and $\sum_{k=1}^{m} c_k\phi_k$ converges to u in $\mathcal{X}$, the trajectory $O_u := \{P_{m,m'}u : 1 \le m \le m' < \infty\}$ is bounded. By the Uniform Boundedness Principle,

$$\sup_{1\le m\le m'<\infty} \|P_{m,m'}\| < \infty.$$

Now, for any $u_* \in \mathcal{X}$, we take a sequence $u_n \in \mathcal{S}(U_2(\Omega))$ such that $u_n \to u_*$ in $\mathcal{X}$, which can be expressed by

$$u_n = \sum_{k=1}^{\infty} c_{nk}\phi_k \text{ in } \mathcal{X}\,.$$

Then, for any $\epsilon > 0$, there exists $n_0 > 0$ such that whenever $n_2 > n_1 \ge n_0$,

$$\sup_{m,m'} \|P_{m,m'}(u_{n_1} - u_{n_2})\| \le C\|u_{n_1} - u_{n_2}\| < \epsilon.$$

This implies that $c_{nk} \to \bar{c}_k$. Moreover, $\|P_{m,m'}u_{n_2}\| \le \|P_{m,m'}u_{n_1}\| + \epsilon$. Fixing n_1, taking m large enough and taking $n_2 \to \infty$, one then has $\|\sum_{k=m}^{m'} \bar{c}_k\phi_k\| < 2\epsilon$. Then, $\sum_{k=1}^{m} \bar{c}_k\phi_k$ is a Cauchy sequence in $\mathcal{X}$. Hence,

$$\bar{u} = \sum_{k=1}^{\infty} \bar{c}_k\phi_k \in \mathcal{X}.$$

It is easy to identify $\bar{u}$ with u_*. This means that $\{\phi_k\}$ is a Schauder basis for $\mathcal{X}$ as well, and the expansion coefficient should be the same as in $U_2(\Omega)$ since the embedding from $\mathcal{X}$ to $U_2(\Omega)$ is continuous. □

4. The Galerkin approximation and precompactness.

In this section, we apply the Galerkin approximations to (2) and perform the energy estimates. Then, we obtain the precompactness of the solutions to the Galerkin systems.

4.1. Galerkin approximation.

To introduce the Galerkin approximation, for any $v_0 \in U_p(\Omega)$, we write it as

$$v_0 = \sum_{n\ge 0} c_{0,n}\phi_n \quad \text{in} \quad U_p(\Omega).$$

Here, $\{\phi_n\}$ is the Schauder basis we constructed in the last section. Since the case for $v_0 = 0$ is trivial, we consider the case $v_0 \ne 0$. Hence, there is a minimum n_* such that $c_{0,,n_*} \ne 0$. For all $N \ge n_*$, let

$$W_N = \mathrm{span}\{\phi_1, \dots, \phi_N\}. \tag{24}$$

We hope to obtain a function $v^N : [0,T] \to W_N \subset W$ of the form

$$v^N(t) = \sum_{n=1}^{N} c_n^N(t)\phi_n,$$

where the coefficients $c_n^N(t) \in \mathbb{R}$ $(0 \le t \le T, n = 1, \cdots, N)$ satisfy the following

(i) The initial conditions hold for $0 \le n \le N$:

$$c_n^N(0) = c_{0,n}. \tag{25}$$

(ii) For any $0 \le t \le T$, $\varphi \in W_N$,

$$\frac{d}{dt}\int_\Omega \varphi\cdot v_p^N dx + \int_\Omega \varphi\cdot(v^N\cdot\nabla v_p^N)dx + \nu\int_\Omega \nabla\varphi : \mathcal{D}(v^N)|\mathcal{D}(v^N)|^{p-2}dx = 0. \tag{26}$$

Here, similar to (2),

$$v_p^N = |v^N|^{p-2} v^N . \tag{27}$$

Clearly, the equation (26) holds if for $i = 1, ..., N$,

$$\frac{d}{dt}\int_\Omega \phi_i \cdot v_p^N dx + \int_\Omega \phi_i \cdot (v^N \cdot \nabla v_p^N) dx + \nu \int_\Omega \nabla\phi_i : \mathcal{D}(v^N)|\mathcal{D}(v^N)|^{p-2} dx = 0. \tag{28}$$

The term $\frac{d}{dt}\int_\Omega \phi_i \cdot v_p^N dx$ is equal to

$$\sum_{j=1}^N \frac{d}{dt} c_j^N(t) \int_\Omega |v^N|^{p-2} \phi_i^T (I + (p-2)\hat{v}^N \otimes \hat{v}^N)\phi_j dx, \tag{29}$$

where $\hat{v}^N$ is the unit vector with the same direction as v^N.

The term $\int_\Omega \phi_i \cdot (v^N \cdot \nabla v_p^N) dx$ is equal to

$$\sum_{j,k=1}^N \int_\Omega |v^N|^{p-2} \phi_{i,k} v_j^N \partial_j v_k^N dx + (p-2) \sum_{j,k,l=1}^N \int_\Omega |v^N|^{p-4} \phi_{i,k} v_j^N \partial_j v_l^N v_l^N v_k^N dx. \tag{30}$$

Define

$$A_{ij}^N(t) := \int_\Omega |v^N|^{p-2} \phi_i^T (I + (p-2)\hat{v}^N \otimes \hat{v}^N)\phi_j dx, \tag{31}$$

and

$$X^N(t) := \begin{pmatrix} c_1^N(t) \\ \vdots \\ c_N^N(t) \end{pmatrix}. \tag{32}$$

Then, the system (26) is reduced to the following equation:

$$\begin{cases} A^N \dot{X}^N(t) = F(X^N(t)), \\ X^N(0) = X_0^N. \end{cases} \tag{33}$$

Here, $F(X^N(t))$ is a vector valued function in $\mathbb{R}^N$ with

$$\begin{aligned}(F(X^N))_i = -\nu \int_\Omega \nabla\phi_i : \mathcal{D}(v^N)|\mathcal{D}(v^N)|^{p-2} dx - \sum_{j,k=1}^N \int_\Omega |v^N|^{p-2} \phi_{i,k} v_j^N \partial_j v_k^N dx \\ - (p-2) \sum_{j,k,l=1}^N \int_\Omega |v^N|^{p-4} \phi_{i,k} v_j^N \partial_j v_l^N v_l^N v_k^N dx.\end{aligned} \tag{34}$$

Lemma 4.1. *As long as $X^N \neq 0$, $F(X^N)$ is locally Lipschitz in X^N and A^N is positive definite .*

Proof. From (34), it is straightforward that $F(X^N)$ is C^1 in X^N as long as $X^N \neq 0$.

