

RIGOROUS JUSTIFICATION OF THE FOKKER–PLANCK EQUATIONS OF NEURAL NETWORKS BASED ON AN ITERATION PERSPECTIVE*

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Abstract. In this work, the primary goal is to establish a rigorous connection between the Fokker–Planck equation of neural networks and its microscopic model: the diffusion-jump stochastic process that captures the mean-field behavior of collections of neurons in the integrate-and-fire model. The proof is based on a novel iteration scheme: with an auxiliary random variable counting the firing events, both the density function of the stochastic process and the solution of the PDE problem admit series representations, and thus the difficulty in verifying the link between the density function and the PDE solution in each subproblem is greatly mitigated. The iteration approach provides a generic framework for integrating the probability approach with PDE techniques, with which we prove that the density function of the diffusion-jump stochastic process is indeed the classical solution of the Fokker–Planck equation with a unique flux-shift structure.

Key words. Fokker–Planck equation, integrate-and-fire model, diffusion-jump stochastic process, mathematical model of neural networks

AMS subject classifications. 35A09, 35Q84, 60H30, 92B20

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1. Introduction. Although various models have emerged in neuroscience [23, 30, 36, 45], which is currently one of the most active disciplines, the level of mathematical rigor in understanding the rational connections between these models is usually formal or empirical. In the case of modeling the dynamics of a large collection of interacting neurons, the integrate-and-fire model for the potential through the neuron cell membrane, which dates back to [30], has received great attention. In this model, the collective behavior of neural networks can be predicted by the stochastic process of a single neuron [3, 4, 14, 16, 25, 29, 31, 34, 35, 41, 42, 44], where the influence of the network is given by an average synaptic input by the mean-field approximation [16, 29, 41, 44]. The time evolution of the probability density function (abbreviated p.d.f.) of the potential voltage is governed by a Fokker–Planck equation on the half space with an unusual structure in which it constantly shifts the boundary flux to an interior point. This equation has been utilized by neuroscientists in exploring the macroscopic behavior of neural networks and, in the past decade, by mathematicians in investigations of the unique solutions structures [5, 6, 7, 8, 9, 10, 28, 37]; these

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studies, in turn, have enriched the scientific interpretation of the integrate-and-fire model.

In this paper, we focus on the single neuron approximation of the celebrated noisy leaky integrate-and-fire (LIF) model for neural networks, where the state variable X_t denotes the membrane potential of a typical neuron within the network. In the LIF model, when the synaptic input of the network (denoted by $I(t)$) vanishes, the membrane potential relaxes to its resting potential V_L , and in the single neuron approximation, the synaptic input $I(t)$, which itself is another stochastic process, is replaced by a continuous-in-time counterpart $I_c(t)$ (see, e.g., [3, 4, 31, 37, 41, 42]), which takes the drift-diffusion form

$$(1) \quad I dt \approx I_c dt = \mu_c dt + \sigma_c dB_t.$$

Here, B_t is the standard Brownian motion, and in principle the two processes $I_c(t)$ and $I(t)$ have the same mean and variance. Thus between the firing events, the evolution of the membrane potential is given by the following stochastic differential equation (SDE):

$$(2) \quad dX_t = (-X_t + V_L + \mu_c) dt + \sigma_c dB_t.$$

The next key component of the model is the firing-and-resetting mechanism: whenever the membrane voltage X_t reaches a threshold value, called the threshold or firing voltage V_F , it is immediately relaxed to a reset value V_R , where $V_R < V_F$. The reader may refer to [41] for a thorough introduction to this subject. It is worth mentioning that numerous mathematical aspects of the LIF model and its variants have been studied (see, e.g., [16, 17, 29, 38, 41, 44]) in addition to its enormous significance in neuroscience.

There has been growing interest in studying the partial differential equation (PDE) problem for the dynamics of the p.d.f. with which the stochastic process X_t is associated [12, 13, 16, 17, 29]. We denote the density of the distribution of neuron potential voltage at time $t \geq 0$ by $f(x, t)$, $x \in (-\infty, V_F]$. At least from a heuristic viewpoint, it is widely accepted that the p.d.f. $f(x, t)$ satisfies the following Fokker–Planck equation on the half line with a singular source term:

$$(3) \quad \frac{\partial f}{\partial t}(x, t) + \frac{\partial}{\partial x}[hf(x, t)] - a \frac{\partial^2 f}{\partial x^2}(x, t) = N(t)\delta(x - V_R), \quad x \in (-\infty, V_F), \quad t > 0,$$

where $N(t)$ denotes the mean firing rate. By formal calculations via Itô’s calculus, we obtain the drift velocity $h = -x + V_L + \mu_c$ and diffusion coefficient $a = \sigma_c^2/2$.

The firing-and-reset mechanism in the stochastic process has led to multiple consequences in the PDE model. First, since the neurons at the threshold voltage have instantaneous discharges where the density is supposed to vanish, and due to the noisy leaky terms, we consider the following Dirichlet boundary conditions:

$$(4) \quad f(V_F, t) = 0, \quad f(-\infty, t) = 0 \quad \forall t \geq 0.$$

Second, due to the Dirichlet boundary condition at $x = V_F$, there is a time-dependent boundary flux escaping the domain, and a Dirac delta source term is added to the reset location $x = V_R$ to compensate for the loss. Noting that (3) is the evolution of a p.d.f., we therefore see that for all $t \geq 0$,

$$\int_{-\infty}^{V_F} f(x, t) dx = \int_{-\infty}^{V_F} f_{\text{in}}(x) dx = 1.$$

The conservation of mass and the boundary condition characterize the magnitude of the mean firing rate

$$(5) \quad N(t) := -a \frac{\partial f}{\partial x}(V_F, t) \geq 0.$$

The PDE problem is completed by an appropriate initial condition $f(x, 0) = f_{\text{in}}(x)$.

Third, the firing events generate currents that propagate within the neural networks, which are incorporated into this PDE model by expressing the drift velocity h and the diffusion coefficient a as functions of the mean firing rate $N(t)$. For example, it is assumed in quite a few works (see, e.g., [5, 6, 10, 28]) that

$$h(x, N) = -x + bN, \quad a(N) = a_0 + a_1N,$$

where $b, a_0 > 0$ and $a_1 \geq 0$ are some modeling parameters. When $b > 0$, the neural network is excitatory on average, and when $b < 0$ the network is inhibitory. In particular, when $b = 0$ and $a_1 = 0$, the PDE problem becomes linear, but the flux-shift structure persists.

We remark that adding the delta source term to the right-hand side of (3) is equivalent to setting the equation on $(-\infty, V_R) \cup (V_R, V_F)$ and imposing the following conditions:

$$f(V_R^-, t) = f(V_R^+, t), \quad a \frac{\partial}{\partial x} f(V_R^-, t) - a \frac{\partial}{\partial x} f(V_R^+, t) = N(t) \quad \forall t \geq 0.$$

The equivalence can be checked by direct integration by parts, and we use this form throughout the rest of the paper.

Due to the unique structure of the PDE problem, most conventional analysis methods do not directly apply, and many recent works are devoted to investigating solution properties of such a model and its various modifications, including finite-time blowup of weak solutions, multiplicity of steady solutions, the relative entropy estimate, existence of classical solutions, structure-preserving numerical approximation, etc. (see, e.g., [5, 6, 7, 9, 10, 28] and the references therein). For the stochastic process (2), as the jumping time for X_t is determined by its hitting time, classical Itô calculus is not directly applicable.

The primary goal of this paper is to show the rigorous derivation of the Fokker–Planck equation from the stochastic process. More specifically, we investigate whether, and in which sense, the p.d.f. $f(x, t)$ of the stochastic process X_t satisfies the PDE model. We choose the model parameters as follows:

$$(6) \quad V_L = V_R = 0, \quad \mu_c = 0, \quad \sigma_c = \sqrt{2}, \quad \text{and} \quad V_F = 1.$$

Let the distribution of X_0 be denoted by ν , which is a probability measure compactly supported on $(-\infty, 1)$, and let $f_{\text{in}}(x)$ denote the density function of ν . Then $X_t \in (-\infty, 1)$ is a stochastic process whose trajectory is *càdlàg* in time, and it evolves as an Ornstein–Uhlenbeck (OU) process,

$$(7) \quad dX_t = -X_{t-} dt + \sqrt{2} dB_t,$$

until it hits 1. Whenever X_t hits 1 at time t , it immediately jumps to 0, i.e.,

$$(8) \quad \text{if } X_{t-} = 1, \quad X_t = 0.$$

Then we restart the OU-like evolution independent of the past. We remark that (7) and (8) serve as a formal definition of the diffusion-jump process only for heuristic purposes, and the rigorous definition shall be presented in section 2.2. We aim to show for any fixed $T > 0$ that the associated density function $f(x, t)$ is indeed a classical solution to the PDE problem

$$(9) \quad \begin{cases} \frac{\partial f}{\partial t} - \frac{\partial}{\partial x}(xf) - \frac{\partial^2 f}{\partial x^2} = 0, & x \in (-\infty, 0) \cup (0, 1), t \in (0, T], \\ f(0^-, t) = f(0^+, t), & \frac{\partial}{\partial x}f(0^-, t) - \frac{\partial}{\partial x}f(0^+, t) = -\frac{\partial}{\partial x}f(1^-, t), & t \in (0, T], \\ f(-\infty, t) = 0, & f(1, t) = 0, & t \in [0, T], \\ f(x, 0) = f_{\text{in}}(x), & x \in (-\infty, 1). \end{cases}$$

The processes of such types (7) and (8) were first introduced by Feller [19, 20] (in terms of transition semigroups). In particular, [20] presents the Fokker-Planck equation of such processes (dubbed “elementary return processes” there) in weak form, the proof of which is based on a Markov semigroup argument in [19]. See Theorem 9 of [20] for details. Such processes have also been studied in later works such as [2, 24, 38, 39, 40, 43]. More specifically, in [1, 2, 38, 39], the authors are concerned with spectral properties of the generator of the stochastic process or related models and have shown exponential convergences in time towards stationary distribution. In particular, [38] studied a neuronal firing model driven by a Wiener process and computed the distribution of the first passage time. In the works [40, 43], the authors made more relaxed or modified assumptions on the stochastic process than those in [24] and proved the existence of pathwise solution of such processes in a generalized sense.

Following in the spirit of the pioneering work of Feller [20], the focus of this paper is to rigorously establish a bridge between the density functions of such processes and the classical solutions of the Fokker-Planck equations to be specified as in (9). From a technical perspective, there are no mathematical tools available for linking the boundary condition at the firing voltage and the jump condition at the reset voltage (or, equivalently, the singular delta source term) of the PDE model to a stochastic model for a single neuron model. In [5, 10], some heuristic arguments are provided to connect $N(t)$ to the rate of change of the expectation of the number of firing events, which is related to the synchronization behavior of the neural networks, whereas such an interpretation is not applicable for a single neuron model. In this paper, we rigorously prove that for a single neuron, the mean firing rate $N(t) = \sum_{n=1}^{\infty} f_{T_n}(t)$ where f_{T_n} stands for the p.d.f. of the n th jumping time of X_t .

The key strategy of our proofs is based on an iterated scheme: with the introduction of an auxiliary random variable counting the number of firing events, the p.d.f. of potential voltage $f(x, t)$ allows for a decomposition as a summation of sub-density functions $\{f_n(x, t)\}_{n=0}^{\infty}$. Each subdensity naturally links to a less singular sub-PDE problem, and all the sub-PDE problems are connected successively by iteration because the escaping boundary flux of $f_n(x, t)$ serves as the singular source for $f_{n+1}(x, t)$. Among all the iterations, the first step from f_0 to f_1 exhibits the strongest singularity at the source of the flux, and thus this turns out to be the major technical difficulty in our proof. In order to tackle this obstacle, elaborated estimates on the regularities of f_0 have to be established. The first sub-PDE problem corresponds to the stochastic process killed at the first hitting time, and there is a vast literature [18, 26, 27, 32, 33] concerned with the stochastic processes with no reset for the killed

particles. In [15, 17], the authors consider the process with firing-and-resetting as in this paper and have established connections between the subdensity function and the PDE solution. They have proved that $f_0(t, x)$ is continuous in (t, x) and continuously differentiable in x on $(0, T] \times (-\infty, 1]$ and admits Sobolev derivatives of order 1 in t and of order 2 in x on any compact subset of $(0, T] \times (-\infty, 1)$. However, these results are not strong enough to guarantee the existence of the classical solution to the whole problem (9). In fact, by analyzing the Green's function for the parabolic equation on the half space, we get estimates for classical derivatives and high order regularity for t in Propositions 3.1 and 3.2, which is essential for the iteration from f_0 to f_1 . In addition, all the desired smoothness properties are maintained by the iteration scheme, and thanks to the decomposition, rigorous justification of the jump condition for each sub-PDE problem becomes tractable. Finally, with the exponential convergence of decomposition, we can pass to the limit and conclude with the preserved properties on the original problem. This iteration scheme is inspired by the renewal nature of the stochastic process, which shares the spirit of Feller's original work [20] and provides a platform on which to combine techniques from both probability theory and differential equations.

It is worth noting that in our first attempt at studying rigorous justification of the Fokker–Planck equations of neural networks from the stochastic model, we have only obtained results for the linear cases. In particular, we could not yet incorporate the dependence on the mean firing rate into the drift velocity or the diffusion coefficient, but we shall investigate those directions in the future.

The rest of the paper is outlined as follows. In section 2, we summarize the main results of this work, give a precise definition of the stochastic process, and lay out the iterated scheme. In section 3, we show that the density function of the stochastic process is indeed the mild solution of the PDE problem with certain smoothing properties, and we make a few remarks on the implications in the weak solution. For the rest of this work, we use C , C_0 , C_k , and C_T to denote generic constants.

2. Preliminaries and main results. In this section, we present our main results in detail and also provide some technical preparations for the proofs, including construction of the stochastic process, which serves as the precise definition, and elaboration of the iterated strategy, accompanied by some elementary estimates.

2.1. Main results. The stochastic process X_t has been formally defined in (7) and (8), but the rigorous construction of such a process can be found in (18) of section 2.2.

We first suppose that the process X_t starts from 0, i.e., the distribution of X_0 is $f_{\text{in}}(x) = \delta(x)$. We state the first main result in the following.

THEOREM 1. *The process X_t as in (18) that starts from 0 has a continuously evolving p.d.f. denoted by $f(x, t)$. $f(x, t)$ is a solution of (9) in the time interval $(0, T]$ for any given $0 < T < +\infty$ and with initial condition $\delta(x)$ in the following sense:*

- (i) $N(t) := -\frac{\partial}{\partial x} f(1^-, t)$ is a continuous function for $t \in [0, T]$.
- (ii) f is continuous in the region $\{(x, t) : -\infty < x \leq 1, t \in (0, T]\}$.
- (iii) f_{xx} and f_t are continuous in the region $\{(x, t) : x \in (-\infty, 0) \cup (0, 1), t \in (0, T]\}$.
- (iv) $f_x(0^-, t)$, $f_x(0^+, t)$ are well defined for $t \in (0, T]$.
- (v) For $t \in (0, T]$, $f_x(x, t) \rightarrow 0$ when $x \rightarrow -\infty$.

- (vi) Equations (9) are satisfied with $f(x, 0) = \delta(x)$ in the following sense: for any $\varphi \in C_b(-\infty, 1)$,

$$(10) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^1 \varphi(x) f(x, t) dx = \varphi(0).$$

The proof of Theorem 1 is shown in section 3 and relies on an iterative approach. In fact, we decompose both the probability density of the stochastic process and the solution to (9) into series and show that there is a one-to-one correspondence between the two series representations.

Next, we let the process start from any fixed $y < 1$; this time we use $f^y(x, t)$ to denote the p.d.f. of the process X_t in (18) starting from y , and now the distribution of X_0 is $f_{in}(x) = \delta(x - y)$. With the same method, we immediately get the following corollary.

COROLLARY 2.1. *For any fixed $y \in (-\infty, 1)$, the process X_t as in (18) that starts from y has a continuously evolving p.d.f. denoted by $f^y(x, t)$. $f^y(x, t)$ is a solution of (9) in the time interval $(0, T]$ for any given $0 < T < +\infty$ and with initial condition $\delta(x - y)$ in the following sense:*

- (i) $N^y(t)$ is a continuous function for $t \in [0, T]$.
- (ii) f^y is continuous in the region $\{(x, t) : -\infty < x \leq 1, t \in (0, T]\}$.
- (iii) $\partial_{xx} f^y$ and $\partial_t f^y$ are continuous in the region $\{(x, t) : x \in (-\infty, 0) \cup (0, 1), t \in (0, T]\}$.
- (iv) $\partial_x f^y(0^-, t)$, $\partial_x f^y(0^+, t)$ are well defined for $t \in (0, T]$.
- (v) For $t \in (0, T]$, $\partial_x f^y(x, t) \rightarrow 0$ when $x \rightarrow -\infty$.
- (vi) Equations (9) are satisfied with $f(x, 0) = \delta(x - y)$ in the following sense: for any $\varphi \in C_b(-\infty, 1)$,

$$(11) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^1 \varphi(x) f^y(x, t) dx = \varphi(y).$$

Moreover, for any fixed $\varepsilon_0 > 0$, the continuity in (i), (ii), (iii) and the convergence in (v) and (vi) are uniform for $y \leq 1 - \varepsilon_0$.

The proof of Corollary 2.1 is the same as that of Theorem 1 and is thus skipped.

The initial condition of the Fokker-Planck equation (9) corresponds to the initial distribution of the stochastic process X_0 . We remark that in the above cases, the majority of arguments below are based on the initial condition of the process $X_0 = y$ for any $y < 1$, and the corresponding initial condition of the PDE problem becomes $f(x, 0) = \delta(x - y)$. Although the initial condition is a singular function, we have shown that the PDE has an instantaneous smoothing effect, while the solution coincides with the density function of the stochastic process. Since the problem is linear, the natural extension to general and proper initial conditions can be obtained by integration against the initial distribution (see, e.g., [15] for a precise discussion).

