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WELL-POSEDNESS FOR THE KELLER-SEGEL EQUATION WITH FRACTIONAL LAPLACIAN AND THE THEORY OF PROPAGATION OF CHAOS

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Abstract. This paper investigates the generalized Keller-Segel (KS) system with a nonlocal diffusion term $-\nu(-\Delta)^{\frac{\alpha}{2}}\rho$ $(1<\alpha<2)$. Firstly, the global existence of weak solutions is proved for the initial density $\rho_0 \in L^1 \cap L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ $(d \geq 2)$ with $\|\rho_0\|_{\frac{d}{\alpha}} < K$, where K is a universal constant only depending on d, α, ν. Moreover, the conservation of mass holds true and the weak solution satisfies some hyper-contractive and decay estimates in L^r for any $1 < r < \infty$. Secondly, for the more general initial data $\rho_0 \in L^1 \cap L^2(\mathbb{R}^d)$ $(d = 2, 3)$, the local existence is obtained. Thirdly, for $\rho_0 \in L^1\big(\mathbb{R}^d, (1+|x|)dx\big) \cap L^\infty(\mathbb{R}^d)(\ d \geq 2)$ with $\|\rho_0\|_{\frac{d}{\alpha}} < K$, we prove the uniqueness and stability of weak solutions under Wasserstein metric through the method of associating the KS equation with a self-consistent stochastic process driven by the rotationally invariant α-stable Lévy process $L_\alpha(t)$. Also, we prove the weak solution is L^∞ bounded uniformly in time. Lastly, we consider the N-particle interacting system with the Lévy process $L_\alpha(t)$ and the Newtonian potential aggregation and prove that the expectation of collision time between particles is below a universal constant if the moment $\int_{\mathbb{R}^d} |x|^\gamma \rho_0 dx$ for some $1 < \gamma < \alpha$ is below a universal constant K_γ and ν is also below a universal constant. Meanwhile, we prove the propagation of chaos as $N \to \infty$ for the interacting particle system with a cut-off parameter $\varepsilon \sim (\ln N)^{-\frac{1}{d}}$, and show that the mean field limit equation is exactly the generalized KS equation.

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1. **Introduction.** In this paper, we study the existence, uniqueness, stability and regularity for the following generalized Keller-Segel model with nonlocal diffusion term $-\nu(-\Delta)^{\frac{\alpha}{2}}\rho$ $(1<\alpha<2)$ in dimension $d\geq 2$

$$\begin{cases} \partial_t\rho = -\nu(-\Delta)^{\frac{\alpha}{2}}\rho - \nabla\cdot(\rho\nabla c),\ x\in\mathbb{R}^d,\ t\geq 0,\\ -\Delta c = \rho,\\ \rho(0,x)=\rho_0(x), \end{cases} \tag{1}$$

where ν is a positive constant. As usual, this model is developed to describe the biological phenomenon chemotaxis with anomalous diffusion. In the context of biological aggregation, $\rho(t,x)$ represents the density of some biology cells, $c(t,x)$ represents the chemical substance concentration and it is given by the fundamental solution

$$c(t,x)=\begin{cases} C_d\displaystyle\int_{\mathbb{R}^d}\frac{\rho(t,y)}{|x-y|^{d-2}}dy, & \text{if } d\geq 3,\\ -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^d}\ln|x-y|\rho(t,y)dy, & \text{if } d=2, \end{cases} \tag{2}$$

where $C_d=\dfrac{1}{d(d-2)\alpha_d}$, $\alpha_d=\dfrac{\pi^{d/2}}{\Gamma(d/2+1)}$, i.e. α_d is the volume of the d-dimensional unit ball.

The motivation of using anomalous diffusion comes from the fact that in many situations found in nature, organisms adopt Lévy process search strategies which have continuous paths interspersed with random jumps (also called Lévy flight) and therefore dispersal is better modelled by the non-local operator such as $-(-\Delta)^{\frac{\alpha}{2}}$ [2, 19, 20, 25, 26]. Indeed, experimental evidences of super-diffusive behaviour have been found in biological systems, such as microzooplankton [25], soil amebas [26] and E. coli [32]. Super-diffusion is characterized by a super-linear dependence in time of the mean square-displacement of the position of the dispersing population. In mathematical description: the normal diffusion's variation satisfies $[X(t),X(t)]\propto t$, and the super-diffusion's variation satisfies $[X(t),X(t)]\propto t^{\beta}$ with $\beta>1$. Moreover, the correct description of a population undergoing super-diffusion is Lévy process.

As the simplest Lévy process, the rotationally invariant α-stable Lévy process has the infinitesimal generator of the form $(-\Delta)^{\frac{\alpha}{2}}, 0<\alpha<2$ [1, 3], see also [38]. For readers' convenience, we give a brief introduction to the α-stable Lévy process in Appendix A. In probabilistic terms, by replacing the Laplacian to its fractional power, we can extend the results for the stochastic equations driven by Brownian motion to those driven by α-stable Lévy process. In [31], the uniqueness and stability under Wasserstein metric of classic KS equation have been proved by associating it with a self-consistent stochastic process driven by Brownian motion. This provides us a similar method to prove the uniqueness and stability for non-local KS equation (1) by considering a self-consistent stochastic process driven by rotationally invariant α-stable Lévy process (see Section 4).

Compare system (1) with the classic model of chemotaxis introduced by Keller and Segel in [24]. The difference is that we replace the Laplacian Δ by its fractional power $-(-\Delta)^{\frac{\alpha}{2}}$ which is a integral operator, namely the fractional diffusion with exponent $0<\alpha<2$. In recent years, there has been a surge of activity focused on the use of this fractional diffusion operator, such as [13, 14, 15] by Caffarelli *et al.*. The main reason for using fractional Laplacian is that we can further extend the theory of diffusion by taking into account the presence of the so-called long range interactions. This nonlocal operator does not act by point-wise differentiation but by a global integration with respect to a singular kernel. We refer to [39, 40] for comprehensive review of recent progress in the theory of fractional Laplacian operator.

Under the mass invariant scaling $\rho_\lambda(t,x) = \lambda^d \rho(\lambda x, \lambda^\alpha t)$, KS model (1) exhibits the supercritical behavior. Namely, the aggregation dominates the diffusion for high density (large λ) and the density may blow up in finite time. While for low density (small λ), the diffusion dominates the aggregation and the density has infinite-time spreading. Also, notice that PDEs (1) possesses L^q norm invariant with $q := \frac{d}{\alpha}$. Indeed if $\rho(t,x)$ is a solution then $\rho_\lambda(t,x) = \lambda^\alpha \rho(\lambda x, \lambda^\alpha t)$ is also a solution, and this scaling preserves the L^q norm $\|\rho_\lambda\|_q = \|\rho\|_q$. This invariant scaling will provide us a sharp initial condition $\|\rho_0\|_q < K$ in the proof of global existence (see Section 2).

The fractional KS system was first studied by Escudero in [20], where the author prove that this model has blowing-up solutions for large initial conditions in dimensions $d \geq 2$. Also, he obtains the global existence with the initial data $\rho_0 \in L^1 \cap H^1(\mathbb{R})$ in dimension $d = 1$, which is a subcritical case. This system has also been studied by Biler *et al.* [6, 7, 8, 9, 10] and Li *et al.* [27, 28, 29]. For example, in [7], authors study the conditions for local and global in time existence of positive weak solutions in dimensions $d = 2, 3$. In [8], authors deal with the so-called mild solutions based on applications of the linear analytic semigroup theory to quasi-linear evolution equations. They prove the existence of local in time mild solutions and global mild solutions under the small initial data $\|\rho_0\|_q < \varepsilon$ in dimensions $d \geq 2$. In [6], authors consider the Keller-Segel model for the chemotaxis with either classical or fractional diffusion in dimension $d = 2$. The blow-up of solutions in terms of suitable Morrey spaces norms is derived. In [29], the authors prove the local existence and uniqueness of solutions by assuming $\rho_0 \in L^p \cup H^s(\mathbb{R}^2)$ with $s > 3$ and $1 < p < 2$. Moreover, they attain further properties of the solutions including mass conservation and non-negativity.

Compared to the former studies, ours has more evolved results:

I. (Global existence, hyper-contractive and decay estimates) For $d \geq 2$, $1 < \alpha < 2$, $0 \leq \rho_0 \in L^1(\mathbb{R}^d)$ and $\|\rho_0\|_{\frac{d}{\alpha}} < K$, where K is a universal constant only depending on d, α, ν. We prove that there exists a global weak solution ρ such that $\|\rho(t,\cdot)\|_{\frac{d}{\alpha}} < K$ for all $t > 0$. The mass conservation, decay estimate and hyper-contractivity are also obtained (see Theorem 2.3).

II. (Local existence) We have proved that under the more general initial density $0 \leq \rho_0 \in L^1 \cap L^2(\mathbb{R}^d)$, for $1 < \alpha < 2$ when $d = 2$ or $\frac{3}{2} < \alpha < 2$ when $d = 3$, there exists a local in time weak solution $\rho(t,x)$ with regularity $\rho \in L^\infty\big(0,T; L^2(\mathbb{R}^d)\big) \cap L^2\big(0,T; H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big)$ and $\partial_t \rho \in L^2\big(0,T; H^{-1}(\mathbb{R}^d)\big)$ (see Theorem 3.1 and Theorem 3.2).

Before we go to further results, we recast c in (2) as $c = \Phi * \rho$ where $\Phi(x)$ is the Newtonian potential, and it can be represented as

$$\Phi(x) = \begin{cases} \dfrac{C_d}{|x|^{d-2}}, & \text{if } d \geq 3, \\ -\dfrac{1}{2\pi} \ln|x|, & \text{if } d = 2. \end{cases} \tag{3}$$

Thus we have the attractive force

$$F(x) = \nabla\Phi(x) = -\frac{C_* x}{|x|^d}, \quad \forall\, x \in \mathbb{R}^d \backslash \{0\}, \tag{4}$$

where $C_* = \frac{\Gamma(d/2)}{2\pi^{2/d}}$. Moreover $\nabla c = F * \rho$.

In this paper, we introduce the following mean-field self-consistent stochastic process $X(t)$ underlying the KS equations:

$$X(t) = X_0 + \int_0^t \int_{\mathbb{R}^d} F\big(X(s) - y\big)\rho(s, y)dyds + \nu L_\alpha(t), \tag{5}$$

where X_0 has density $\rho_0(x)$, and $L_\alpha(t)$ is a rotationally invariant α-stable Lévy process. Furthermore, we require the process $X(t)$ has the density $\rho(t,x)$ and the drift term $\int_{\mathbb{R}^d} F(x-y)\rho(s,y)dy$ is self-determined. Next we introduce the following notion of strong solution of (5) by requiring $\rho \in L^\infty\big(0,T; L^1 \cap L^\infty(\mathbb{R}^d)\big)$ for any $T > 0$ to make sure the log-Lipschitz continuity of the self-consistent term $\int_{\mathbb{R}^d} F(x-y)\rho(s,y)dy$. This kind of log-Lipschitz continuity also appeared in the 2D incompressible Euler equation and the uniqueness was proved by Yudovich [42].

Definition 1.1. We say that $\big(X(t),\rho\big)$ is a strong solution to (5) if there is a stochastic process $X(t)$ and it has the density $\rho \in L^\infty\big(0,T; L^1 \cap L^\infty(\mathbb{R}^d)\big)$ for any $T > 0$ such that

$$X(t) = X_0 + \int_0^t \int_{\mathbb{R}^d} F\big(X(s) - y\big)\rho(s, y)dyds + \nu L_\alpha(t) \quad a.s.,$$

We will utilize the strong solution of (5) as a characteristic line to prove the uniqueness and stability for the KS equation (1) under the following assumptions:

Assumption 1. *For $1 < \alpha < 2$, the initial data $\rho_0(x)$ satisfies:*

1. $0 \le \rho_0(x) \in L^1 \cap L^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \rho_0(x)dx = 1$ and $\int_{\mathbb{R}^d} |x|\rho_0(x)dx < \infty$;

2. $\|\rho_0\|_{\frac{d}{\alpha}} < \frac{4\alpha\nu}{dS^2_{\alpha,d}}$,

where $S^2_{\alpha,d} = 2^{-\alpha}\pi^{\frac{\alpha}{2}}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d+\alpha}{2})}\big[\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\big]^{\frac{\alpha}{d}} = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d+\alpha}{2})}|\mathbb{S}^{d-1}|^{-\frac{\alpha}{d}}$, and $|\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

III. (Uniqueness) For $d \ge 2$, the initial data ρ_0 satisfies Assumption 1. We obtain a unique global weak solution $\rho(t,x)$ to (1) with regularity

$$\rho \in L^\infty\left(0,T; L^\infty(\mathbb{R}^d) \cap L^1\big(\mathbb{R}^d, (1+|x|)dx\big)\right);$$

$$\rho^{\frac{q}{2}} \in L^2\big(\mathbb{R}_+; H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big);\ \partial_t\rho \in L^2\big(0,T; W^{-\alpha,\frac{2(q+1)}{q+3}}(\mathbb{R}^d)\big),$$

for any $T > 0$. Moreover, the corresponding self-consistent stochastic equation (5) has a unique strong solution $\big(X(t),\rho\big)$ with initial data (X_0,ρ_0), and ρ is the unique weak solution to (1) (see Theorem 4.2).

IV. (Dobrushin's type Stability) With the help of self-consistent stochastic process (5), we also obtain the stability with initial data in Wasserstein distance $\mathcal{W}_1$ for (1). Namely, for any fixed $T > 0$, there exists two constants C $\big($depending on $\|\rho_t^1\|_{L^\infty\left(0,T;L^1\cap L^\infty(\mathbb{R}^d)\right)}$ and $\|\rho_t^2\|_{L^\infty\left(0,T;L^1\cap L^\infty(\mathbb{R}^d)\right)}\big)$ and C_T $\big($depending only on $T\big)$ such that for any $t \in [0,T]$

$$\sup_{t\in[0,T]} \mathcal{W}_1(\rho_t^1,\rho_t^2) \le C_T \max\left\{\mathcal{W}_1(\rho_0^1,\rho_0^2), \big\{\mathcal{W}_1(\rho_0^1,\rho_0^2)\big\}^{e^{-CT}}\right\},$$

where ρ_t^1, ρ_t^2 are weak solutions to (1) with initial data $\rho_0^1(x), \rho_0^2(x)$ respectively (see Theorem 4.3).

Our last result will deal with the following N-particle interacting system of many indistinguishable individuals $\{X^i(t)\}_{i=1}^N$ with Newtonian potential aggregation and N independent rotationally invariant α-stable Lévy process $\{L_\alpha^i(t)\}_{i=1}^N$:

$$dX^i(t) = \frac{1}{N-1}\sum_{j\neq i}^N F\big(X^i(t)-X^j(t)\big)dt + \nu\, dL_\alpha^i(t), \quad i=1,\cdots,N.$$

under the condition that the initial data $\{X_0^i\}_{i=1}^N$ are independent and identically distributed (i.i.d.) with a common probability density function $\rho_0(x)$.

For $d\geq 2$, $\alpha\in(1,2)$ and some $1<\gamma<\alpha$, suppose the initial data satisfies $\rho_0\in L^1(\mathbb{R}^d,(1+|x|^\gamma)dx)$ and $\|\rho_0\|_1=1$. In [8], Biller *et al.* have proved that there exists a universal constant $K_\gamma>0$, such that the weak solution to the non-local KS equation (1) with initial density $\int_{\mathbb{R}^d}|x|^\gamma\rho_0(x)dx\leq K_\gamma$ will blow up at a finite time. Inspired by this, we prove the following result:

V. (Collision between particles) For $d\geq 2$, $\alpha\in(1,2)$, $\nu<\frac{K_1}{K_2}$ (will be specified) and some $\gamma\in(1,\alpha)$, suppose initial data satisfies $\rho_0\in L^1\big(\mathbb{R}^d,(1+|x|^\gamma)dx\big)$ and $\|\rho_0\|_1=1$. Then there exists two universal constants K_γ , $T^c>0$, such that if $\int_{\mathbb{R}^d}|x|^\gamma\rho_0(x)dx<K_\gamma$, the expectation of the collision time $\mathbb{E}(\tau)$ satisfies

$$\mathbb{E}(\tau)\leq T^c$$

(see Theorem 5.4).