Pick any vector $a \in \mathbb{R}^N$, $a \neq 0$, and one has the following expression of $a^T A^N a$

$$\int_\Omega |v^N|^{p-2} \sum_{i=1}^N a_i \phi_i^T (I + (p-2)\hat{v}^N \otimes \hat{v}^N) \sum_{j=1}^N a_j \phi_j \, dx.$$

Define $\alpha = \sum_{i=1}^N a_i \phi_i$, and we thus have

$$\int_\Omega |v^N|^{p-2} \alpha^T (I + (p-2)\hat{v}^N \otimes \hat{v}^N)\alpha \, dx.$$

We notice that the engenvectors of $\hat{v}^N \otimes \hat{v}^N$ are vectors parallel to v^N and vectors perpendicular to v^N, so the eigenvalues of $\hat{v}^N \otimes \hat{v}^N$ are 1 and 0. Consequently the eigenvalues of $(I + (p-2)\hat{v}^N \otimes \hat{v}^N)$ are $p-1$ and 1. Hence, we have

$$\int_\Omega |v^N|^{p-2} \alpha^T (I + (p-2)\hat{v}^N \otimes \hat{v}^N)\alpha \, dx \geq \min\{p-1, 1\} \int_\Omega |v^N|^{p-2}|\alpha|^2 \, dx > 0,$$

as long as $v^N \neq 0$, i.e, $X^N \neq 0$. □

Proposition 4.2. *Given $|X| \neq 0$, there exists $\delta > 0$ and a unique $X^N(t) \in C^1([0,\delta))$ such that $|X^N(t)| > 0$ satisfying (33), and $X^N(0) = X$.*

Proof. First, by Lemma 4.1, we can rewrite ODE (33) as $\dot{X}^N(t) = (A^N)^{-1}F(X^N)$. By *Cramer's rule*,

$$(A^N)^{-1} = \frac{1}{\det(A^N)} M^T,$$

where M is the matrix of *cofactors* of A^N. Since $\det(A^N)$ and M are both C^1 in X^N as long as $X^N \neq 0$, we have $(A^N)^{-1}F(X^N)$ is C^1 as long as $X^N \neq 0$. This ensures $(A^N)^{-1}F(X^N)$ is locally Lipchitz as long as $X^N \neq 0$. Hence, following from classical ODE theory, we conclude that there exists $\delta > 0$ such that ODE system (33) has a unique solution on $[0, \delta)$. □

Note that by the argument in the proof of Proposition 4.2, as long as $0 < |X_N(t)| < \infty$, the solution can be extended. The largest existence time t_* before $|X_N|$ touching 0 is thus defined by

$$t_* := \sup\{t \geq 0 : (33) \text{ has a unique solution } X^N \in C^1[0,t], |X^N(s)| \neq 0\,,\ \forall s \in [0,t)\}. \tag{35}$$

Clearly, at least one of the following must happen if $t_* < \infty$:

- $\limsup_{t \to t_*} |X_N(t)| = +\infty$;
- $\liminf_{t \to t_*} |X_N(t)| = 0$.

Next, we prove that $X^N(t)$ is never 0 and does not blow up . Once this has been proved, the solution to ODE (33) is defined globally.

Proposition 4.3. *Suppose $v_0 \in U_p(\Omega)$. For $T < t_*$, one has*

$$\frac{d}{dt}\int_\Omega |v^N|^p dx = -q\nu \int_\Omega |\mathcal{D}(v^N)|^p dx, \tag{36}$$

and thus there exists a constant $C(p,\nu,v_0)$ independent of N and T such that

$$\begin{aligned} &\|v^N\|_{L^\infty(0,T;L^p(\Omega))} \leq \|v_0^N\|_{L^p(\Omega)}, \\ &\|v^N\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C(p,\nu,v_0). \end{aligned} \tag{37}$$

Moreover, there are positive constants C, C_N such that $\int_\Omega |v^N|^p dx \geq Ce^{-C_N t}$ for any $t \leq T$. Consequently, the solution v^N exists globally (i.e., $t_* = \infty$).

Proof. First, take $\varphi = v^N$ in (26) (equivalently, multiply $X^N(t)^T$ on both sides of (33)). As v^N is divergence free and disappears on the boundary, one has $\langle v^N, v^N \cdot \nabla v_p^N\rangle = 0$. Moreover,

$$\nabla v^N : \mathcal{D}(v^N) = \mathcal{D}(v^N) : \mathcal{D}(v^N).$$

Hence, we have

$$\frac{d}{dt}\int_\Omega |v^N|^p dx = -q\nu \int_\Omega |\mathcal{D}(v^N)|^p dx,$$

where q satisfies $\frac{1}{p}+\frac{1}{q}=1$. As a result, we have $\|v^N(t)\|_{L^p(\Omega)} \le \|v_0\|_{L^p(\Omega)}$ for any $0 \le t \le T$, or $\|v^N\|_{L^\infty(0,T;L^p(\Omega))} \le \|v_0\|_{L^p(\Omega)}$. Integrating equation (36) over time interval $[0,T]$, one has

$$\|v^N(T)\|^p_{L^p(\Omega)} - \|v^N(0)\|^p_{L^p(\Omega)} = -q\nu\|\mathcal{D}(v^N)\|_{L^p(0,T;L^p(\Omega))}.$$

This implies

$$\begin{aligned}\|v^N\|_{L^p(0,T;W_0^{1,p}(\Omega))} &\le C\|\mathcal{D}(v^N)\|_{L^p(0,T;L^p(\Omega))}\\ &= \frac{C}{q\nu}\big(\|v^N(0)\|^p_{L^p(\Omega)} - \|v^N(T)\|^p_{L^p(\Omega)}\big) \le \frac{C}{q\nu}\|v_0\|^p_{L^p(\Omega)}.\end{aligned}$$

Next, we show that $t_* = \infty$. In fact, define

$$\|X^N\|_1 := \left(\int_\Omega |v^N|^p dx\right)^{1/p}$$

and

$$\|X^N\|_2 := \left(\int_\Omega |\mathcal{D}(v^N)|^p dx\right)^{1/p}.$$

It is easy to see by Minkowski inequality that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms for X^N. Since X^N is in a finite dimensional Euclidean space, one thus can find a constant $c_N^1 > 0, c_N^2 > 0$ such that

$$c_N^1\|X^N\|_2 \le \|X^N\|_1 \le c_N^2\|X^N\|_2, \quad \forall X^N \in \mathbb{R}^N.$$

Hence,

$$\frac{d}{dt}\int_\Omega |v^N|^p dx \ge -\frac{q\nu}{(c_N^1)^p}\int_\Omega |v^N|^p dx.$$

Hence, v^N is never zero. Moreover, since $\|v^N\|_{L^\infty(0,T;L^p(\Omega))} \le \|v_0^N\|_{L^p(\Omega)}$, X^N never blows up so that one can in fact take $t_* = \infty$. $\square$

By the fact that $v_p^N = |v^N|^{p-2}v^N$, it is easy to obtain the following corollary.

