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LARGE TIME BEHAVIORS OF UPWIND SCHEMES AND B-SCHEMES FOR FOKKER-PLANCK EQUATIONS ON $\mathbb{R}$ BY JUMP PROCESSES

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Abstract. We revisit some standard schemes, including upwind schemes and some B-schemes, for linear conservation laws from the viewpoint of jump processes, allowing the study of them using probabilistic tools. For Fokker-Planck equations on $\mathbb{R}$, in the case of weak confinement, we show that the numerical solutions converge to some stationary distributions. In the case of strong confinement, using a discrete Poincaré inequality, we prove that the $O(h)$ numeric error under ℓ^1 norm is uniform in time, and establish the uniform exponential convergence to the steady states. Compared with the traditional results of exponential convergence of these schemes, our result is in the whole space without boundary. We also establish similar results on the torus for which the stationary solution of the scheme does not have detailed balance. This work could motivate better understanding of numerical analysis for conservation laws, especially parabolic conservation laws, in unbounded domains.

1. Introduction

It is well known that for numerically solving the partial differential equations (PDEs), suitable discretization must be used to preserve correct physics. For example, in discretizing hyperbolic equations or the convection terms in mixed type equations such as the Navier-Stokes equations, the upwind scheme is usually used to numerically simulate the direction of propagation of information and to ensure desired stability [31]. For nonlinear hyperbolic conservation laws, the upwind scheme (and generally the so-called monotone schemes [26]) can guarantee that the numerical solutions converge to the entropy weak solution [9, 26], important for physical phenomena like shocks. Even for parabolic conservation laws where the solutions are smooth, like Fokker-Planck equations, correct discretization must be adopted so that the correct equilibrium can be recovered [42]. Such type of discretizations are often nonlinear, which is necessary even for linear parabolic equations.

We are interested in spatial discretization of linear conservation laws while keeping the time variable continuous (for simulation, one can use the methods in [48] to get fully discretized schemes or just leave the time variable continuous as in

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section 7). In particular, we apply the upwind schemes, and the B-schemes [6] for linear parabolic equations to the Fokker-Planck equations on $\mathbb{R}$, given by

$$\partial_t \rho = -\partial_x (b(x)\rho) + \frac{1}{2}\partial_{xx}(\sigma^2 \rho), \tag{1.1}$$

where $\rho \geq 0$ often describes the density while $b(\cdot)$ and $\sigma(\cdot)$ are given functions. The Fokker-Planck equations are closely related to the stochastic differential equations (SDEs) (see [41] and section 2 below for more details). We are interested to see whether these schemes can capture the correct equilibrium for large time in unbounded domains.

Let us first focus on the upwind schemes for conservation laws and we take the one dimensional case as the example. In general, the scalar conservation law for $\rho : (x,t) \mapsto \rho(x,t)$ in 1D space is given by

$$\partial_t \rho + \partial_x (f(x,\rho)) = \partial_x (D(x)\partial_x \rho). \tag{1.2}$$

Here $\partial_x(f(x,\rho)) = \partial_x f(x,\rho) + \partial_\rho f(x,\rho)\partial_x \rho$. We will assume all functions are smooth enough, $f(x,0) = 0$, and $D(x) \geq 0$. If $D(x) = 0$, we have the hyperbolic conservation laws. For $D = 0$, Kružkov proved in [29] that if $\partial_x f(x,\rho)$ is locally Lipschitz in ρ, the bounded weak solution satisfying an entropy condition ([29, Definition 1]) is unique. The existence result of such solutions in [29] requires that the derivatives of $f(x,\rho)$ satisfy some boundedness conditions uniform in x so that the vanishing viscosity method works. In particular, if $f(x,\rho) = f_1(\rho)$ with f_1 being locally Lipschitz, the existence result holds. With suitable assumptions on the flux $f(x,\rho)$, like $f(x,\rho) = f_1(\rho)$, or some confinement conditions, $\int_{\mathbb{R}} \rho\, dx$ is a constant (see, for example, [47, Proposition 2.3.6]). For general fluxes that can depend on x, even if the equation is well-posed, the total mass can decay because some mass can escape to infinity, like $\rho_t + \partial_x((1+x^2)\rho) = 0$. For upwind discretization, we decompose the flux as

$$f = f_+ - f_-,\ \partial_\rho f_\pm(x,\rho) \geq 0,\ f_\pm(x,0) = 0,\ i = 1,2. \tag{1.3}$$

Clearly, we can set

$$f_\pm(x,\rho) = \int_0^\rho (\partial_\rho f(x,v))^\pm\, dv, \tag{1.4}$$

where we have used $z^+ = z \vee 0$ and $z^- = -(z \wedge 0)$ for $z \in \mathbb{R}$. If $f \in C^1$, $f_\pm$ is also C^1.