Although we can only prove the collision happens when $\int_{\mathbb{R}^d}|x|^\gamma\rho_0(x)dx$ is below a certain constant, we believe that the collision for (1) is generic since the initial data ρ_0 may concentrate in a local region. Therefore in order to obtain a global strong solution to the interacting particle system, we regularize the force $F(x)$ by a blob function $J(x)\in C^2(\mathbb{R}^d)$, supp $J(x)\subset B(0,1)$, $J(x)\geq 0$ and $\int_{B(0,1)}J(x)dx=1$. Let $J_\varepsilon(x)=\frac{1}{\varepsilon^d}J(\frac{x}{\varepsilon})$, $\Phi_\varepsilon(x)=J_\varepsilon*\Phi(x)$ for $x\in\mathbb{R}^d$ and $F_\varepsilon(x)=\nabla\Phi_\varepsilon(x)$. In this article we take a cut-off function $J(x)\geq 0$, $J(x)\in C_0^3(\mathbb{R}^d)$,

$$J(x)=\begin{cases} C(1+\cos\pi|x|)^2, & \text{if } |x|\leq 1,\\ 0, & \text{if } |x|>1.\end{cases} \tag{6}$$

where C is a constant such that $C|\mathbb{S}^{d-1}|\int_0^1(1+\cos\pi r)^2r^{d-1}dr=1$. Then we have $F_\varepsilon(x)=F(x)g(\frac{|x|}{\varepsilon})$ for any $x\neq 0$, where $g(r)=|\mathbb{S}^{d-1}|\int_0^r J(s)s^{d-1}ds$. Moreover $F_\varepsilon(x)=F(x)$ for any $|x|\geq\varepsilon$ and $|F_\varepsilon(x)|\leq|F(x)|$(see [31, Lemma 2.1]).

The regularized particle system is given by

$$dX_\varepsilon^i(t)=\frac{1}{N-1}\sum_{j\neq i}^N F_\varepsilon\big(X_\varepsilon^i(t)-X_\varepsilon^j(t)\big)dt+\nu\, dL_\alpha^i(t), \quad i=1,\cdots,N,$$

with i.i.d. initial random variables $\{X_0^i\}_{i=1}^N$. This system has a unique global strong solution $\{X_\varepsilon^i(t)\}_{i=1}^N$ by a standard theorem for stochastic differential equations (SDEs) [33, pp.249, Theorem 6].

VI. (Propagation of chaos) Assume the initial density ρ_0 satisfies Assumption 1. Let $\{X^i(t)\}_{i=1}^N$ be the unique strong solution to (5). We prove the propagation of chaos for the interacting system with a cutoff parameter $\varepsilon\sim(\ln N)^{-\frac{1}{d}}$, i.e.

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_{\varepsilon(N)}^i(t)-X^i(t)|\right]\to 0, \quad \text{as } N\to\infty,$$

(see Theorem 5.5).

Noticing that in the case $\alpha = 2$, the generalized KS equation (1) reduces to the classic KS equation and the stable Lévy process $L_\alpha(t)$ reduces to the Brownian motion which has been studied in [31]. In the following sections, our discussion will focus on the case $1 < \alpha < 2$, but the same results for $\alpha = 2$ can be obtained similarly.

Concluding this introduction, we present the outline of the paper.

In Section 2, we start with the definition of fractional Laplacian and its basic properties. As a preliminaries, some useful functional inequalities are introduced too. The main results in this section are the global existence and hyper-contractive estimates. Then, the local existence is given in Section 3. Section 4 is devoted to the well-posedness for the generalized KS equation and its corresponding self-consistent stochastic equation. In Section 5.1, we show that the expectation of the collision time for the particle systems is bounded by a universal constant, and then we prove the propagation of chaos in Section 5.2. In the Appendix A, we introduce the definition and basic properties of the rotationally invariant α-stable Lévy process, and the proof of L^∞ uniform bound is given in Appendix B.

2. Global existence with initial data $0 \le \rho_0 \in L^1 \cap L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ and $\|\rho_0\|_{\frac{d}{\alpha}} < K$.

2.1. Preliminaries. According to Stein, Chapter V in [35], the definition of the nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$, known as the Laplacian of order $\frac{\alpha}{2}$, is given by means of the Fourier multiplier

$$D^\alpha \rho(x) := (-\Delta)^{\frac{\alpha}{2}} \rho(x) = \mathcal{F}^{-1}\big(|\xi|^\alpha \hat{\rho}(\xi)\big)(x),$$

where $\hat{\rho}(\xi) = \mathcal{F}\big(\rho(x)\big)$ is the Fourier transformation of $\rho(x)$.

Also, we will use the following formula as in [13], which is useful to study local properties of equations involving the fractional Laplacian operator

$$-(-\Delta)^{\frac{\alpha}{2}} h(x) = C_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{h(y) - h(x)}{|x-y|^{d+\alpha}} dy = C_{d,\alpha} P.V. \int_{|y|>0} \frac{h(x+y) - h(x)}{|y|^{d+\alpha}} dy,$$

where $C_{d,\alpha} = \frac{2^{\alpha-1}\alpha\Gamma((d+\alpha)/2)}{\pi^{2/d}\Gamma(1-\alpha/2)}$ is a normalization constant and $P.V.$ denotes the Cauchy principle value. Then observe that the following properties hold:

$$\begin{aligned} -(-\Delta)^{\frac{\alpha}{2}} h(x) &= C_{d,\alpha} P.V. \int_{|y|<1} \frac{[h(x+y) - h(x)]}{|y|^{d+\alpha}} dy \\ &\quad + C_{d,\alpha} \int_{|y|\ge 1} \frac{[h(x+y) - h(x)]}{|y|^{d+\alpha}} dy \\ &= C_{d,\alpha} P.V. \int_{|y|<1} \frac{[h(x+y) - h(x) - y \cdot \nabla h(x)]}{|y|^{d+\alpha}} dy \\ &\quad + C_{d,\alpha} \int_{|y|\ge 1} \frac{[h(x+y) - h(x)]}{|y|^{d+\alpha}} dy. \end{aligned} \tag{7}$$

Next, we give the definition of weak solution to the KS equation (1).

Definition 2.1. Assume the initial data $0 \le \rho_0(x) \in L^1(\mathbb{R}^d)$, and $T > 0$. We say $\rho(t,x)$ is a weak solution to (1) with initial data $\rho_0(x)$ if it satisfies

1. Regularity:

$$\rho(t,x) \in L^\infty\big(0,T; L^1(\mathbb{R}^d)\big) \cap L^2\big(0,T; L^2(\mathbb{R}^d)\big), \tag{8}$$

$$\partial_t \rho \in L^2\big(0,T;W^{-p_1,p_2}(\mathbb{R}^d)\big) \quad \text{for some } p_1,\, p_2 \geq 1. \tag{9}$$

2. For all $\varphi(x) \in C_c^\infty(\mathbb{R}^d)$, $0 < t \leq T$, it has

$$\begin{aligned}\int_{\mathbb{R}^d} \rho(t,x)\varphi(x)dx \quad &= \quad \int_{\mathbb{R}^d} \rho_0(x)\varphi(x)dx - \nu \int_0^t \int_{\mathbb{R}^d} [\rho(s,x)D^\alpha \varphi(x)]dxds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \rho(s,x)\big(\nabla c\big)\cdot \nabla\varphi(x)dxds. \end{aligned}\tag{10}$$

3. c is the chemical substance concentration associated with ρ and given by

$$\nabla c = \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy. \tag{11}$$

Remark 1. Notice that the regularity (8) is enough to make sense of each term in (10). By the Hardy-Littelwood-Sobolev inequality one has

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho(s,x)\big(\nabla c\big)\cdot\nabla\varphi(x)dx \\ \leq& C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\rho(t,x)\rho(t,y)}{|x-y|^{d-1}}dxdy \leq C\|\rho\|_{\frac{2d}{d+2}}\|\rho\|_2. \end{aligned}\tag{12}$$

Now, We recall here some useful inequalities which will be used throughout the paper:

(Stroock-Varopoulos' inequality)[12] Let $0 < \frac{\alpha}{2} < 1$, $p > 1$, then

$$-\int_{\mathbb{R}^d} |f|^{p-2} f D^\alpha f dx \leq -\frac{4(p-1)}{p^2}\|D^{\frac{\alpha}{2}} f^{\frac{p}{2}}\|_2^2, \tag{13}$$

for all $f \in L^p(\mathbb{R}^d)$ such that $D^\alpha f \in L^p(\mathbb{R}^d)$.

(Sobolev inequality)[12] Let $0 < \frac{\alpha}{2} \leq 1$ and $\alpha < d$, for any $f \in H_{\frac{\alpha}{2}}(\mathbb{R}^d)$, then

$$\|f\|_{\frac{2d}{d-\alpha}} \leq S_{\alpha,d}\|D^{\frac{\alpha}{2}} f\|_2, \tag{14}$$

where the best constant is given by

$$S_{\alpha,d}^2 := 2^{-\alpha}\pi^{\frac{\alpha}{2}}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d+\alpha}{2})}\left[\frac{\Gamma(d)}{\Gamma(\frac{d}{2})}\right]^{\frac{\alpha}{d}} = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d+\alpha}{2})}|\mathbb{S}^{d-1}|^{-\frac{\alpha}{d}}.$$

Lemma 2.2. [5, Lemma 2.6] *Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq a - by(t)^c$ for $a \geq 0, b > 0$ and $c > 0$, then*

(i) For $c > 1, y(t)$ has the following hyper-contractive property

$$y(t) \leq \left(\frac{a}{b}\right)^{\frac{1}{c}} + \left[\frac{1}{b(c-1)t}\right]^{\frac{1}{c-1}}, \quad \textit{for } t > 0.$$

Furthermore, if $y(0)$ is bounded, then

$$y(t) \leq \max\left\{y(0), (\frac{a}{b})^{\frac{1}{c}}\right\}.$$

(ii) For $c = 1, y(t)$ uniformly bounded

$$y(t) \leq \frac{a}{b} + y(0)e^{-bt}, \quad \textit{for } t > 0.$$

Notation. Without confusion, we denote the L^p norm of a function by $\|f\|_p$. Also, we will use the Sobolev space of non-integer power $W^{s,p}(\mathbb{R}^d)$, $s \in \mathbb{R}$, which is defined via the Fourier transform $\mathcal{F}$:

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} := \big\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(\xi)]\big\|_p.$$

Specifically, when $p=2$, we have $H^s(\mathbb{R}^d)$ with the norm defined as

$$\|f\|_{H^s} := \big\|(1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\big\|_2.$$

Inessential constants will be denoted generically by C, even if it is different from line to line.

2.2. **Global existence and hyper-contractivity.** In this subsection, we derive the global existence of weak solutions in a standard approach. Firstly, we define two constants which are related to the initial condition for the existence results:

$$q := \frac{d}{\alpha}; \quad K := \frac{4\alpha\nu}{dS_{\alpha,d}^2}.$$

Theorem 2.3. *Denote* $q=\frac{d}{\alpha}$ $(1<\alpha<2)$, $\zeta = K - \|\rho_0\|_q$. *Assume* $0\le\rho_0\in L^1\cap L^q(\mathbb{R}^d)$ *and* $\zeta>0$, *then there exists a global weak solution* ρ *such that* $\|\rho(t,x)\|_q<K$ *for all* $t>0$. *Furthermore,*

(i) For any $T>0$, *we have the following regularity*

$$\rho\in L^\infty\big(\mathbb{R}_+;L^1\cap L^q(\mathbb{R}^d)\big)\cap L^{q+1}\big(\mathbb{R}_+;L^{q+1}(\mathbb{R}^d)\big);\ \rho^{\frac{q}{2}}\in L^2\big(\mathbb{R}_+;H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big);$$

$$\partial_t\rho\in L^2\big(0,T;W^{-\alpha,\frac{2(q+1)}{q+3}}(\mathbb{R}^d)\big).$$

(ii) The weak solution satisfies mass conservation and the following hyper contractive estimates hold true for any $t>0$ *and any* $1<r<\infty$:

$$\|\rho\|_r^r\le C(r,\nu,d,\alpha,\|\rho_0\|_1,\zeta)t^{-q(q-1)}, \quad 1<r\le q;$$

$$\|\rho\|_r^r\le C(r,\nu,d,\alpha,\|\rho_0\|_1,\zeta)\left(t^{-\frac{q^2(q+\epsilon_0-1)(1+r-q)(r-1)}{(qr-q+1)\epsilon_0}}+t^{-q(r-1)}\right), \quad q<r<\infty,$$

where ϵ_0 *satisfies* $\frac{4\nu}{(q+\epsilon_0)S_{\alpha,d}^2}-\|\rho_0\|_q=\frac{\zeta}{2}$.

Proof. The proof can be divided into 9 steps. Steps 1-6 give some crucial priori estimates for the statement $(i),(ii)$. In Steps 7-9, a regularized equation is constructed to make these priori estimates of Steps 1-6 rigorous and obtain the global existence of a weak solution to (1).

For the rigorous proof, we follow the method in [4] by taking a cutoff function $0\le\psi_1(x)\le1, \psi_1(x)\in C_0^\infty(\mathbb{R}^d)$, which satisfies

$$\psi_1(x)=\begin{cases}1, & \text{if } |x|\le1,\\ 0, & \text{if } |x|\ge2.\end{cases}$$

Define $\psi_R(x):=\psi_1(\frac{x}{R})$, then $\psi_R(x)\to1$, as $R\to\infty$, and there exist constants C_1,C_2 such that $|\nabla\psi_R(x)|\le\frac{C_1}{R}$, $|D^\alpha\psi_R(x)|\le\frac{C_2}{R^\alpha}$ for $x\in\mathbb{R}^d$. Indeed, if we set $x'=\frac{x}{R}$, then

$$\begin{aligned} D^\alpha\psi_R(x) &= C_{d,\alpha}P.V.\int_{\mathbb{R}^d}\frac{\psi_1(\frac{x}{R})-\psi_1(\frac{y}{R})}{|x-y|^{d+\alpha}}dy \\ &= C_{d,\alpha}P.V.\int_{\mathbb{R}^d}\frac{\psi_1(\frac{x}{R})-\psi_1(y)}{R^\alpha|\frac{x}{R}-y|^{d+\alpha}}dy=\frac{1}{R^\alpha}D^\alpha\psi_1(x'), \end{aligned} \tag{15}$$

and $D^\alpha\psi_1(x')$ is finite for $x'\in\mathbb{R}^d$. This cutoff function will be used to derive the existence of the weak solution.