THEOREM 2. *Let ν be a cumulative distribution function (c.d.f.) whose p.d.f. $f_{in}(x) \in C_c(-\infty, 1)$. We assume that $f_{in}(x)$ is continuous and supported in $(-\infty, 1 - \varepsilon_0)$ for some $\varepsilon_0 > 0$. Then the process X_t as in (18) that starts from p.d.f. $f_{in}(x)$ has a continuously evolving p.d.f. denoted by $f^\nu(x, t)$, with*

$$(12) \quad f^\nu(x, t) = \int_{-\infty}^{1-\varepsilon_0} f^y(x, t) \nu(dy), \quad x \in (-\infty, 1], \quad t > 0,$$

and $f^\nu(x, t)$ is a classical solution of (9) in the time interval $(0, T]$ for any given $0 < T < +\infty$ with initial condition $f_{in}(x)$ in the following sense:

- (i) $N^\nu(t) := -\frac{\partial}{\partial x} f^\nu(1^-, t)$ is a continuous function for $t \in [0, T]$.
- (ii) f^ν is continuous in the region $\{(x, t) : -\infty < x \leq 1, t \in [0, T]\}$.
- (iii) $\partial_{xx} f^\nu$ and $\partial_x f^\nu$ are continuous in the region $\{(x, t) : x \in (-\infty, 0) \cup (0, 1), t \in [0, T]\}$.
- (iv) $\partial_x f^\nu(0^-, t)$, $\partial_x f^\nu(0^+, t)$ are well defined for $t \in [0, T]$.
- (v) For $t \in (0, T]$, $\partial_x f^\nu(x, t) \rightarrow 0$ when $x \rightarrow -\infty$.
- (vi) Equations (9) are satisfied with the L^2 convergence to the initial condition as $t \rightarrow 0^+$, i.e.,

$$(13) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^1 |f^\nu(x, t) - f_{in}(x)|^2 dx = 0.$$

A proof can be found at the end of section 3.1.

Remark 1. It is not clear yet how to get the uniform estimates near the boundary of the domain, and thus we suppose that the initial distribution is compactly supported on $(-\infty, 1)$. Actually, some recent work [27] concerning related models progressed towards more general assumptions, from compactly supported to $o(1-x)$ decay near 1, and more recently, $\mathcal{O}((1-x)^\beta)$ with $\beta \in (0, 1)$. Usually, the literature assumes $\mathcal{O}(1-x)$ decay near 1 (see, e.g., [11]) and in Theorem 1.1 of [18], this boundary decay is linked to short-term regularity of the solutions. Thus the hypothesis of a compactly supported initial condition has deep consequences on the smoothness of the solution in the short term.

2.2. Construction of the process. For the rest of this section, we shall present some preliminaries of the stochastic process. First, we should give the process X_t a precise definition in probability by following the construction of Gihman and Skorohod [24]. We emphasize that an additional process n_t is introduced to count the number of jumping events of a trajectory that have taken place before time t .

On a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we consider a sequence of independent OU processes,

$$\left\{ Y_t^{(n)} \right\}_{n=1}^{\infty},$$

with $Y_0^{(n)} = 0$ for all $n \geq 1$. Note that an OU process Y_t starting from initial value y_0 is an SDE with an almost surely (a.s.) pathwise continuous strong solution. That is,

$$(14) \quad Y_t = e^{-t} y_0 + \sqrt{2} \int_0^t e^{-(t-s)} dB_s$$

with a normal p.d.f.,

$$(15) \quad N(e^{-t} y_0, 1 - e^{-2t}).$$

For each $n \in \mathbb{N}$, $t \in [0, \infty]$, define the natural filtration

$$\mathcal{F}_t^{(n)} = \sigma \left(Y_s^{(n)} : s \in [0, t] \right).$$

In other words, $\mathcal{F}_t^{(n)}$ represents the information carried by the path of the n th copy of the OU process by time t . For all n , $\mathcal{F}_\infty^{(n)}$ are abbreviated as $\mathcal{F}^{(n)}$, which are easily seen to be jointly independent. Now define their filtration

$$\mathcal{G}_n = \sigma(\mathcal{F}^{(k)}, k \leq n), \quad \mathcal{G}^n = \sigma(\mathcal{F}^{(k)}, k \geq n)$$

with the convention $\mathcal{G}_\infty = \mathcal{G}$.

For each n , let

$$(16) \quad \tau_n = \inf \left\{ t \geq 0 : Y_t^{(n)} = 1 \right\} = \inf \left\{ t \geq 0 : \lim_{h \rightarrow t^-} Y_h^{(n)} = 1 \right\}$$

be the first time $Y_t^{(n)}$ hits 1, with the convention $\tau_0 = 0$. Moreover, for all $n \geq 0$ and $k \leq n$, define

$$(17) \quad T_n = \sum_{i=0}^n \tau_i, \quad T_{n,k} = \sum_{i=k+1}^n \tau_i.$$

By definition, τ_n is a stopping time with respect to the natural filtration $\{\mathcal{F}_t^{(n)}\}_{t \geq 0}$. Also, we have that $\{\tau_n\}_{n=1}^\infty$ is a sequence of independent and identically distributed random variables with strictly positive expectation. Thus by the law of large numbers, $(\sum_{i=1}^n \tau_i)/n \rightarrow E[\tau_1] > 0$ a.s., which implies that

$$\mathbf{P} \left(\sum_{i=k}^\infty \tau_i = \infty \forall k \geq 1 \right) = 1.$$

Particularly, we have $T_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Then within the almost sure event $A_0 = \{\sum_{i=k}^\infty \tau_i = \infty \text{ for all } k \geq 1\}$, we define (X_t, n_t) as follows: for any $k \geq 1$,

$$(18) \quad (X_t, n_t) = (Y_{t-T_{k-1}}^{(k)}, k-1)$$

on $[T_{k-1}, T_k)$. Thus T_k is interpreted as the k th jumping time associated with X_t .

By definition, we have constructed a piecewise continuous path on $[0, \infty)$ for each $\omega \in A_0$, and thus a mapping from A_0 to $(D[0, \infty) \times \mathbb{N}, \mathcal{D} \times \mathcal{N})$ is clearly measurable with respect to \mathcal{G} , where $D[0, \infty)$ is the space of càdlàg paths. Here \mathcal{D} is the smallest sigma field generated by all coordinate mappings, and \mathcal{N} is the trivial sigma field on \mathbb{N} . In the rest of this paper, we will use the construction above as the formal definition of (X_t, n_t) , which is the stochastic process of interest.

Similarly, we can define the process X_t that starts from $y < 1$ or starts from a distribution ν . We denote the probability measure of (X_t, n_t) by $\mathbf{P}^y(\cdot)$ and the expectation by $\mathbf{E}^y[\cdot]$. The meanings of $\mathbf{P}^\nu(\cdot)$ and $\mathbf{E}^\nu[\cdot]$ are analogous. Using F_{τ_k}/F_{T_k} to denote the c.d.f. of τ_k/T_k , we immediately see that for any k and t , $\mathbf{P}(\tau_k = t) \leq \mathbf{P}(Y_t^{(k)} = 1) = 0$. So F_{τ_k} and F_{T_k} are always continuous.

2.3. Properties of the process and the iterated approach. We derive some preliminary estimates for the process (X_t, n_t) , which manifest the solution properties and also motivate us to propose the iterated scheme.

It has been shown in [24] that the process X_t constructed above is always Markovian. Now we are ready to show the following “strong Markovian” result that allows us to later calculate the probability distribution of (X_t, n_t) in an iterative fashion: for each integer $k \geq 0$, define

$$(19) \quad F_k(x, t) = \mathbf{P}^0(X_t \leq x, n_t = k);$$

then we have the following proposition.

PROPOSITION 2.1. For any $x < 1$, $k \geq 1$, and $t > 0$,

$$(20) \quad F_k(x, t) = \mathbf{E}_0 \left[\mathbf{P} \left(Y_{t-T_k}^{(k+1)} \leq x, \tau_{k+1} > t - T_k \right) \mathbb{1}_{T_k < t} \right].$$

Thus,

$$(21) \quad F_k(x, t) = \int_0^t F_0(x, t-s) dF_{T_k}(s).$$

Proof. We only prove (20); (21) is obvious. First, note that $T_{k+1} = T_k + \tau_{k+1}$ and that

$$\{n_t = k\} = \{T_k \leq t, T_{k+1} > t\}.$$

By Fubini's formula,

$$\mathbf{P}^0(n_t = k) = \mathbf{E}_0 [\mathbf{P}(\tau_{k+1} > t - T_k) \mathbb{1}_{T_k < t}].$$

Thus it suffices to prove

$$\mathbf{P}^0(X_t > x, n_t = k) = \mathbf{E}_0 \left[\mathbf{P} \left(Y_{t-T_k}^{(k+1)} > x, \tau_{k+1} > t - T_k \right) \mathbb{1}_{T_k < t} \right].$$

Let $A = \{X_t > x, n_t = k\}$ be our event of interest. For any $n \geq 1$ and any $0 \leq i \leq 2^n - 1$, we define the interval

$$I_n^{(i)}(t) = (2^{-n}it, 2^{-n}(i+1)t].$$

Moreover, for any $s \in (0, t]$ and any n , one may define $\text{Id}(n, s)$ as the unique $i \leq 2^n - 1$ such that $s \in I_n^{(i)}(t)$. Now we define the event

$$A_n^{(i)} = \left\{ \inf_{s \in t - I_n^{(i)}(t)} Y_s^{(k+1)} > x, \tau_{k+1} > (1 - 2^{-n}i)t \right\} \cap \{T_k \in I_n^{(i)}(t)\}$$

and define $A_n = \cup_{i=0}^{2^n-1} A_n^{(i)}$. By definition, $A_n^{(i)} \subset A$ for every feasible n and i . Thus $\mathbf{P}(A_n) \leq \mathbf{P}(A)$. On the other hand, for any $\omega \in \bar{A} = \{X_t > x, n_t = k, T_k < t\}$, the continuity of the path in $Y^{(k+1)}$ guarantees that there has to be some $N < \infty$ such that for all $n \geq N$, $\omega \in A_n^{(\text{Id}(n, T_k(\omega)))}$, and thus $\mathbf{P}^0(A_n) \rightarrow \mathbf{P}^0(\bar{A}) = \mathbf{P}^0(A)$ as $n \rightarrow \infty$. The last equality follows from the fact that F_{T_k} is continuous.

Meanwhile, note that T_k is independent of $Y^{(k+1)}$. We have

$$\begin{aligned} \mathbf{P}^0(A_n) &= \sum_{i=0}^{2^n-1} \mathbf{P}^0 \left(T_k \in I_n^{(i)}(t) \right) \mathbf{P} \left(\inf_{s \in t - I_n^{(i)}(t)} Y_s^{(k+1)} > x, \tau_{k+1} > (1 - 2^{-n}i)t \right) \\ &= \mathbf{E}^0 \left[\mathbf{P} \left(\inf_{s \in t - I_n^{(\text{Id}(n, T_k))}(t)} Y_s^{(k+1)} > x, \tau_{k+1} > (1 - 2^{-n}\text{Id}(n, T_k))t \right) \mathbb{1}_{T_k < t} \right]. \end{aligned}$$

Now, noting that for any $0 < h < t$, one may similarly have from the continuity of $Y^{(k+1)}$

$$\begin{aligned} &\mathbf{P} \left(\inf_{s \in t - I_n^{(\text{Id}(n, T_k))}(t)} Y_s^{(k+1)} > x, \tau_{k+1} > (1 - 2^{-n}\text{Id}(n, h))t \right) \\ &\quad \rightarrow \mathbf{P} \left(Y_{t-h}^{(k+1)} > x, \tau_{k+1} > t - h \right), \end{aligned}$$

we have that (20) follows from monotone convergence. □

For any $t > 0$, we first consider the case where no jumps have been made by time t . Note that $F_0(x, t) = P(X_t \leq x, T_1 > t) = \mathbf{P}(Y_t^{(1)} \leq x, \tau_1 > t)$ for all $x \in (-\infty, 1)$. It is clear that $F_0(\cdot, t)$ induces a measure on $((-\infty, 1), \mathcal{B})$, which is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} . The assertion above can be seen from the facts that for any measurable A , $\mathbf{P}(Y_t^{(1)} \in A, \tau_1 > t) \leq \mathbf{P}(Y_t^{(1)} \in A)$ and that $Y_t^{(1)}$ is a continuous random variable. Here we also use $F_0(\cdot, t)$ to denote the corresponding measure on $((-\infty, 1), \mathcal{B})$ and let $f_0(x, t)$ be its density. Also, $p_{\text{ou}}(x, t)$ denotes the p.d.f. of $Y_t^{(1)}$. Thus we have

$$(22) \quad f_0(x, t) \leq p_{\text{ou}}(x, t) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp\left\{\frac{-x^2}{2(1 - e^{-2t})}\right\},$$

which, together with (15), derives

$$(23) \quad f_0(x, t) \leq \frac{1}{\sqrt{2\pi(1 - e^{-2t})}}.$$

LEMMA 2.1. $F_0(x, t)$ is a bivariate continuous function on $(-\infty, 1] \times (0, \infty)$. Moreover, for any bounded continuous function $\varphi(x)$,

$$(24) \quad \lim_{t \rightarrow 0^+} \mathbf{E}^0[\varphi(X_t) \mathbb{1}_{n_t=0}] = \lim_{t \rightarrow 0^+} \int_{-\infty}^1 \varphi(x) f_0(x, t) dx = \varphi(0).$$

Proof. In order to prove this lemma, one may first show that for any $(x, t) \in (-\infty, 1) \times (0, \infty)$, $F_0(\cdot, \cdot)$ is continuous at (x, t) on both directions.

The continuity on the direction of x is obvious since for all $x' > x$,

$$0 \leq F_0(x', t) - F_0(x, t) \leq \mathbf{P}\left(Y_t^{(1)} \in [x, x']\right) = \int_x^{x'} p_{\text{ou}}(y, t) dy,$$

and the last term goes to 0 as $x' \rightarrow x^+$.

Thus one may concentrate on proving continuity on the direction of t . Let Δ be the symmetric difference between events. One may first note that for any events $A = A_1 \cap A_2$ and $B = B_1 \cap B_2$,

$$(25) \quad \begin{aligned} A\Delta B &= (A_1 \cap A_2 \cap B_1^c) \cup (A_1 \cap A_2 \cap B_2^c) \cup (A_1^c \cap B_1 \cap B_2) \cup (A_2^c \cap B_1 \cap B_2) \\ &\subset (A_1 \cap B_1^c) \cup (A_2 \cap B_2^c) \cup (A_1^c \cap B_1) \cup (A_2^c \cap B_2) \\ &= (A_1 \Delta B_1) \cup (A_2 \Delta B_2). \end{aligned}$$

For any $t > 0$, any fixed x_0 , and any Δt sufficiently close to 0 (without loss of generality, one may assume $\Delta t > 0$),

$$\begin{aligned} F_0(x_0, t) &= \mathbf{P}(Y_t^{(1)} \leq x_0, \tau_1 > t), \\ F_0(x_0, t + \Delta t) &= \mathbf{P}(Y_{t+\Delta t}^{(1)} \leq x_0, \tau_1 > t + \Delta t). \end{aligned}$$

Now let $A_1 = \{Y_t^{(1)} \leq x_0\}$, $A_2 = \{\tau_1 > t\}$ and $B_1 = \{Y_{t+\Delta t}^{(1)} \leq x_0\}$, $B_2 = \{\tau_1 > t + \Delta t\}$. By (25) we have

$$\begin{aligned} &|F_0(x_0, t) - F_0(x_0, t + \Delta t)| \\ &\leq \mathbf{P}(A\Delta B) \leq \mathbf{P}(A_1 \Delta B_1) + \mathbf{P}(A_2 \Delta B_2) \\ &= \mathbf{P}\left(Y_t^{(1)} \leq x_0, Y_{t+\Delta t}^{(1)} > x_0\right) + \mathbf{P}\left(Y_t^{(1)} > x_0, Y_{t+\Delta t}^{(1)} \leq x_0\right) + \mathbf{P}(\tau_1 \in (t, t + \Delta t]) \\ &\leq \mathbf{P}\left(\exists s \in [t, t + \Delta t] \text{ such that } Y_s^{(1)} = x_0\right) + F_{\tau_1}(t + \Delta t) - F_{\tau_1}(t). \end{aligned}$$

Recalling that $F_{\tau_1}(\cdot)$ is continuous, we have

$$\lim_{\Delta t \rightarrow 0} F_{\tau_1}(t + \Delta t) - F_{\tau_1}(t) = 0.$$

At the same time, for any positive integer n , define event

$$\Delta_n = \left\{ \exists s \in [t, t + n^{-1}] \text{ such that } Y_s^{(1)} = x_0 \right\}.$$

Note that

$$\mathbf{P}(\Delta_n) \rightarrow \mathbf{P}(Y_t^{(1)} = x_0) = 0 \quad \text{as } n \rightarrow \infty.$$

We obtain the continuity of t .

Thus, one can show that $F_0(x, t)$ is binary continuous at (x, t) as follows: given $(x, t) \in (-\infty, 1) \times (0, +\infty)$ and any $\epsilon > 0$, $\exists 0 < \delta < \frac{t}{2}$ such that for any $|t' - t| \leq \delta$,

$$|F_0(x, t') - F_0(x, t)| < \frac{\epsilon}{2}.$$

Also, for any $s > \frac{t}{2}$ and any $|x' - x| \leq \delta$ (here, without loss of generality, we ask for $x < x'$),

$$|F_0(x', s) - F_0(x, s)| \leq \mathbf{P}(Y_s^{(1)} \in [x, x']) < \frac{\epsilon}{2}.$$

(The last inequality occurs because when $s < \frac{t}{2}$, the density of $Y_s^{(1)}$ can be bounded by a big enough constant C .)

Then for all $(x', t') \in (-\infty, 1) \times (0, \infty)$ such that $|t' - t| \leq \delta$, $|x' - x| \leq \delta$, we have

$$|F_0(x', t') - F_0(x, t)| \leq |F_0(x', t') - F_0(x, t')| + |F_0(x, t') - F_0(x, t)| < \epsilon.$$

Finally, we show that $F_0(x, t)$ is continuous at $x = 1$. It suffices to prove that for any $t_n \rightarrow t$ and $\varepsilon_n \rightarrow 0^+$, we have $\lim_{n \rightarrow \infty} F_0(1 - \varepsilon_n, t_n) = F_0(1, t) = \mathbf{P}(\tau_1 > t)$, i.e., $\lim_{n \rightarrow \infty} \mathbf{P}(X_{t_n} \leq 1 - \varepsilon_n, \tau_1 > t_n) = \mathbf{P}(\tau_1 > t)$, which is equivalent to

$$\lim_{n \rightarrow \infty} \mathbf{P}(X_{t_n} > 1 - \varepsilon_n, \tau_1 > t_n) = 0.$$

Setting event $A_n = \{X_{t_n} > 1 - \varepsilon_n, \tau_1 > t_n\}$, we have

$$\mathbf{P}(\cup_{m \geq n}^\infty A_m) \leq \mathbf{P}\left(\exists s \in \left[\max_{m \geq n} t_m, \max_{m \geq n} t_m\right] \text{ such that } X_s > 1 - \varepsilon_n, \tau_1 > \min_{m \geq n} t_m\right).$$

Note that $\limsup_{n \rightarrow \infty} \mathbf{P}(A_n) \leq \mathbf{P}(\limsup A_n) \leq \mathbf{P}(X_t \geq 1, \tau_1 \geq t) = 0$. Thus we get $\lim_{n \rightarrow \infty} \mathbf{P}(A_n) = 0$ and the desired result.