Corollary 4.4. *It holds that*

$$v_p^N \in L^\infty(0,T;L^q(\Omega;\mathbb{R}^d)) \cap L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^d)).$$

Moreover,

$$\sup_N \|v_p^N\|_{L^\infty(0,T;L^q)} + \|v_p^N\|_{L^q(0,T;W^{1,q})} < \infty.$$

4.2. **Compactness.** In this section, we prove the precompactness of the sequences generated by the Galerkin approximation (25) and (28).

Later, we will need the time regularity of the sequences. To this end, we introduce a time-shift operator:

$$\tau_h v^N(x,t) = v^N(x,t+h). \tag{38}$$

First, we state a lemma which is useful in proving the convergence of time-shift operator. For a detailed proof of the lemma, one can read [9] or [13, Lemma 2].

Lemma 4.5. *Let $p > 1$. Then, there exists $C(p) > 0$ such that for any $\eta_1, \eta_2 \in \mathbb{R}^d$, it holds that*

$$(|\eta_1|^{p-2}\eta_1 - |\eta_2|^{p-2}\eta_2)\cdot(\eta_1-\eta_2) \ge C(p)(|\eta_1|+|\eta_2|)^{p-2}|\eta_1-\eta_2|^2.$$

In the following lemma, we would study the asymptotic behavior of sequence $\tau_h v^N(x,t)$ as h goes to 0.

Lemma 4.6. *Let $p \geq d \geq 2$ and Ω be a bounded domain. then, it holds that $\|\tau_h v^N - v^N\|_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))} \to 0$ uniformly in N as $h \to 0+$.*

Proof. For any fixed $t \leq T-h$ and any $\varphi \in W_N$, one has

$$\left\langle \tau_h v_p^N(t) - v_p^N(t), \varphi \right\rangle + \left\langle \int_t^{t+h} v^N \cdot \nabla v_p^N ds, \varphi \right\rangle = \nu \left\langle \int_t^{t+h} \mathcal{L}_p(v^N) ds, \varphi \right\rangle. \quad (39)$$

Taking $\varphi = \tau_h v^N(t) - v^N(t)$, we now estimate each term in (39) in detail.

First, by Lemma 4.5, it holds that

$$\begin{aligned}\int_\Omega (\tau_h v_p^N(t) - v_p^N(t)) \cdot (\tau_h v^N(t) - v^N(t)) dx &\geq C(p) \int_\Omega (|\tau_h v^N| + |v^N|)^{p-2} |\tau_h v^N - v^N|^2 dx \\ &\geq C(p) \|(\tau_h v^N - v^N)(t)\|_p^p.\end{aligned} \quad (40)$$

For the term $\int_\Omega \int_t^{t+h} \tau_h v^N(t) \cdot \mathcal{L}_p(v^N) ds dx$, Young's inequality yields:

$$\begin{aligned}\int_\Omega \int_t^{t+h} \tau_h v^N(t) \cdot \mathcal{L}_p(v^N) ds dx &= -\int_t^{t+h} \int_\Omega (\mathcal{D}(\tau_h v^N) : \mathcal{D}(v^N)) |\mathcal{D}(v^N)|^{p-2} dx ds \\ &\leq \int_t^{t+h} (\frac{1}{p} \|\mathcal{D}(\tau_h v^N)(t)\|_p^p + \frac{1}{q} \|\mathcal{D}(v^N)(s)\|_p^p) ds \\ &= \frac{h}{p} \|\mathcal{D}(\tau_h v^N)(t)\|_p^p + \frac{1}{q} \int_t^{t+h} \|\mathcal{D}(v^N)(s)\|_p^p ds,\end{aligned} \quad (41)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Similarly for the integral term $\int_\Omega \int_t^{t+h} v^N(t) \cdot \mathcal{L}_p(v^N) ds dx$, one has

$$\int_\Omega \int_t^{t+h} v^N(t) \cdot \mathcal{L}_p(v^N) ds dx \leq \frac{h}{p} \|\mathcal{D}(v^N)(t)\|_p^p + \frac{1}{q} \int_t^{t+h} \|\mathcal{D}(v^N)(s)\|_p^p ds. \quad (42)$$

To estimate the term $\int_\Omega \int_t^{t+h} v^N \cdot \nabla v_p^N \cdot \tau_h v^N(t) ds dx$, we need the Gagliardo-Nirenberg inequality, which tells us that on a bounded domain $\Omega \times [0,T]$, for any function $f \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^d)) \cap L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^d))$,

$$\|f\|_{2p}^{2p} \leq C \|\nabla f\|_p^d \|f\|_p^{2p-d} \leq C(\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^d))}) \|\nabla f\|_p^d.$$

Therefore,

$$\begin{aligned}&\int_\Omega \int_t^{t+h} v^N \cdot \nabla v_p^N \cdot \tau_h v^N(t) ds dx \\ &\leq \int_t^{t+h} \frac{1}{2p} (\|\tau_h v^N(t)\|_{2p}^{2p} + \|v^N(s)\|_{2p}^{2p}) + \frac{1}{q} \|v_p^N(s)\|_{W^{1,p}}^q ds \\ &\leq C \left(h \|\nabla \tau_h v^N(t)\|_p^d + \int_t^{t+h} \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q ds \right).\end{aligned} \quad (43)$$

Similarly, it holds that

$$\int_\Omega \int_t^{t+h} v^N \cdot \nabla v_p^N \cdot v^N(t) ds dx \leq C \left(h \|\nabla v^N(t)\|_p^d + \int_t^{t+h} \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q ds \right). \quad (44)$$

Overall, we have the final estimate:

$$\begin{aligned}\|(\tau_h v^N - v^N)(t)\|_p^p \leq & Ch(\|\nabla\tau_h v^N(t)\|_p^p + \|\nabla v^N(t)\|_p^p + |\nabla\tau_h v^N(t)\|_p^d + \|\nabla v^N(t)\|_p^d) \\ & + C\int_t^{t+h} \|\nabla v^N(s)\|_p^p + \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q ds.\end{aligned} \tag{45}$$

Integrating both sides over time t from 0 to $T-h$, one has

$$\begin{aligned}&\|\tau_h v^N - v^N\|_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))}^p \\ \leq & C_1 h\int_0^{T-h} \|\nabla\tau_h v^N(t)\|_p^p + \|\nabla v^N(t)\|_p^p + |\nabla\tau_h v^N(t)\|_p^d + \|\nabla v^N(t)\|_p^d dt \\ & + C_2\int_0^{T-h}\int_t^{t+h} \|\nabla v^N(s)\|_p^p + \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q dsdt \\ \leq & C_1 h\int_0^T \|\nabla v^N(s)\|_p^p + \|\nabla v^N(s)\|_p^d dt + C_2 h\int_0^T \|\nabla v^N(s)\|_p^p \\ & + \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q ds. \\ \leq & \tilde{C}h\int_0^T \|\nabla v^N(s)\|_p^p + \|\nabla v^N(s)\|_p^d + \|v_p^N(s)\|_{W^{1,q}}^q ds.\end{aligned} \tag{46}$$

With assumption $d \leq p$, $\int_0^T \|\nabla v^N(s)\|_p^d$ is bounded above by

$$\left(\int_0^T \|\nabla v^N(s)\|_p^p ds\right)^{\frac{d}{p}} T^{\frac{p-d}{p}}.$$

Following Proposition 4.3 and Remark 4.4, we conclude

$$\|\tau_h v^N - v^N\|_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))}^p \leq Ch, \tag{47}$$

which is a bound uniform in N. Thus, the lemma is proved. □

Next, we are going to derive some compactness results from the previous estimates. To reach this goal, we need the help from a variant of the Aubin-Lions Lemma [6, 18].