Step 1. (Uniform L^q estimates) Firstly, it is obtained by multiplying (1) with $q\rho^{q-1}$ and using (13), then integrate over $\mathbb{R}^d$

$$\frac{d}{dt}\|\rho\|_q^q + 4\frac{(q-1)\nu}{q}\|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2 \le (q-1)\|\rho\|_{q+1}^{q+1}. \tag{16}$$

Compute the right side and use (14)

$$\|\rho\|_{q+1}^{q+1} \le \|\rho\|^q_{\frac{qd}{d-\alpha}}\|\rho\|_q = \|\rho^{\frac{q}{2}}\|^2_{\frac{2d}{d-\alpha}}\|\rho\|_q \le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2\|\rho\|_q, \tag{17}$$

which implies

$$\frac{d}{dt}\|\rho\|_q^q + (q-1)S_{\alpha,d}^2(K-\|\rho\|_q)\|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2 \le 0.$$

Since $\|\rho_0\|_q < K$, so the following estimates hold true

$$\|\rho(t,\cdot)\|_q < \|\rho_0\|_q < K,$$

$$(q-1)S_{\alpha,d}^2(K-\|\rho\|_q)\int_0^\infty \|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2 dt \le K.$$

Recall we denote $K = \frac{4}{qS_{\alpha,d}^2}$ and $\zeta = K - \|\rho_0\|_q$, from the equations above one has

$$\int_0^\infty \|\rho\|_{q+1}^{q+1}dt \le S_{\alpha,d}^2 K\int_0^\infty \|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2 dt \le \frac{K^2}{(q-1)\zeta},$$

which leads to the following estimates

$$\rho \in L^{q+1}\big(\mathbb{R}_+; L^{q+1}(\mathbb{R}^d)\big), \quad D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}} \in L^2\big(\mathbb{R}_+; L^2(\mathbb{R}^d)\big).$$

Step 2. (L^q decay estimates) By using $\|\rho\|_1 \le \|\rho_0\|_1$ compute

$$\|\rho\|_q^{\frac{q^2}{q-1}} \le \|\rho\|_{q+1}^{q+1}\|\rho\|_1^{\frac{1}{q-1}} \le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2\|\rho\|_q\|\rho\|_1^{\frac{1}{q-1}}, \tag{18}$$

which leads to

$$(\|\rho\|_q^q)^{1+\frac{1}{q(q-1)}} \le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{q}{2}}\|_2^2\|\rho\|_1^{\frac{1}{q-1}}.$$

Thus one has

$$\frac{d}{dt}\|\rho\|_q^q + \frac{(q-1)\zeta}{\|\rho_0\|_1^{\frac{1}{q-1}}}(\|\rho\|_q^q)^{1+\frac{1}{q(q-1)}} \le 0,$$

which leads to the decay property

$$\|\rho\|_q^q \le (\frac{\zeta}{q})^{-q(q-1)}\|\rho_0\|_1^q t^{-q(q-1)}.$$

Step 3. (Uniform L^{r_0} estimates with $r_0 = q+\epsilon_0$ for ϵ_0 small enough) As we have done before

$$\begin{aligned}\frac{d}{dt}\|\rho\|_{r_0}^{r_0} + 4\frac{(r_0-1)\nu}{r_0}\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2 &\le (r_0-1)\|\rho\|_{r_0+1}^{r_0+1} \\ &\le (r_0-1)S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2\|\rho_0\|_q. \end{aligned} \tag{19}$$

If we choose ϵ_0 such that

$$\frac{\zeta}{2} := \frac{4\nu}{(q+\epsilon_0)S_{\alpha,d}^2} - \|\rho_0\|_q < \zeta, \tag{20}$$

then one has

$$\frac{d}{dt}\|\rho\|_{r_0}^{r_0} + S_{\alpha,d}^2(r_0-1)\frac{\zeta}{2}\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2 \le 0. \tag{21}$$

On the other hand

$$\begin{aligned}\|\rho\|_{r_0}^{\frac{r_0^2}{r_0-1}} &\le \|\rho\|_{r_0+1}^{r_0+1}\|\rho\|_1^{\frac{1}{r_0-1}} \le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2\|\rho\|_q\|\rho\|_1^{\frac{1}{r_0-1}} \\ &\le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2(\|\rho\|_{r_0}^{\theta}\|\rho\|_1^{1-\theta})\|\rho\|_1^{\frac{1}{r_0-1}}, \end{aligned} \tag{22}$$

where $\theta = \frac{r_0(q-1)}{q(r_0-1)}$, and it leads to

$$(\|\rho\|_{r_0}^{r_0})^{\delta} \le S_{\alpha,d}^2\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2\|\rho\|_1^{\frac{r_0}{q(r_0-1)}},$$

and here $\delta = 1 + \frac{1}{q(r_0-1)}$, which implies

$$\frac{d}{dt}\|\rho\|_{r_0}^{r_0} + \frac{\zeta(r_0-1)}{2\|\rho_0\|_1^{\frac{r_0}{q(r_0-1)}}}(\|\rho\|_{r_0}^{r_0})^{\delta} \le 0.$$

Now denote $C_{r_0} = \frac{\zeta(r_0-1)}{2\|\rho_0\|_1^{\frac{r_0}{q(r_0-1)}}}$, then one computes

$$\|\rho\|_{r_0}^{r_0} \le [C_{r_0}(\delta-1)]^{-q(r_0-1)}t^{-q(r_0-1)}. \tag{23}$$

Step 4. (Hyper-contractive estimates of L^r norm for $r > r_0$) For $r > r_0$ we compute as before by using the Young's inequality

$$\begin{aligned}&\frac{d}{dt}\|\rho\|_r^r + 4\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2 \\ \le&(r-1)\|\rho\|_{r+1}^{r+1} \le (r-1)S_{\alpha,d}^{\frac{2\theta(r+1)}{r}}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^{\frac{2\theta(r+1)}{r}}\|\rho\|_{r_0}^{(1-\theta)(r+1)} \\ \le&2\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2 + C(r,\nu,r_0,d)(\|\rho\|_{r_0}^{r_0})^{\frac{1+r-q}{r_0-q}}, \end{aligned} \tag{24}$$

where $\theta = \frac{qr[r_0-(r+1)]}{(r+1)[r_0(q-1)-qr]}$ satisfying $\frac{2\theta(r+1)}{r} < 1$ for $r_0 > q$. Collecting (23) yields

$$\begin{aligned}\frac{d}{dt}\|\rho\|_r^r \le& -\frac{2(r-1)\nu}{rS_{\alpha,d}^2\|\rho_0\|_1^{\frac{r}{q(r-1)}}}(\|\rho\|_r^r)^{1+\frac{1}{q(r-1)}} \\ &+C(r,\nu,r_0,d,\alpha,\|\rho_0\|_1)t^{-\frac{q(r_0-1)(1+r-q)}{r_0-q}}. \end{aligned} \tag{25}$$

Thus we have for any $t > 0$ with ϵ_0 satisfying (20)

$$\|\rho\|_r^r \le C(r,\nu,d,\alpha,\|\rho_0\|_1,\zeta)\left(t^{-\frac{q^2(q+\epsilon_0-1)(1+r-q)(r-1)}{(qr-q+1)\epsilon_0}} + t^{-q(r-1)}\right). \tag{26}$$

Step 5. (Decay estimates on $\|\rho\|_r$) In this step, based on the decay of $\|\rho\|_q$ with time evolution, $\|\rho\|_r$ decays for large time. Divide r into two cases $1 < r < q$ and $q < r < \infty$. Recalling that in Step 2 we have gotten

$$\|\rho\|_q^q \le (\frac{\zeta}{q})^{-q(q-1)}\|\rho_0\|_1^q t^{-q(q-1)}. \tag{27}$$

(1) For $1 < r < q$, it follows from (27) by applying the interpolation inequality that for any $t > 0$,

$$\|\rho\|_r^r \le \|\rho\|_q^{\frac{q(r-1)}{q-1}}\|\rho\|_1^{\frac{q-r}{q-1}} \le (\frac{\zeta}{q})^{-q(r-1)}\|\rho_0\|_1^{\frac{q(r-1)}{q-1}+\frac{q-r}{q-1}}t^{-q(q-1)}. \tag{28}$$

(2) For $q < r < \infty$, since $\|\rho\|_q$ decays to zero as time goes to infinity, then for t larger than some T_r one has

$$(r-1)S_{\alpha,d}^2\|\rho\|_q \leq \frac{2(r-1)\nu}{r},$$

which leads to

$$\frac{d}{dt}\|\rho\|_r^r \leq -\frac{2(r-1)\nu}{rS_{\alpha,d}^2\|\rho_0\|_1^{\frac{r}{q(r-1)}}}(\|\rho\|_r^r)^{1+\frac{1}{q(r-1)}}, \qquad t > T_r.$$

Solving this ordinary differential inequality, the large time decay of $\|\rho\|_r$ has been obtained

$$\|\rho\|_r^r \leq C(r,\nu,d,\alpha,\|\rho_0\|_1)(t-T_r)^{-q(r-1)}, \qquad t > T_r. \tag{29}$$

Step 6. (Mass conservation) Observe that for any $t > 0$

$$\begin{aligned}
&\left|\frac{d}{dt}\int_{\mathbb{R}^d}\rho(t,x)\psi_R(x)dx\right| \\
=&\left|-\int_{\mathbb{R}^d}\nu\rho(t,x)D^\alpha\psi_R(x)dx + \int_{\mathbb{R}^d}\rho(t,x)\nabla c\cdot\nabla\psi_R(x)dx\right| \\
\leq&\frac{C}{R^\alpha} + \frac{C}{R}\|\rho\|_{\frac{2d}{d+1}}^2.
\end{aligned} \tag{30}$$

Using the interpolation inequality, we have

$$\int_0^t \|\rho\|_{\frac{2d}{d+1}}^2 ds \leq \int_0^t \|\rho\|_{q+1}^{2\theta}\|\rho\|_1^{2(1-\theta)}ds \leq C(t),$$

which implies

$$-\frac{C}{R^\alpha} - \frac{C(t)}{R} \leq \int_{\mathbb{R}^d}\rho(t,x)\psi_R(x)dx - \int_{\mathbb{R}^d}\rho_0(x)\psi_R(x)dx \leq \frac{C}{R^\alpha} + \frac{C(t)}{R}.$$

Thus as $R \to \infty$ by the dominated convergence theorem one has

$$\int_{\mathbb{R}^d}\rho(t,x)dx = \int_{\mathbb{R}^d}\rho_0(x)dx.$$

Combing the virtue of (26), (28) and (29), the statement (ii) has been proved.

Step 7. (Regularization) In order to show the existence of a weak solution with the above prosperities and make the proof rigorous, we consider the following regularized problem for $\varepsilon > 0$:

$$\begin{cases}
\partial_t\rho_\varepsilon = -\nu(-\Delta)^{\frac{\alpha}{2}}\rho_\varepsilon - \nabla\cdot(\rho_\varepsilon\nabla c_\varepsilon), \\
-\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon, \\
\rho_\varepsilon(0,x) = \rho_0(x).
\end{cases} \tag{31}$$

Here $J_\varepsilon(x) = \frac{1}{\varepsilon^d}J(\frac{x}{\varepsilon})$ is defined by $J(x)$ as in (6). From parabolic theory, the regularized problem has a global smooth positive solution ρ_ε with the regularity $\|\rho_\varepsilon(t,x)\|_r \leq C_\varepsilon$ for all $r \geq 1, t > 0$. By taking the similar arguments as in Step 6 arrives at the mass conservation of ρ_ε.

Multiply equation (31) with $r\rho_\varepsilon^{r-1}\psi_R(x)$, then integrate over $\mathbb{R}^d$, one has

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho_\varepsilon^r\psi_R(x)dx$$

$$
\begin{aligned}
&\le -\frac{4(r-1)\nu}{r}\int_{\mathbb{R}^d}|D^{\frac{\alpha}{2}}\rho_\varepsilon^{\frac{r}{2}}|_2^2\psi_R(x)dx - r\int_{\mathbb{R}^d}\rho_\varepsilon^r D^\alpha\psi_R(x)dx\\
&\quad + (r-1)\int_{\mathbb{R}^d}(J_\varepsilon * \rho_\varepsilon)\rho_\varepsilon^r\psi_R(x)dx + \int_{\mathbb{R}^d}\nabla c_\varepsilon\cdot\nabla\psi_R(x)\rho_\varepsilon^r dx\\
&\le -\frac{4(r-1)\nu}{r}\int_{\mathbb{R}^d}|D^{\frac{\alpha}{2}}\rho_\varepsilon^{\frac{r}{2}}|_2^2\psi_R(x)dx + (r-1)\int_{\mathbb{R}^d}(J_\varepsilon * \rho_\varepsilon)\rho_\varepsilon^r\psi_R(x)dx\\
&\quad + \frac{rC}{R^\alpha}\|\rho\|_r + \frac{C}{R}\int_{\mathbb{R}^d}|\nabla c_\varepsilon|\rho_\varepsilon^r dx. \qquad (32)
\end{aligned}
$$

By using the Hardy-Littlewood-Sobolev inequality, we know

$$
\frac{C}{R}\int_{\mathbb{R}^d}|\nabla c_\varepsilon|\rho_\varepsilon^r dx \le \frac{C}{R}\|\rho_\varepsilon\|_{\frac{d(r+1)}{d+1}}^{r+1}.
$$

Combine this with $\|\rho_\varepsilon(t,x)\|_r \le C_\varepsilon$ and $\|\rho_\varepsilon(t,x)\|_{\frac{d(r+1)}{d+1}} \le C_\varepsilon$, then the last two terms of (32) will vanish as $R\to\infty$. Thus the following inequality holds

$$
\frac{d}{dt}\|\rho_\varepsilon\|_r^r + 4\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho_\varepsilon^{\frac{r}{2}}\|_2^2 \le (r-1)\|\rho_\varepsilon\|_{r+1}^{r+1}.
$$

Therefore all the estimates in Steps 1-5 hold true.

For the initial density $0\le\rho_0\in L^1(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$, the following basic estimates are obtained:

$$
\|\rho_\varepsilon\|_{L^\infty\left(\mathbb{R}_+;L^1\cap L^q(\mathbb{R}^d)\right)} \le C, \qquad (33)
$$

$$
\|\rho_\varepsilon\|_{L^{q+1}\left(\mathbb{R}_+;L^{q+1}(\mathbb{R}^d)\right)} \le C, \qquad (34)
$$

$$
\|D^{\frac{\alpha}{2}}\rho_\varepsilon^{\frac{r}{2}}\|_{L^2\left(\mathbb{R}_+;L^2(\mathbb{R}^d)\right)} \le C,\ 1<r\le q. \qquad (35)
$$

In addition, for any $T>0$, applying the weak Young's inequality, one has

$$
\begin{aligned}
\int_0^T\|\nabla c_\varepsilon\|_2^{\frac{2(q+1)}{q-1}}dt &\le C\int_0^T\|\rho_\varepsilon\|_{\frac{2d}{d+2}}^{\frac{2(q+1)}{q-1}}\||x|^{-(d-1)}\|_{L_w^{\frac{d}{d-1}}}^{\frac{2(q+1)}{q-1}}dt\\
&\le C\int_0^T\|\rho_\varepsilon\|_{\frac{2d}{d+2}}^{\frac{2(q+1)}{q-1}}dt \le C(T). \qquad (36)
\end{aligned}
$$

Step 8. (Time regularity and application of Aubin-Lions-Dubinskiĭ lemma) In order to get the regularity of $\partial_t\rho_\varepsilon$, one takes any test function

$$
h\in W_c^{\alpha,\frac{2(q+1)}{q-1}}(\mathbb{R}^d),\ \|h\|_{W_c^{\alpha,\frac{2(q+1)}{q-1}}(\mathbb{R}^d)} \le 1,
$$

and estimate $\langle\partial_t\rho_\varepsilon, h\rangle$. We have

$$
\begin{aligned}
|\langle\partial_t\rho_\varepsilon,h\rangle| &= |-\nu\langle\rho_\varepsilon, D^\alpha h\rangle + \langle\rho_\varepsilon\nabla c_\varepsilon,\nabla h\rangle|\\
&\le \nu\|\rho_\varepsilon\|_{\frac{2(q+1)}{q+3}} + \|\rho_\varepsilon\nabla c_\varepsilon\|_{\frac{2(q+1)}{q+3}}. \qquad (37)
\end{aligned}
$$

Thus for any $T>0$

$$
\begin{aligned}
&\int_0^T\|\partial_t\rho_\varepsilon\|^2_{W^{-\alpha,\frac{2(q+1)}{q+3}}(\mathbb{R}^d)}dt\\
\le& 2\left(\int_0^T\nu\|\rho_\varepsilon\|^2_{\frac{2(q+1)}{q+3}}dt + \int_0^T\|\rho_\varepsilon\nabla c_\varepsilon\|^2_{\frac{2(q+1)}{q+3}}dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(T) + 2\int_0^T \|\rho_\varepsilon\|_{q+1}^2 \|\nabla c_\varepsilon\|_2^2 dt \\
&\leq C(T) + 2\left(\int_0^T \|\rho_\varepsilon\|_{q+1}^{q+1} dt\right)^{\frac{2}{q+1}} \left(\int_0^T \|\nabla c_\varepsilon\|_2^{\frac{2(q+1)}{q-1}} dt\right)^{\frac{q-1}{q+1}} \\
&\leq C(T). \qquad (38)
\end{aligned}
$$

Finally the regularity of $\partial_t \rho_\varepsilon$ follows

$$
\|\partial_t \rho_\varepsilon\|_{L^2\left(0,T;W^{-\alpha,\frac{2(q+1)}{q+3}}(\mathbb{R}^d)\right)} \leq C(T).
$$

Before we use the Aubin-Lions-Dubinskiĭ lemma, we introduce the so called seminormed nonnegative cone M_+ in Banach space B: $M_+ \in B$; for all $u \in M_+$ and $c \geq 0$, $cu \in M_+$; there exist a function $[\cdot] : M_+ \to [0, +\infty)$ such that $[u] = 0$ if and only if $u = 0$, and $[cu] = c[u]$ for all $c \geq 0$.

