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A Simple Construction of Fat Cantor Sets

Let $\mathcal{C} \subset [0,1]$ be the standard Cantor set, and let $c\colon [0,1] \to [0,1]$ be the standard Cantor function—continuous, increasing, and constant on each of the countably many components of the open set $\mathcal{C}^c = [0,1] \setminus \mathcal{C}$. Applying c expands $\mathcal{C}$, a set of measure zero, to $c(\mathcal{C}) = [0,1]$, the unit interval.

For any $t \in (0,1)$, the function $c_t\colon [0,1] \to [0,1]$ defined by

$$c_t(x) = (1-t)x + tc(x),$$

is continuous, strictly increasing, and bijective. By consequence, the set $c_t(\mathcal{C})$, like $\mathcal{C}$, is closed and nowhere dense. For any component interval I of $\mathcal{C}^c$, its image $c_t(I)$ is an open interval with length $\lambda(c_t(I)) = (1-t)\lambda(I)$, where λ is Lebesgue measure. It follows the image $c_t(\mathcal{C}^c)$ has measure $1-t$, hence $c_t(\mathcal{C}) = c_t(\mathcal{C}^c)^c$ is a fat Cantor set with measure $t \in (0,1)$.

This construction is motivated by the theory of optimal transport [1]. The maps c_t push forward λ to a family of measures ν_t which provide *displacement interpolants* between λ and the pure point measure $\nu = \sum_I \lambda(I)\delta_{c(I)}$, where the sum is over the countably many components of $\mathcal{C}^c$.

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