The operator $\mathcal{B}: X \to Y$ is called a (nonlinear) compact operator; if it maps bounded subsets of X to relatively compact subsets of Y. Let $L^1_{\text{loc}}(0,T;X)$ be the set of functions f such that for any $0 < t_1 < t_2 < T$, $f \in L^1(t_1,t_2;X)$, equipped with the semi-norms $\|f\|_{L^1(t_1,t_2;X)}$. A subset F of $L^1_{\text{loc}}(0,T;X)$ is called bounded, if for any $0 < t_1 < t_2 < T$, F is bounded in $L^1(t_1,t_2;X)$.

Lemma 4.7. *[Aubin-Lions] Let X, Y be Banach spaces, $1 \leq p < \infty$ and $\mathcal{B}: X \to Y$ be a (nonlinear) compact operator. Assume that F is a bounded subset of $L^1_{\text{loc}}(0,T;X)$ such that $E = \mathcal{B}(F) \subset L^p(0,T;Y)$, and*

- *E is bounded in $L^1_{\text{loc}}(0,T;Y)$,*
- *$\|\tau_h u - u\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0+$, uniformly for $u \in E$.*

Then, E is relatively compact in $L^p(0,T;Y)$.

Now, we are ready to get a candidate of weak solutions through the limit of subsequence of $\{v^N\}_{N\geq 1}$.

Proposition 4.8. *Let v^N be the solution to the ODE system (33). There exists a subsequence $\{N_k\}_{k\geq 1}$, $v \in L^\infty(0,T;U_p(\Omega)) \cap L^p(0,T;W)$ and a symmetric matrix $\chi \in L^q(0,T;L^q(\Omega;\mathbb{R}^{d\times d}))$, such that as $k \to \infty$,*

$$\begin{aligned} &v^{N_k} \to v, \ \textit{strongly in } L^p(0,T;L^p(\Omega;\mathbb{R}^d)),\\ &v_p^{N_k} \to |v|^{p-2}v =: v_p, \ \textit{strongly in } L^q(0,T;L^q(\Omega;\mathbb{R}^d)),\\ &\nabla v^{N_k} \rightharpoonup \nabla v, \ \textit{weakly in } L^p(0,T;L^p(\Omega;\mathbb{R}^d)),\\ &|\mathcal{D}(v^{N_k})|^{p-2}\mathcal{D}(v^{N_k}) \rightharpoonup \chi, \ \textit{weakly in } L^q(0,T;L^q(\Omega;\mathbb{R}^d)). \end{aligned} \tag{48}$$

Proof. In Lemma 4.7, take $X = W_0^{1,p}(\Omega;\mathbb{R}^d)$, $Y = L^p(\Omega;\mathbb{R}^d)$, $E = F = \{v^N\}_{N\geq n_*}$ and $\mathcal{B}$ to be the embedding map from X to Y. By Proposition 4.3 and the fact that $W_0^{1,p}(\Omega;\mathbb{R}^d)$ is compactly embedded to $L^p(\Omega;\mathbb{R}^d)$, $\{v^N\}_{N\geq n_*}$ is bounded in $L^\infty(0,T;L^p(\Omega;\mathbb{R}^d))$, and hence E is a bounded set in $L^1(0,T;L^p(\Omega;\mathbb{R}^d))$. In addition, by Lemma 4.6,

$$\|\tau_h v^N - v^N\|_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))} \to 0$$

as $h \to 0+$ uniformly for N. Then, by Lemma 4.7, $E = \{v^N\}$ is relatively compact in $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$. Hence, there is a subsequence $\{v^{N_k}\}$ and $v \in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$ such that

$$v^{N_k} \to v, \ \text{strongly in } L^p(0,T;L^p(\Omega;\mathbb{R}^d)).$$

Since $v^N \in L^\infty(0,T;U_p(\Omega))$ with the uniform bound $\|v_0\|_{L^p}$, one has $v \in L^\infty(0,T;U_p(\Omega))$ with the same bound.

The strong convergence of v^{N_k} in $L^p(0,T;L^p)$ implies the almost everywhere convergence, and thus

$$v_p^{N_k} \to |v|^{p-2}v := v_p, \ \text{a.e in } \Omega\times[0,T].$$

Combining with the fact that

$$\|v_p^{N_k}\|_{L^q(0,T;L^q(\Omega;\mathbb{R}^d))} = \|v^{N_k}\|^{p/q}_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} \to \|v\|^{p/q}_{L^p(0,T;L^p(\Omega;\mathbb{R}^d))} = \|v_p\|_{L^q(0,T;L^q(\Omega;\mathbb{R}^d))},$$

one has

$$v_p^{N_k} \to |v|^{p-2}v := v_p, \ \text{strongly in } L^q(0,T;L^q(\Omega;\mathbb{R}^d)).$$

From Proposition 4.3, we know that $\{\nabla v^{N_k}\}$ is bounded in $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$, which is a reflexive space. Then, for a subsequence (without relabeling), $\nabla v^{N_k} \rightharpoonup \zeta \in L^p(0,T;L^p(\Omega;\mathbb{R}^d))$. Take $\phi \in C^\infty(\Omega\times[0,T);\mathbb{R}^{d\times d})$. then,

$$\begin{aligned} \int_0^T \int_\Omega \zeta : \phi dxdt &= \lim_{k\to\infty} \int_0^T \int_\Omega \nabla v^{N_k} : \phi dxdt\\ &= -\lim_{k\to\infty} \int_0^T \int_\Omega v^{N_k}\cdot(\nabla\cdot\phi)dxdt = -\int_0^T\int_\Omega v\cdot(\nabla\cdot\phi)dxdt, \end{aligned} \tag{49}$$

where the last equality follows from the fact that $v^{N_k} \to v$ in $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$. Hence, we have $\nabla v = \zeta$. Then, v^{N_k} converges weakly to v in $L^p(0,T;W^{1,p})$ with $v^{N_k} \in L^p(0,T;W)$. So, $v \in L^p(0,T;W)$. Note that

$$\||\mathcal{D}(v^{N_k})|^{p-2}\mathcal{D}(v^{N_k})\|^q_{L^q(0,T;L^q(\Omega;\mathbb{R}^d))} = \int_0^T\int_{\mathbb{R}^d} |\mathcal{D}(v^{N_k})|^p \, dxdt < C,$$

which then yields that $|\mathcal{D}(v^{N_k})|^{p-2}\mathcal{D}(v^{N_k}) \rightharpoonup \chi$, for some $\chi \in L^q(0,T;L^q(\Omega;\mathbb{R}^d))$. □