For any bounded domain Ω, we choose $B = L^q(\Omega)$, and define $M_+(\Omega) := \{\rho : [\rho] \leq C\}$ with $[\rho] = \|D^{\frac{\alpha}{2}} \rho^{\frac{q}{2}}\|_2^{\frac{2}{q}} + \|\rho\|_1 + \|\rho\|_q$. It is easy to check that $M_+(\Omega)$ defined here is a seminormed nonnegative cone in $L^q(\Omega)$.

Next we will prove that $M_+(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, i.e. for any bounded sequence $\{\rho_\varepsilon\}$ in $M_+(\Omega)$, there exists a subsequence converging in $L^q(\Omega)$. Since $H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we know there is a subsequence $\{\rho_\varepsilon^{\frac{q}{2}}\}$ without relabeling such that

$$
\rho_\varepsilon^{\frac{q}{2}} \to \rho^{\frac{q}{2}} \quad \text{in } L^2(\Omega), \qquad \text{as } \varepsilon \to 0.
$$

For $q \geq 2$, one has

$$
\begin{aligned}
\int_\Omega |\rho_\varepsilon - \rho|^q dx &= \int_\Omega \left|\rho_\varepsilon^{\frac{q}{2}\frac{2}{q}} - \rho^{\frac{q}{2}\frac{2}{q}}\right|^q dx \leq \int_\Omega \left|\rho_\varepsilon^{\frac{q}{2}} - \rho^{\frac{q}{2}}\right|^{q\frac{2}{q}} dx \\
&= \int_\Omega \left|\rho_\varepsilon^{\frac{q}{2}} - \rho^{\frac{q}{2}}\right|^2 dx \to 0, \qquad \text{as } \varepsilon \to 0. \qquad (39)
\end{aligned}
$$

For $1 < q < 2$, we set $u_\varepsilon = \rho_\varepsilon^{\frac{q}{2}}$ and $u = \rho^{\frac{q}{2}}$, by the mean value theorem and Hölder inequality, one has

$$
\begin{aligned}
\int_\Omega |\rho_\varepsilon - \rho|^q dx &= \int_\Omega \left|u_\varepsilon^{\frac{2}{q}} - u^{\frac{2}{q}}\right|^q dx \leq C\int_\Omega \left[|u_\varepsilon + u|^{\frac{2}{q}-1}|u_\varepsilon - u|\right]^q dx \\
&\leq C\left(\int_\Omega u_\varepsilon^{\frac{2}{q}} dx\right)^{\frac{2-q}{2}} \left(\int_\Omega |u_\varepsilon - u|^2 dx\right)^{\frac{q}{2}} \\
&\leq C\|\rho_\varepsilon\|_1^{\frac{2-q}{2}} \|\rho_\varepsilon^{\frac{q}{2}} - \rho^{\frac{q}{2}}\|_2^q \to 0, \qquad \text{as } \varepsilon \to 0. \qquad (40)
\end{aligned}
$$

Thus, we get

$$
M_+(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).
$$

Recall that

$$
\|\rho_\varepsilon\|_{L^q\left(0,T;M_+(\Omega)\right)} \leq C,
$$

$$
\|\rho_\varepsilon\|_{L^q\left(0,T;L^q(\Omega)\right)} \leq C,
$$

$$
\|\partial_t \rho_\varepsilon\|_{L^2\left(0,T;W^{-\alpha,\frac{2(q+1)}{q+3}}(\Omega)\right)} \leq C,
$$

and $M_+(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \hookrightarrow W^{-\alpha,\frac{2(q+1)}{q+3}}(\Omega)$. By Aubin-Lions-Dubinskiĭ lemma as in [17], one arrives at that $\{\rho_\varepsilon\}_{\varepsilon>0}$ is compact in $L^q\big(0,T;L^q(\Omega)\big)$. Consequently, there exists a subsequence ρ_ε without relabeling such that

$$\rho_\varepsilon \to \rho \quad \text{in } L^q\big(0,T;L^q(\Omega)\big), \qquad \text{as } \varepsilon \to 0.$$

Let $\{B_k\}_{k=1}^\infty \in \mathbb{R}^d$ be a sequence of balls centered at 0 with radius $R_k, R_k \to \infty$. By a standard diagonal argument, there exists a subsequence ρ_ε without relabeling the following uniform strong convergence holds true

$$\rho_\varepsilon \to \rho \quad \text{in } L^q\big(0,T;L^q(B_k)\big), \qquad \text{as } \varepsilon \to 0, \quad \forall k.$$

Step 9. (Existence of a global weak solution) Now, we will prove that ρ is a weak solution to (1). Indeed, the weak formulation for ρ_ε is that for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} \rho_0(x)\varphi(x)dx \\
&= -\int_0^t \int_{\mathbb{R}^d} \nu[\rho_\varepsilon(s,x)D^\alpha\varphi(x)]dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \rho_\varepsilon(s,x)\big(\nabla c_\varepsilon\big)\cdot\nabla\varphi(x)dxds.
\end{aligned} \tag{41}$$

For the first term of the right side of (41), it is obvious that

$$-\int_0^t \int_{\mathbb{R}^d} \nu[\rho_\varepsilon(s,x)D^\alpha\varphi(x)]dxds \to -\int_0^t \int_{\mathbb{R}^d} \nu[\rho(s,x)D^\alpha\varphi(x)]dxds, \qquad \text{as } \varepsilon \to 0. \tag{42}$$

For the second term, since $F_\varepsilon(x) = F(x)$ for any $|x| \geq \varepsilon$, one has

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \rho_\varepsilon(s,x)\big(\nabla c_\varepsilon\big)\cdot\nabla\varphi(x)dxds - \int_0^t \int_{\mathbb{R}^d} \rho(s,x)\big(\nabla c\big)\cdot\nabla\varphi(x)dxds \\
&= \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big[\rho_\varepsilon(s,x)F_\varepsilon(x-y)\rho_\varepsilon(s,y) - \rho(s,x)F(x-y)\rho(s,y)\big]\cdot\nabla\varphi(x)dxdyds \\
&= \int_0^t \iint_{|x-y|\geq\varepsilon} \big[\rho_\varepsilon(s,x)F(x-y)\rho_\varepsilon(s,y) - \rho(s,x)F(x-y)\rho(s,y)\big]\cdot\nabla\varphi(x)dxdyds \\
&\quad + \int_0^t \iint_{|x-y|<\varepsilon} \big[\rho_\varepsilon(s,x)F_\varepsilon(x-y)\rho_\varepsilon(s,y) \\
&\qquad\qquad - \rho(s,x)F(x-y)\rho(s,y)\big]\cdot\nabla\varphi(x)dxdyds \\
&=: \int_0^t I_1(s)ds + \int_0^t I_2(s)ds.
\end{aligned} \tag{43}$$

Firstly, by using the fact $F(x-y) = -\frac{C_*(x-y)}{|x-y|^d}$ and $|\nabla\varphi(x)| \leq C$, we calculate $I_1(s)$

$$\begin{aligned}
|I_1(s)| &\leq C\iint_{|x-y|\geq\varepsilon} \frac{|\rho_\varepsilon(s,x)\rho_\varepsilon(s,y) - \rho(s,x)\rho(s,y)|}{|x-y|^{d-1}}dxdy \\
&\leq C\iint_{|x-y|\geq\varepsilon} \frac{\rho_\varepsilon(s,y)|\rho_\varepsilon(s,x) - \rho(s,x)|}{|x-y|^{d-1}}dxdy
\end{aligned}$$

$$
\begin{aligned}
&\quad +C\iint_{|x-y|\geq\varepsilon}\frac{\rho(s,x)|\rho_\varepsilon(s,y)-\rho(s,y)|}{|x-y|^{d-1}}dxdy\\
&=: \quad I_{11}+I_{12}.
\end{aligned}\tag{44}
$$

For I_{11}, by using the Hardy-Littlewood-Sobolev inequality, one has

$$
\begin{aligned}
I_{11} &\leq C\|\rho_\varepsilon-\rho\|_{\frac{2d}{d+2}}\|\rho_\varepsilon\|_2 \quad \left(1\leq\frac{2d}{d+2}\leq q\right)\\
&\leq C\|\rho_\varepsilon-\rho\|_q^\theta\|\rho_\varepsilon-\rho\|_1^{1-\theta}\|\rho_\varepsilon\|_{q+1}^{\theta'}\|\rho_\varepsilon\|_1^{1-\theta'}\\
&\leq C\|\rho_\varepsilon-\rho\|_q^\theta\|\rho_\varepsilon\|_{q+1}^{\theta'}.
\end{aligned}\tag{45}
$$

By using regularity (34) of ρ_ε, from (45), we know

$$
\int_0^t I_{11}ds \quad\leq\quad C(t)\left[\int_0^t\|\rho_\varepsilon-\rho\|_q^q ds\right]^{\frac{\theta}{q}}\to 0, \quad \text{as } \varepsilon\to 0. \tag{46}
$$

Similarly, we can obtain $\int_0^t I_{12}ds\to 0$ as $\varepsilon\to 0$, which leads to $\int_0^t I_1(s)ds\to 0$.

For $I_2(s)$, since $F(-x)=-F(x)$, $F_\varepsilon(-x)=-F_\varepsilon(x)$ and $|\nabla\varphi(x)-\nabla\varphi(y)|\leq C|x-y|\leq C\varepsilon$, we have

$$
\begin{aligned}
|I_2(s)| &= \frac{1}{2}\Big|\iint_{|x-y|<\varepsilon}\big(\rho_\varepsilon(s,x)F_\varepsilon(x-y)\rho_\varepsilon(s,y)\\
&\qquad\qquad -\rho(s,x)F(x-y)\rho(s,y)\big)\cdot\big(\nabla\varphi(x)-\nabla\varphi(y)\big)dxdy\Big|\\
&\leq C\iint_{|x-y|<\varepsilon}|\rho_\varepsilon(x)-\rho(x)||F_\varepsilon(x-y)|\rho_\varepsilon(y)dxdy\\
&\quad + C\varepsilon\iint_{|x-y|<\varepsilon}\rho(x)\rho_\varepsilon(y)|F_\varepsilon(x-y)-F(x-y)|dxdy\\
&\quad + C\iint_{|x-y|<\varepsilon}\rho(x)|\rho_\varepsilon(y)-\rho(y)||F(x-y)|dxdy\\
&=: I_{21}+I_{22}+I_{23}.
\end{aligned}\tag{47}
$$

Using $|F_\varepsilon(x)|\leq|F(x)|$, and calculate I_{21}

$$
\begin{aligned}
I_{21} &\leq C\iint_{|x-y|<\varepsilon}\frac{|\rho_\varepsilon(x)-\rho(x)|\rho_\varepsilon(y)}{|x-y|^{d-1}}dxdy\\
&\leq C\|\rho_\varepsilon-\rho\|_{\frac{2d}{d+2}}\|\rho_\varepsilon\|_2.
\end{aligned}\tag{48}
$$

Same as the discussion of I_{11}, (48) leads to $\int_0^t I_{21}ds\to 0$, as $\varepsilon\to 0$. And similarly, we have $\int_0^t I_{23}ds\to 0$. As for I_{22}, we have

$$
\begin{aligned}
|I_{22}| &\leq C\varepsilon\iint_{|x-y|<\varepsilon}\frac{\rho(x)\rho_\varepsilon(y)}{|x-y|^{d-1}}dxdy\\
&\leq C\varepsilon\|\rho\|_{\frac{2d}{d+1}}\|\rho_\varepsilon\|_{\frac{2d}{d+1}} \quad \left(1<\frac{2d}{d+1}<q+1\right)\\
&\leq C\varepsilon\|\rho\|_{q+1}^\theta\|\rho\|_1^{1-\theta}\|\rho_\varepsilon\|_{q+1}^\theta\|\rho_\varepsilon\|_1^{1-\theta}\\
&\leq C\varepsilon\|\rho\|_{q+1}^\theta\|\rho_\varepsilon\|_{q+1}^\theta.
\end{aligned}\tag{49}
$$

From (49), one has

$$\begin{aligned}\int_0^t I_{22}ds &\leq C(t)\varepsilon\left[\int_0^t \|\rho\|_{q+1}^{q+1}ds\right]^{\frac{\theta}{q+1}}\left[\int_0^t \|\rho_\varepsilon\|_{q+1}^{q+1}ds\right]^{\frac{\theta}{q+1}}\\ &\leq C(t)\varepsilon \to 0, \quad \text{as } \ \varepsilon\to 0,\end{aligned} \tag{50}$$

where we have used the regularity (34) for ρ_ε and ρ. Hence we have $\int_0^t I_2 ds \to 0$ as $\varepsilon \to 0$.

From all the discussion above, one obtain

$$\int_0^t\int_{\mathbb{R}^d}\rho_\varepsilon(s,x)\big(\nabla c_\varepsilon\big)\cdot\nabla\varphi(x)dxds \to \int_0^t\int_{\mathbb{R}^d}\rho(s,x)\big(\nabla c\big)\cdot\nabla\varphi(x)dxds, \quad \text{as } \varepsilon\to 0. \tag{51}$$

Combining (42)and (51), we conclude ρ is a weak solution to (1). By now we have given the existence of a global weak solution, and the conservation of mass is easy to attain by the similar arguments as in Step 6. □

3. **Local existence.**

For more general initial data, the following local in time existence holds true.

Theorem 3.1. *([10, Theorem 2.2]) Suppose that either $d = 2$, $(1 < \alpha < 2)$, or $d = 3$, $(\frac{3}{2} < \alpha < 2)$, and the initial density satisfies $0 \leq \rho_0(x) \in L^2(\mathbb{R}^d)$, $\|\rho_0\|_1 = 1$. Then there exists $T > 0$, such that a weak solution to (1) $\rho(t,x)$ exits in $[0,T]$ with regularity*

$$\rho \in L^\infty\big(0,T;L^2(\mathbb{R}^d)\big)\cap L^2\big(0,T;H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big);\partial_t\rho\in L^2\big(0,T;H^{-1}(\mathbb{R}^d)\big),$$

and the conservation of mass

$$\|\rho_0\|_1 = \|\rho(t,\cdot)\|_1 = 1, \quad t\in[0,T].$$

Proof. The result can be found in [10], but for the completeness, we will give the sketch of the proof here.