Finally, to prove (24) we recall that φ is a bounded and continuous function. Thus $|\varphi(x)| \leq M$ for all x , and for each $\varepsilon > 0$, there is a $0 < \delta < 1$ such that for all $x \in [-\delta, \delta]$, $|\varphi(x) - \varphi(0)| < \varepsilon$. So we have

$$|\mathbf{E}_0[\varphi(X_t) \mathbb{1}_{n_t=0}] - \varphi(0)| \leq \varepsilon + 2M \mathbf{P}\left(\max_{s \leq t} |Y_s^{(1)}| \geq \delta\right).$$

Now recalling (14), we have

$$(26) \quad |Y_t^{(1)}| \stackrel{d}{=} \left| \sqrt{2} \int_0^t e^{-(t-s)} dB_s \right| \stackrel{d}{\leq} \left| \sqrt{2} \int_0^t e^s dB_s \right|,$$

where the d means the probability distribution. Note that the right-hand side of (26) forms a martingale. One immediately has

$$\lim_{t \rightarrow 0^+} \mathbf{P} \left(\max_{s \leq t} |Y_s^{(1)}| \geq \delta \right) = 0$$

by Doob's inequality. Thus we have shown (24), and this completes the proof. \square

Remark 2. With Lemma 2.1, one may immediately have that $F(x_0, t)$ is a bounded and measurable function of $t \in [0, \infty)$.

Moreover, the following corollary follows directly from Proposition 2.1, Lemma 2.1, and a standard measure theory argument.

COROLLARY 2.2. *For any bounded measurable function f , any integer $k \geq 1$, and any $t > 0$, $\mathbf{E}[f(Y_t^{(1)}) \mathbb{1}_{\tau_1 > t}]$ is measurable with respect to t , and*

$$(27) \quad \mathbf{E}^0 [f(X_t) \mathbb{1}_{n_t = k}] = \mathbf{E}^0 \left[\mathbf{E} \left[f(Y_{t-T_k}^{(k+1)}) \mathbb{1}_{\tau_{k+1} > t - T_k} \right] \mathbb{1}_{T_k < t} \right].$$

Note that

$$(28) \quad F_{\tau_1}(t) = 1 - P(\tau_1 > t) = 1 - F_0(1, t) = 1 - \int_{-\infty}^1 f_0(x, t) dx$$

and

$$(29) \quad F_{T_n} = F_{\tau_1} * F_{\tau_2} * \dots * F_{\tau_n}.$$

Moreover, for each n , $F_n(\cdot, t)$ is absolutely continuous, and we let $f_n(x, t)$ denote its density.

In the rest of this section, we use Proposition 2.1 and a renewal argument similar to that in [20] to calculate the distribution of X_t . First one has the following lemma.

LEMMA 2.2. *For all $n \geq 1$, $t > 0$, and $x < 1$,*

$$(30) \quad F_n(x, t) = \int_0^t F_{n-1}(x, t-s) dF_{\tau_1}(s).$$

Moreover, $F_n(x, t)$ is also bivariate continuous on $(-\infty, 1] \times (0, \infty)$.

Proof. Suppose the lemma holds for $n - 1 \geq 0$, which has been shown true for $n = 1$. By Proposition 2.1, Lemma 2.1, and Fubini's formula on the independent random variables T_{n-1} and τ_n ,

$$\begin{aligned} F_n(x, t) &= \mathbf{P}(X_t \leq x, n_t = n) = \mathbf{E}_0 \left[\mathbf{P} \left(Y_{t-T_n}^{(n+1)} \leq x, \tau_{n+1} > t - T_n \right) \mathbb{1}_{T_n < t} \right] \\ &= \mathbf{E}_0 [F_0(x, t - T_n) \mathbb{1}_{T_n < t}] = \mathbf{E}_0 [F_0(x, t - T_{n-1} - \tau_n) \mathbb{1}_{T_{n-1} + \tau_n < t}] \\ &= \int_0^t \int_0^{t-s} F_0(x, t - s - h) dF_{T_{n-1}}(h) dF_{\tau_1}(s) \\ &= \int_0^t F_{n-1}(x, t - s) dF_{\tau_1}(s), \end{aligned}$$

and thus we have (30). With (30), for any $t_0 > 0$ and $x_0 < 1$, the continuity of $F_n(x, t)$ at (x_0, t_0) with respect to t can be shown as follows: for any $\varepsilon > 0$, by the continuity of $F_{\tau_1}(t)$, there is a $\delta_1 \in (0, t_0)$ such that

$$F_{\tau_1}(t_0 + \delta_1) - F_{\tau_1}(t_0 - \delta_1) < \varepsilon.$$

Now note that $F_{n-1}(x_0, t)$ is continuous on $(0, \infty)$ and thus uniformly continuous on $[\delta_1/2, t_0 + \delta_1]$. Thus there is a $\delta_2 > 0$ such that for all $t_1, t_2 \in [\delta_1/2, t_0 + \delta_1]$, $|t_1 - t_2| < \delta_2$,

$$|F_{n-1}(x_0, t_1) - F_{n-1}(x_0, t_2)| < \varepsilon.$$

Thus for any t such that $|t - t_0| < \min\{\delta_1/2, \delta_2\}$ (here, without loss of generality, we may assume that $t < t_0$), one has

$$\begin{aligned} |F_n(x_0, t_0) - F_n(x_0, t)| &\leq \int_0^{t_0 - \delta_1} |F_{n-1}(x_0, t_0 - s) - F_{n-1}(x_0, t - s)| dF_{\tau_1}(s) \\ &\quad + \int_{t_0 - \delta_1}^t F_{n-1}(x_0, t - s) dF_{\tau_1}(s) + \int_{t_0 - \delta_1}^{t_0} F_{n-1}(x_0, t_0 - s) dF_{\tau_1}(s) \\ &\leq \varepsilon + 2[F_{\tau_1}(t_0 + \delta) - F_{\tau_1}(t_0 - \delta)] \leq 3\varepsilon. \end{aligned}$$

Similarly, the continuity of $F_n(x, t)$ at (x_0, t_0) with respect to x is guaranteed by the facts that $F_{n-1}(x, t)$ is continuous and thus uniformly continuous on $[x, x'] \times [\varepsilon, t]$ for all $\varepsilon > 0$ and that $F_{\tau_1}(\cdot)$ puts no mass on point t_0 . Also, by an argument similar to the last lemma, we show that $F_n(\cdot, \cdot)$ is bivariate continuous and complete the proof. \square

With the same argument as before, we have the following corollary.

COROLLARY 2.3. *For any bounded measurable function f , any integer $k \geq 1$, and any $t > 0$,*

$$\mathbf{E}_0 [f(X_t) \mathbb{1}_{n_t=k}] = \int_{-\infty}^1 f(x) dF_k(x, t)$$

is measurable with respect to t , and

$$(31) \quad \mathbf{E}_0 [f(X_t) \mathbb{1}_{n_t=k}] = \int_{-\infty}^1 f(x) dF_k(x, t) = \int_0^t \int_{-\infty}^1 f(x) dF_{k-1}(x, t-s) dF_{\tau_1}(s).$$

Our next lemma gives the exponential decay of $F_n(x, t)$ on a compact set of t , which is useful in our later calculations, especially when we need to deal with the convergence of some series.

LEMMA 2.3. *There is a $\theta > 0$ such that for $T \in (0, \infty)$,*

$$(32) \quad F_n(x, t) \leq \exp(-\theta n + T)$$

for all $n \in \mathbb{N}$, $t \leq T$, and $x \in (-\infty, 1]$.

Proof. For any $t \leq T$ and $x \in (-\infty, 1]$,

$$F_n(x, t) = \mathbf{P}(X_t \leq x, n_t = n) \leq \mathbf{P}(n_t \geq n) = \mathbf{P}(T_n \leq t) \leq \mathbf{P}(T_n \leq T).$$

Thus it suffices to show that

$$\mathbf{P}(T_n \leq T) \leq \exp(-\theta n + T).$$

Now recalling that $T_n = \sum_{i=1}^n \tau_i \in (0, \infty)$, define

$$Y_n = \exp(-T_n) \in (0, 1),$$

where, by the independence of $\{\tau_i, i \geq 1\}$,

$$\mathbf{E}[Y_n] = (\mathbf{E}[\exp(-\tau_1)])^n.$$

Note that for a.s. ω , $Y_t^{(1)}(\omega)$ is a continuous trajectory, which implies $\tau_1(\omega) > 0$ a.s. Thus we have $\mathbf{P}(\tau_1 > 0) = 1$, which implies

$$\mathbf{E}[\exp(-\tau_1)] = \exp(-\theta) < 1$$

for some $\theta > 0$. Then the desired result follows from the Markov inequality for Y_n and the fact that $\{T_n \leq T\} = \{Y_n \geq \exp(-T)\}$. \square

Remark 3. The upper bound found in Lemma 2.3 is clearly not sharp, although it suffices for our purpose later in the paper.

In light of the properties of joint process (X_t, n_t) defined in (18) above, we have a new perspective on investigating the distribution of X_t . Let $F(x, t)$ denote the c.d.f. of X_t . Based on the number of jumping times, it admits the decomposition

$$(33) \quad F(x, t) = \sum_{n=0}^{\infty} F_n(x, t).$$

There are two major types of results that we could obtain from the decomposition above.

On one hand, we immediately get the well-posedness and regularity properties of the distribution of X_t at a given time, which are not easily achievable due to the complication of jumps. We observe that the right-hand side of (33) converges by the bounded convergence theorem, and, moreover, it is clear that by the previous lemmas, $F(x, t)$ is continuous on $(-\infty, 1] \times (0, \infty)$. In addition, due to the exponential decay of $F_n(x, t)$ with respect to n , we know that the measure induced by $F(\cdot, t)$ is absolutely continuous with respect to the Lebesgue measure, whose density function we shall denote by $f(x, t)$.

On the other hand, such a decomposition provides an auxiliary degree of freedom in the representation of the density function, which facilitates analyzing the time evolution of the density function. While the flux-shift mechanism makes the evolution of $F(x, t)$ nonlocal, the decomposition unfolds the distribution by adding one more dimension such that the evolution has a simpler structure: the evolution of F_0 is self-contained without any nonlocality, and for $n \geq 1$, the evolution of F_n is also local, although it has a tractable dependence on F_{n-1} . Recall that we have used $f_n(x, t)$ to denote the density function of $F_n(x, t)$. In fact, we are able to show that $f_n(x, t)$ is a solution to a sub-PDE problem, and, eventually, the exponential convergence in n can help us conclude that

$$(34) \quad f(x, t) = \sum_{n=0}^{\infty} f_n(x, t)$$

is a solution of the PDE problem of interest, satisfying the properties in Theorem 1.

3. Iteration approach. In this section we aim to prove the theorems in section 2.1. First, we prove that the density of the process X_t that starts from 0 is an instantaneous smooth mild solution of (9) with initial condition $f_{\text{in}}(x) = \delta(x)$. Then with a similar treatment we can easily get Corollary 2.1, which, together with the integral representation (12), derives Theorem 2. Finally, we show that the mild solution is consistent with the definition of the weak solution of (9) defined in [5].

3.1. Solutions in iteration. Recalling the process (X_t, n_t) defined in (18), we first focus on the case $X_0 = 0$, i.e., the initial condition of PDE (9) is $f(x, 0) = \delta(x)$. In

the previous section, we decomposed the distribution $F(x, t)$ of the stochastic process X_t into a summation of series $\{F_n(x, t)\}_{n=0}^{+\infty}$ according to (19) and (33). We also decompose the original PDE problem (9) into a sequence of sub-PDE problems: for $n = 0$,

$$(35) \quad \begin{cases} \frac{\partial f_0}{\partial t} - \frac{\partial}{\partial x}(xf_0) - \frac{\partial^2 f_0}{\partial x^2} = 0, & x \in (-\infty, 1), t \in (0, T], \\ f_0(-\infty, t) = 0, \quad f_0(1, t) = 0, & t \in [0, T], \\ f_0(x, 0) = \delta(x) & \text{in } \mathcal{D}'(-\infty, 1), \end{cases}$$

where $\mathcal{D}(-\infty, 1) = C_c^\infty(-\infty, 1)$, and for $n \geq 1$ defining $N_{n-1}(t) = -\frac{\partial}{\partial x}f_{n-1}(1, t)$, we solve

$$(36) \quad \begin{cases} \frac{\partial f_n}{\partial t} - \frac{\partial}{\partial x}(xf_n) - \frac{\partial^2 f_n}{\partial x^2} = 0, & x \in (-\infty, 0) \cup (0, 1), t \in (0, T], \\ f_n(0^-, t) = f_n(0^+, t), \quad \frac{\partial}{\partial x}f_n(0^-, t) - \frac{\partial}{\partial x}f_n(0^+, t) = N_{n-1}(t), & t \in (0, T], \\ f_n(-\infty, t) = 0, \quad f_n(1, t) = 0, & t \in [0, T], \\ f_n(x, 0) = 0, & x \in (-\infty, 1). \end{cases}$$

In particular, we find that the PDE problem (35) for f_0 is self-contained with singular initial data, and thus only a mild solution can be expected, which, however, can be shown to be instantaneously smooth. For $n \geq 1$ the PDE problems (36) for f_n are defined when $x \in (-\infty, 0) \cup (0, 1)$, and the time-dependent interface boundary data N_{n-1} at $x = 1$ is determined by f_{n-1} , the solution to the previous PDE problem in the sequence, but the classical solution of such problems can be understood in the usual sense.

Here, there is a bit of ambiguity in the notation, since we have used $f_n(x, t)$ to denote the subdensity function of the stochastic process and also the solution to the PDE problem. In fact, we shall show that these two functions coincide, the precise meaning of which shall be specified. In the following, we show that subdensity function f_0 with delta initial data is an instantaneous smooth mild solution of (35), and then following the iteration scheme, we prove that for each $n \geq 1$, the subdensity function f_n is the classical solution of (36). We conclude with the proof of Theorem 1 at the end of this subsection.

Before we prove our main theorem, we first discuss the Green's function of the Fokker–Planck equation (35). According to Theorem 1.10 in Chapter VI of Garroni and Menaldi [22], we know that the generator of the OU process (14), i.e.,

$$\mathcal{L}_y := (-y)\partial_y \cdot + \partial_{yy}^2,$$

admits a Green's function $G : (-\infty, 1] \times [0, T] \times (-\infty, 1] \times [0, T] \ni (y, s, x, t) \mapsto G(y, s, x, t)$. For a given $(x, t) \in (-\infty, 1] \times [0, T]$, the function $(-\infty, 1] \times [0, t) \ni (y, s) \mapsto G(y, s, x, t)$ is a solution of the PDE

$$(37) \quad \begin{cases} \partial_s G(y, s, x, t) + \mathcal{L}_y G(y, s, x, t) = 0, & y \in (-\infty, 1), s \in [0, t), \\ G(1, s, x, t) = 0, & s \in [0, t], \\ G(y, t, x, t) = \delta(y - x) & \text{in } \mathcal{D}'(-\infty, 1). \end{cases}$$

Following Theorem 5 in Chapter 9 of [21], for a given $(y, s) \in (-\infty, 1) \times [0, T)$, the function $(-\infty, 1] \times (s, T] \ni (x, t) \mapsto G(y, s, x, t)$ is also known to be Green's function

of the adjoint operator

$$\mathcal{L}_x^* = \partial_x[x \cdot] + \partial_{xx}^2 \cdot;$$

i.e., the function $(-\infty, 1] \times (s, T] \ni (x, t) \mapsto G(y, s, x, t)$ is a classical solution of the PDE

$$(38) \quad \begin{cases} \partial_t G(y, s, x, t) = \mathcal{L}_x^* G(y, s, x, t), & x \in (-\infty, 1), t \in (s, T], \\ G(y, s, 1, t) = 0, & t \in [s, T], \\ G(y, s, x, s) = \delta(x - y) \text{ in } \mathcal{D}'(-\infty, 1), \end{cases}$$

which is consistent with (35). Now we give an important lemma that connects the density function of the stochastic process before the first jumping time with the Green's function of PDE problem (35), which is the starting point of our iteration strategy. Also, for Green's function G , although we cannot find a closed formula for it, there exists the following estimation.

LEMMA 3.1. *There exists a unique Green's function $G : (-\infty, 1] \times [0, T] \times (-\infty, 1] \times [0, T] \ni (y, s, x, t) \mapsto G(y, s, x, t)$ for (35). Let $f_0(x, t)$ denote the density of the distribution $F_0(x, t)$ defined in (19); then $f_0(x, t) = G(0, 0, x, t)$, i.e., it is a mild solution of (35) on $(-\infty, 1] \times [0, T]$. In addition, we have the estimation*

$$(39) \quad |\partial^\ell G(y, s, x, t)| \leq C(t - s)^{-\frac{1+\ell}{2}} \exp\left(-C_0 \frac{(x - y)^2}{t - s}\right), \quad 0 \leq s < t \leq T,$$

where $\ell = 0, 1, 2$, $\partial^\ell = \partial_{tx}^\ell = \partial_t^m \partial_x^n$, $\ell = 2m + n$, for $m, n \in \mathbb{N}_0$.