5. **Existence of weak solutions.** To establish the existence of the weak solutions, we need to identify χ with $|\mathcal{D}(v)|^{p-2}\mathcal{D}(v)$. Now, define $G : L^p(0,T;L^p(\Omega;\mathbb{R}^{d\times d})) \to L^q(0,T;L^q(\Omega;\mathbb{R}^{d\times d}))$ by

$$G(\theta) := |\theta|^{p-2}\theta. \tag{50}$$

For any $\theta_1, \theta_2 \in L^p(0,T;L^p(\Omega;\mathbb{R}^{d\times d}))$ (see Lemma 4.5, where the product of two matrices is $A : B$),

$$\langle \theta_1 - \theta_2, G(\theta_1) - G(\theta_2)\rangle_{L^2_t(0,T;L^2_x(\Omega))} \geq 0, \tag{51}$$

which indicates that G is a monotone operator. We also note that the mapping $\lambda \to \langle v_2, G(v_1 + \lambda v_2)\rangle$ is continuous. To verify that $\chi = G(\mathcal{D}v)$, we will basically apply the following Browder-Minty theorem ([17, Theorem 10.49]).

Lemma 5.1. *Let X be a real reflexive Banach space. Let $G\colon X \to X'$ be a nonlinear, bounded monotone operator satisfying that $\forall v_1, v_2 \in X$, the mapping $\lambda \to \langle v_2, G(v_1 + \lambda v_2)\rangle$ is continuous. If $w_n \rightharpoonup w$ in X, and $G(w_n) \rightharpoonup \beta$ in X', and*

$$\limsup_{n\to\infty} \langle w_n, G(w_n)\rangle \leq \langle w, \beta\rangle,$$

then $G(w) = \beta$.

Below, we will set $X = L^p(0,T;L^q)$ and $X' = L^q(0,T;L^q)$. Moreover, recall that

$$V = \{v \in L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^d)), \nabla\cdot v = 0\}. \tag{52}$$

We note that V is reflexive since it is a closed subspace of $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^d))$, which is reflexive.

To establish the conditions in this lemma, we need to estimate the time regularity of the solutions.

5.1. **Time regularity and finite difference approximation.** In the following lemma, we discuss the time regularity. We aim to figure out the convergence of subsequence $\{\partial_t v_p^{N_k}\}$. However, $\partial_t v_p^{N_k}$ is in W_N', which decreases as N becomes large. Hence, we introduce the projection operator Q_N so that $Q_N : W \to W_N$. Then, we can talk about the convergence of $Q_N^*\partial_t v_p^{N_k}$, where Q_N^* is the conjugate operator of Q_N.

Particularly, we will introduce the following. For any $u \in W$, in terms of the Schauder basis, one has

$$u = \sum_{k=1}^{\infty} c_k\phi_k.$$

Define $Q_N : W \to W_N$ to be the projection operator

$$Q_N u = \sum_{k=1}^{N} c_k\phi_k.$$

Similar to the proof of Proposition 3.2, the Uniform Boundedness principle implies that $Q_N : W \to W_N \subset W$ is uniformly bounded in N, i.e.,

$$\sup_{N\in\mathbb{N}} \|Q_N\|_{W\to W} < \infty.$$

For a function v in V, for a.e. t, $v(t) \in W$. Hence, Q_N is well-defined on V as well, and Q_N is also uniformly bounded in V. Let $Q_N^* : V' \longrightarrow V'$ be the conjugate operator of Q_N, i.e., for any $u \in V$, $w \in V'$, it holds that

$$\langle Q_N u, w\rangle_{L_t^2(0,T;L_x^2(\Omega))} = \langle u, Q_N^* w\rangle_{L_t^2(0,T;L_x^2(\Omega))}.$$

We are now ready to show the time regularity results for the sequence $\{Q_N^* \partial_t v_p^{N_k}\}$.

Lemma 5.2. *For the subsequence $\{v^{N_k}\}$ in Proposition 4.8, we can further get a subsequence of v^{N_k} (without relabeling) such that*

$$Q_{N_k}^* \partial_t v_p^{N_k} \rightharpoonup \partial_t v_p, \text{ weakly in } V'. \tag{53}$$

Here, V' is the dual space of V as in (21). Moreover, for any $w \in V$, we have

$$\langle w, \partial_t \mathcal{P} v_p\rangle_{L_t^2(0,T;L_x^2(\Omega))} - \int_0^T \int_\Omega \nabla w : v \otimes v_p \, dxdt + \nu \int_0^T \int_\Omega \nabla w : \chi dxdt = 0. \tag{54}$$

Proof. For any fixed $\varphi \in V$ with $\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^d))} \le 1$, note that

$$\langle \varphi, Q_{N_k}^* \partial_t v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))} = \langle Q_{N_k}\varphi, \partial_t v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))}.$$

Since $Q_{N_k}\varphi(t) \in W_N$ for a.e. t, and Q_{N_k} is uniformly bounded, one then has by (26) that

$$\begin{aligned}
&\langle \varphi, Q_{N_k}^* \partial_t v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))} \\
&= -\langle Q_{N_k}\varphi, v^{N_k} \cdot \nabla v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))} + \nu \langle Q_{N_k}\varphi, \mathcal{L}_p(v^{N_k})\rangle_{L_t^2(0,T;L_x^2(\Omega))} \\
&= \int_0^T \int_\Omega \nabla Q_{N_k}\varphi : (v^{N_k} \otimes v_p^{N_k}) dxdt - \nu \int_0^T \int_\Omega \nabla Q_{N_k}\varphi : \mathcal{D}(v^{N_k})|\mathcal{D}(v^{N_k})|^{p-2} dxdt.
\end{aligned}$$

Now, we estimate the right hand side term by term. For the first term, by the Hölder inequality,

$$\left|\int_0^T \int_\Omega \nabla Q_{N_k}\varphi : (v^{N_k} \otimes v_p^{N_k}) dxdt\right| \le \int_0^T \int_\Omega \frac{|\nabla Q_{N_k}\varphi|^p}{p} + \frac{|v^{N_k}|^{2p}}{2p} + \frac{2p-3}{2p} |v_p^{N_k}|^{\frac{2p}{2p-3}} dxdt. \tag{55}$$

Since Q_{N_k} is bounded in $L^p(0,T;W_0^{1,p})$, $\int_0^T \int_\Omega |\nabla Q_{N_k}\varphi|^p \, dxdt$ is bounded uniformly in N_k. Applying the Gagliardo-Nirenberg inequality, one has

$$\|v^{N_k}\|_{2p}^{2p} \le C \|\nabla v^{N_k}\|_p^d \|v^{N_k}\|_p^{2p-d}, \tag{56}$$

and

$$\|v_p^{N_k}\|_{\frac{2p}{2p-3}}^{2p/(2p-3)} \le C \|\nabla v_p^{N_k}\|_q^{d/(2p-3)} \|v_p^{N_k}\|_q^{(2p-d)/(2p-3)}. \tag{57}$$

Using the estimates in Proposition 4.3, and the fact that $\frac{d}{2p-3} \le q$ and $\frac{2p-d}{2p-3} \le q$ for $p \ge d \ge 2$, one concludes the boundedness.