Multiply equation (31) with $2\rho_\varepsilon\psi_R(x)$, then integrate over $\mathbb{R}^d$, one has

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\rho_\varepsilon^2\psi_R(x)dx &= -2\nu\int_{\mathbb{R}^d}|D^{\frac{\alpha}{2}}\rho_\varepsilon|^2\psi_R(x)dx - \int_{\mathbb{R}^d}\rho_\varepsilon^2 D^\alpha\psi_R(x)dx\\ &\quad+\int_{\mathbb{R}^d}(J_\varepsilon*\rho_\varepsilon)\rho_\varepsilon^2\psi_R(x)dx + \int_{\mathbb{R}^d}\nabla c_\varepsilon\cdot\nabla\psi_R(x)\rho_\varepsilon^2dx\\ &\leq -2\nu\int_{\mathbb{R}^d}|D^{\frac{\alpha}{2}}\rho_\varepsilon|^2\psi_R(x)dx + \int_{\mathbb{R}^d}(J_\varepsilon*\rho_\varepsilon)\rho_\varepsilon^2\psi_R(x)dx\\ &\quad+\frac{C}{R^\alpha}\int_{\mathbb{R}^d}\rho_\varepsilon^2dx + \frac{C}{R}\int_{\mathbb{R}^d}|\nabla c_\varepsilon|\rho_\varepsilon^2dx.\end{aligned} \tag{52}$$

By using the Hardy-Littlewood-Sobolev inequality, we know

$$\frac{C}{R}\int_{\mathbb{R}^d}|\nabla c_\varepsilon|\rho_\varepsilon^2dx \leq \frac{C}{R}\|\rho_\varepsilon\|^3_{\frac{3d}{d+1}}.$$

As we have done in the last section, the last two terms of (32) will vanish as $R\to\infty$. Thus the following inequality holds

$$\frac{d}{dt}\|\rho_\varepsilon\|_2^2 + 2\nu\|D^{\frac{\alpha}{2}}\rho_\varepsilon\|_2^2 \leq \int_{\mathbb{R}^d}\rho_\varepsilon^2 J_\varepsilon*\rho_\varepsilon dx.$$

For $\alpha \geq \frac{d}{3}$, we can use the interpolation inequality and the Sobolev imbedding theorem ($H^s \hookrightarrow L^{\frac{2d}{d-2s}}$)

$$\int_{\mathbb{R}^d} \rho_\varepsilon^2 J_\varepsilon * \rho_\varepsilon dx \leq \|\rho_\varepsilon\|_3^3 \leq \|\rho_\varepsilon\|_{\frac{2d}{d-\alpha}}^{\frac{d}{\alpha}} \|\rho_\varepsilon\|_2^{3-\frac{d}{\alpha}} \leq C\|\rho_\varepsilon\|_{H^{\frac{\alpha}{2}}}^{\frac{d}{\alpha}} \|\rho_\varepsilon\|_2^{3-\frac{d}{\alpha}}.$$

For $\alpha > \frac{d}{2}$, we imply the Young's inequality

$$\frac{d}{dt}\|\rho_\varepsilon\|_2^2 + 2\nu\|D^{\frac{\alpha}{2}}\rho_\varepsilon\|_2^2 \leq \nu\|\rho_\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 + C\|\rho_\varepsilon\|_2^{\frac{6\alpha-2d}{2\alpha-d}}.$$

Thus we have

$$\begin{aligned}\frac{d}{dt}\|\rho_\varepsilon\|_2^2 + \nu\|D^{\frac{\alpha}{2}}\rho_\varepsilon\|_2^2 &\leq \nu\|\rho_\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 - \nu\|D^{\frac{\alpha}{2}}\rho_\varepsilon\|_2^2 + C\|\rho_\varepsilon\|_2^{\frac{6\alpha-2d}{2\alpha-d}} \\ &\leq C\left(\|\rho_\varepsilon\|_2^2 + \|\rho_\varepsilon\|_2^{\frac{6\alpha-2d}{2\alpha-d}}\right). \end{aligned} \tag{53}$$

Solving above ordinary differential inequality, we obtain

$$\|\rho_\varepsilon\|_2^2 + 1 \leq \frac{1}{\left[(\|\rho_0\|_2^2+1)^{-\frac{\alpha}{2\alpha-d}} - \frac{\alpha}{2\alpha-d}Ct\right]^{\frac{2\alpha-d}{\alpha}}}, \tag{54}$$

which implies there exists a $T(\|\rho_0\|_2^2)$ independent of ε such that for $t \in [0, T]$, the following estimates hold

$$\|\rho_\varepsilon\|_{L^\infty\left(0,T;L^2(\mathbb{R}^d)\right)} < C;$$
$$\|\rho_\varepsilon\|_{L^2\left(0,T;H^{\frac{\alpha}{2}}(\mathbb{R}^d)\right)} < C;$$
$$\|\partial_t\rho_\varepsilon\|_{L^2\left(0,T;H^{-1}(\mathbb{R}^d)\right)} < C.$$

Now we can use the Lions-Aubin lemma, there exists a subsequence ρ_ε without relabeling such that for any ball B_R,

$$\rho_\varepsilon \to \rho \text{ in } L^2\big(0,T;L^2(B_R)\big), \quad \text{as } \varepsilon \to 0$$

and $\rho(t,x)$ is a weak solution to (1). The regularity of ρ follows:

i) $\rho \in L^\infty\big(0,T;L^2(\mathbb{R}^d)\big)$;
ii) $\rho \in L^2\big(0,T;H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big)$;
iii) $\partial_t\rho \in L^2\big(0,T;H^{-1}(\mathbb{R}^d)\big)$.

Moreover the conservation of mass can be proved as we have done in the Step 6 from last section. □

Theorem 3.2. *Let $d \geq 2$ and $1 < \alpha < 2$. Assume $0 \leq \rho_0 \in L^1 \cap L^r(\mathbb{R}^d)$ for some $r > \frac{d}{\alpha} := q$, then there are $T > 0$ and a weak solution $\rho(t,x)$ in $0 < t < T$ to (1) with mass conservation.*

Proof. As in (16), it yields

$$\frac{d}{dt}\|\rho\|_r^r + 4\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2 \leq (r-1)\|\rho\|_{r+1}^{r+1}. \tag{55}$$

And notice that

$$\begin{aligned}\|\rho\|_{r+1}^{r+1} &\leq \|\rho\|_{\frac{rq}{d-\alpha}}^q \|\rho\|_r^{r+1-q} \leq S_{\alpha,d}^{\frac{2q}{r}}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^{\frac{2q}{r}}\|\rho\|_r^{r+1-q} \\ &\leq \frac{2(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2 + C\left(\|\rho\|_r^r\right)^{1+\frac{1}{r-q}}. \end{aligned} \tag{56}$$

Solving above ordinary differential inequality, we obtain

$$\|\rho\|_r^r \le \left(\frac{1}{\|\rho_0\|_r^{\frac{r}{q-r}} - \frac{Ct}{r-q}}\right)^{r-q}, \tag{57}$$

which implies the local in time estimate. The proof for the regularization, existence of a weak solution and mass conservation is the same as the proof of Theorem 2.3. □

4. Well-posedness for the self-consistent stochastic differential equation (SDE) and KS equation.

4.1. Well-posedness of the regularized self-consistent SDE.

In this subsection, we claim the global existence and uniqueness of strong solutions to the following regularized self-consistent stochastic equation:

$$\begin{cases} X_\varepsilon(t) = X_0 + \int_0^t \int_{\mathbb{R}^d} F_\varepsilon\big(X_\varepsilon(s) - y\big) df_\varepsilon(t,y) ds + \nu L_\alpha(t), \\ f_\varepsilon(t,x) = \mathcal{L}(X_\varepsilon(t)), \end{cases} \tag{58}$$

where X_0 has the density $\rho_0(x)$ satisfying $\int_{\mathbb{R}^d} \rho_0(x)dx = 1$ and $\mathcal{L}(X_\varepsilon(t))$ denotes the law of $X_\varepsilon(t)$. Moreover, $L_\alpha(t)$ is the rotationally invariant α-stable Lévy process (see Appendix A), which has following expression as in (103)

$$L_\alpha(t) = P.V. \int_{|x|<1} x\tilde{N}(t,dx) + \int_{|x|\ge 1} xN(t,dx), \tag{59}$$

where $N(t,dx)$ is a Poisson random measure generated by Lévy measure $\mu'(dx) = \frac{C_{d,\alpha}}{|x|^{d+\alpha}}dx$ as in (102) and $\tilde{N}(t,dx)$ is the corresponding compensator as in (101).

The PDE associated to (58) is the following regularized KS equation (as will be proved in the theorem below):

$$\begin{cases} \partial_t \rho_\varepsilon = -\nu(-\Delta)^{\frac{\alpha}{2}}\rho_\varepsilon - \nabla\cdot(\rho_\varepsilon \nabla c_\varepsilon), \\ -\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon, \\ \rho_\varepsilon(0,x) = \rho_0(x), \end{cases} \tag{60}$$

which has a unique global weak solution ρ_ε, and $\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)dx \equiv 1$.

Theorem 4.1. *Given ρ_0 satisfying $\int_{\mathbb{R}^d} \rho_0(x)dx = 1$ and X_0 is a random variable with density $\rho_0(x)$. Then for any $T > 0$ and $\varepsilon > 0$, (58) has a unique strong solution $(X_\varepsilon(t), \rho_\varepsilon(t,x))$ with initial data (X_0, ρ_0). Furthermore, $\rho_\varepsilon(t,x)$ is the unique weak solution to (60).*

Proof. Suppose $\rho_\varepsilon(t,x)$ is the unique weak solution to (60). Let

$$G_\varepsilon(t,x) = \int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy = \nabla c_\varepsilon(t,x),$$

then $G_\varepsilon(t,x)$ is bounded and Lipschitz continuous. So the following stochastic equation

$$X_\varepsilon(t) = X_0 + \int_0^t G_\varepsilon\big(s, X_\varepsilon(s)\big) ds + \nu L_\alpha(t),$$

has a unique strong solution $X_\varepsilon(t)$, which admits a time marginal density denoted by $\tilde{\rho}_\varepsilon(t,x)$ [33, pp.249, Theorem 6]. Then we use Itô formula [1, Theorem 4.4.7]: for each $h \in C_c^\infty(\mathbb{R}^d)$, with probability 1 we have

$$\begin{aligned} h(X_\varepsilon(t)) \quad = \quad & h(X(0)) + \int_0^t \nabla h(X_\varepsilon(s-)) \cdot G_\varepsilon(s, X_\varepsilon(s))ds \\ & +\nu \int_0^t P.V. \int_{|x|<1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-))]\, \tilde{N}(ds,dx) \\ & +\nu \int_0^t \int_{|x|\geq 1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-)]N(ds,dx) \\ & +\nu \int_0^t P.V. \int_{|x|<1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-)) \\ & \qquad\qquad -x \cdot \nabla h(X_\varepsilon(s-))]\mu'(dx)ds. \end{aligned} \tag{61}$$

Substitute $\tilde{N}(ds,dx) = N(ds,dx) - \mathbb{E}[N(ds,dx)] = N(ds,dx) - \mu'(dx)ds$ in (61), we obtain

$$\begin{aligned} h(X_\varepsilon(t)) \quad = \quad & h(X(0)) + \int_0^t \nabla h(X_\varepsilon(s-)) \cdot G_\varepsilon(s, X_\varepsilon(s))ds \\ & +\nu \int_0^t P.V. \int_{|x|<1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-))]N(ds,dx) \\ & -\nu \int_0^t P.V. \int_{|x|<1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-))]\mu'(dx)ds \\ & +\nu \int_0^t \int_{|x|\geq 1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-)]N(ds,dx) \\ & +\nu \int_0^t P.V. \int_{|x|<1} [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-)) \\ & \qquad\qquad -x \cdot \nabla h(X_\varepsilon(s-))]\mu'(dx)ds. \end{aligned} \tag{62}$$

To better understand the Poisson stochastic integrals [1, P.231], let A be an arbitrary Borel set in $\mathbb{R}^d - \{0\}$ which satisfies $\inf_{x\in A} |x| \geq C > 0$, and denote $P_A(t) = \int_A xN(t,dx)$. Then

$$\begin{aligned} & \int_0^t \int_A [h(X_\varepsilon(s-)+x) - h(X_\varepsilon(s-)]N(ds,dx) \\ = & \sum_{0\leq s\leq t} [h(X_\varepsilon(s-) + \Delta P_A(s)) - h(X_\varepsilon(s-)]1_A(\Delta P_A(s)). \end{aligned} \tag{63}$$

where $\Delta P_A(s) = P_A(s) - P_A(s-)$ is the jump increment. Since Lévy process has independent increments, we know $X_\varepsilon(s-)$ and $\Delta P_A(s)$ are independent. Moreover, it follows from [1, Theorem 2.3.7] that

$$\mathbb{E}_{\Delta P_A} \left[\sum_{0\leq s\leq t} f(\Delta P_A(s))1_A(\Delta P_A(s)) \right]$$

$$=\mathbb{E}\left[\int_0^t\int_A f(x)N(ds,dx)\right]=\int_0^t\int_A f(x)\mu'(dx)ds. \tag{64}$$

Hence we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_A[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]N(ds,dx)\right]\\ =&\mathbb{E}_{X_\varepsilon}\mathbb{E}_{\Delta P_A}\left[\sum_{0\le s\le t}[h(X_\varepsilon(s-)+\Delta P_A(s))-h(X_\varepsilon(s-)]1_A(\Delta P_A(s))\right]\\ =&\mathbb{E}_{X_\varepsilon}\left[\int_0^t\int_A[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]\mu'(dx)ds\right].\end{aligned} \tag{65}$$

We take $A=\{x:|x|\ge 1\}$ in (65), thus

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_{|x|\ge 1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]N(ds,dx)\right]\\ =&\mathbb{E}_{X_\varepsilon}\left[\int_0^t\int_{|x|\ge 1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]\mu'(dx)ds\right].\end{aligned} \tag{66}$$

Denote $A_\delta=\{x:0<\delta\le|x|<1\}$, and

$$g_\delta=\int_0^t\int_{A_\delta}f(x)N(ds,dx).$$

Then

$$\lim_{\delta\to 0}g_\delta=\int_0^t P.V.\int_{|x|<1}f(x)N(ds,dx). \tag{67}$$

Moreover, we can prove that g_δ is a Cauchy sequence. Actually, for $0<\delta_1<\delta_2$, we have

$$\mathbb{E}[|g_{\delta_2}-g_{\delta_1}|]\le\mathbb{E}\left[\int_0^t\int_{\delta_1\le|x|<\delta_2}|f(x)|N(ds,dx)\right]=\int_0^t\int_{\delta_1\le|x|<\delta_2}|f(x)|\mu'(dx)ds, \tag{68}$$

by (64). It follows from (67) and (68) that

$$\lim_{\delta\to 0}\mathbb{E}[g_\delta]=\mathbb{E}\left[\int_0^t P.V.\int_{|x|<1}f(x)N(ds,dx)\right]. \tag{69}$$

Now we apply (65) and (69)

$$\begin{aligned}&\mathbb{E}\left[\int_0^t P.V.\int_{|x|<1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]N(ds,dx)\right]\\ =&\lim_{\delta\to 0}\mathbb{E}\left[\int_0^t\int_{A_\delta}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]N(ds,dx)\right]\\ =&\lim_{\delta\to 0}\mathbb{E}_{X_\varepsilon}\left[\int_0^t\int_{A_\delta}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]\mu'(dx)ds\right]\\ =&\mathbb{E}_{X_\varepsilon}\left[\int_0^t P.V.\int_{|x|<1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-)]\mu'(dx)ds\right],\end{aligned} \tag{70}$$

where we have used the Dominated convergence theorem in the last equality.