Proof. Set

$$p(x, t) := G(0, 0, x, t), \quad x \in (-\infty, 1], t \in (0, T].$$

Now we prove that $p(x, t)$ coincides with $f_0(x, t)$, which immediately derives that $f_0(x, t)$ is a mild solution of (35). Given a smooth function $\phi : (-\infty, 1] \times [0, T] \rightarrow \mathbb{R}$ with a compact support, noting that Green's function satisfies (38), we have that the PDE

$$(40) \quad \begin{cases} \partial_s u(y, s) - y \partial_y u(y, s) + \partial_{yy} u(y, s) + \phi(y, s) = 0, & (y, s) \in (-\infty, 1) \times (0, T], \\ u(1, s) = 0, & s \in [0, T], \\ u(y, T) = 0, & y \in (-\infty, 1), \end{cases}$$

admits a (unique) classical solution

$$(41) \quad u(y, s) = \int_s^T \int_{-\infty}^1 G(y, s, x, t) \phi(x, t) dx dt, \quad s \in [0, T], y \leq 1.$$

Moreover, u is bounded and continuous on $(-\infty, 1] \times [0, T]$ and is once continuously differentiable in time and twice differentiable in space on $(-\infty, 1) \times [0, T]$. Let (X_t, n_t) be the process defined in (18), and let $\tau := \inf\{t \geq 0 : X_{t \wedge T} \geq 1\}$. By Itô's formula, we have

$$du(X_{t \wedge \tau}, t \wedge \tau) = -\phi(X_{t \wedge \tau}, t \wedge \tau) dt + \sqrt{2} u_x(X_{t \wedge \tau}, t \wedge \tau) dB_t.$$

Integrating the above formula from 0 to T and taking the expectation, with the boundary condition in (40), we then have the representation formula

$$(42) \quad u(0, 0) = \mathbf{E} \left[\int_0^{T \wedge \tau} \phi(X_t, t) dt \right].$$

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Also, with the two presentations for $u(0, 0)$ above, i.e., (41) and (42), we obtain

$$\mathbf{E} \left[\int_0^{T \wedge \tau} \phi(X_t, t) dt \right] = \int_0^T \int_{-\infty}^1 p(x, t) \phi(x, t) dx dt.$$

We further rewrite (42) as

$$\mathbf{E} \left[\int_0^{T \wedge \tau} \phi(X_t, t) dt \right] = \int_0^T \mathbf{E} [\phi(X_t, t) \mathbb{1}_{\{t \leq \tau\}}] dt = \int_0^T \int_{-\infty}^1 \phi(x, t) \mathbf{P}(X_t \in dx, \tau > t) dt.$$

Clearly, for $t \in [0, T]$, $\{\tau > t\} = \{T_1 > t\} = \{n_t = 0\}$, and thus

$$(43) \quad \int_0^T \int_{-\infty}^1 \phi(x, t) f_0(x, t) dx dt = \int_0^T \int_{-\infty}^1 \phi(x, t) p(x, t) dx dt.$$

By (22) and (39), $p(x, t)$ and $f_0(x, t)$ decay at $-\infty$, and thus (43) is also valid for any smooth function ϕ that is only bounded, which derives that the density function $f_0(x, t)$ coincides with $p(x, t)$. With (24), we conclude that $f_0(x, 0) = \delta(x)$, and thus $f_0(x, t)$ is a mild solution of (35). The complete proof of estimation (39) can be found in Theorem 1.10 in Chapter VI of Garroni and Menaldi [22]. The proof is complete. \square

Remark 4. The proof of Lemma 3.1 is essentially implied from the results in [15, 17, 22], in particular, Lemma 2.1 of [15] and Theorem 1.10 in Chapter VI of [22].

Next, we prove some regularities of the subdensity $f_0(x, t)$ that are useful in our later calculations.

PROPOSITION 3.1. *Let X_t be the process defined in (18), and let T_1 be the stopping time defined in (17). Let $F_0(x, t)$ be defined in (19) and its density be denoted as $f_0(x, t)$. Let $f_{T_1}(t)$ denote the p.d.f. of T_1 . For any fixed $T > 0$, we have*

(i)

$$(44) \quad \lim_{x \rightarrow -\infty} \partial_x f_0(x, t) = 0, \quad t \in (0, T].$$

(ii) *For any $x_0 \in (0, 1)$, $f_0(x, t) \in C^{2,1}((-\infty, -x_0] \cup [x_0, 1] \times [0, T])$. Moreover, for all $|x| \geq |x_0|$, $\lim_{t \rightarrow 0^+} f_0(x, t) = 0$.*

(iii) *For any $0 < \varepsilon_0 < T < \infty$, $f_0(x, t) \in C^{2,1}((-\infty, 1] \times [\varepsilon_0, T])$, with the uniform gradient estimations*

(45)

$$\begin{aligned} \sup_{(-\infty, 1] \times [\varepsilon_0, T]} |f_0| < \infty, \quad \sup_{(-\infty, 1] \times [\varepsilon_0, T]} \left| \frac{\partial f_0}{\partial t} \right| < \infty, \quad \sup_{(-\infty, 1] \times [\varepsilon_0, T]} \left| \frac{\partial f_0}{\partial x} \right| < \infty, \\ \sup_{(-\infty, 1] \times [\varepsilon_0, T]} \left| \frac{\partial(xf_0)}{\partial x} \right| < \infty, \quad \sup_{(-\infty, 1] \times [\varepsilon_0, T]} \left| \frac{\partial^2 f_0}{\partial x^2} \right| < \infty. \end{aligned}$$

(iv) *We have the coupling relation between $f_{T_1}(t)$ and $f_0(x, t)$: for all $t \in (0, T]$, it satisfies*

$$(46) \quad f_{T_1}(t) = - \int_{-\infty}^1 \frac{\partial f_0(x, t)}{\partial t} dx = - \frac{\partial}{\partial x} f_0(1, t),$$

and $f_{T_1}(t) \in C[0, T]$ with $f_{T_1}(0) = 0$.

Proof. (i) is the direct corollary of estimate (39). From (39), we know that the Green’s function of (35) is continuously differentiable and decays exponentially fast as t tends to 0^+ when x stay away from 0. Thus we immediately obtain the properties in (ii). Also by the estimation (39) for the Green’s function, we can easily get the bound for f_0 in (iii) when t stays away from 0. Finally, to prove (iv), recall that $f_0(x, t)dx = \mathbb{P}(X_t \in dx, T_1 > t)$; thus the c.d.f. of T_1 is given by

$$\mathbf{P}(T_1 \leq t) = 1 - \mathbf{P}(T_1 > t) = 1 - \int_{-\infty}^1 f_0(x, t)dx.$$

By (39), we can differentiate the above formula with respect to t and exchange the derivative and the integral. Using (i) and the boundary condition of f_0 , we have for any $t \in (0, T]$,

$$f_{T_1}(t) = \frac{d}{dt} \mathbf{P}(T_1 \leq t) = - \int_{-\infty}^1 \frac{\partial f_0(x, t)}{\partial t} dx = - \int_{-\infty}^1 \frac{\partial}{\partial x} (x f_0) + \frac{\partial^2 f_0}{\partial x^2} dx = - \frac{\partial}{\partial x} f_0(1, t).$$

With Lemma 3.1,

$$|f_{T_1}(t)| = |\partial_x f_0(1, t)| \leq \frac{C}{t} \exp\left(-\frac{C_0}{t}\right),$$

and we conclude that $f_{T_1}(t) \in C(0, T]$, $\lim_{t \rightarrow 0^+} f_{T_1}(t) = 0$, and thus $f_{T_1}(t) \in C[0, T]$. \square

In order to make the iteration strategy successful, we need to further show that $f_{T_1}(t)$ is continuously differentiable, which is not a direct consequence of estimating Green’s function. Thus next we shall prove that $f_{T_1}(t) \in C^1[0, T]$ and the following estimation is useful in the further calculations.

COROLLARY 3.1. *For any $T > 0$ and for all $0 < \varepsilon_0 < \min\{\frac{1}{T}, T\}$, $f_{T_1}(t) \in C^1(0, T]$ and for any $t \geq \varepsilon_0$, we have*

$$(47) \quad |f'_{T_1}(t)| \leq C\varepsilon_0^{-3}.$$

Proof. By Proposition 3.1, we know that $f_0(x, t) \in C^{2,1}((-\infty, 1] \times [\varepsilon_0, T])$ and $f_{T_1}(t) = -\frac{\partial}{\partial x} f_0(1, t) \in C[0, T]$. Then for any $x \in (-\infty, 1]$, $t \in [\varepsilon_0, T]$, set $g_0(x, t) = \frac{\partial}{\partial t} f_0(x, t)$ and it satisfies

$$(48) \quad \begin{cases} \frac{\partial g_0}{\partial t} - \frac{\partial}{\partial x} (xg_0) - \frac{\partial^2 g_0}{\partial x^2} = 0, & x \in (-\infty, 1), t \in (\varepsilon_0, T], \\ g_0(-\infty, t) = 0, \quad g_0(1, t) = 0, & t \in [\varepsilon_0, T], \\ g_0(x, \varepsilon_0) = \frac{\partial}{\partial t} f_0(x, \varepsilon_0), & x \in (-\infty, 1). \end{cases}$$

Defining $\varphi(x) := \frac{\partial}{\partial t} f_0(x, \varepsilon_0)$, we immediately get that $\varphi(x) \in C^2(-\infty, 1] \cap L^\infty(-\infty, 1]$ and by (39)

$$|\varphi(x)| \leq C\varepsilon_0^{-\frac{3}{2}}.$$

For any $t \geq 0$, $x \in (-\infty, 1]$, define $h(x, t) := g_0(x, t + \varepsilon_0)$ and then $h(x, 0) = \varphi(x)$. Recalling the Green’s function $G(s, y, x, t)$ in PDE (38), we have

$$h(x, t) = \int_{-\infty}^1 G(y, 0, t, x)\varphi(y)dy, \quad t \geq 0.$$

Then

$$(49) \quad g_0(x, t) = \int_{-\infty}^1 G(y, 0, t - \varepsilon_0, x) \varphi(y) dy, \quad t \geq \varepsilon_0.$$

By (39) and Lemma 3.1, we have

$$(50) \quad f'_{T_1}(t) = -\frac{\partial}{\partial t} \frac{\partial}{\partial x} f_0(1, t) = -\frac{\partial}{\partial x} g_0(1, t) = -\int_{-\infty}^1 \frac{\partial}{\partial x} G(y, 0, t - \varepsilon_0, 1) \varphi(y) dy, \quad t > \varepsilon_0,$$

and thus $f_{T_1}(t) \in C^1(\varepsilon_0, T]$.

When $t \geq 2\varepsilon_0$,

$$(51) \quad \begin{aligned} |f'_{T_1}(t)| &\leq \int_{-\infty}^1 \left| \frac{\partial}{\partial x} G(y, 0, t - \varepsilon_0, 1) \right| C \varepsilon_0^{-\frac{3}{2}} dy \\ &\leq C \varepsilon_0^{-\frac{3}{2}} \int_{-\infty}^1 \frac{C}{t - \varepsilon_0} \exp\left(-C_0 \frac{(1-y)^2}{t - \varepsilon_0}\right) dy \\ &= C \varepsilon_0^{-\frac{5}{2}} \int_0^{+\infty} \exp\left(-C_0 \frac{\xi^2}{t - \varepsilon_0}\right) d\xi \\ &\leq C \varepsilon_0^{-\frac{5}{2}} \frac{\sqrt{T - \varepsilon_0} \sqrt{\pi}}{\sqrt{C_0} \cdot 2} \\ &\leq C \varepsilon_0^{-3}, \end{aligned}$$

where the second inequality follows by the change of variable $\xi = 1 - y$, and the third inequality results from the fact that $\varepsilon_0 \leq \frac{1}{T}$. Also, because ε_0 can be arbitrarily small, we can complete the proof. \square

Now we focus on the behavior of $f'_{T_1}(t)$ when t is small. This proof is partially inspired by the reformulation and the representation proposed in [10].

PROPOSITION 3.2. *The p.d.f. $f_{T_1}(t)$ of the first hitting time T_1 is $C^1[0, T]$ for any fixed $T > 0$.*

Proof. By Proposition 3.1 and Corollary 3.1, we know $f_{T_1}(t) \in C^1(0, T] \cap C[0, T]$, and thus we need only prove that $\lim_{t \rightarrow 0^+} f'_{T_1}(t)$ exists. We prove this in the following steps. First, we lay out our strategy.

Step 1. We rewrite the problem (35) as a moving boundary problem and rewrite $f_{T_1}(t)$ as $M(s)$. With the heat kernel Γ , we derive an integral representation of $M(s)$.

Step 2. We analyze the decay rate of $M(s)$ and $M'(s)$ at 0 by utilizing the decay property of heat kernel Γ .

Step 3. Using the estimations of $M(s)$ and $M'(s)$ and heat kernel Γ , we derive $\lim_{t \rightarrow 0^+} f'_{T_1}(t) = 0$.

Second, we give details of the proof.

Step 1. Inspired by [10], we introduce a change of variable to transform (35) into a moving boundary problem. Let

$$(52) \quad y = e^t x, \quad s = (e^{2t} - 1)/2, \quad u(y, s) = e^{-t} f(x, t).$$

Note that PDE (35) corresponds to the OU process killed at a stopping time and thus has the Dirichlet boundary condition. By the standard change of variable (52), we can transform (35) into a heat equation with the moving boundary $b(s) = \sqrt{2s + 1}$.

Actually, we have the new equation

$$(53) \quad \begin{cases} u_s = u_{yy}, & y \in (-\infty, b(s)), s > 0, \\ u(-\infty, s) = 0, & u(b(s), s) = 0, \quad s \geq 0, \\ u(y, 0) = \delta(y) & \text{in } \mathcal{D}'(-\infty, b(s)). \end{cases}$$

Let Γ be the Green's function for the heat equation on the real line as follows:

$$(54) \quad \Gamma(y, s, \xi, \tau) = \frac{1}{\sqrt{4\pi(s-\tau)}} \exp\left\{-\frac{(y-\xi)^2}{4(s-\tau)}\right\}, \quad s > \tau.$$

In the region $-\infty < \xi < b(\tau)$, $0 < \tau < h$, recall the Green's identity

$$(55) \quad \frac{\partial}{\partial \xi}(\Gamma u_\xi - u \Gamma_\xi) - \frac{\partial}{\partial \tau}(\Gamma u) = 0.$$

To derive an expression of u , we consider the integration of (55) over such a region and let

$$I = \int_0^s \int_{-\infty}^{b(\tau)} (\Gamma u_\xi)_\xi d\xi d\tau, \quad II = \int_0^s \int_{-\infty}^{b(\tau)} (u \Gamma_\xi)_\xi d\xi d\tau, \quad III = \int_0^s \int_{-\infty}^{b(\tau)} (\Gamma u)_\tau d\xi d\tau.$$

We have

$$I = \int_0^s \Gamma u_\xi|_{\xi=b(\tau)} d\tau.$$

Using the boundary condition of $u(y, s)$ in (53), we have

$$II = 0$$

and

$$III = \int_{-\infty}^{b(s)} \Gamma u|_{\tau=s} d\xi - \int_{-\infty}^{b(0)} \Gamma u|_{\tau=0} d\xi = u(y, s) - \int_{-\infty}^{b(0)} \Gamma u|_{\tau=0} d\xi.$$

Plugging in (55), we obtain

$$(56) \quad \begin{aligned} u(y, s) &= \int_{-\infty}^{b(0)} \Gamma(y, s, \xi, 0) \delta(\xi) d\xi + \int_0^s \Gamma(y, s, b(\tau), \tau) u_\xi(b(\tau), \tau) d\tau \\ &= \Gamma(y, s, 0, 0) - \int_0^s \Gamma(y, s, b(\tau), \tau) M(\tau) d\tau, \end{aligned}$$

where $M(\tau) = -u_\xi(b(\tau), \tau)$. Note that the Green's function Γ is infinitely continuously differentiable, and thus the regularity of u depends on M . Using Lemma 1 of [21, p. 217], we know that for any continuous function ρ , the following limit holds:

$$\lim_{y \rightarrow b(s)^-} \frac{\partial}{\partial y} \int_0^s \rho(\tau) \Gamma(y, s, b(\tau), \tau) d\tau = \frac{1}{2} \rho(s) + \int_0^s \rho(\tau) \Gamma_y(b(s), s, b(\tau), \tau) d\tau.$$

So, by differentiating (56) at $y = b(s)^-$, we can get the following integral equation:

$$-M(s) = \Gamma_y(b(s), s, 0, 0) - \frac{1}{2} M(s) - \int_0^s \Gamma_y(b(s), s, b(\tau), \tau) M(\tau) d\tau.$$

That is,

$$(57) \quad \begin{aligned} M(s) &= -2\Gamma_y(b(s), s, 0, 0) + 2 \int_0^s \Gamma_y(b(s), s, b(\tau), \tau) M(\tau) d\tau \\ &=: 2J_1(s) + 2J_2(s). \end{aligned}$$

Recalling the change of variable in (52) and taking derivatives directly, we know that

$$(58) \quad f_{T_1}(t) = e^{2t}M(s) \quad \text{and} \quad f'_{T_1}(t) = 2e^{2t}M(s) + e^{4t}M'(s).$$

Step 2. We shall analyze the decay rate of $M(s)$ at 0. By the heat kernel (54), we have $\Gamma(y, s, 0, 0) = \frac{1}{\sqrt{4\pi s}} \exp(-\frac{y^2}{4s})$ and $b(s) = \sqrt{2s+1}$, we have that for any $n \geq 0$, $\lim_{s \rightarrow 0^+} \frac{J_1(s)}{s^n} = 0$, and thus there exists a constant C such that for $s \in [0, T]$, $n \geq 0$,

$$(59) \quad |J_1(s)| \leq Cs^n.$$

Note that

$$(60) \quad \Gamma_y(b(s), s, b(\tau), \tau) = \frac{1}{\sqrt{4\pi(s-\tau)}} \exp\left\{-\frac{(b(s)-b(\tau))^2}{4(s-\tau)}\right\} \left\{\frac{b(s)-b(\tau)}{-2(s-\tau)}\right\},$$

and thus we have

$$|\Gamma_y(b(s), s, b(\tau), \tau)| \leq \frac{C}{(s-\tau)^{\frac{1}{2}}}.$$

By (iv) of Proposition 3.1 and (58), there exists another big enough constant K such that $|M(s)| \leq K$ for all $s \in [0, T]$. Thus

$$|J_2(s)| \leq C \int_0^s \frac{K}{(s-\tau)^{\frac{1}{2}}} = C\sqrt{s}.$$

Combining this with (59), we also have $|M(s)| \leq |J_1(s)| + |J_2(s)| \leq C\sqrt{s}$, and thus

$$|J_2(s)| \leq C \int_0^s \frac{\sqrt{\tau}}{(s-\tau)^{\frac{1}{2}}} = Cs.$$

Using (59) again, we have $|M(s)| \leq Cs$, and thus

$$|J_2(s)| \leq C \int_0^s \frac{\tau}{(s-\tau)^{\frac{1}{2}}} = Cs^{\frac{3}{2}}.$$

Using (59) for the third time, we can get $|M(s)| \leq Cs^{\frac{3}{2}}$, which, together with $M(0) = 0$, leads to the fact that the right derivative of M at 0 exists and that

$$M'(0^+) = \lim_{s \rightarrow 0^+} \frac{M(s)}{s} = 0.$$

Repeating the above calculations step by step, we find that for any $n \geq 0$, there exists a constant that depends on n such that

$$(61) \quad |M(s)| \leq Cs^n.$$

By (47) and (58), we know that for any sufficiently small $\varepsilon_0 > 0$, there is a constant $C < +\infty$ such that

$$(62) \quad |M'(s)| \leq C\varepsilon_0^{-3} \quad \forall s \in [\varepsilon_0, 1].$$

Step 3. In order to prove $f_{T_1}(t) \in C^1[0, T]$, which is equivalent to proving that $\lim_{s \rightarrow 0^+} M'(s)$ exists by (58), now we prove that $\lim_{s \rightarrow 0^+} M'(s) = 0$. Using (57) and the fact that $\lim_{s \rightarrow 0^+} J'_1(s) = 0$, we need only prove that

$$(63) \quad \lim_{s \rightarrow 0^+} J'_2(s) = 0.$$

Using the estimations (61), (62) and heat kernel Γ , we compute the difference between $A := \int_0^s \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau$ and $B := \int_0^{s+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau$. A can have the decomposition

$$A := \left(\int_0^{\frac{s}{2}} + \int_{\frac{s}{2}}^s \right) \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau,$$

and for B ,

$$B := \left(\int_0^{\frac{s}{2}} + \int_{\frac{s}{2}}^{\frac{s}{2}+\Delta s} + \int_{\frac{s}{2}+\Delta s}^{s+\Delta s} \right) \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau.$$

Define

$$\frac{J_2(s+\Delta s) - J_2(s)}{\Delta s} =: I_1 + I_2 + I_3,$$

where

$$I_1 := \int_0^{\frac{s}{2}} \left[\frac{\Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau) - \Gamma_y(b(s), s, b(\tau), \tau)}{\Delta s} \right] M(\tau)d\tau,$$

$$I_2 := \frac{1}{\Delta s} \int_{\frac{s}{2}}^{\frac{s}{2}+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau,$$

and

$$I_3 := \frac{1}{\Delta s} \left[\int_{\frac{s}{2}+\Delta s}^{s+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau - \int_{\frac{s}{2}}^s \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau \right].$$

Thus to get (63), now it suffices to show that

$$(64) \quad \lim_{\Delta s \rightarrow 0} |I_1| \leq \int_0^{\frac{s}{2}} |\partial_s \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)| d\tau = o(1),$$

$$(65) \quad \lim_{\Delta s \rightarrow 0} |I_2| = o(1),$$

and

$$(66) \quad \lim_{\Delta s \rightarrow 0} |I_3| = o(1).$$

The above “ $= o(1)$ ” means that the left side goes to 0 as $s \rightarrow 0^+$.