For the second term, it is easy to check the boundedness, since the Hölder inequality yields that

$$\left|\int_0^T \int_\Omega \nabla Q_{N_k}\varphi : \mathcal{D}(v^{N_k})|\mathcal{D}(v^{N_k})|^{p-2} dxdt\right| \le \int_0^T \int_\Omega \frac{|\nabla Q_{N_k}\varphi|^p}{p} + \frac{|\mathcal{D}(v^{N_k})|^p}{q} dxdt.$$

Again, using the estimates in Proposition 4.3, one gets the boundedness.

Therefore, $Q_{N_k}^* \partial_t v_p^{N_k}$ is bounded in V'. Since V is reflexive, there are a subsequence (without relabeling) and $\alpha \in V'$ such that

$$Q_{N_k}^* \partial_t v_p^{N_k} \rightharpoonup \alpha \text{ weakly in } V'. \tag{58}$$

For any $\psi \in C_c^1(\Omega \times [0,T)])$ with $\nabla \cdot \psi = 0$, one has

$$\begin{aligned}\langle \psi, Q_{N_k}^* \partial_t v_p^{N_k} \rangle_{L_t^2(0,T;L_x^2(\Omega))} &= \langle Q_{N_k}\psi, \partial_t v_p^{N_k} \rangle_{L_t^2(0,T;L_x^2(\Omega))} \\ &= -\langle \partial_t Q_{N_k}\psi, v_p^{N_k} \rangle_{L_t^2(0,T;L_x^2(\Omega))} - \int_\Omega Q_{N_k}\psi(x,0) v_p^{N_k}(x,0)dx \\ &\to -\langle \partial_t \psi, v_p \rangle_{L_t^2(0,T;L_x^2(\Omega))} - \int_\Omega \psi(x,0) v_p(x,0)dx,\end{aligned}$$

as $k \to \infty$. Note that the completion of $C_c^1(\Omega \times [0,T)])$ with zero divergence in $W_0^{1,p}$ is V. Hence, $\alpha = \partial_t v_p$ in V'.

Now, for any $\varphi \in C_c^1(\Omega \times [0,T))$ with $\nabla \cdot \varphi = 0$, by the convergence in Proposition 4.8, one clearly has $v^{N_k} \otimes v_p^{N_k} \to v \otimes v_p$ strongly in $L^1(0,T;L^1(\Omega;\mathbb{R}^{d\times d}))$. Hence it holds that

$$\langle \varphi, \partial_t v_p \rangle_{L_t^2(0,T;L_x^2(\Omega))} - \int_0^T \int_\Omega \nabla\varphi : v \otimes v_p \, dxdt + \nu \int_0^T \int_\Omega \nabla \varphi : \chi \, dxdt = 0.$$

Similar to the estimate in (55), $v \otimes v_p \in L^q(0,T;L^q(\Omega;\mathbb{R}^{d\times d}))$. Hence, by density argument, we can replace φ by any $w \in V$. ☐

Remark 5.3. We do not have the convergence $v^N \otimes v_p^N \to v \otimes v_p$ in $L^q(0,T;L^q(\Omega;\mathbb{R}^{d\times d}))$.

Next, we need a technical result to obtain the chain rule for the weak time derivative of the L^p integral for v. In the proof of the original work [13], there is a gap to establish the chain rule (specifically, in the paragraph and equation below Equation (97)). The method here can be used to fill that gap there. Define the energy function

$$H(t) := \frac{1}{q}\int |v(x,t)|^p \, dx = \frac{1}{q}\int |v_p(x,t)|^q \, dx. \tag{59}$$

Consider the finite time differences

$$D_h^+ g(t) := \frac{1}{h}(\tau_h g(t) - g(t)) \tag{60}$$

and

$$D_h^- g(t) := \frac{1}{h}(g(t) - \tau_{-h} g(t)). \tag{61}$$

We have

Proposition 5.4. *The time differences $D_h^+ v_p(t) 1_{[0,T-h]}(t)$ and $D_h^- v_p(t) 1_{[h,T]}(t)$ are bounded uniformly in V', and both have subsequences (without labeling) converging weakly to $\partial_t v_p$ as $h \to 0$ in V'. Moreover, there is a version of the mapping $t \mapsto H(t)$ that is continuous with $H(0) = \frac{1}{q}\|v_0\|_{L^p}^p$ and satisfies the following for any $0 \le s \le t \le T$:*

$$\int_s^t \int_\Omega v(\tau)\partial_t v_p(\tau)\, dxd\tau = H(t) - H(s). \tag{62}$$

Proof. Take $\varphi \in V$, $T > h > 0$. Let Q_N^* be the conjugate operator of Q_N as defined before. Then, by (26), one has

$$\begin{aligned}\int_0^T \int_\Omega \varphi(t) \cdot Q_N^* D_h^- v_p \, dxdt &= \int_h^T \frac{1}{h}\int_{t-h}^t \int_\Omega \nabla_x Q_N \varphi(t) : v^N(\tau) \otimes v_p^N(\tau) dx d\tau dt \\ &- \nu \int_h^T \frac{1}{h} \int_{t-h}^t \int_\Omega \nabla_x Q_N\varphi(t) : \mathcal{D}(v^N(\tau))|\mathcal{D}v^N(\tau)|^{p-2} dx d\tau dt =: I_1 + I_2. \end{aligned}\tag{63}$$