Now combing (66) and (70), we can take expectation on both side of (62) , then one has

$$
\begin{aligned}
&\mathbb{E}\big[h(X_\varepsilon(t))\big]\\
=&\mathbb{E}\left[h(X(0))\right]+\mathbb{E}\left[\int_0^t \nabla h(X_\varepsilon(s-))\cdot G_\varepsilon(s,X_\varepsilon(s))ds\right]\\
&+\nu\mathbb{E}\left[\int_0^t\int_{|x|\geq 1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-))]\mu'(dx)ds\right]\\
&+\nu\mathbb{E}\left[\int_0^t P.V.\int_{|x|<1}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-))-x\cdot\nabla h(X_\varepsilon(s-))]\mu'(dx)ds\right].
\end{aligned}
\tag{71}
$$

Substitute $\mu'(dx)=\frac{C_{d,\alpha}}{|x|^{d+\alpha}}dx$ in (71), and it leads to

$$
\begin{aligned}
&\mathbb{E}\big[h(X_\varepsilon(t))\big]\\
=&\mathbb{E}\big[h(X(0))\big]+\mathbb{E}\left[\int_0^t \nabla h(X_\varepsilon(s-))\cdot G_\varepsilon(s,X_\varepsilon(s))ds\right]\\
&+\nu\mathbb{E}\bigg[\int_0^t\big[\int_{|x|\geq 1}\frac{C_{d,\alpha}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-))]}{|x|^{d+\alpha}}dx\\
&\qquad+P.V.\int_{|x|<1}\frac{C_{d,\alpha}[h(X_\varepsilon(s-)+x)-h(X_\varepsilon(s-))-x\cdot\nabla h(X_\varepsilon(s-))]}{|x|^{d+\alpha}}dx\big]ds\bigg].
\end{aligned}
$$

Then we use the properties of fractional Laplacian in (7), one has

$$
\begin{aligned}
\mathbb{E}\big[h(X_\varepsilon(t))\big] \;=\;& \mathbb{E}\left[h(X(0))\right]+\mathbb{E}\left[\int_0^t \nabla h(X_\varepsilon(s-))\cdot G_\varepsilon(s,X_\varepsilon(s))ds\right]\\
&+\nu\mathbb{E}\left[\int_0^t\big[-(-\Delta)^{\frac{\alpha}{2}}h(X_\varepsilon(s-))ds\right],
\end{aligned}
\tag{72}
$$

which leads to

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\tilde\rho_\varepsilon(t,x)h(x)dx-\int_{\mathbb{R}^d}\rho_0(x)h(x)dx\\
=&\int_0^t\int_{\mathbb{R}^d}\tilde\rho_\varepsilon(s,x)\left(\int_{\mathbb{R}^d}F_\varepsilon(x-y)\tilde\rho_\varepsilon(s,y)dy\right)\cdot\nabla h(x)dxds\\
&+\nu\int_0^t\big[-(-\Delta)^{\frac{\alpha}{2}}h(x)\tilde\rho_\varepsilon(s,x)dx\big]ds.
\end{aligned}
\tag{73}
$$

Thus we know $\tilde\rho_\varepsilon(t,x)$ satisfies the following equation in distribution sense

$$
\begin{cases}
\partial_t\tilde\rho_\varepsilon(t,x)=-\nu(-\Delta)^{\frac{\alpha}{2}}\tilde\rho_\varepsilon(t,x)-\nabla\cdot[\tilde\rho_\varepsilon(t,x)\nabla c_\varepsilon(t,x)],\\
\tilde\rho_\varepsilon(0,x)=\rho_0(x).
\end{cases}
\tag{74}
$$

Since $\rho_\varepsilon(t,x)$ is also a weak solution to (74) and the weak solution of (74) is unique, then we get $\tilde\rho_\varepsilon(t,x)=\rho_\varepsilon(t,x)$. It means that $\big(X_\varepsilon(t),\rho_\varepsilon(t,x)\big)$ is a strong solution to (58). The uniqueness of the strong solutions to (58) comes from the uniqueness of the solutions to (60). In fact, suppose $\big(X^1_\varepsilon(t),\rho^1_\varepsilon\big)$ and $\big(X^1_\varepsilon(t),\rho^2_\varepsilon\big)$ are two solutions to (58). By the Itô formula we have used before, one knows the ρ^1_ε and ρ^2_ε both are weak solutions to (60) with the same initial data $\rho_0(x)$. Since the weak solution to (60) is unique, one has $\rho^1_\varepsilon=\rho^2_\varepsilon$ which leads to $X^1_\varepsilon(t)=X^2_\varepsilon(t)$. □

4.2. **Existence, uniqueness and stability with initial data** $0 \leq \rho_0 \in L^1 \cap L^q \cap L^\infty(\mathbb{R}^d)$ **and** $\|\rho_0\|_q < K$**.** The uniqueness of weak solutions to the KS model has been concerned by many scholars. The optimal transport method [16] and the renormalizing argument [18] have been used to prove the uniqueness of weak solutions to the classical KS model with normal Laplacian term. Here we will follow the method in [31] to prove the uniqueness for the generalized KS model (1).

Now we introduce a topology of the Wasserstein space which is useful in proving the following theorem. Consider the space of probability measure

$$\mathcal{P}_1(\mathbb{R}^d) = \left\{ f | f \text{ is a probability measure on } \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d} |x| df(x) < \infty \right\}.$$

We define the Kantorovich-Rubinstein distance in $\mathcal{P}_1(\mathbb{R}^d)$ as follows

$$\mathcal{W}_1(f,g) := \inf_{\pi \in \Lambda(f,g)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\pi(x,y) \right\},$$

where $\Lambda(f,g)$ is the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and g. And it has been proved that $\mathcal{P}_1(\mathbb{R}^d)$ endowed with this distance is a complete metric space in [41, Theorem 6.18].

Also we will use the following time dependent measure space $L^\infty\big(0,T;\mathcal{P}_1(\mathbb{R}^d)\big)$

$$\left\{ f(t,x) | f(t,\cdot) \text{ is a probability measure on } \mathbb{R}^d \text{ with } \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x| df(t,x) < \infty \right\}.$$

Moreover, it is a complete metric space equipped with metric

$$\mathcal{M}_T(f_t^1, f_t^2) = \sup_{t\in[0,T]} \mathcal{W}_1(f_t^1, f_t^2),$$

for any two elements $f_t^1, f_t^2 \in L^\infty\big(0,T;\mathcal{P}_1(\mathbb{R}^d)\big)$.

Theorem 4.2. *Assume the initial density ρ_0 satisfies Assumption 1, then for any $T > 0$ and $t \in [0,T]$,*

(i) There exists a unique weak solution $\rho(t,x)$ to (1) with initial density ρ_0 and regularity

$$\rho \in L^\infty\left(0,T;L^\infty(\mathbb{R}^d) \cap L^1\big(\mathbb{R}^d,(1+|x|)dx\big)\right); \rho^{\frac{q}{2}} \in L^2\big(\mathbb{R}_+;H^{\frac{\alpha}{2}}(\mathbb{R}^d)\big);$$

$$\partial_t \rho \in L^2\big(0,T;W^{-\alpha,\frac{2(q+1)}{q+3}}(\mathbb{R}^d)\big).$$

(ii) There exists a unique strong solution $\big(X(t),\rho\big)$ to (5) with initial data (X_0,ρ_0), and ρ is the unique weak solution to (1).

Proof. The sketch of the proof will be divided into 4 steps and we refer to [31, Theorem 1.1] for more details.

Step 1. (Global existence and $L^\infty(\mathbb{R}^d)$ uniform bound) The global existence of weak solution to (1) has been proved in Theorem 2.3. Following the method in [30], we leave the proof of the uniform L^∞ estimate in Appendix B.

Step 2. (Existence of strong solution to (5)) Firstly, we give some uniform estimates for the regularized equation. For $\varepsilon > \varepsilon' > 0$, consider equation (58), and suppose $\big(X_\varepsilon(t),\rho_\varepsilon(t,x)\big)$, $\big(X_{\varepsilon'}(t),\rho_{\varepsilon'}(t,x)\big)$ are two strong solutions in Theorem 4.1 starting from the same initial data X_0. One can show that there exists a constant C_T and $\varepsilon_0(T)$ such that if $\varepsilon < \varepsilon_0(T)$, then

$$\sup_{t\in[0,T]} \mathcal{W}_1(f_\varepsilon, f_{\varepsilon'}) \le \mathbb{E}\left[\sup_{t\in[0,T]} |X_\varepsilon(t) - X_{\varepsilon'}(t)|\right] \le C_T \varepsilon^{e^{-CT}}, \tag{75}$$

where $df_\varepsilon = \rho_\varepsilon(t,x)dx$, $df_\varepsilon^{'} = \rho_{\varepsilon'}(t,x)dx$.

Consequently, there exists a stochastic process $X(t) \in L^\infty\big(0,T; L^1(\Omega,\mathbb{P})\big)$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_\varepsilon(t) - X(t)|\right] \le C_T \varepsilon^{e^{-CT}}. \tag{76}$$

On the other hand there exists a unique $f(t,x) \in L^\infty\big(0,T; \mathcal{P}_1(\mathbb{R}^d)\big)$ such that

$$\mathcal{M}_T(f_\varepsilon, f) \le C_T \varepsilon^{e^{-CT}}, \tag{77}$$

since $L^\infty\big(0,T;\mathcal{P}_1(\mathbb{R}^d)\big)$ is a complete metric space.

Secondly, one can show the limiting density is the weak solution to KS equations (1), i.e.

$$df(t,x) = \rho(t,x)dx, \tag{78}$$

where $\rho(t,x)$ is a weak solution of (1).

Lastly, we conclude that the limiting stochastic process $X(t)$ is the strong solution to (5).

Step 3. (Uniqueness of strong solutions to (5)) Assume

$$\big(X(t), \rho(t,x)\big), \big(X'(t), \rho'(t,x)\big),$$

are strong solutions to (5) with the same initial data. Then one can deduce that

$$\mathbb{E}[|X'(t) - X(t)|] \le C \int_0^t \omega\big(\mathbb{E}[|X'(t) - X(t)|]\big) ds. \tag{79}$$

Here $\omega(x)$ is defined as

$$\omega(x) := \begin{cases} 1, & \text{if } x \ge 1, \\ x(1-\ln x), & \text{if } 0 < r < 1, \end{cases} \tag{80}$$

which is related to log-Lipschitz continuity of the field $\int F(x-y)\rho(s,y)dy$, seen in [31, Lemma 2.2].

By $\mathbb{E}[|X'(0) - X(0)|] = 0$ and Gronwall inequality, we have

$$\mathcal{M}_T(\rho', \rho) \le \mathbb{E}[|X'(t) - X(t)|] \equiv 0. \tag{81}$$

Hence $\rho' = \rho$ and $X'(t) = X(t)$ a.s. for all $t \ge 0$.

Step 4. (Uniqueness of weak solutions to (1)) Suppose ρ', ρ are two weak solutions with the same initial density ρ_0. For any fixed random variable X_0 with density ρ_0, by the following Proposition, there exists two processes $X(t)$ and $X'(t)$ such that $\big(X(t), \rho(t,x)\big)$, $\big(X'(t), \rho'(t,x)\big)$ both are strong solution to (5) with the same initial data $\big(X_0, \rho_0\big)$. Therefore (81) holds, the uniqueness is proved. □

Proposition 1. *The relation between weak solution to* (1) *and strong solution to* (5) *can be described*

(i) *If* $\big(X(t), \rho\big)$ *is a strong solution to* (5) *with initial data* $\big(X_0, \rho_0\big)$*, then* $\rho(t,x)$ *is a weak solution to* (1) *with initial data* ρ_0*.*

(ii) If $\rho(t,x)$ is a weak solution to (1) with initial data $\rho_0(x)$, then for any X_0 with density $\rho_0(x)$, there is a unique process $X(t)$ with density $\rho(t,x)$ and $\big(X(t),\rho\big)$ is a strong solution to (5) with initial data $\big(X_0,\rho_0\big)$.

Proof. The proof of this proposition is similar to [31, Proposition 2.3] except that we use a different Itô formula as we have done in Theorem 4.1. □

Furthermore, with the help of the self-consistent stochastic process of (5), we also obtain the following stability with initial data in the Wasserstein distance for (1).

Theorem 4.3. *For any fixed $T>0$, suppose $\rho_t^1,\rho_t^2\in L^\infty\big(0,T;L^\infty(\mathbb{R}^d)\cap L^1\big(\mathbb{R}^d,(1+|x|)dx\big)\big)$ be two weak solutions to (1) with initial data $\rho_0^1(x),\rho_0^2(x)$ respectively and they satisfy Assumption 1. Then there exists two constants C $\big($depending on $\|\rho_t^1\|_{L^\infty\left(0,T;L^1\cap L^\infty(\mathbb{R}^d)\right)}$ and $\|\rho_t^2\|_{L^\infty\left(0,T;L^1\cap L^\infty(\mathbb{R}^d)\right)}\big)$ and C_T $\big($depending only on $T\big)$ such that*

$$\sup_{t\in[0,T]}\mathcal{W}_1(\rho_t^1,\rho_t^2)\le C_T\max\left\{\mathcal{W}_1(\rho_0^1,\rho_0^2),\left\{\mathcal{W}_1(\rho_0^1,\rho_0^2)\right\}^{e^{-CT}}\right\},$$

where $\mathcal{W}_1$ is the Wasserstein distance.

Proof. The proof of this theorem is similar to [31, Theorem 1.2] except that we change Brownian motion into rotationally invariant α-stable Lévy process.. □

5. Interacting particle system and mean-field limit.

Inspired by [31], we introduce the stochastic system of interacting particles with singular force kernel and rotationally invariant α-stable Lévy process described as follows. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t,t\ge 0)$ which satisfies the usual hypothesis of right continuity and completion, i.e. $\mathcal{F}$ is complete and $\mathcal{F}_t$ is right continuous. We suppose that the space is endowed with N independent d-dimensional rotationally invariant α-stable Lévy process $\{L_\alpha^i(t)\}_{i=1}^N$. Furthermore, every Lévy process $L_\alpha^i(t)$ will be assumed to be $\mathcal{F}_t$-adapted which have *càdlàg* (right continuous with left limits) simple paths, and $L_\alpha^i(t)-L_\alpha^i(s)$ is independent of $\mathcal{F}_s$ for all $0\le s<t<\infty$. And with the assumption $\alpha\in(1,2]$, it allows us to freely use expectation of the α-stable process. Denote $\{X^i(t)\}_{i=1}^N$ be the positions of N-particles at time t, where $X^i(t)\in\mathbb{R}^d$. The initial data $\{X_0^i\}_{i=1}^N$ are the i.i.d. random variables with a common probability density function $\rho_0(x)$. Moreover we assume the particles in the system interact with each other by Newtonian potential $\Phi(x)$ as in (3), and we have the interacting force $F(x)=\nabla\Phi(x)$. Thus the dynamics of the interacting particle system can be described by a system of stochastic differential equations

$$\begin{cases} dX^i(t)=\dfrac{1}{N-1}\displaystyle\sum_{j\ne i}^N F\big(X^i(t)-X^j(t)\big)dt+\nu\,dL_\alpha^i(t), \quad i=1,\cdots,N,\\ X^i|_{t=0}=X_0^i. \end{cases} \tag{82}$$

Various interesting particle systems in physical and biological science can described by this equation. We can see the first term in the right hand side of (82) represents the attractive force on $X^i(t)$ by all particles. Moreover, we assume the initial data $\{X_0^i\}_{i=1}^N$ are i.i.d. random variables with the same distribution $\mathcal{L}(X_0^i)=f_0(x)$ and density $\rho_0(x)$.

For (82), particles may collide to each other due to attractive force. Hence we consider a standard smoothing kernel $J_\varepsilon(x)$ satisfying $J_\varepsilon(x) = \frac{1}{\varepsilon^d} J(\frac{x}{\varepsilon})$, where $J(x)$ is defined as in (6). Let $F_\varepsilon = J_\varepsilon * F$, then regularized system

$$\begin{cases} dX_\varepsilon^i(t) = \dfrac{1}{N-1} \displaystyle\sum_{j\neq i}^{N} F_\varepsilon\big(X_\varepsilon^i(t) - X_\varepsilon^j(t)\big)dt + \nu\, dL_\alpha^i(t), \quad i = 1, \cdots, N, \\ X_\varepsilon^i|_{t=0} = X_0^i, \end{cases} \tag{83}$$

has a unique strong solution $\{X_\varepsilon^i(t)\}_{i=1}^N$ and $F_\varepsilon(x) = F(x)$ for any $|x| > \varepsilon$ [31, Lemma 2.1].

5.1. **Collision between particles.** In this subsection, we show that the expectation of the collision time for the interacting particle system (82) is below a constant.

Lemma 5.1. [8, Lemmma 4.1] *For $d \geq 2$, $\alpha \in (1, 2)$ and some $\gamma \in (1, \alpha)$, and define $h(x) := (1 + |x|^2)^{\frac{\gamma}{2}} - 1$. Then*

$$K_1 := \| - (-\Delta)^{\frac{\alpha}{2}} h\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

The following lemma is a useful result from the process of proving Theorem 2.3 in [8].