Note that for $\tau \leq \frac{3}{4}s$, the Γ_y and $\partial_s \Gamma_y$ terms in (64) and (65) can be bounded by a polynomial order with respect to s^{-1} , which, together with (61), immediately

derives (64) and (65). Thus we need only focus on proving (66). With a simple change of variable, we have

$$\begin{aligned} & \int_{\frac{s}{2} + \Delta s}^{s + \Delta s} \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau), \tau) M(\tau) d\tau \\ &= \int_{\frac{s}{2}}^s \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) M(\tau + \Delta s) d\tau \\ &= \int_{\frac{s}{2}}^s \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) M(\tau) d\tau \\ &+ \int_{\frac{s}{2}}^s \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) [M(\tau + \Delta s) - M(\tau)] d\tau. \end{aligned}$$

We define

$$I_3 := I_{3,1} + I_{3,2},$$

where

$$I_{3,1} = \frac{1}{\Delta s} \int_{\frac{s}{2}}^s [\Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) - \Gamma_y(b(s), s, b(\tau), \tau)] M(\tau) d\tau$$

and

$$I_{3,2} = \int_{\frac{s}{2}}^s \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) \frac{M(\tau + \Delta s) - M(\tau)}{\Delta s} d\tau.$$

Thus to show (66), it suffices to prove

$$(67) \quad \lim_{\Delta s \rightarrow 0} |I_{3,1}| = o(1)$$

and

$$(68) \quad \lim_{\Delta s \rightarrow 0} |I_{3,2}| = o(1).$$

For (67), by (60) we have

$$\begin{aligned} & \Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) - \Gamma_y(b(s), s, b(\tau), \tau) \\ &= \frac{1}{\sqrt{4\pi(s - \tau)}} \exp \left\{ -\frac{1}{2} \frac{b(s + \Delta s) - b(\tau + \Delta s)}{b(s + \Delta s) + b(\tau + \Delta s)} \right\} \left\{ \frac{-1}{b(s + \Delta s) + b(\tau + \Delta s)} \right\} \\ &- \frac{1}{\sqrt{4\pi(s - \tau)}} \exp \left\{ -\frac{1}{2} \frac{b(s) - b(\tau)}{b(s) + b(\tau)} \right\} \left\{ \frac{-1}{b(s) + b(\tau)} \right\}, \end{aligned}$$

and thus there exists a constant $C < +\infty$ independent of the choices of s , τ , and Δs such that

$$|\Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s) - \Gamma_y(b(s), s, b(\tau), \tau)| \leq C \cdot \Delta s \cdot \frac{1}{\sqrt{s - \tau}},$$

which, together with (61), derives $\lim_{\Delta s \rightarrow 0} |I_{3,1}| = o(1)$.

Finally, for (68), note that $M(\tau) \in C^1[\frac{s}{2}, s]$ and that $|\Gamma_y(b(s + \Delta s), s + \Delta s, b(\tau + \Delta s), \tau + \Delta s)| \leq \frac{C}{\sqrt{s - \tau}}$. By the dominated convergence theorem, we have that the limit in (68) exists and is equal to

$$(69) \quad I_{3,3} := \int_{\frac{s}{2}}^s \Gamma_y(b(s), s, b(\tau), \tau) M'(\tau) d\tau.$$

To prove $|I_{3,3}| = o(1)$, one may further decompose it as

$$I_{3,3} = \int_{\frac{s}{2}}^{s-s^7} \Gamma_y(b(s), s, b(\tau), \tau) M'(\tau) d\tau + \int_{s-s^7}^s \Gamma_y(b(s), s, b(\tau), \tau) M'(\tau) d\tau$$

$$=: I_4 + I_5.$$

For I_4 , note that $\Gamma_y(b(s), s, b(\tau), \tau)$ and $M(\tau)$ are both smooth on $[\frac{s}{2}, s - s^7]$; we may use integration by parts to obtain

$$|I_4| \leq \left| \Gamma_y(b(s), s, b(s - s^7), s - s^7) \cdot M(s - s^7) \right| + \left| \Gamma_y(b(s), s, b\left(\frac{s}{2}\right), \left(\frac{s}{2}\right)) \cdot M\left(\frac{s}{2}\right) \right|$$

$$+ \left| \int_{\frac{s}{2}}^{s-s^7} \partial_\tau \Gamma_y(b(s), s, b(\tau), \tau) M'(\tau) d\tau \right|,$$

where all the terms are small since $|M(\tau)|$ is much less than any polynomial of τ , and thus $I_4 = o(1)$. For I_5 , recall that $|\Gamma_y(b(s), s, b(\tau), \tau)| \leq \frac{C}{\sqrt{s-\tau}}$ and $|M'(\tau)| \leq Cs^{-3}$ on $[s - s^7, s]$, and we have

$$|I_5| \leq s^{-3} \int_{s-s^7}^s \frac{C}{\sqrt{s-\tau}} d\tau \leq C\sqrt{s} = o(1),$$

which derives $\lim_{\Delta s \rightarrow 0} |I_{3,2}| = o(1)$ and thus $\lim_{\Delta s \rightarrow 0} |I_3| = o(1)$. Combining (64), (65), and (66), we get $\lim_{s \rightarrow 0^+} J_2'(s) = 0$ and then $\lim_{s \rightarrow 0^+} M'(s) = 0$, which, together with (58), derive $\lim_{t \rightarrow 0^+} f_{T_1}'(t) = 0$ and $f_{T_1}(t) \in C^1[0, T]$. \square

Next, we can do the first iteration.

PROPOSITION 3.3. *Let $f_1(x, t)$ be the density function of the measure induced by $F_1(\cdot, t)$ defined in (19); it satisfies the following initial condition and the recursive relation:*

$$(70) \quad \begin{aligned} f_1(x, 0) &= 0 \quad \forall x \in (-\infty, 1), \\ f_1(x, t) &= \int_0^t f_0(x, t-s) f_{T_1}(s) ds \quad \forall x \in (-\infty, 0) \cup (0, 1), t > 0. \end{aligned}$$

For any fixed $T > 0$, we have the following:

(i) $f_1(x, t)$ is the classical solution of the following PDE on $(-\infty, 1] \times [0, T]$:

$$(71) \quad \frac{\partial f_1}{\partial t} - \frac{\partial}{\partial x}(x f_1) - \frac{\partial^2}{\partial x^2} f_1 = 0, \quad x \in (-\infty, 0) \cup (0, 1), t \in (0, T],$$

$$(72) \quad f_1(0^-, t) = f_1(0^+, t), \quad \frac{\partial}{\partial x} f_1(0^-, t) - \frac{\partial}{\partial x} f_1(0^+, t) = f_{T_1}(t), \quad t \in (0, T],$$

$$(73) \quad f_1(-\infty, t) = 0, \quad f_1(1, t) = 0, \quad t \in [0, T],$$

$$(74) \quad f_1(x, 0) = 0, \quad x \in (-\infty, 1),$$

with

$$(75) \quad \lim_{x \rightarrow -\infty} \partial_x f_1(x, t) = 0, \quad t \in [0, T].$$

(ii) There is a big enough constant C_T depending only on T such that

$$(76) \quad |f_1(x, t)| \leq C_T \quad \forall x \in (-\infty, 0) \cup (0, 1), t \in [0, T],$$

$$(77) \quad \left| \frac{\partial}{\partial x} f_1(x, t) \right| \leq C_T \quad \forall x \in (-\infty, 0) \cup (0, 1), t \in [0, T].$$

Also, at the domain boundary,

$$(78) \quad \left| \frac{\partial}{\partial x} f_1(0^-, t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_1(0^+, t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_1(1^-, t) \right| \leq C_T, \quad t \in [0, T].$$

(iii) For $t > 0$, recalling that the density of the second jumping time is

$$(79) \quad f_{T_2}(t) = \int_0^t f_{T_1}(t-s)f_{T_1}(s)ds,$$

we have

$$(80) \quad -\frac{\partial f_1}{\partial x}(1, t) = f_{T_2}(t).$$

Proof. By (30) and the Fubini formula, we immediately get (70). As we already know that $f_0(x, t)$ satisfies PDE (35), from iteration relationship (70) and the regularities for $f_0(x, t)$ in Proposition 3.1, we can check that $f_1(x, t)$ satisfies PDE (36) with $n = 1$, and the estimations for $f_1(x, t)$ are valid.

To prove (i), by the regularities of f_0 in Proposition 3.1, we have for all $x \in (-\infty, 0) \cup (0, 1)$,

$$\frac{\partial}{\partial x} f_1(x, t) = \int_0^t \frac{\partial}{\partial x} f_0(x, t-s)f_{T_1}(s)ds \quad \text{and} \quad \frac{\partial^2}{\partial x^2} f_1(x, t) = \int_0^t \frac{\partial^2}{\partial x^2} f_0(x, t-s)f_{T_1}(s)ds,$$

which, together with the decay property (44) for f_0 , derive (75). Moreover,

$$\begin{aligned} \frac{\partial}{\partial t} f_1(x, t) &= \frac{\partial}{\partial t} \int_0^t f_0(x, t-s)f_{T_1}(s)ds \\ &= \lim_{\Delta t \rightarrow 0} \int_0^t \frac{f_0(x, t+\Delta t-s) - f_0(x, t-s)}{\Delta t} f_{T_1}(s)ds \\ &\quad + \lim_{\Delta t \rightarrow 0} \frac{\int_t^{t+\Delta t} f_0(x, t+\Delta t-s)f_{T_1}(s)ds}{\Delta t} \\ &= \int_0^t \frac{\partial}{\partial t} f_0(x, t-s)f_{T_1}(s)ds. \end{aligned}$$

Thus we have checked (71) and also gotten the continuity of $\frac{\partial^2}{\partial x^2} f_1(x, t)$ and $\frac{\partial}{\partial t} f_1(x, t)$. At the same time, (73) and (74) are obvious because of the boundary conditions of f_0 and the formula (70). So, for the rest of the proof we concentrate on verifying (72), which is composed of

$$(81) \quad f_1(0^-, t) = f_1(0^+, t), \quad t \in (0, T],$$

and

$$(82) \quad \frac{\partial}{\partial x} f_1(0^-, t) - \frac{\partial}{\partial x} f_1(0^+, t) = f_{T_1}(t), \quad t \in (0, T].$$

To show (81), note that $\int_0^1 \frac{1}{\sqrt{1-e^{-2s}}} ds < \infty$, and thus for any $\varepsilon > 0$, $\exists 0 < \delta < t$ such that $\int_0^\delta \frac{1}{\sqrt{1-e^{-2s}}} ds < \frac{\varepsilon}{c}$, where the constant c is the same as in (23). With (70),

we have for any $x \neq 0$,
 (83)

$$f_1(x, t) = \int_0^t f_0(x, t-s) f_{T_1}(s) ds = \int_0^{t-\delta} f_0(x, t-s) f_{T_1}(s) ds + \int_{t-\delta}^t f_0(x, t-s) f_{T_1}(s) ds.$$

For the second term above, using (23), we obtain

$$\int_{t-\delta}^t f_0(x, t-s) f_{T_1}(s) ds \leq \|f_{T_1}\|_{L^\infty[0,t]} \int_0^\delta \frac{c}{\sqrt{1-e^{-2s}}} ds \leq \|f_{T_1}\|_{L^\infty[0,t]} \cdot \varepsilon,$$

while for the first term, one may use (45) and see that

$$\lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} \int_0^{t-\delta} |f_0(x_1, t-s) - f_0(-x_2, t-s)| f_{T_1}(s) ds = 0.$$

Since ε is arbitrary, we get (81).

Now we prove (82). Note that

$$\frac{\partial}{\partial x} f_1(x_1, t) = \int_0^t \frac{\partial}{\partial x} f_0(x_1, t-s) f_{T_1}(s) ds, \quad x_1 \in (0, 1),$$

and for any $t-s \neq 0$,

$$\begin{aligned} \frac{\partial}{\partial x} f_0(x_1, t-s) &= - \int_{x_1}^1 \frac{\partial^2}{\partial x^2} f_0(x, t-s) dx + \frac{\partial}{\partial x} f_0(1, t-s) \\ &= - \int_{x_1}^1 \left[\frac{\partial f_0}{\partial t}(x, t-s) - \frac{\partial}{\partial x}(x f_0)(x, t-s) \right] dx + \frac{\partial}{\partial x} f_0(1, t-s) \\ &= \int_{x_1}^1 \left[\frac{\partial}{\partial x}(x f_0)(x, t-s) - \frac{\partial f_0}{\partial t}(y, t-s) \right] dx + \frac{\partial}{\partial x} f_0(1, t-s) \\ &= f_0(1, t-s) - x_1 f_0(x_1, t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x, t-s) dx + \frac{\partial}{\partial x} f_0(1, t-s) \\ &= -x_1 f_0(x_1, t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x, t-s) dx - f_{T_1}(t-s). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial x} f_1(x_1, t) &= \int_0^t \frac{\partial}{\partial x} f_0(x_1, t-s) f_{T_1}(s) ds \\ (84) \quad &= \int_0^t \left[-x_1 f_0(x_1, t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x, t-s) dx - f_{T_1}(t-s) \right] f_{T_1}(s) ds. \end{aligned}$$

Similarly, for any $x_2 > 0$, we have

$$\begin{aligned} \frac{\partial}{\partial x} f_0(-x_2, t-s) &= \int_{-\infty}^{-x_2} \frac{\partial^2}{\partial x^2} f_0(x, t-s) dx + 0 \\ &= \int_{-\infty}^{-x_2} \left[\frac{\partial f_0}{\partial t}(x, t-s) - \frac{\partial}{\partial z}(x f_0)(x, t-s) \right] dx \\ &= x_2 f_0(-x_2, t-s) + \int_{-\infty}^{-x_2} \frac{\partial f_0}{\partial t}(x, t-s) dx. \end{aligned}$$

Thus

$$(85) \quad \frac{\partial}{\partial x} f_1(-x_2, t) = \int_0^t \left[x_2 f_0(-x_2, t-s) + \int_{-\infty}^{-x_2} \frac{\partial f_0}{\partial t}(x, t-s) dx \right] f_{T_1}(s) ds.$$

Combining (84) and (85), we have for all $x_1 \in (0, 1)$ and $x_2 > 0$,

$$(86) \quad \begin{aligned} & \frac{\partial}{\partial x} f_1(-x_2, t) - \frac{\partial}{\partial x} f_1(x_1, t) \\ &= x_2 \int_0^t f_0(-x_2, t-s) f_{T_1}(s) ds + x_1 \int_0^t f_0(x_1, t-s) f_{T_1}(s) ds \\ &+ \int_0^t \left[\int_{\mathbb{R} \setminus [-x_2, x_1]} \frac{\partial f_0}{\partial t}(x, t-s) dx + f_{T_1}(t-s) \right] f_{T_1}(s) ds \\ &=: I_6 + I_7 + I_8. \end{aligned}$$

For I_6 , we have by (23),

$$I_6 \leq \|f_{T_1}\|_{L^\infty[0,t]} \cdot x_2 \cdot \int_0^t \frac{c}{\sqrt{1-e^{-2s}}} ds \rightarrow 0 \quad \text{as } x_2 \rightarrow 0^+.$$

Also, $I_7 \rightarrow 0$ by the same argument, and it now suffices to show

$$(87) \quad I_8 \rightarrow f_{T_1}(t) \quad \text{as } x_1, x_2 \rightarrow 0^+.$$

In the rest of our calculations, the integrand of I_8 will be called $H(s)$. As a result of Proposition 3.2, for any $\varepsilon > 0$, we let the chosen δ be small enough such that

$$(88) \quad \delta \|f_{T_1}\|_{L^\infty[0,t]}^2 < \varepsilon,$$

$$(89) \quad \int_{t_1}^{t_2} |f'_{T_1}(s)| ds < \varepsilon \quad \forall t_1 < t_2 < t, \quad t_2 - t_1 < \delta,$$

$$(90) \quad P(T_1 < \delta) < \varepsilon.$$

Then for the fixed $\delta > 0$ defined above,

$$(91) \quad I_8 = \int_0^{t-\delta} H(s) ds + \int_{t-\delta}^t H(s) ds =: I_{8,1} + I_{8,2}.$$