Next, we estimate these two terms. By Young's inequality,

$$
\begin{aligned}
|I_1| &\le \int_h^T \int_\Omega \frac{|\nabla_x Q_N \varphi(t)|^p}{p} dt + \int_h^T \frac{1}{h}\int_{t-h}^t \int_\Omega \frac{|v^N(\tau)\otimes v_p^N(\tau)|^q}{q}\, dx d\tau dt \\
&\le \frac{C_p}{p}\|\varphi\|^p_{L^p(0;T;W_0^{1,p})} + \int_0^T \int_\Omega \frac{|v^N(s)\otimes v_p^N(s)|^q}{q}\, ds,
\end{aligned}
\tag{64}
$$

which is bounded as in (55), (56) and (57). In addition, one similarly has

$$
\begin{aligned}
|I_2| &\le \int_h^T \int_\Omega \frac{|\nabla_x Q_N \varphi(t)|^p}{p} dt + \int_h^T \frac{1}{h}\int_{t-h}^t \int_\Omega \frac{\left(|\mathcal{D}v^N(\tau)|^{p-1}\right)^q}{q}\, dx d\tau dt \\
&\le \frac{C_p}{p}\|\varphi\|^p_{L^p(0;T;W_0^{1,p})} + \int_0^T \int_\Omega \frac{\left|\mathcal{D}v^N(s)\right|^{(p-1)q}}{q}\, dx ds,
\end{aligned}
\tag{65}
$$

which is also bounded due to the fact that $\mathcal{D}(v^N)$ is bounded in $L^p(0,T;L^p(\Omega))$. Hence, $\frac{1}{h}(Q_N^*\mathcal{P}v_p^N(t) - Q_N^*\mathcal{P}v_p^N(t-h))1_{[h,T]}(t)$ is uniformly (in N and h) bounded in V'. Letting $N\to\infty$, by the strong convergence of v_p^N to v_p, we have that $D_h^- v_p(t)1_{[h,T]}(t)$ is uniformly (in h) bounded in V'. Similar arguments hold for $D_h^+ v_p(t)1_{[0,T-h]}$.

Now, up to a subsequence, as $h\to 0$, $D_h^- v_p(t)1_{[h,T]}(t)$ would have a weak limit in V', denoted by γ. By pairing with a smooth function, it is not hard to identify that γ is just $\partial_t v_p$. Similarly, $D_h^+ v_p(t)1_{[0,T-h]}$ would have a subsequence converging to $\partial_t v_p$ as well.

Now, since $v\in V$, let $s,t\in(h,T)$, and we have

$$
\begin{aligned}
\int_s^t \int_\Omega v(\tau) D_h^- v_p(\tau)\, dx d\tau &= \frac{1}{h}\int_s^t \int_\Omega |v(\tau)|^p - v(\tau)v_p(\tau-h)\, dx d\tau \\
&\ge \frac{1}{h}\int_s^t \int_\Omega |v(\tau)|^p - \frac{|v(\tau)|^p}{p} - \frac{1}{q}|v_p(\tau-h)|^q\, dx d\tau \\
&= \frac{1}{qh}\int_t^{t-h} \int_\Omega |v(\tau)|^p\, dx d\tau - \frac{1}{qh}\int_s^{s-h}\int_\Omega |v(\tau)|^p\, dx d\tau.
\end{aligned}
\tag{66}
$$

Then, as $h\to 0^+$, the left hand side converges to $\int_s^t\int_\Omega v(\tau)\partial_t v_p(\tau)\, dx d\tau$ due to the weak convergence of $D_h^- v_p 1_{[h,T]}$ and thus $D_h^- v_p 1_{[s,t]}$ in V'. The right hand side tends to $\frac{1}{q}\|v(t)\|^p_{L^p} - \frac{1}{q}\|v(s)\|^p_{L^p}$ as h tends to zero, for almost every $t,s\in(0,T)$. Hence, for almost every $s<t, s,t\in(0,T)$, one has

$$
\int_s^t \int_\Omega v(\tau)\partial_t v_p(\tau)\, dx d\tau \ge H(t) - H(s).
$$

Moreover, one may do the same thing for $D_h^+ v_p(t)$ to get for almost every $s,t\in(0,T-h)$ that

$$
\int_s^t \int_\Omega v(\tau) D_h^+ v_p(\tau)\, dx d\tau \le \frac{1}{qh}\int_t^{t+h}\int_\Omega |v(\tau)|^p\, dx d\tau - \frac{1}{qh}\int_s^{s+h}\int_\Omega |v(\tau)|^p\, dx d\tau.
$$

By the same argument, for almost every $s<t, s,t\in(0,T)$, one has

$$
\int_s^t \int_\Omega v(\tau)\partial_t v_p(\tau)\, dx d\tau \le H(t) - H(s).
$$

Hence, we have the chain to hold

$$H(t)-H(s)=\int_s^t\int_\Omega v(\tau)\partial_t v_p(\tau)\,dxd\tau. \tag{67}$$

for almost every $s,t\in(0,T)$ with $s<t$. Since the right hand side is continuous in s,t H can be made into a continuous function.

Moreover, to see $H(0^+)=\frac{1}{q}\|v_0\|_{L^p}^p$, we note that

$$\left|\frac{1}{q}\int_\Omega |v^N(t)|^p\,dx-\frac{1}{q}\int_\Omega |v^N(0)|^p\,dx\right|\le\left|\int_0^t\int_\Omega v^N(\tau)\partial_t v_p^N(\tau)\,d\tau\right|\le Ct,$$

where C is uniform in N. As $N\to\infty$, $\frac{1}{q}\int_\Omega |v^N(0)|^p\,dx\to\frac{1}{q}\|v_0\|_{L^p}^p$, and for almost every t, $\frac{1}{q}\int_\Omega |v^N(t)|^p\,dx$ converges to $H(t)$. Hence, $H(0+)$ is given as mentioned. This also means in (67) we can take $t=T$ and $s=0$. $\square$

5.2. **Existence of weak solutions and the energy dissipation equality.** In this subsection, we first identify χ and then prove the existence of weak solutions.

Lemma 5.5. *In $L^q(0,T;L^q(\Omega;\mathbb{R}^{d\times d}))$, we have*

$$\chi=|\mathcal{D}v|^{p-2}\mathcal{D}v. \tag{68}$$

In other words, for any $\varphi\in C_c^\infty([0,T)\times\Omega)$,

$$\int_0^T\int_\Omega \nabla\varphi:\chi dxdt=\int_0^T\int_\Omega\nabla\varphi:\mathcal{D}(v)|\mathcal{D}(v)|^{p-2}dxdt. \tag{69}$$

Proof. Taking $w=v$ in (54), due to the fact that χ is symmetric, one has

$$\langle v,\partial_t v_p\rangle_{L_t^2(0,T;L_x^2(\Omega))}=-\nu\int_0^T\int_\Omega\nabla v:\chi dxdt=-\nu\int_0^T\int_\Omega\mathcal{D}(v):\chi dxdt.$$

In fact, we need to show

$$\int_0^T\int_\Omega\nabla v:v\otimes v_p dxdt=0$$

To justify this, first recalling (56) and (57), one has $v\in L^{2p}(0,T;L^{2p}(\Omega))$ and $v_p\in L^r(0,T;L^r(\Omega))$ with $r=\frac{2p}{2p-3}$. We can extend v to be defined in $\mathbb{R}^d$ such that $v=0$ for $x\notin\Omega$. Then, $v\in L^{2p}(0,T;L^{2p}(\mathbb{R}^d))$, and $\nabla v\in L^p(0,T;L^p(\mathbb{R}^d))$. Then, $v_\epsilon:=v*J_\epsilon$ for a mollifier J_ϵ. Then, one has $v_\epsilon\to v$ in $L^{2p}(0,T;L^{2p}(\mathbb{R}^d))$, $\nabla v_\epsilon\to\nabla v$ in $L^p(0,T;L^p(\mathbb{R}^d))$ and $(v_\epsilon)_p\to v_p$ in $L^r(0,T;L^r(\Omega))$. Hence, due to $\nabla\cdot v_\epsilon=0$, one has

$$0=\int_0^T\int_\Omega\nabla v_\epsilon:v_\epsilon\otimes(v_\epsilon)_p dxdt\to\int_0^T\int_\Omega\nabla v:v\otimes v_p dxdt.$$