Lemma 5.2. [8] *Define the generalized momentum $M_\gamma(t) := \int_{\mathbb{R}^d} h(x)\rho(x,t)dx$ with $h(x) = (1+|x|^2)^{\frac{\gamma}{2}} - 1$. For $d \geq 2$, $\alpha \in (1, 2)$, $\gamma \in (1, \alpha)$, there exist suitable constants $K_2, s > 0$, such that*

$$-\frac{C_*}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\nabla h(x) - \nabla h(y)\big) \cdot (x - y)\frac{\rho(t, x)\rho(t, y)}{|x - y|^d} dxdy \leq -K_2 \frac{1}{\big(1 + 2M_\gamma(t)\big)^s},$$

where $h(x) = (1 + |x|^2)^{\frac{\gamma}{2}} - 1$ and C_* is from the definition of $F(x)$, see (4).

Lemma 5.3. [8, Theorem 2.3] *For $d \geq 2$, $\alpha \in (1, 2)$ and some $\gamma \in (1, \alpha)$, suppose initial data ρ_0 satisfies $\rho_0 \in L^1\big(\mathbb{R}^d, (1+|x|^\gamma)dx\big)$ and $\|\rho_0\|_1 = 1$. Define the generalized momentum $M_\gamma(t) := \int_{\mathbb{R}^d} h(x)\rho(x,t)dx$ with $h(x) = (1+|x|^2)^{\frac{\gamma}{2}} - 1$. For certain (sufficiently small) universal constant $K_\gamma > 0$, if $\nu < \frac{K_1}{K_2}$ (K_1, K_2 as in Lemma 5.1 and Lemma 5.2) and*

$$\int_{\mathbb{R}^d} |x|^\gamma \rho_0(x)dx \leq K_\gamma,$$

then $M_\gamma(t)$ is strictly decreasing and the solution to (1) has a concentration at finite time.

Theorem 5.4. *For $d \geq 2$, given N i.i.d. random variables $\{X_0^i\}_{i=1}^N$ with common density ρ_0 satisfying $\rho_0 \in L^1(\mathbb{R}^d, (1+|x|^\gamma)dx)$ and $\|\rho_0\|_1 = 1$. Let $\{X^i(t)\}_{i=1}^N$ be the strong solution of (82) with initial data $\{X_0^i\}_{i=1}^N$. For any fixed $T > 0$ and $\nu < \frac{K_2}{K_1}$ (K_1, K_2 as in Lemma 5.1 and Lemma 5.2), define*

$$A(t) := \inf_{0\leq s\leq t} \min_{i\neq j} |X^i(s) - X^j(s)|, \tag{84}$$

$$\tau_\varepsilon = \begin{cases} 0, & \text{if } \varepsilon \geq A(0), \\ \sup\{t \wedge T : A(t) \geq \varepsilon\}, & \text{if } \varepsilon < A(0), \end{cases} \tag{85}$$

and let $\tau = \lim_{\varepsilon\to 0} \tau_\varepsilon$. There exist universals two constants $K_\gamma, T^c > 0$, such that if $\int_{\mathbb{R}^d} |x|^\gamma \rho_0(x)dx < K_\gamma$, then

$$\mathbb{E}(\tau) \leq T^c.$$

Proof. Adapting the method of proof of in [31, Theorem 3.1], we know the system (82) has a unique strong solution until the explosion time $\tau = \sup\{t \wedge T : \inf_{0\le s\le t} \min_{i\neq j} |X^i(s) - X^j(s)| = 0\}$. Since $F_\varepsilon(x) = F(x)$ for any $|x| > \varepsilon$, we get $X^i(t) \equiv X^i_\varepsilon(t)$ for $1 \le i \le N$, when $t \le \tau_\varepsilon$, where $X^i_\varepsilon(t)$ is the global unique solution to the regularized interacting system (83). By Itô formula, we choose $h(x) = (1+|x|^2)^{\frac{\gamma}{2}} - 1$ in (72), then one has

$$\begin{aligned} h(X^i_\varepsilon(t)) &= h(X^i_0) + \frac{1}{N-1}\int_0^t \nabla h(X^i_\varepsilon(s)) \cdot \sum_{j\neq i}^N F_\varepsilon\big(X^i_\varepsilon(s) - X^j_\varepsilon(s)\big)ds \\ &\quad -\nu \int_0^t (-\Delta)^{\frac{\alpha}{2}} h(X^i_\varepsilon(s))ds \\ &\le h(X^i_0) + \frac{1}{N-1}\int_0^t \nabla h(X^i_\varepsilon(s)) \cdot \sum_{j\neq i}^N F_\varepsilon\big(X^i_\varepsilon(s) - X^j_\varepsilon(s)\big)ds \\ &\quad +K_1\nu t. \end{aligned} \tag{86}$$

where we have used Lemma 5.1. Sum all of (86), we get

$$\begin{aligned} \sum_{i=1}^N h(X^i_\varepsilon(t)) &\le \sum_{i=1}^N h(X^i_0) + \frac{1}{N-1}\int_0^t \sum_{\substack{i,j=1\\ j\neq i}}^N \nabla h\big(X^i_\varepsilon(s)\big) \cdot F_\varepsilon\big(X^i_\varepsilon(s) - X^j_\varepsilon(s)\big)ds \\ &\quad +K_1 N\nu t. \end{aligned} \tag{87}$$

Since $X^i_\varepsilon(t)$ is the unique solution to (82) and $F_\varepsilon = F$ on $[0,\tau_\varepsilon]$, one has

$$\begin{aligned} &\mathbb{E}\left[\sum_{\substack{i,j=1\\ j\neq i}}^N \nabla h\big(X^i_\varepsilon(s)\big) \cdot F_\varepsilon\big(X^i_\varepsilon(s) - X^j_\varepsilon(s)\big)\right] \\ &= -\frac{C_* N(N-1)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\nabla h(x) - \nabla h(y)\big)\cdot (x-y)\frac{\rho(s,x)\rho(s,y)}{|x-y|^d}dxdy \\ &\le -K_2\frac{N(N-1)}{\big(1+2M_\gamma(t)\big)^s}, \quad \text{for all } s\in[0,\tau_\varepsilon], \end{aligned} \tag{88}$$

where in the second inequality we have used Lemma 5.2. Take expectation of (87) and by exchangeability of $X^i_\varepsilon(t)$, one has

$$\begin{aligned} \mathbb{E}[h(X^i(\tau_\varepsilon)] &\le \mathbb{E}[h(X^i_0)] + \left(\nu K_1 - K_2\frac{1}{(1+2M_\gamma(t))^s}\right)\mathbb{E}[\tau_\varepsilon] \\ &= M_\gamma(0) + \left(\nu K_1 - K_2\frac{1}{(1+2M_\gamma(t))^s}\right)\mathbb{E}[\tau_\varepsilon]. \end{aligned} \tag{89}$$

From Lemma 5.3, For certain (sufficiently small) universal constant $K_\gamma > 0$, if $M_\gamma(t) \le M_\gamma(0) \le \int_{\mathbb{R}^d} |x|^\gamma \rho_0(x)dx \le K_\gamma$, we have

$$\mathbb{E}[h(X^i(\tau_\varepsilon)] \le M_\gamma(0) + \left(\nu K_1 - K_2\frac{1}{(1+2K_\gamma)^s}\right)\mathbb{E}[\tau_\varepsilon]. \tag{90}$$

If we choose $0 < K_\gamma < \frac{1}{2}(\frac{K_2}{\nu K_1})^{\frac{1}{s}} - \frac{1}{2}$, then

$$\left(\nu K_1 - K_2\frac{1}{(1+2K_\gamma)^s}\right) < 0. \tag{91}$$

By the positivity of the left handside of (90), one has

$$\mathbb{E}[\tau_\varepsilon] \leq \frac{M_\gamma(0)}{\left(K_2 \frac{1}{(1+2K_\gamma)^s} - \nu K_1\right)} \leq \frac{K_\gamma(1+2K_\gamma)^s}{K_2 - (1+2K_\gamma)^s \nu K_1} =: T^c. \tag{92}$$

Finally, by the monotone convergence theorem, we concludes the proof. □

5.2. **Propagation of chaos.** The concept of the propagation of chaos was originated by Kac [23]. It is important for the kinetic theory that serves to relate the kinetic equations, such as the Fokker-Planck, Boltzmann and Vlasov equations. In this subsection we prove the propagation of chaos for the KS equations (1) following the method in [31]. We refer to [11, 22, 36, 37] for more instances of the propagation of chaos.

Theorem 5.5. *Assume that the initial density $\rho_0(x)$ satisfies Assumption 1 and $\{X^i_\varepsilon(t)\}_{i=1}^N$ is the unique strong solution to (83) with i.i.d. initial data $\{X^i_0\}_{i=1}^N$ and Lévy motions $\{L^i_\alpha(t)\}_{i=1}^N$, $\mathcal{L}\{X^i_0\} = f_0$, $df_0 = \rho_0(x)dx$. Let $\{(X^i(t), \rho^i)\}_{i=1}^N$ be the unique solution to (5) with the same initial data $\{X^i_0\}_{i=1}^N$ and Lévy motions $\{L^i_\alpha(t)\}_{i=1}^N$. Then $\{X^i_\varepsilon(t)\}_{i=1}^N$ are exchangeable, $\{X^i(t)\}_{i=1}^N$ are i.i.d. and there is a list of regularized parameters $\varepsilon(N) \sim (\ln N)^{-\frac{1}{d}} \to 0$ as $N \to \infty$, such that for any $T > 0$ and all $1 \leq i \leq N$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^i_{\varepsilon(N)}(t) - X^i(t)|\right] \to 0, \quad \text{as } N \to \infty.$$

Proof. We will only give a sketch of proof here since it is similar to [31, Theorem 1.3]. The main idea is to link (83) with (5) through (58). In Theorem 4.1, we stated the existence and uniqueness for strong solutions to (58), which derives that if initial data $\{X^i_0\}_{i=1}^N$ are i.i.d. and Lévy process $\{L^i_\alpha(t)\}_{i=1}^N$ are independent, then the following nonlinear stochastic differential equations

$$X^i_\varepsilon(t) = X^i_0 + \int_0^t \int_{\mathbb{R}^d} F_\varepsilon\big(X^i_\varepsilon(s) - y\big) df^i_\varepsilon(t,y) ds + \nu L^i_\alpha(t), \quad i = 1, \cdots, N, \tag{93}$$

have a unique strong solution $\{\bar{X}^i_\varepsilon(t)\}_{i=1}^N$ and they are i.i.d.. Suppose $\{X^i_\varepsilon(t)\}_{i=1}^N$ is the unique strong solution to (83) with the same initial data $\{X^i_0\}_{i=1}^N$ and Lévy process $\{L^i_\alpha(t)\}_{i=1}^N$. Then for any $\varepsilon > 0, 1 \leq i \leq N$ and $T > 0$, we can prove

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^i_\varepsilon(t) - \bar{X}^i_\varepsilon(t)|\right] \leq \frac{C_T}{\sqrt{N-1}\varepsilon^{d-1}} e^{\frac{C_T}{\varepsilon^d}}, \tag{94}$$

where C_T is a constant independent of ε. The detail of the proof to (94) can be find in [31, Proposition 3.1].

On the other hand, similar to (76), there exists a constant C_T and $\varepsilon_0(T) > 0$ such that if $\varepsilon < \varepsilon_0(T)$ for any $\varepsilon > 0, 1 \leq i \leq N$ and $T > 0$, one has

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\bar{X}^i_\varepsilon(t) - X^i(t)|\right] \leq C_T \varepsilon^{e^{-CT}}. \tag{95}$$

Combine (94) and (95) together, one has

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^i_\varepsilon(t) - X^i(t)|\right]$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\sup_{t\in[0,T]} |X_\varepsilon^i(t) - \bar{X}_\varepsilon^i(t)|\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |\bar{X}_\varepsilon^i(t) - X^i(t)|\right] \\
&\leq \frac{C_T}{\sqrt{N-1}\varepsilon^{d-1}} e^{\frac{C_T}{\varepsilon^d}} + C_T \varepsilon^{e^{-CT}}.
\end{aligned} \tag{96}
$$

We choose $\varepsilon = \varepsilon(N) = \lambda(\ln N)^{-\frac{1}{d}} \to 0$ as $N \to \infty$ in (96), where λ is a large enough positive constant. And then

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |X_{\varepsilon(N)}^i(t) - X^i(t)|\right] &\leq \frac{C_T N^{\frac{C_T}{\lambda^d}} (\ln N)^{\frac{d-1}{d}}}{\sqrt{N-1}\lambda^{d-1}} + C_T \varepsilon^{e^{-CT}} \\
&\to 0, \quad \text{as } N \to \infty,
\end{aligned} \tag{97}
$$

which ends the proof. □

Appendices.

Appendix A. **Introduction to rotationally invariant α-stable Lévy process.** In this subsection we refer to [1] for some basic definitions in probability theory. A random variable X is said to be stable if there exist real-valued sequences $(c_n, n \in \mathbb{N})$ and $(d_n, n \in \mathbb{N})$ with each $c_n \geq 0$ such that

$$
X_1 + X_2 + \cdots + X_n \overset{d}{=} c_n X + d_n, \tag{98}
$$

where $X_1, X_2, \ldots, X_n$ are independent copies of X. In fact, it can be shown in [21, pp.166] that only possible choice of c_n in (98) is $\sigma n^{\frac{1}{\alpha}}$,where the parameter α ($0 < \alpha \leq 2$) is called the index of stability which plays a key role in the investigation of stable random variables. An alternative characterization of stable random variable is defined by its characteristic function. A real-valued random variable X is stable if and only if there exist $\sigma > 0, -1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that the characteristic function for X is

$$
\phi_X(u) = \mathbb{E}[e^{iuX}] = \begin{cases} \exp\left(i\mu u - \dfrac{1}{2}\sigma^2 u^2\right), & \text{if } \alpha = 2, \\ \exp\left\{i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan(\dfrac{\pi\alpha}{2})\right]\right\}, & \text{if } \alpha \neq 1, 2, \\ \exp\left\{i\mu u - \sigma|u|\left[1 + i\beta \dfrac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]\right\}, & \text{if } \alpha = 1. \end{cases} \tag{99}
$$

It can be shown $\mathbb{E}(|X|) < \infty$ if and only if $1 < \alpha \leq 2$. We define $\eta_X(u) := \log(\phi_X(u))$ which is called the Lévy symbol. In particular, we are interested in the case whose Lévy symbol is given by

$$
\eta_X(u) = -\sigma^\alpha |u|^\alpha, \tag{100}
$$

which is called the rotationally invariant α-stable random variable. The generalisation of stability to random vectors is straightforward, readers can find more details in [34, Theorem 14.3, Theorem 14.10]. And in the following, we will always talk about the d-dimensional case.

A stochastic process $X(t)$ is called Lévy process if:

i) $X(0) = 0 \quad a.s.$;
ii) X has independent and stationary increments;

iii) X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Then we have to mention the famous Lévy-Itô decomposition: If $X(t)$ is a Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion B_A with covariance matrix A, an independent Poisson Random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ and corresponding compensator

$$\tilde{N} = N - \mathbb{E}(N), \tag{101}$$

such that, for each $t \geq 0$,

$$X(t) = bt + B_A(t) + P.V. \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

An important by-product of the Lévy-Itô decomposition is the Lévy-Khintchine formula: If X is a Lévy process then for each $u \in \mathbb{R}^d, t \geq 0$,

$$\mathbb{E}\left[e^{i(u,X(t))}\right] = \exp\left(t\{i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} [e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}}(y)]\mu'(dy)\}\right).$$

where $\hat{B} = \{y \in \mathbb{R}^d, |y| < 1\}$, μ' is the Lévy measure. The triple (b, A, μ') is called the characteristics of the Lévy process.

For a Lévy process $X(t)$ we have

$$\phi_{X(t)}(u) = e^{t\eta_{X(1)}(u)},$$

where $\eta_{X(1)}$ is the Lévy symbol of $X(1)$, which can be seen in [1, Therorem 1.3.3]. When we say the Lévy symbol of a Lévy process $X(t)$, it means the Lévy symbol of the random variable $X(1)$.