For $I_{8,1}$, we have by (45) and (46),

$$\begin{aligned} |I_{8,1}| &= \left| \int_0^{t-\delta} \left[\int_{\mathbb{R} \setminus [-x_2, x_1]} \frac{\partial f_0}{\partial t}(y, t-s) dy + f_{T_1}(t-s) \right] f_{T_1}(s) ds \right| \\ &= \left| \int_0^{t-\delta} \int_{-x_2}^{x_1} \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) dy ds \right| \\ &\leq \|f_{T_1}\|_{L^\infty[0,t]} \int_0^{t-\delta} \int_{-x_2}^{x_1} \left\| \frac{\partial f_0}{\partial t} \right\|_{L^\infty(-\infty, 1] \times [\delta, T]} dy ds \\ &\leq t \cdot \|f_{T_1}\|_{L^\infty[0,T]} \cdot \left\| \frac{\partial f_0}{\partial t} \right\|_{L^\infty(-\infty, 1] \times [\delta, T]} \cdot (x_1 + x_2), \end{aligned}$$

which $\rightarrow 0$ as $x_1, x_2 \rightarrow 0$. As for $I_{8,2}$,

$$I_{8,2} = \int_{t-\delta}^t \left[\int_{\mathbb{R} \setminus [-x_2, x_1]} \frac{\partial f_0}{\partial t}(y, t-s) dy + f_{T_1}(t-s) \right] f_{T_1}(s) ds.$$

One may first see by (88) that we have $\int_{t-\delta}^t f_{T_1}(t-s) f_{T_1}(s) ds \leq \varepsilon$. Moreover, for any $x_1, x_2 > 0$, note that function $\frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s)$ is bounded and continuous on the region $(\mathbb{R} \setminus [-x_2, x_1]) \times [t-\delta, t]$. One may apply Fubini's formula and obtain

$$(92) \quad I_{8,2} = \int_{\mathbb{R} \setminus [-x_2, x_1]} \int_{t-\delta}^t \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) ds dy.$$

At the same time, by (39) we have for any fixed $t > 0, y \notin [-x_2, x_1]$,

$$f_0(y, t-s) f_{T_1}(s) \in C^1[t-\delta, t]$$

and

$$\lim_{s \rightarrow t^-} f_0(y, t-s) f_{T_1}(s) = 0.$$

Thus, one may apply integration by parts and obtain

$$(93) \quad \begin{aligned} & \int_{t-\delta}^t \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) ds \\ &= (-f_0(y, t-s) f_{T_1}(s)) \Big|_{t-\delta}^t + \int_{t-\delta}^t f_0(y, t-s) f'_{T_1}(s) ds \\ &= f_0(y, \delta) f_{T_1}(t-\delta) + \int_{t-\delta}^t f_0(y, t-s) f'_{T_1}(s) ds. \end{aligned}$$

Plugging (93) back into (92) and applying the Fubini theorem once again, we have

$$(94) \quad \begin{aligned} I_{8,2} &= \left[\int_{\mathbb{R} \setminus [-x_2, x_1]} f_0(y, \delta) dy \right] f_{T_1}(t-\delta) + \int_{t-\delta}^t \int_{\mathbb{R} \setminus [-x_2, x_1]} f_0(y, t-s) dy f'_{T_1}(s) ds \\ &=: I_9 + I_{10}. \end{aligned}$$

First, for I_{10} , noting that f_0 is a p.d.f., for any $s \in (t-\delta, t)$ we have

$$\int_{\mathbb{R} \setminus [-x_2, x_1]} f_0(y, t-s) dy \leq 1,$$

which, together with (89), derives

$$(95) \quad |I_{10}| \leq \int_{t-\delta}^t |f'_{T_1}(s)| ds < \varepsilon.$$

Then for I_9 , by (90) we have

$$\lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} \int_{\mathbb{R} \setminus [-x_2, x_1]} f_0(y, \delta) dy = \int_{-\infty}^1 f_0(y, \delta) dy = P(T_1 > \delta) \in [1-\varepsilon, 1]$$

and

$$|f_{T_1}(t-\delta) - f_{T_1}(t)| < \varepsilon.$$

Thus we have for all sufficiently small $x_1 > 0, x_2 > 0$,

$$(96) \quad |I_9 - f_{T_1}(t)| = |I_9 - f_{T_1}(t - \delta)| + |f_{T_1}(t - \delta) - f_{T_1}(t)| < (\|f_{T_1}\|_{L^\infty[0,t]} + 1)\varepsilon.$$

Now combining (91)–(96), we conclude that $I_8 \rightarrow f_{T_1}(t)$ as $x_1, x_2 \rightarrow 0^+$, which, together with $I_6 \rightarrow 0$ and $I_7 \rightarrow 0$, derives (82).

As for (ii), we first derive (76) and (77) which are essential to getting (80), and we then set the basis for subsequent iterations. First, we verify (76), and without loss of generality, one may assume $T > 1$. So when $t \in (0, T]$ and $x \in (-\infty, 0) \cup (0, 1)$,

$$\begin{aligned} f_1(x, t) &= \int_0^t f_0(x, t-s)f_{T_1}(s)ds \\ &\leq \int_0^{t-1} f_0(x, t-s)f_{T_1}(s)ds + \|f_{T_1}\|_{L^\infty[0,T]} \int_0^1 \frac{1}{\sqrt{1-e^{-2s}}}ds \\ &\leq \|f_0(x, t)\|_{L^\infty(-\infty, 1] \times [1, \infty)} + C'_T = C_T. \end{aligned}$$

For (77), without loss of generality, one may assume that $x > 0$, and by (84) we have

$$\begin{aligned} &\frac{\partial f_1}{\partial x}(x, t) \\ &= \int_0^t \left[-xf_0(x, t-s) - \int_x^1 \frac{\partial f_0}{\partial t}(y, t-s)dy - f_{T_1}(t-s) \right] f_{T_1}(s)ds \\ &=: I_{11} + I_{12} + I_{13}. \end{aligned}$$

Using the estimate (22) for $f_0(x, t)$, one has

$$\begin{cases} |I_{11}| \leq C \cdot F_{T_1}(t) \leq C_T, \\ |I_{13}| \leq \|f_{T_1}\|_{L^\infty[0,T]} F_{T_1}(t) \leq C_T. \end{cases}$$

For the remaining I_{12} , by using Fubini's theorem twice and integrating by parts, together with the fact that $f_0(\cdot, t)$ is a p.d.f., we have

$$\begin{aligned} |I_{12}| &= \left| \int_x^1 \int_0^t \frac{\partial f_0}{\partial t}(y, t-s)f_{T_1}(s)dsdy \right| \\ &= \left| \int_x^1 \int_0^t f_0(y, t-s)f'_{T_1}(s)dsdy \right| \\ &= \left| \int_0^t \int_x^1 f_0(y, t-s)dyf'_{T_1}(s)ds \right| \\ &\leq \int_0^t |f'_{T_1}(s)| ds \\ &\leq C_T. \end{aligned}$$

Because of the proof of (72) in (i), property (ii) of Proposition 3.1, and representation (70), we know that $\frac{\partial}{\partial x}f_1(0^-, t)$, $\frac{\partial}{\partial x}f_1(0^+, t)$, and $\frac{\partial}{\partial x}f_1(1^-, t)$ are well defined, and thus by taking the one-sided limit in (77), we immediately get (78), and thus we complete the proof of (ii).

Finally, for (iii), using integral representation (29), we immediately get (79). Recalling that $f_1(1, t) = 0$ for all $t > 0$, it suffices to prove

$$(97) \quad \lim_{x_1 \rightarrow 0^+} \frac{f_1(1-x_1, t)}{x_1} = f_{T_2}(t) \quad \forall t > 0.$$

Now note that for all $0 < x_1 < \frac{1}{2}$,

$$f_1(1 - x_1, t) = \int_0^t f_0(1 - x_1, t - s) f_{T_1}(s) ds,$$

while at the same time, by the mean value theorem on f_0 , for all $s \in [0, t]$, $\exists \xi_{t-s}(x_1) \in [1 - x_1, 1] \subset [\frac{1}{2}, 1]$ such that

$$\begin{aligned} \frac{f_0(1 - x_1, t - s)}{x_1} &= - \frac{f_0(1, t - s) - f_0(1 - x_1, t - s)}{x_1} \\ &= - \frac{\partial}{\partial x} f_0(\xi_{t-s}(x_1), t - s). \end{aligned}$$

Note that for all $0 < x_1 < \frac{1}{2}$, by (ii) of Proposition 3.1 for f_0 ,

$$\frac{\partial}{\partial x} f_0(\xi_{t-s}(x_1), t - s) \leq \left\| \frac{\partial f_0}{\partial x} \right\|_{L^\infty_{[\frac{1}{2}, 1] \times [0, T]}} ,$$

and we have

$$\lim_{x_1 \rightarrow 0^+} \frac{\partial}{\partial x} f_0(\xi_{t-s}(x_1), t - s) = \frac{\partial}{\partial x} f_0(1, t - s).$$

By the dominated convergence theorem,

$$\lim_{x_1 \rightarrow 0^+} \frac{f_1(1 - x_1, t)}{x_1} = - \int_0^t \frac{\partial}{\partial x} f_0(1, t - s) f_{T_1}(s) ds = \int_0^t f_{T_1}(t - s) f_{T_1}(s) ds = f_{T_2}(t),$$

and thus

$$- \frac{\partial f_1}{\partial x}(1, t) = f_{T_2}(t).$$

The proof of Proposition 3.3 is complete. □

Similarly, by (30), for all $n \geq 1$, we have

$$(98) \quad \begin{aligned} f_n(x, 0) &= 0 \quad \forall x \in (\infty, 1), \\ f_n(x, t) &= \int_0^t f_{n-1}(x, t - s) f_{T_1}(s) ds \quad \forall x \in (-\infty, 0) \cup (0, 1), t > 0, \end{aligned}$$

and

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(t - s) f_{T_1}(s) ds.$$

Hence, the iterative construction is feasible, and we can show the next proposition.

PROPOSITION 3.4. *For each $n \geq 1$, let $f_n(x, t)$ be the density function of the measure induced by $F_n(\cdot, t)$ defined in (19). For any fixed $T > 0$, we have the following:*

(i) f_n is the classic solution of the following PDE:

$$(99) \quad \frac{\partial f_n}{\partial t} - \frac{\partial}{\partial x} (x f_n) - \frac{\partial^2}{\partial x^2} f_n = 0, \quad x \in (-\infty, 0) \cup (0, 1), t \in [0, T],$$

(100)

$$f_n(0^-, t) = f_n(0^+, t), \quad \frac{\partial}{\partial x} f_n(0^-, t) - \frac{\partial}{\partial x} f_n(0^+, t) = f_{T_n}(t), \quad t \in (0, T],$$

(101)

$$f_n(-\infty, t) = 0, \quad f_n(1, t) = 0, \quad t \in [0, T],$$

(102)

$$f_n(x, 0) = 0, \quad x \in (-\infty, 1),$$

with

$$(103) \quad \lim_{x \rightarrow -\infty} \partial_x f_n(x, t) = 0, \quad t \in [0, T].$$

(ii) There is a C_T that depends only on T such that

$$(104) \quad |f_n(x, t)| \leq C_T \quad \forall x \in (-\infty, 0) \cup (0, 1), t \in [0, T],$$

$$(105) \quad \left| \frac{\partial}{\partial x} f_n(x, t) \right| \leq C_T \quad \forall x \in (-\infty, 0) \cup (0, 1), t \in [0, T],$$

and at the domain boundary

$$(106) \quad \left| \frac{\partial}{\partial x} f_n(0^-, t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_n(0^+, t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_n(1^-, t) \right| \leq C_T.$$

(iii) For $t > 0$, f_n is differentiable at $x = 1$, and

$$(107) \quad -\frac{\partial f_n}{\partial x}(1, t) = f_{T_{n+1}}(t).$$

Proof. The proof of Proposition 3.4 follows from induction. By Proposition 3.3, we have presented the inductive basis at $n = 1$. Now, assuming the inductive hypothesis holds up to $n > 1$, to prove (i), by

$$f_{n+1}(x, t) = \int_0^t f_n(x, t-s) f_{T_1}(s) ds,$$

one may immediately see that (99), (101), (102), and (103) hold. For (100), note that $f_n(0^-, t) = f_n(0^+, t)$ for all $t > 0$ and that $|f_n(x, t)| \leq C_T$ for all $x \in (-\infty, 0) \cup (0, 1), t \leq T$. By the dominated convergence theorem, we have

$$\begin{aligned} & \lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} |f_{n+1}(x_1, t) - f_{n+1}(-x_2, t)| \\ & \leq \lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} \int_0^t |f_n(x_1, t-s) - f_n(-x_2, t-s)| f_{T_1}(s) ds \\ & = 0. \end{aligned}$$

So we have

$$f_{n+1}(0^-, t) = f_{n+1}(0^+, t).$$

Similarly,

$$\begin{aligned} & \lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} \frac{\partial f_{n+1}}{\partial x}(x_1, t) - \frac{\partial f_{n+1}}{\partial x}(-x_2, t) \\ & = \lim_{x_1 \rightarrow 0^+, x_2 \rightarrow 0^+} \int_0^t \frac{\partial f_n}{\partial x}(x_1, t-s) - \frac{\partial f_n}{\partial x}(-x_2, t-s) f_{T_1}(s) ds. \end{aligned}$$

By the inductive hypothesis and the dominated convergence theorem, we have

$$\frac{\partial}{\partial x} f_{n+1}(0^-, t) - \frac{\partial}{\partial x} f_{n+1}(0^+, t) = f_{T_{n+1}}(t).$$

As for (ii), to check the additional regularity conditions, note that by the inductive hypothesis,

$$0 \leq f_{n+1}(x, t) = \int_0^t f_n(x, t - s) f_{T_1}(s) ds \leq C_T.$$

For any $y \in (-\infty, 0) \cup (0, 1)$ and $t \leq T$,

$$\left| \frac{\partial f_{n+1}}{\partial x}(y, t) \right| \leq \int_0^t \left| \frac{\partial f_n}{\partial x}(y, t - s) \right| f_{T_1}(s) ds \leq C_T.$$

Using arguments similar to Proposition 3.3, we have that $\frac{\partial}{\partial x} f_{n+1}(0^-, t)$, $\frac{\partial}{\partial x} f_{n+1}(0^+, t)$, and $\frac{\partial}{\partial x} f_{n+1}(1^-, t)$ are individually bounded by C_T . Finally, for (iii), noting that $|\frac{\partial f_n}{\partial x}(y, t)| \leq C_T$ for all $t \leq T$, $0 < y < 1$, the proof of

$$-\frac{\partial f_n}{\partial x}(1, t) = f_{T_{n+1}}(t) \quad \forall t > 0$$

follows from the same treatment as in Proposition 3.3. □

Now we can finish the proof of Theorem 1.

Proof of Theorem 1. Based on the previous analysis in Propositions 3.1–3.4, we have shown that for $n \geq 0$, f_n is the density function of the measure induced by $F_n(\cdot, t)$ defined in (19) as well as the solution to the sub-PDE problems (35) and (36). Next, we consider the density function of the stochastic process X_t as in (18) that admits the series representation $f(x, t) = \sum_{n=0}^{+\infty} f_n(x, t)$.

In order to prove that $f(x, t)$ satisfies the properties in Theorem 1, we first show that the relevant derivatives of $f(x, t)$ also have the series representations, and the series converge uniformly so that we can pass the regularity from $f_n(x, t)$ to $f(x, t)$. In addition, noting that f_n is the solution to the sub-PDE problems (35) and (36), we can show that $f = \sum_{n=0}^{+\infty} f_n$ satisfies (9), which is the summation of sub-PDE problems (35) and (36).

For any fixed $T > 0$, we first show the uniform convergence of the relevant derivatives of $\sum_{n=0}^{+\infty} f_n(x, t)$ on $((-\infty, 0) \cup (0, 1]) \times [0, T]$. By (98), for all $x_0 \in (-\infty, 0) \cup (0, 1]$, we have for any $0 \leq t \leq T$ and $n \geq 1$,

$$(108) \quad \left| \frac{\partial}{\partial x} f_n(x_0, t) \right| \leq \int_0^T f_{T_1}(s) ds \cdot \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_{n-1}(x_0, t) \right| \leq \rho_T \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_{n-1}(x_0, t) \right|,$$

where

$$(109) \quad \rho_T = \int_0^T f_{T_1}(s) ds = \mathbf{P}_0(T_1 \leq T) \in (0, 1)$$

is a constant that depends only on T . The proof of (109) is quite standard in probability, and thus we defer the whole proof to the appendix. With (108), we have

$$(110) \quad \sum_{n=0}^{+\infty} \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_n(x_0, t) \right| \leq \sum_{n=0}^{+\infty} \rho_T^n \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_0(x_0, t) \right| = \frac{1}{1 - \rho_T} \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_0(x_0, t) \right|,$$

which implies that to show the uniform convergence of such a series, it suffices to check the regularities of $f_0(x, t)$. In fact, with (ii) of Proposition 3.1, we know that for any $\varepsilon_0 \in (0, 1)$, $f_0(x, t) \in C^{2,1}(((-\infty, -\varepsilon_0] \cup [\varepsilon_0, 1]) \times [0, T])$, and thus the last

term in (110) has a uniform bound on any compact subset of $(-\infty, 0) \cup (0, 1]$; i.e., for any compact subset I of $(-\infty, 0) \cup (0, 1]$,

$$\sum_{n=0}^{+\infty} \max_{t \in [0, T]} \max_{x \in I} \left| \frac{\partial}{\partial x} f_n(x_0, t) \right| \leq \frac{1}{1 - \rho_T} \max_{t \in [0, T]} \max_{x \in I} \left| \frac{\partial}{\partial x} f_0(x_0, t) \right| < +\infty.$$

With the same treatment, we know that

$$(111) \quad \sum_{n=0}^{+\infty} f_n(x, t), \quad \sum_{n=0}^{+\infty} \frac{\partial}{\partial t} f_n(x, t), \quad \sum_{n=0}^{+\infty} \frac{\partial}{\partial x} (x f_n(x, t)), \quad \text{and} \quad \sum_{n=0}^{+\infty} \frac{\partial^2}{\partial x^2} f_n(x, t)$$

are inner closed uniformly convergent on $((-\infty, 0) \cup (0, 1]) \times [0, T]$, and thus we can exchange the derivative and the summation in (111). By (110), we have

$$\max_{t \in [0, T]} |\partial_x f(x_0, t)| \leq \sum_{n=0}^{+\infty} \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_n(x_0, t) \right| \leq \frac{1}{1 - \rho_T} \max_{t \in [0, T]} \left| \frac{\partial}{\partial x} f_0(x_0, t) \right|.$$

With the same treatment, we can get the same bounds for the series in (111), from which we can analyze the regularities of $f(x, t)$ by estimating $f_0(x, t)$.