On the other hand, with the same notation as in Lemma 5.2 and Proposition 5.4,

$$\frac{\|v^{N_k}(t)\|_{L^p}^p}{q}-\frac{\|v^{N_k}(s)\|_{L^p}^p}{q}\to\frac{\|v(t)\|_{L^p}^p}{q}-\frac{\|v(s)\|_{L^p}^p}{q}=\int_s^t\int_\Omega v(\tau)\partial_t v_p(\tau)\,dxd\tau. \tag{70}$$

By the continuity argument, same as in the end of the proof of Proposition 5.4, we can take $t=T$ and $s=0$. Also, the left hand side of (70) equals

$$\langle v^{N_k},Q_N^*\partial_t v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))}=\langle v^{N_k},\partial_t v_p^{N_k}\rangle_{L_t^2(0,T;L_x^2(\Omega))}=-\int_0^T\int_\Omega|\mathcal{D}v^{N_k}|^p dxdt. \tag{71}$$

Hence, one actually has,

$$\lim_{k\to\infty}\int_0^T\int_\Omega |\mathcal{D}v^{N_k}|^p dxdt = \int_0^T\int_\Omega \mathcal{D}(v):\chi dxdt.$$

By the property of G and Lemma 5.1, one has

$$\chi = G(\mathcal{D}(v)).$$

□

We get the existence of weak solution to p-Navier-Stokes equations.

Theorem 5.6. *Let Ω be a bounded domain in $\mathbb{R}^d$ with C^∞ boundary and $p \ge d \ge 2$. Let $v_0 \in U_p(\Omega)$. There exists a global weak solution to initial/boundary value problem of the p-Navier-Stokes equations (Equations (2)) in the sense of Definition 2.3.*

Proof. For any $\psi \in C_c^\infty(\Omega\times[0,T))$, one has

$$\int_0^T\int_\Omega \nabla\psi\cdot v^{N_k} dxdt = 0,$$

since v^{N_k} is divergence free and disappears on the boundary. Using Lemma 4.8, sending k to ∞, one has

$$\int_0^T\int_\Omega \nabla\psi\cdot v dxdt = 0.$$

For any $\varphi \in C_c^\infty(\Omega\times[0,T),\mathbb{R}^d)$, one has

$$\begin{aligned}\int_0^T\int_\Omega & v_p^{N_k}\cdot\partial_t\varphi dxdt + \int_0^T\int_\Omega \nabla\varphi:(v^{N_k}\otimes v_p^{N_k})dxdt \\ &- \nu\int_0^T\int_\Omega \nabla\varphi:\mathcal{D}(v^{N_k})|\mathcal{D}(v^{N_k})|^{p-2}dxdt + \int_\Omega |v_0^{N_k}|^{p-2}v_0^{N_k}\cdot\varphi(x,0)dx = 0.\end{aligned}$$

Again using Lemma 4.8, sending k to ∞, one gets

$$\begin{aligned}\int_0^T\int_\Omega & v_p\cdot\partial_t\varphi dxdt + \int_0^T\int_\Omega \nabla\varphi:(v\otimes v_p)dxdt \\ &- \nu\int_0^T\int_\Omega \nabla\varphi:\mathcal{D}(v)|\mathcal{D}(v)|^{p-2}dxdt + \int_\Omega |v_0|^{p-2}v_0\cdot\varphi(x,0)dx = 0.\end{aligned}$$

For the time regularity, note that for some C independent of N_k,

$$\|\tau_h v^{N_k} - v^{N_k}\|^p_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))} \le Ch,$$

by Lemma 4.6. Using the fact that $v^{N_k}\to v$ strongly in $L^p(0,T;L^p(\Omega;\mathbb{R}^d))$, one then has

$$\|\tau_h v - v\|^p_{L^p(0,T-h;L^p(\Omega;\mathbb{R}^d))} \le Ch.$$

Thus, the time regularity is proved. From all the above, we proved that v is a weak solution to the p-Navier-Stokes problem for given T.

Note that T is arbitrary, and one may use a diagonal argument to extract a subsequence of v^{N_k} that converges in the sense listed in Proposition 4.8 for every $[0,n]$. The limit of this subsequence is then a weak solution on any bounded interval and thus a global weak solution. □

Moreover, we have the following the energy dissipation law for the weak solutions, which follows directly from Proposition 5.4 and the argument in the proof of Lemma 5.5.

Proposition 5.7. *Suppose that $p \geq d \geq 2$. The weak solutions considered in Theorem 5.6 satisfy the energy dissipation equality*

$$H(t) - H(s) = -\nu \int_s^t \int_\Omega |\mathcal{D}(v)|^p \, dx d\tau. \tag{72}$$

As well-known, the usual Navier-Stokes equations have energy dissipation equality for $d = 2$ only and this may be lost for $d = 3$, related to the famous Onsager conjecture [7]. Now, for the p-Navier-Stokes equations, when $d \geq 3$, if p is large enough, the energy dissipation equality still holds.

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Appendix A. **Equivalence of the norms.** In this section, we show the following equivalency of the $W^{1,p}$ norm:

$$C'(\|u\|_{L^p} + \|\mathcal{D}(u)\|_{L^p}) \leq \|u\|_{L^p} + \|Du\|_{L^p} \leq C(\|u\|_{L^p} + \|\mathcal{D}(u)\|_{L^p}), \tag{73}$$

where C, C' are constants, $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The proof can be found in [16], Proposition 1.1, for completion. We sketch the main steps here.

The first inequality of (73) is trivial. For the second inequality, observe that one has the following relations between u and $\mathcal{D}(u)$:

$$\mathcal{D}(u)_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial \mathcal{D}(u)_{ik}}{\partial x_j} + \frac{\partial \mathcal{D}(u)_{ij}}{\partial x_k} - \frac{\partial \mathcal{D}(u)_{jk}}{\partial x_i}.$$

Hence, suppose $\|u\|_{L^p} + \|\mathcal{D}(u)\|_{L^p}$ is finite, and then we have $\frac{\partial^2 u_i}{\partial x_j \partial x_k} \in W^{-1,p}$. By Theorem 1.1 in [16],

$$\|\partial_j u_i\|_{L^p} \leq C(\|\partial_j u_i\|_{W^{-1,p}} + \|\nabla \partial_j u_i\|_{W^{-1,p}}),$$

so that $\partial_j u_i \in L^p$. By the Open Mapping Theorem, these two norms are equivalent.

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