A Lévy process $X(t)$ is called stable if in which each $X(t)$ is a stable random variable. Of particular interest is the so called rotationally invariant α-stable Lévy process, where the Lévy symbol is defined by (100), i.e. $\eta_{X(1)}(u) = -\sigma^\alpha |u|^\alpha (0 < \alpha < 2)$. For simplicity we choose $\sigma = 1$ in the sequel. And we denote this particular rotationally invariant α-stable Lévy process as $L_\alpha(t)$, and the characteristics of it is $(0, 0, \mu')$, where

$$\mu'(dx) = \frac{C}{|x|^{d+\alpha}} dx, C > 0 \tag{102}$$

[1, pp.37]. Specifically, we choose C to be $C_{d,\alpha}$ as in the definition of fractional Laplacian operator. Moreover we have

$$L_\alpha(t) = P.V. \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx). \tag{103}$$

Appendix B. **Uniform L^∞ estimate.** First, we will give a proof for the $L^r(\mathbb{R}^d)$ $(q < r < +\infty)$ bound uniformly in time in the following lemma.

Lemma B.1. *Denote $q = \frac{d}{\alpha}$ $(1 < \alpha < 2)$, $\zeta = K - \|\rho_0\|_q$. Assume $0 \leq \rho_0 \in L^1 \cap L^q(\mathbb{R}^d)$ and $\zeta > 0$, then the weak solution ρ to (1) satisfies $\|\rho\|_q < K$. Furthermore,*

$$\|\rho\|_r^r \leq \|\rho_0\|_r^r + C(\alpha, d, \nu, \|\rho_0\|_1, r)(\|\rho_0\|_r^r)^{\frac{q+\epsilon_0-1}{\epsilon_0}\frac{q(r-q+1)}{q(r-1)+1}}, \quad q < r < +\infty,$$

where ϵ_0 satisfies

$$\frac{4\nu}{(q+\epsilon_0)S_{\alpha,d}^2} - \|\rho_0\|_q = \frac{\zeta}{2}.$$

Proof. Recall in (21) we have already got
$$\frac{d}{dt}\|\rho\|_{r_0}^{r_0}+S_{\alpha,d}^2(r_0-1)\frac{\zeta}{2}\|D^{\frac{\alpha}{2}}\rho^{\frac{r_0}{2}}\|_2^2\le 0, \tag{104}$$
with $r_0=q+\epsilon_0$ and ϵ_0 small enough satisfying
$$\frac{4\nu}{(q+\epsilon_0)S_{\alpha,d}^2}-\|\rho_0\|_q=\frac{\zeta}{2}.$$
Hence, it follows from (104) that $\frac{d}{dt}\|\rho\|_{r_0}^{r_0}\le 0$, which leads to the uniform estimate for $\|\rho\|_{r_0}$:
$$\|\rho\|_{r_0}\le\|\rho_0\|_{r_0}. \tag{105}$$
For $r>r_0$ we compute as before by using Young's inequality
$$\begin{aligned}&\frac{d}{dt}\|\rho\|_r^r+4\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2\le(r-1)\|\rho\|_{r+1}^{r+1}\\ \le&(r-1)S_{\alpha,d}^{\frac{2\theta(r+1)}{r}}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^{\frac{2\theta(r+1)}{r}}\|\rho\|_{r_0}^{(1-\theta)(r+1)}\\ \le&2\frac{(r-1)\nu}{r}\|D^{\frac{\alpha}{2}}\rho^{\frac{r}{2}}\|_2^2+C(r,r_0,\alpha,d,\nu)(\|\rho\|_{r_0}^{r_0})^{\frac{1+r-q}{r_0-q}},\end{aligned} \tag{106}$$
where $\theta=\frac{qr[r_0-(r+1)]}{(r+1)[r_0(q-1)-qr]}$ satisfying $\frac{2\theta(r+1)}{r}<1$ for $r_0>q$. Collecting (105) yields
$$\begin{aligned}\frac{d}{dt}\|\rho\|_r^r\quad\le\quad&-\frac{2(r-1)\nu}{rS_{\alpha,d}^2\|\rho_0\|_1^{\frac{r}{q(r-1)}}}(\|\rho\|_r^r)^{1+\frac{1}{q(r-1)}}\\ &+C(r,r_0,\alpha,d,\nu)(\|\rho_0\|_{r_0}^{r_0})^{\delta},\end{aligned} \tag{107}$$
with $\delta=\frac{1+r-q}{r_0-q}$. Solving the above ODE inequality we have
$$\begin{aligned}\|\rho\|_r^r\quad\le\quad&\max\left\{\|\rho_0\|_r^r,C(\alpha,d,\nu,\|\rho_0\|_1,r)(\|\rho_0\|_{r_0}^{r_0})^{\frac{q\delta(r-1)}{q(r-1)+1}}\right\}\\ \le\quad&\|\rho_0\|_r^r+C(\alpha,d,\nu,\|\rho_0\|_1,r)(\|\rho_0\|_r^r)^{\frac{r_0-1}{r_0-q}\frac{q(r-q+1)}{q(r-1)+1}},\end{aligned} \tag{108}$$
here we have used the interpolation inequality in the second inequality. Recall $r_0=q+\epsilon_0$, thus the theorem has been proved. □

Now, we will get the uniform estimate in $L^\infty(\mathbb{R}_+,L^\infty(\mathbb{R}^d))$ of the solution by utilizing a bootstrap iterative technique [5] in the following theorem.

Theorem B.2. *Assume initial density ρ_0 satisfies Assumption 1, then the weak solution ρ of (1) has the uniform estimate in $L^\infty\left(\mathbb{R}_+,L^\infty(\mathbb{R}^d)\right)$, i.e. for any $t>0$*
$$\|\rho\|_\infty\le C(\alpha,d,\nu,A_0),$$
where $A_0=\max\{1,\|\rho_0\|_1,\|\rho_0\|_\infty\}$.

Proof. Define $p_k:=2^k+q+1$ with $k\ge 0$. For $k=0$, $p_0=q+2>q$, from Lemma B.1, we have
$$\|\rho\|_{p_0}^{p_0}\le\|\rho_0\|_{p_0}^{p_0}+C(\alpha,d,\nu,\|\rho_0\|_1,q)(\|\rho_0\|_{p_0}^{p_0})^{\frac{q+\epsilon_0-1}{\epsilon_0}\frac{3q}{q(q+1)+1}}\le C(\alpha,d,\nu,A_0).$$
For $k\ge 1$, take $p_k\rho^{p_k-1}$ as a test function in the first equation of (1), we have
$$\begin{aligned}\frac{d}{dt}\|\rho\|_{p_k}^{p_k}\quad\le\quad&-4\frac{(p_k-1)\nu}{p_k}\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^2+(p_k-1)\|\rho\|_{p_k+1}^{p_k+1}\\ \le\quad&-2C_{p_k}\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^2+p_k\|\rho\|_{p_k+1}^{p_k+1},\end{aligned} \tag{109}$$
where $0<C_{p_k}\le 4\frac{(p_k-1)\nu}{p_k}$ is a fixed constant.

Now we will focus on estimating the last term $\|\rho\|_{p_k+1}^{p_k+1}$

$$\begin{aligned} \|\rho\|_{p_k+1}^{p_k+1} &= \|\rho^{\frac{p_k}{2}}\|_{\frac{2(p_k+1)}{p_k}}^{\frac{2(p_k+1)}{p_k}} \leq \|\rho^{\frac{p_k}{2}}\|_{\frac{2d}{d-\alpha}}^{\theta\frac{2(p_k+1)}{p_k}} \|\rho^{\frac{p_k}{2}}\|_r^{(1-\theta)\frac{2(p_k+1)}{p_k}} \\ &\leq S_{\alpha,d}^{\theta\frac{2(p_k+1)}{p_k}} \|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^{\theta\frac{2(p_k+1)}{p_k}} \|\rho^{\frac{p_k}{2}}\|_r^{(1-\theta)\frac{2(p_k+1)}{p_k}}, \end{aligned} \tag{110}$$

with

$$\frac{p_k}{2}r = p_{k-1}, \quad \theta = \frac{\frac{1}{r} - \frac{p_k}{2(p_k+1)}}{\frac{1}{r} - \frac{d-\alpha}{2d}},$$

where in the first inequality of (110) the interpolation inequality has been used and Sobolev inequality (14) has been used in the second inequality.

The Young's inequality tells that

$$\begin{aligned} \frac{d}{dt}\|\rho\|_{p_k}^{p_k} &\leq -2C_{p_k}\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^2 + \sigma_1\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^{q_1\theta\frac{2(p_k+1)}{p_k}} \\ &\quad + C(\sigma_1)(p_k)^{q_2} S_{\alpha,d}^{q_2\theta\frac{2(p_k+1)}{p_k}} \|\rho^{\frac{p_k}{2}}\|_r^{q_2(1-\theta)\frac{2(p_k+1)}{p_k}}, \end{aligned} \tag{111}$$

where $C(\sigma_1) = (\sigma_1 q_1)^{-q_2/q_1} q_2^{-1}$, $q_1 = \frac{p_k}{\theta(p_k+1)}$, i.e. $q_1\theta\frac{2(p_k+1)}{p_k} = 2$, and

$$q_2 = \frac{p_k}{p_k - \theta(p_k+1)} = \frac{2^{k-1} + \frac{\alpha}{d}p_{k-1}}{\frac{\alpha}{d}p_{k-1} - 1} \leq d+1.$$

By taking $\sigma_1 = C_{p_k}$ in (111), we get

$$\frac{d}{dt}\|\rho\|_{p_k}^{p_k} \leq -C_{p_k}\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^2 + C(\sigma_1)(p_k)^{q_2} S_{\alpha,d}^{q_2\theta\frac{2(p_k+1)}{p_k}} (\|\rho\|_{p_{k-1}}^{p_{k-1}})^{\eta_1}, \tag{112}$$

where

$$\eta_1 = \frac{q_2(1-\theta)(p_k+1)}{p_{k-1}} = \frac{\frac{\alpha}{d}(p_k+1) - 1}{\frac{\alpha}{d}p_{k-1} - 1} \leq 2.$$

On the other hand,

$$\|\rho\|_{p_k}^{p_k} = \|\rho^{\frac{p_k}{2}}\|_2^2 \leq S_{\alpha,d}^{2\theta_1}\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^{2\theta_1}\|\rho^{\frac{p_k}{2}}\|_r^{2(1-\theta_1)}, \tag{113}$$

where r is the same as before, and

$$\theta_1 = \frac{\frac{1}{r} - \frac{1}{2}}{\frac{1}{r} - \frac{d-\alpha}{2d}}.$$

Similar to (111), we have

$$\|\rho\|_{p_k}^{p_k} \leq \sigma_1\|D^{\frac{\alpha}{2}}\rho^{\frac{p_k}{2}}\|_2^2 + \bar{C}(\sigma_1)S_{\alpha,d}^{2\theta_1 l_2}\|\rho\|_{p_{k-1}}^{p_k(1-\theta_1)l_2}, \tag{114}$$

where $\bar{C}(\sigma_1) = (\sigma_1 l_1)^{-l_2/l_1} l_2^{-1}$, $l_1 = \frac{1}{\theta_1}$, and $l_2 = \frac{1}{1-\theta_1}$. Hence from (112) and (114), we deduce

$$\frac{d}{dt}\|\rho\|_{p_k}^{p_k} \leq -\|\rho\|_{p_k}^{p_k} + C(\sigma_1)(p_k)^{q_2} S_{\alpha,d}^{q_2\theta\frac{2(p_k+1)}{p_k}} (\|\rho\|_{p_{k-1}}^{p_{k-1}})^{\eta_1} + \bar{C}(\sigma_1)S_{\alpha,d}^{2\theta_1 l_2}(\|\rho\|_{p_{k-1}}^{p_{k-1}})^{\eta_2}, \tag{115}$$

where $\eta_2 = \frac{p_k(1-\theta_1)l_2}{p_{k-1}} = \frac{p_k}{p_{k-1}} \leq 2$. Define

$$C_1(p_k) := C(\sigma_1)S_{\alpha,d}^{q_2\theta\frac{2(p_k+1)}{p_k}}; \quad C_2(p_k) := \bar{C}(\sigma_1)S_{\alpha,d}^{2\theta_1 l_2}.$$

It is easy to know that $C_1(p_k)$ and $C_2(p_k)$ is uniformly bounded for any $k \geq 1$. So, we let $C(d,\alpha,\nu) > 1$ be a common upper bound of $C_1(p_k)$ and $C_2(p_k)$, we obtain the following inequality

$$\frac{d}{dt}\|\rho\|_{p_k}^{p_k} \leq -\|\rho\|_{p_k}^{p_k} + C(d,\nu,\alpha)p_k^{q_2}\left((\|\rho\|_{p_{k-1}}^{p_{k-1}})^{\eta_1} + (\|\rho\|_{p_{k-1}}^{p_{k-1}})^{\eta_2}\right). \tag{116}$$

Let $y_k(t) := \|\rho\|_{p_k}^{p_k}$, solving the ODE inequality (116), we get

$$\begin{aligned}(e^t y_k(t))' &\leq C(\nu,d,\alpha)p_k^{q_2}(y_{k-1}^{\eta_1} + y_{k-1}^{\eta_2})e^t \\ &\leq 2C(\nu,d,\alpha)4^{d+1}2^{k(d+1)}\max\left\{1, \sup_{t\geq 0} y_{k-1}^2(t)\right\}e^t, \end{aligned}\tag{117}$$

where the last inequality used $1 < q_2 \leq d+1$. Let $a_k := 2C(\nu,d,\alpha)4^{d+1}2^{k(d+1)} > 1$ and integrate (117), then one has

$$y_k(t) \leq a_k \max\left\{1, \sup_{t\geq 0} y_{k-1}^2(t)\right\}(1-e^{-t}) + y_k(0)e^{-t}. \tag{118}$$

Notice that $y_k(0) = \|\rho_0\|_{p_k}^{p_k} \leq \|\rho_0\|_1\|\rho_0\|_\infty^{p_k-1}$, so we have

$$\max_{k\geq 1}\{y_k(0), 1\} \leq A^{p_k},$$

where constant $A > 1$ is independent of k but depends on $\|\rho_0\|_1, \|\rho_0\|_\infty$. Hence it follows from (118) that

$$y_k(t) \leq a_k \max\left\{\sup_{t\geq 0} y_{k-1}^2(t), A^{p_k}\right\}$$

After some iterative steps, we have

$$\begin{aligned} y_k(t) &\leq a_k(a_{k-1})^2(a_{k-2})^{2^2}\cdots a_1^{2^{k-1}}\max\left\{\sup_{t\geq 0} y_0^{2^k}(t), \sum_{i=0}^{k-1} A^{p_{k-i}2^i}\right\} \\ &\leq (2C(\nu,d,\alpha)4^{d+1})^{2^k-1}(2^{d+1})^{2^{k+1}-k-2}\max\left\{\sup_{t\geq 0} y_0^{2^k}(t), \sum_{i=0}^{k-1} A^{p_{k-i}2^i}\right\}. \end{aligned}$$

Since $A^{p_{k-i}2^i} \leq \tilde{A}^{p_k}$, one concludes that

$$\|\rho\|_{p_k}^{p_k} \leq (2C(\nu,d,\alpha)4^{d+1})^{2^k-1}(2^{d+1})^{2^{k+1}-k-2}\max\left\{\sup_{t\geq 0} y_0^{2^k}(t), k\tilde{A}^{p_k}\right\},$$

where $\tilde{A} > 1$ is constant independent of k but depends on $\|\rho_0\|_1, \|\rho_0\|_\infty$.

Taking the power $\frac{1}{p_k}$ to above inequality, then

$$\|\rho\|_{p_k} \leq 2C(\nu,d,\alpha)4^{d+1}2^{2(d+1)}\max\left\{\sup_{t\geq 0} y_0(t), k^{1/p_k}\tilde{A}\right\}. \tag{119}$$

Recall that $y_0(t) = \|\rho\|_{p_0}^{p_0} \leq C(\alpha,d,\nu,A_0)$, then the estimate is obtained by passing to the limit $k \to \infty$ in (119),

$$\|\rho\|_\infty \leq C(\alpha,d,\nu,A_0). \tag{120}$$

□

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