To check (i), we show that $N(t) = -\frac{\partial}{\partial x} f(1^-, t)$ is well defined and $N(t)$ has a series representation in terms of the densities of jumping times. In fact, by uniform convergence, it is clear that $\sum_{n=0}^{+\infty} \frac{\partial}{\partial x} f_n(1^-, t)$ uniformly converges on $[0, T]$. In particular,

$$\frac{\partial f}{\partial x}(1^-, t) = \sum_{n=0}^{\infty} \frac{\partial f_n}{\partial x}(1^-, t).$$

Then by (46) and (107), we also have

$$(112) \quad N(t) = -\frac{\partial f}{\partial x}(1^-, t) = \sum_{n=0}^{\infty} f_{T_n}(t).$$

Note that $f_{T_n}(t) \in C[0, T]$, and thus $N(t) \in C[0, T]$. Hence, (i) is completely proved.

With the uniform convergence of the series representations and the regularities of $f_0(x, t)$ in Proposition 3.1, we can easily show (ii), (iii), (iv), and (v) of Theorem 1. By (44) and (103), we have

$$\lim_{x \rightarrow -\infty} \partial_x f(x, t) = \sum_{n=0}^{+\infty} \lim_{x \rightarrow -\infty} \partial_x f_n(x, t) = 0, \quad t \in (0, T],$$

and thus (v) is valid. Similarly, the uniform convergence, together with the continuity of f_n , $\partial_{xx} f_n$, and $\partial_t f_n$ on $((-\infty, 0) \cup (0, 1]) \times (0, T]$ implies (ii) and (iii). To check (iv), we aim to show that $f_x(0^-, t)$ and $f_x(0^+, t)$ are well defined for $t \in (0, T]$. With a similar analysis, we can prove that for fixed $0 < t \leq T$, $\sum_{n=0}^{\infty} \frac{\partial f_n}{\partial x}(x, t)$ uniformly converge on $[-1, 0)$ and $(0, 1]$, which, together with Lemma 3.1 and the existence of one-sided limits given in (45) and (106), derives (iv) of Theorem 1.

Finally, to prove (vi), that is, density f satisfies PDE problem (9), we need to show that the equation is satisfied and that all the conditions are met as well. With uniform convergence, we can sum (74) from $n = 0$ to $+\infty$, and thus for any

$(x, t) \in ((-\infty, 0) \cup (0, 1)) \times (0, T]$,

$$\begin{aligned}
 & \frac{\partial f}{\partial t} - \frac{\partial}{\partial x}(xf) - \frac{\partial^2 f}{\partial x^2} \\
 (113) \quad &= \frac{\partial}{\partial t} \left(\sum_{n=0}^{+\infty} f_n(x, t) \right) - \frac{\partial}{\partial x} \left(\sum_{n=0}^{+\infty} x f_n(x, t) \right) - \frac{\partial^2}{\partial x^2} \left(\sum_{n=0}^{+\infty} f_n(x, t) \right) \\
 &= \sum_{n=0}^{+\infty} \left(\frac{\partial f_n}{\partial t} - \frac{\partial}{\partial x}(x f_n) - \frac{\partial^2}{\partial x^2} f_n \right) \\
 &= 0.
 \end{aligned}$$

With the regularities of f proved above, all the initial and boundary conditions in (9) are trivially satisfied, but we need to prove the jump condition on f_x at $x = 0$. Given any fixed $t > 0$, for any $\epsilon > 0$, due to the uniform convergence, there is a constant $N < \infty$ such that

$$(114) \quad \left| \sum_{n=N+1}^{\infty} \frac{\partial f_n}{\partial x}(x, t) \right| < \epsilon \quad \forall x \in (-\infty, 0] \cup [0, 1],$$

where at 0, 1 the derivatives are understood in the one-sided sense. Moreover, for the now fixed N , by (100), there exists $\delta > 0$, such that for all $y < 0 < x$, $|x|, |y| \leq \delta$,

$$(115) \quad \left| \frac{\partial f_0}{\partial x}(x, t) - \frac{\partial f_0}{\partial x}(y, t) \right| \leq \epsilon$$

and

$$(116) \quad \sum_{n=1}^N \left| \frac{\partial f_n}{\partial x}(x, t) - \frac{\partial f_n}{\partial x}(y, t) + f_{T_n}(t) \right| < \epsilon.$$

Combining (114)–(116), we have

$$\begin{aligned}
 & \left| \frac{\partial f}{\partial x}(x, t) - \frac{\partial f}{\partial x}(y, t) + \sum_{n=1}^{\infty} f_{T_n}(t) \right| \\
 &= \left| \sum_{n=0}^{\infty} \left(\frac{\partial f_n}{\partial x}(x, t) - \frac{\partial f_n}{\partial x}(y, t) \right) + \sum_{n=1}^{\infty} f_{T_n}(t) \right| \\
 &\leq \left| \frac{\partial f_0}{\partial x}(x, t) - \frac{\partial f_0}{\partial x}(y, t) \right| + \left| \sum_{n=1}^N \left(\frac{\partial f_n}{\partial x}(x, t) - \frac{\partial f_n}{\partial x}(y, t) + f_{T_n}(t) \right) \right| \\
 &+ \left| \sum_{n=N+1}^{\infty} \frac{\partial f_n}{\partial x}(x, t) \right| + \left| \sum_{n=N+1}^{\infty} \frac{\partial f_n}{\partial x}(y, t) \right| + \left| \sum_{n=N+1}^{\infty} f_{T_n}(t) \right| \leq 5\epsilon,
 \end{aligned}$$

and thus we conclude that for $t > 0$,

$$\frac{\partial}{\partial x} f(0^-, t) - \frac{\partial}{\partial x} f(0^+, t) = -\frac{\partial}{\partial x} f(1^-, t).$$

Similarly, we can get, for $t > 0$,

$$f(0^-, t) = f(0^+, t).$$

Now that we have thoroughly checked (vi), the proof of Theorem 1 is complete. \square

With the same steps as in the proof of Theorem 1, we can show Corollary 2.1. Next, we only focus on proving Theorem 2. Due to the results for the process X_t as in (18) that, starting from $y < 1$, are largely parallel to the result starting from 0, we only provide a sketch of the proof for those parts.

Now, note that ν is a c.d.f. whose p.d.f. $f_{\text{in}}(x) \in C_c(-\infty, 1)$, and that $f_{\text{in}}(x)$ is continuous and compactly supported in $(-\infty, 1 - \varepsilon_0]$ for some $\varepsilon_0 > 0$. Without loss of generality, we assume $f_{\text{in}}(x)$ is supported in $[-C_0, 1 - \varepsilon_0]$ for some $C_0 > 0$. Thus for the fixed $T > 0$ we have the following:

- (1) By conditional distribution, we have, for any $x \in (-\infty, 1]$, $t \in (0, T]$,

$$f^\nu(x, t) = \int_{-\infty}^{1-\varepsilon_0} f^y(x, t) f_{\text{in}}(y) dy.$$

- (2) For all $t \in (0, T]$, $x \neq 0$ or 1 , $f^y(x, t)$ is continuous with respect to y .
- (3) All the regularities and convergences in Corollary 2.1 are uniform with respect to $y \in (-\infty, 1 - \varepsilon_0]$. Actually, for all $\varepsilon_1 > 0$, $t_0 > 0$, and any $x \in (-\infty, -\varepsilon_1] \cup [-\varepsilon_1, 1)$, $t \in [t_0, T]$, $y \in (-\infty, 1 - \varepsilon_0]$, we have

$$\begin{aligned} |f^y(x, t)| &\leq C_{\varepsilon_0, \varepsilon_1, t_0, T}^{(0)}, & |\partial_x f^y(x, t)| &\leq C_{\varepsilon_0, \varepsilon_1, t_0, T}^{(1)}, \\ |\partial_t f^y(x, t)| &\leq C_{\varepsilon_0, \varepsilon_1, t_0, T}^{(2)}, & |\partial_{xx} f^y(x, t)| &\leq C_{\varepsilon_0, \varepsilon_1, t_0, T}^{(3)}. \end{aligned}$$

Moreover, for all $t \in [t_0, T]$, $y \in (-\infty, 1 - \varepsilon_0]$, and $x \in [-1, 0) \cup (0, 1)$,

$$|\partial_x f^y(x, t)| \leq C_{\varepsilon_0, t_0, T}.$$

- (4) Then we can take the derivative into the integral in (12), i.e., for $\ell = 0, 1, 2$, $\partial^\ell = \partial_{tx}^\ell = \partial_t^m \partial_x^n$, $\ell = 2m + n$,

$$\partial^\ell f^\nu(x, t) = \int_{-\infty}^{1-\varepsilon_0} \partial^\ell f^y(x, t) \nu(dy), \quad x \in (-\infty, 1], \quad t > 0,$$

and thus,

$$N^\nu(t) := -\partial_x f^\nu(1^-, t) = -\int_{-\infty}^1 \partial_x f^y(1^-, t) \nu(dy) = \int_{-\infty}^1 N^y(t) \nu(dy).$$

By the regularities and convergences for $f^y(x, t)$ in Corollary 2.1, we get properties (i), (ii), (iii), (iv), and (v) for $f^\nu(x, t)$.

- (5) Finally, we check the L^2 convergence (13). We first turn the problem into proving L^1 convergence by showing the uniform boundedness of $f^\nu(x, t)$ when t is sufficiently small. In fact, similar to the decomposition in (34), we have

$$(117) \quad f^y(x, t) = \sum_{n=0}^{+\infty} f_n^y(x, t),$$

where $f_n^y(x, t) dx = \mathbf{P}(X_t^y \in dx, n_t = n)$ as in (19). With (22), we have

$$(118) \quad f_0^y(x, t) \leq f_{\text{ou}}^y(x, t) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp \left\{ \frac{-(x - e^{-t}y)^2}{2(1 - e^{-2t})} \right\}.$$

By the same method as in Lemma 2.2, we get the iteration relationship for any $n \geq 1$,

$$(119) \quad f_n^y(x, t) = \int_0^t f_{T_1^y}(t-s) f_{n-1}(x, s) ds.$$

Using (23), we know that for any $t > 0$, $f_0(x, t) \leq f_{ou}(x, t) \leq \frac{C}{\sqrt{t}}$ and with an estimation similar to that in Proposition 3.2, we have for any $k \in \mathbb{N}$, all sufficiently small t , and $s \leq t$,

$$f_{T_1^y}(t-s) \leq C_k t^k,$$

where the constant C_k is independent of all $y \leq 1 - \varepsilon_0$. Thus

$$(120) \quad f_1^y(x, t) \leq C_k t^k \int_0^t \frac{1}{\sqrt{s}} ds \leq C_k t^{k+\frac{1}{2}}.$$

Repeat calculations in (120); with the iteration (119), one has for all sufficiently small t ,

$$f_n^y(x, t) \leq (Ct)^n,$$

and thus for all sufficiently small t ,

$$(121) \quad \sum_{n=1}^{+\infty} f_n^y(x, t) \leq \frac{Ct}{1-Ct} \leq C.$$

Combining (117), (118), and (121), we have

$$\begin{aligned} f^\nu(x, t) &\leq \int_{-\infty}^{1-\varepsilon_0} [f_{ou}^y(x, t) + C] f_{in}(y) dy \\ &\leq C + \|f_{in}(y)\|_{L^\infty(-\infty, 1-\varepsilon_0]} \int_{-\infty}^{1-\varepsilon_0} f_{ou}^y(x, t) dy. \end{aligned}$$

Note that by (118), $\int_{-\infty}^{1-\varepsilon_0} f_{ou}^y(x, t) dy$ is uniformly bounded for any x and sufficiently small t , and so is f^ν . Note that both $f_{in}(x)$ and $f^\nu(x, t)$ are uniformly bounded for all sufficiently small t ; thus to prove (13), it suffices to prove

$$(122) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} |f^\nu(x, t) - f_{in}(x)| dx = 0.$$

To get (122), for a suitable constant M_0 whose value will be specified in the following, we introduce the following decomposition:

$$(123) \quad \int_{-\infty}^{+\infty} |f^\nu(x, t) - f_{in}(x)| dx = \left(\int_{-\infty}^{-M_0} + \int_{-M_0}^1 \right) |f^\nu(x, t) - f_{in}(x)| dx =: P_1 + P_2.$$

First, to bound P_1 , we have the following lemma.

LEMMA 3.2. *Consider the process X_t as in (18) that starts from y . For any $\varepsilon > 0$, there exist $t_0 > 0$ and $M_0 < \infty$ such that for any $t \in [0, t_0]$ and any $y \in \text{supp}(f_{in}) = [-C_0, 1 - \varepsilon_0]$,*

$$(124) \quad \mathbf{P}^y(X_t \leq -M_0) \leq \varepsilon.$$

Proof. Note that according to the construction of the process X_t as in (18) that starts from y , we have

$$\{X_t > -M_0\} \supset \{Y_t^{(1)} > -M_0\} \cap \{T_1 > t\},$$

which immediately implies

$$(125) \quad \mathbf{P}^y(X_t \leq -M_0) \leq \mathbf{P}^y(Y_t^{(1)} \leq -M_0) + \mathbf{P}^y(T_1 > t) := Q_1 + Q_2.$$

For Q_2 when $t \leq t_0$,

$$\mathbf{P}^y(T_1 \leq t) = \int_0^t f_{T_1^y}(s) ds \leq C_k \int_0^t s^k ds.$$

So, letting $k = 1$ and $t_0 = \sqrt{\frac{\varepsilon}{C_1}}$, we have for all $t \leq t_0$

$$(126) \quad \mathbf{P}^y(T_1 \leq t) \leq C_1 \int_0^t s ds \leq \frac{1}{2}\varepsilon.$$

For Q_1 , noting that $Y_t^{(1)}$ is Gaussian, we can choose M_0 large enough to control Q_1 , and we complete the proof. \square

Remark 5. Without loss of generality, we choose the constant M_0 in Lemma 3.2 larger than C_0 .

Lemma 3.2 immediately implies that

$$(127) \quad F^\nu(-M_0, t) = \mathbf{P}^\nu(X_t \leq -M_0) = \int_{-C_0}^{1-\varepsilon_0} \mathbf{P}^y(X_t \leq -M_0) f_{\text{in}}(y) dy \leq \varepsilon.$$

For any $\varepsilon > 0$, $\exists t_0 > 0$ and $M_0 < \infty$ such that for all $t < t_0$,

$$(128) \quad P_1 = \int_{-\infty}^{-M_0} f^\nu(x, t) dx = \mathbf{P}^\nu(X_t \leq -M_0) < \varepsilon.$$

To estimate P_2 in (123), we show the following.

LEMMA 3.3. *For any $\varepsilon > 0$, there is a $t_1 > 0$ such that for any $t \in (0, t_1]$ and $x \in \mathbb{R}$,*

$$f^\nu(x, t) \leq f_{\text{in}}(x) + \varepsilon.$$

Proof. Note that when $x > 1$, $f^\nu(x, t) = f_{\text{in}}(x) = 0$; thus we need only focus on $x \in (-\infty, 1]$. By (117), (118), and (121), we already have

$$(129) \quad f^\nu(x, t) \leq \int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy + \frac{Ct}{1 - Ct},$$

and $\frac{Ct}{1 - Ct} \rightarrow 0$ as $t \rightarrow 0^+$. Thus we need only bound $\int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy$. To do this we will separate the cases $x \in [-C_0 - 1, 1]$ and $x \in (-\infty, -C_0 - 1)$.

(i) When x belongs to the compact set $[-C_0 - 1, 1]$, by (118) we have

$$(130) \quad f_{\text{ou}}^y(x, t) = e^t \frac{1}{\sqrt{2\pi(1 - e^{-2t})e^{2t}}} \exp\left\{-\frac{(y - xe^t)^2}{2(1 - e^{-2t})e^{2t}}\right\},$$

which is equal to the multiply of e^t and the p.d.f. of the normal distribution $N(xe^t, (1 - e^{-2t})e^{2t})$. Note that $f_{\text{in}}(y)$ is uniformly continuous; thus for any $\varepsilon > 0$, $\exists \delta > 0$ such that for all $|x_1 - x_2| \leq \delta$, we have $|f_{\text{in}}(x_1) - f_{\text{in}}(x_2)| \leq \varepsilon$, and $\exists t_2 > 0$ such that for all $t < t_2$ and $x \in [-C_0 - 1, 1]$, $|x - e^t x| < \frac{\delta}{2}$. Moreover, for the fixed δ above, $\exists t_3 > 0$ such that for all $t \in (0, t_3)$,

$$(131) \quad \mathbf{P} \left(|N(0, 1)| \geq \frac{\delta}{2\sqrt{(1 - e^{-2t})e^{2t}}} \right) \leq \frac{\varepsilon}{\|f_{\text{in}}\|_{L^\infty}},$$

where $N(0, 1)$ stands for the standard normal distribution. Thus for $t_1 = t_2 \wedge t_3$,

$$(132) \quad \int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy = \left(\int_{xe^t - \frac{\delta}{2}}^{xe^t + \frac{\delta}{2}} + \int_{\mathbb{R} \setminus [xe^t - \frac{\delta}{2}, xe^t + \frac{\delta}{2}]} \right) f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy =: K_1 + K_2.$$

For K_1 , we have by (130)

$$(133) \quad K_1 \leq \max_{y \in [x-\delta, x+\delta]} f_{\text{in}}(y) \cdot e^t \leq \|f_{\text{in}}\|_{L^\infty} (e^t - 1) + \max_{y \in [x-\delta, x+\delta]} f_{\text{in}}(y) \leq f_{\text{in}}(x) + \varepsilon.$$

For K_2 , we have by (131)

$$(134) \quad K_2 \leq \|f_{\text{in}}\|_{L^\infty} \int_{\mathbb{R} \setminus [xe^t - \frac{\delta}{2}, xe^t + \frac{\delta}{2}]} f_{\text{ou}}^y(x, t) dy \leq e^t \frac{\varepsilon}{\|f_{\text{in}}\|_{L^\infty}} = e^t \cdot \varepsilon.$$

Combining (133) and (134), we see that the proof of case (i) is complete.

- (ii) Note that $f_{\text{in}}(x) = 0$ on $x \in (-\infty, -C_0 - 1)$ and $f^\nu(x, t) = 0$ on $x \geq 1$. We need only prove that for all $\varepsilon > 0$, $\exists t_1 > 0$ such that for all $t \in (0, t_1]$ and any $x < -C_0 - 1$,

$$(135) \quad \int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy < \varepsilon.$$

By (118) and noting that for any $x < -C_0 - 1$ and $y \in [-C_0, 1]$, we have $|x - e^{-t}y| \geq 1$, and thus

$$\begin{aligned} & \int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy \\ & \leq (C_0 + 1) \|f_{\text{in}}\|_{L^\infty} \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp\left(-\frac{1}{2(1 - e^{-2t})}\right) \\ & \leq (C_0 + 1) \|f_{\text{in}}\|_{L^\infty} \frac{u}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right), \end{aligned}$$

where $u := (1 - e^{-2t})^{-\frac{1}{2}}$. Thus we know that $\int_{-\infty}^{+\infty} f_{\text{ou}}^y(x, t) f_{\text{in}}(y) dy \rightarrow 0$ as $t \rightarrow 0^+$, and the proof of Lemma 3.3 is complete. \square

With Lemma 3.3, now we conclude the proof of (13). For the fixed M_0 in Lemma 3.2, there exists $t_2 \geq 0$ such that for all $t \in (0, t_2]$ and $x \in \mathbb{R}$,

$$f^\nu(x, t) \leq f_{\text{in}}(x) + \frac{\varepsilon}{M_0 + 1}.$$

Noting that $|a - b| \leq b - a + 2 \max\{a - b, 0\}$, we have

$$\begin{aligned} P_2 &\leq \int_{-M_0}^1 \left[f_{\text{in}}(x) - f^\nu(x, t) + \frac{2\varepsilon}{M_0 + 1} \right] dx \\ (136) \quad &= \int_{-M_0}^1 f_{\text{in}}(x) dx - \int_{-M_0}^1 f^\nu(x, t) dx + 2\varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

Combining (128) and (136), we get (13). The proof of Theorem 2 is complete.

3.2. Weak solution. In this section, we show that the density of X_t , which we denote by $f(x, t)$ and $N(t) = \sum_{n=1}^{+\infty} F'_{T_n}(t)$, is the weak solution of PDE problem (9). We adopt the definition of the weak solution of (9) as in [5]. The main theorem in this section is as follows.

THEOREM 3. *Let $f^\nu(x, t)$ be the p.d.f. of the process X_t as in (18) that starts from p.d.f. $f_{\text{in}}(x) \in C_c(-\infty, 1)$, and let $N^\nu(t) := \sum_{n=1}^{+\infty} F'_{T_n}(t)$. The pair (f, N) is a weak solution of (9) in the following sense: for any test function $\phi(x, t) \in C^\infty((-\infty, 1] \times [0, T])$ such that $\frac{\partial^2 \phi}{\partial x^2}, x \frac{\partial \phi}{\partial x} \in L^\infty((-\infty, 1] \times [0, T])$, we have*

$$\begin{aligned} (137) \quad &\int_0^T \int_{-\infty}^1 \left(\frac{\partial}{\partial t} \phi - x \frac{\partial}{\partial x} \phi + \frac{\partial^2}{\partial x^2} \phi \right) f^\nu(x, t) dx dt \\ &= \int_0^T (\phi(1, t) - \phi(0, t)) N^\nu(t) dt - \int_{-\infty}^1 \phi(x, 0) f_{\text{in}}(x) dx + \int_{-\infty}^1 \phi(x, T) f^\nu(x, T) dx. \end{aligned}$$

The convergence of the series $\sum_{n=1}^{+\infty} F'_{T_n}(t)$ relies on the proof of Theorem 2, by which we already know that $f^\nu(x, t)$ is a solution to PDE problem (9). To prove that (f^ν, N^ν) is also a weak solution of (9), one simply multiplies the equation by the test function ϕ and carries out the integration by parts in space and in time, respectively. Since the calculation is rather straightforward, we choose to omit the details in this work, but we remark that the weak-strong uniqueness is still an open problem for such a Fokker–Planck equation with a flux-shift structure, and we will continue research along this line in the future.

4. Appendix. Now we shall show (109). In the following, we let X_t as in (14) denote an OU process starting from 0, and we define the stopping time T_1 as the first time that X_t hits 1, i.e., $T_1 = \inf\{t \geq 0, X_t = 1\}$. Now it suffices to prove that for all fixed $T \in (0, +\infty)$,

$$(138) \quad \mathbf{P}(T_1 > T) > 0.$$

In order to show (138), we show that the probability of an event being included in $\{T_1 > T\}$ is positive. Actually, we construct a sequence of stopping times and use the strong Markov property to decompose the process X_t such that at each time $|X_t| > 1$, it escapes from -1 . By showing that the product of the probability of an event sequence is positive, we complete the proof. Now we show a useful lemma.

LEMMA 4.1. For the OU process X_t defined above, define a stopping time $\tau_1 = \inf\{t \geq 0, |X_t| = 1\}$; then

$$(139) \quad \begin{cases} \mathbf{P}(\tau_1 < +\infty) = 1, \\ (140) \quad \mathbf{P}(\tau_1 > \frac{1}{16}, X_{\tau_1} = -1) = \mathbf{P}(\tau_1 > \frac{1}{16}, X_{\tau_1} = 1) > 0. \end{cases}$$

Proof. Line (139) follow from the fact that $\tau_1 < \inf\{n \in \mathbb{N}, |X_n| > 1\}$, the Markov property, and the Gaussian transition distribution of X_t . As for (140), by symmetry, we need only prove

$$(141) \quad \mathbf{P}\left(\tau_1 > \frac{1}{16}\right) > 0.$$

By (14), $X_t = \sqrt{2} \int_0^t e^{-(t-s)} dB_s$, and thus

$$\left\{ \tau_1 \leq \frac{1}{16} \right\} = \left\{ \max_{t \leq \frac{1}{16}} |X_t| \geq 1 \right\} \subset \left\{ \max_{t \leq \frac{1}{16}} \left| \sqrt{2} \int_0^t e^{-(t-s)} dB_s \right| \geq 1 \right\}.$$

Now note that $\int_0^t e^s dB_s$ is a martingale, and then

$$(142) \quad \begin{aligned} \mathbf{P}\left(\tau_1 \leq \frac{1}{16}\right) &\leq \mathbf{P}\left(\max_{t \leq \frac{1}{16}} \left| \sqrt{2} \int_0^t e^{-(t-s)} dB_s \right| \geq 1\right) \\ &\leq 2\mathbf{E}\left(\max_{t \leq \frac{1}{16}} \left| \int_0^t e^s dB_s \right|\right)^2 \leq 8\mathbf{E}\left(\int_0^{\frac{1}{16}} e^s dB_s\right)^2, \end{aligned}$$

where the last two inequalities follow from the Markov inequality and Doob’s inequality, respectively. Note that

$$\mathbf{E}\left(\int_0^{\frac{1}{16}} e^s dB_s\right)^2 = \int_0^{\frac{1}{16}} e^{2s} ds = \frac{1}{2}(e^{\frac{1}{8}} - 1) < \frac{1}{8},$$

and thus (141) is valid. □

With the above lemma, now we prove that (138) is equivalent to (109).

Proof of (109). We let Y_t be an OU process starting at -1 and derive stopping time $\tau'_1 = \inf\{t \geq 0, Y_t = 0\}$. Then by the recurrence of the OU process,

$$(143) \quad \mathbf{P}(\tau'_1 < +\infty) = 1.$$

Next we define an increasing sequence of stopping times as follows:

$$\begin{aligned} S'_0 &= 0, \\ S_1 &= \inf\{t \geq 0, |X_t| = 1\}, \\ S'_1 &= \inf\{t \geq S_1, X_t = 0\}, \\ S_2 &= \inf\{t \geq S'_1, |X_t| = 1\}, \\ S'_2 &= \inf\{t \geq S_2, X_t = 0\}, \\ &\vdots \end{aligned}$$

Combining (139), (143), and the strong Markov property of the OU process, we have that $S_n, S'_n < +\infty$ for all n . At the same time,

$$S_1 - S'_0, S'_1 - S_1, S_2 - S'_1, \dots$$

are independent of each other, while

$$\begin{aligned} S_n - S'_{n-1} &\stackrel{d}{=} \tau_1, \\ S'_n - S_n &\stackrel{d}{=} \tau'_1. \end{aligned}$$

Thus for the fixed $T \in (0, +\infty)$ above, let $N_0 = \lfloor T \rfloor + 1$, and then

$$\{T_1 > T\} \supset \cap_{i=1}^{16N_0} \left\{ S_i - S'_{i-1} > \frac{1}{16}, X_{S_i} = -1, S'_i - S_i < +\infty \right\}.$$

Using the strong Markov property, we have

$$\begin{aligned} \mathbf{P}(T_1 > T) &\geq \mathbf{P} \left(\cap_{i=1}^{16N_0} \left\{ S_i - S'_{i-1} > \frac{1}{16}, X_{S_i} = -1, S'_i - S_i < +\infty \right\} \right) \\ &= \prod_{n=1}^{16N_0} \mathbf{P} \left(\tau_1 > \frac{1}{16}, X_{\tau_1} = -1 \right) > 0, \end{aligned}$$

which completes the proof of (109). \square

REFERENCES

- [1] I. BEN-ARI AND R. G. PINSKY, *Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure*, J. Funct. Anal., 251 (2007), pp. 122–140, <https://doi.org/10.1016/j.jfa.2007.05.019>.
- [2] I. BEN-ARI AND R. G. PINSKY, *Ergodic behavior of diffusions with random jumps from the boundary*, Stochastic Process. Appl., 119 (2009), pp. 864–881, <https://doi.org/10.1016/j.spa.2008.05.002>.
- [3] N. BRUNEL, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons*, J. Comput. Neurosci., 8 (2000), pp. 183–208.
- [4] N. BRUNEL AND V. HAKIM, *Fast global oscillations in networks of integrate-and-fire neurons with low firing rates*, Neural Comput., 11 (1999), pp. 1621–1671.
- [5] M. J. CÁCERES, J. A. CARRILLO, AND B. PERTHAME, *Analysis of nonlinear noisy integrate & fire neuron models: Blow-up and steady states*, J. Math. Neurosci., 1 (2011), 7.
- [6] M. J. CÁCERES, J. A. CARRILLO, AND L. TAO, *A numerical solver for a nonlinear Fokker–Planck equation representation of neuronal network dynamics*, J. Comput. Phys., 230 (2011), pp. 1084–1099.
- [7] M. J. CÁCERES AND B. PERTHAME, *Beyond blow-up in excitatory integrate and fire neuronal networks: Refractory period and spontaneous activity*, J. Theoret. Biol., 350 (2014), pp. 81–89, <https://doi.org/10.1016/j.jtbi.2014.02.005>.
- [8] M. J. CÁCERES, P. ROUX, D. SALORT, AND R. SCHNEIDER, *Global-in-time solutions and qualitative properties for the NNLF neuron model with synaptic delay*, Comm. Partial Differential Equations, 44 (2019), pp. 1358–1386.
- [9] M. J. CÁCERES AND R. SCHNEIDER, *Analysis and numerical solver for excitatory-inhibitory networks with delay and refractory periods*, ESAIM Math. Model. Numer. Anal., 52 (2018), pp. 1733–1761, <https://doi.org/10.1051/m2an/2018014>.
- [10] J. A. CARRILLO, M. D. M. GONZÁLEZ, M. P. GUALDANI, AND M. E. SCHONBEK, *Classical solutions for a nonlinear Fokker–Planck equation arising in computational neuroscience*, Comm. Partial Differential Equations, 38 (2013), pp. 385–409.
- [11] J. A. CARRILLO, B. PERTHAME, D. SALORT, AND D. SMETS, *Qualitative properties of solutions for the noisy integrate and fire model in computational neuroscience*, Nonlinearity, 28 (2015), pp. 3365–3388.

- [12] J. CHEVALLIER, *Mean-field limit of generalized Hawkes processes*, Stochastic Process Appl., 127 (2017), pp. 3870–3912.
- [13] J. CHEVALLIER, M. J. CÁCERES, M. DOUMIC, AND P. REYNAUD-BOURET, *Microscopic approach of a time elapsed neural model*, Math. Models Methods Appl. Sci., 25 (2015), pp. 2669–2719.
- [14] A. COMPTE, N. BRUNEL, P. S. GOLDMAN-RAKIC, AND X.-J. WANG, *Synaptic mechanisms and network dynamics underlying spatial working memory in a cortical network model*, Cerebral Cortex, 10 (2000), pp. 910–923.
- [15] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ, *First Hitting Times for General Non-homogeneous 1d Diffusion Processes: Density Estimates in Small Time*, preprint, hal-00870991f, 2013.
- [16] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stochastic Process. Appl., 125 (2015), pp. 2451–2492, <https://doi.org/10.1016/j.spa.2015.01.007>.
- [17] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ, *Global solvability of a networked integrate-and-fire model of McKean–Vlasov type*, Ann. Appl. Probab., 25 (2015), pp. 2096–2133.
- [18] F. DELARUE, S. NADTOCHIY, AND M. SHKOLNIKOV, *Global Solutions to the Supercooled Stefan Problem with Blow-Ups: Regularity and Uniqueness*, preprint, <https://arxiv.org/abs/1902.05174>, 2019.
- [19] W. FELLER, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. of Math. (2), 55 (1952), pp. 468–519, <https://doi.org/10.2307/1969644>.
- [20] W. FELLER, *Diffusion processes in one dimension*, Trans. Amer. Math. Soc., 77 (1954), pp. 1–31, <https://doi.org/10.2307/1990677>.
- [21] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Dover Publications, Inc., 2008.
- [22] M. G. GARRONI AND J. L. MENALDI, *Green Functions for Second Order Parabolic Integro-differential Problems*, Pitman Res. Notes Math. Ser. 275, Longman Scientific & Technical, 1992.
- [23] W. GERSTNER AND W. M. KISTLER, *Spiking Neuron Models: Single Neurons, Populations, Plasticity*, Cambridge University Press, 2002.
- [24] Ī. Ī. GHĪMAN AND A. V. SKOROHOD, *Stochastic Differential Equations*, translated from the Russian by K. Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 72, Springer-Verlag, 1972.
- [25] T. GUILLAMON I GRABOLOSÀ, *An introduction to the mathematics of neural activity*, Butl. Soc. Catalana Mat., 19 (2004), pp. 25–45.
- [26] B. HAMBLY AND S. LEDGER, *A stochastic McKean–Vlasov equation for absorbing diffusions on the half-line*, Ann. Appl. Probab., 27 (2017), pp. 2698–2752.
- [27] B. HAMBLY, S. LEDGER, AND A. SØJMARK, *A McKean–Vlasov equation with positive feedback and blow-ups*, Ann. Appl. Probab., 29 (2019), pp. 2338–2373.
- [28] J. HU, J.-G. LIU, Y. XIE, AND Z. ZHOU, *A structure preserving numerical scheme for Fokker–Planck equations of neuron networks: Numerical analysis and exploration*, J. Comput. Phys., 433 (2021), 110195.
- [29] J. INGLIS AND D. TALAY, *Mean-field limit of a stochastic particle system smoothly interacting through threshold hitting-times and applications to neural networks with dendritic component*, SIAM J. Math. Anal., 47 (2015), pp. 3884–3916, <https://doi.org/10.1137/140989042>.
- [30] L. LAPIQUE, *Recherches quantitatives sur l’excitation électrique des nerfs traitée comme une polarisation*, J. Physiol. Pathol., 9 (1907), pp. 620–635.
- [31] M. MATTIA AND P. DEL GIUDICE, *Population dynamics of interacting spiking neurons*, Phys. Rev. E, 66 (2002), 051917.
- [32] S. NADTOCHIY AND M. SHKOLNIKOV, *Particle Systems with Singular Interaction through Hitting Times: Application in Systemic Risk Modeling*, preprint, <https://arxiv.org/abs/1705.00691>, 2017.
- [33] S. NADTOCHIY AND M. SHKOLNIKOV, *Mean field systems on networks, with singular interaction through hitting times*, Ann. Probab., 48 (2020), pp. 1520–1556.
- [34] K. A. NEWHALL, G. KOVAČIĆ, P. R. KRAMER, AND D. CAI, *Cascade-induced synchrony in stochastically driven neuronal networks*, Phys. Rev. E, 82 (2010), 041903.
- [35] K. A. NEWHALL, G. KOVACIC, P. R. KRAMER, D. ZHOU, A. V. RANGAN, AND D. CAI, *Dynamics of current-based, Poisson driven, integrate-and-fire neuronal networks*, Comm. Math. Sci., 8 (2010), pp. 541–600.
- [36] D. Q. NYKAMP, *A Population Density Approach That Facilitates Large-Scale Modeling of Neural Networks*, Ph.D. thesis, New York University ProQuest Dissertation Publishing, 2000, <https://www.proquest.com/docview/304608460>.

- [37] A. OMURTAG, B. W. KNIGHT, AND L. SIROVICH, *On the simulation of large populations of neurons*, J. Comput. Neurosci., 8 (2000), pp. 51–63.
- [38] J. PENG, *A note on the first passage time of diffusions with holding and jumping boundary*, Statist. Probab. Lett., 93 (2014), pp. 58–64, <https://doi.org/10.1016/j.spl.2014.06.012>.
- [39] J. PENG AND W. V. LI, *Diffusions with holding and jumping boundary*, Sci. China Math., 56 (2013), pp. 161–176, <https://doi.org/10.1007/s11425-012-4416-9>.
- [40] P. PROTTER, *Stochastic differential equations with jump reflection at the boundary*, Stochastics, 3 (1980), pp. 193–201, <https://doi.org/10.1080/17442508008833144>.
- [41] A. RENART, N. BRUNEL, AND X.-J. WANG, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, in Computational Neuroscience: A Comprehensive Approach, J. Feng., ed., Chapman & Hall/CRC, 2004, pp. 425–484.
- [42] L. SIROVICH, A. OMURTAG, AND K. LUBLINER, *Dynamics of neural populations: Stability and synchrony*, Network Comput. Neural Syst., 17 (2006), pp. 3–29.
- [43] L. SŁOMIŃSKI AND T. WOJCIECHOWSKI, *Stochastic differential equations with jump reflection at time-dependent barriers*, Stochastic Process. Appl., 120 (2010), pp. 1701–1721, <https://doi.org/10.1016/j.spa.2010.04.008>.
- [44] J. TOUBOUL, G. HERMANN, AND O. FAUGERAS, *Noise-induced behaviors in neural mean field dynamics*, SIAM J. Appl. Dyn. Syst., 11 (2012), pp. 49–81, <https://doi.org/10.1137/110832392>.
- [45] H. C. TUCKWELL, *Introduction to Theoretical Neurobiology: Volume 2: Nonlinear and Stochastic Theories*, Cambridge Stud. Math. Biol. 8, Cambridge University Press, 1988.