We discretize the space with step size $h > 0$ and set $x_j = jh$. Let $\rho_j(t)$ be the numerical solution at site x_j, with $\rho_j(0)$ being some approximation for $\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho(x,0)\, dx$. Then, the upwind scheme for (1.2) can be constructed based on the flux splitting [9, 50]

$$\begin{aligned}\frac{d}{dt}\rho_j = -\Big(\frac{f_+(x_j,\rho_j) - f_+(x_{j-1},\rho_{j-1})}{h} - \frac{f_-(x_{j+1},\rho_{j+1}) - f_-(x_j,\rho_j)}{h}\Big)\\ + \frac{1}{h^2}\Big(D_{j+1/2}\rho_{j+1} - (D_{j+1/2} + D_{j-1/2})\rho_j + D_{j-1/2}\rho_{j-1}\Big),\end{aligned} \tag{1.5}$$

where $D_{j+1/2} = D(x_j + \frac{h}{2})$. We denote $f_{\pm,j} := f_\pm(x_j,\rho_j)$. The upwind scheme (1.5) can be rearranged to a conservative scheme

$$\frac{d}{dt}\rho_j + \frac{1}{h}[J_{j+1/2} - J_{j-1/2}] = 0, \tag{1.6}$$

where

$$J_{j+1/2} = h\alpha_j \rho_j - h\beta_{j+1}\rho_{j+1}, \tag{1.7}$$

with

$$\alpha_j = \frac{f_{+,j}/\rho_j}{h} + \frac{1}{h^2}D_{j+1/2},\ \beta_j = \frac{f_{-,j}/\rho_j}{h} + \frac{1}{h^2}D_{j-1/2}. \tag{1.8}$$

Hence, it can be further written as the discrete form

$$\frac{d}{dt}\rho_j = \alpha_{j-1}\rho_{j-1} + \beta_{j+1}\rho_{j+1} - (\alpha_j + \beta_j)\rho_j. \tag{1.9}$$

According to (1.4), we have for any $j \in \mathbb{Z}$, $f_{\pm,j}/\rho_j \ge 0$ and is bounded for bounded ρ_j. If $\rho_j = 0$, the quotient is understood as the partial derivative of $f_\pm$ on ρ at $(x_j, 0)$. Hence, the upwind scheme ensures that α_j, β_j are nonnegative. We can then interpret the upwind scheme as the master equation of some transition phenomena. In particular, α_j can be understood as the rate of moving the mass from site j to site $j+1$ while β_j is the rate of moving mass from j to $j-1$. Then (1.9) describes the evolution of mass. Due to this physical understanding, if the upwind scheme (1.6)-(1.7) is well-posed, we expect that (1.9) is nonnegativity preserving, and ℓ^1 is nonexpansive (i.e., $\|\rho^1(t) - \rho^2(t)\|_{\ell^1} \le \|\rho^1(0) - \rho^2(0)\|_{\ell^1}$).

We remark that the time continuous upwind scheme (1.5) is total variation diminishing (TVD) for bounded ℓ^1 solutions that decay fast enough (of course, whether the true solutions of (1.5) decay fast enough depends on concrete conditions on $f(x,\rho)$ and $D(x)$). In other words, if $\rho \in L^\infty(0,T;\ell^1 \cap \ell^\infty)$ is a solution that decays fast enough, $\sum_j |\rho_{j+1} - \rho_j|$ is nonincreasing. Here, $L^\infty(0,T;X)$ means the $\|\cdot\|_X$ norm is essentially bounded on $[0,T]$ while ℓ^p refers to the usual Banach spaces in numerical analysis (note that there is h involved)

$$\ell^p := \begin{cases} \{\rho : \mathbb{Z} \to \mathbb{R} \,\big|\, \|\rho\|_{\ell^p} := (\sum_{j\in\mathbb{Z}} h|\rho_j|^p)^{1/p} < \infty\},\ p \in [1,\infty), \\ \{\rho : \mathbb{Z} \to \mathbb{R} \,\big|\, \|\rho\|_{\ell^\infty} := \sup_{j\in\mathbb{Z}} |\rho_j| < \infty\},\ p = \infty. \end{cases} \tag{1.10}$$

The reason that the scheme is TVD is that the numbers

$$a_j^+ := \frac{f_+(x_j,\rho_j) - f_+(x_{j-1},\rho_{j-1})}{\rho_j - \rho_{j-1}},\ a_j^- := \frac{f_-(x_{j+1},\rho_{j+1}) - f_-(x_j,\rho_j)}{\rho_{j+1} - \rho_j},$$

are bounded for given j (since we have assumed ρ is bounded) and nonnegative (see [24, 25]). One can also use a similar technique as in the proof of Proposition 4.1 to conclude the TVD property. The significance of TVD property is that it ensures the boundedness of variation and L^1 norms, which imply compactness in $L^1([0,T] \times K)$ for any compact domain K. Then one can obtain the convergence of the numerical scheme by compactness in $L^1_{\rm loc}(\mathbb{R})$. This is particularly important for nonlinear hyperbolic conservation laws because TVD schemes satisfying entropy inequality can recover the unique entropy weak solution [9, 24, 25]. The monotone schemes, including upwind schemes, are TVD schemes with no surprise.

In the case of linear parabolic conservation laws with nondegenerate diffusivity (advection diffusion equations), the so-called B-schemes (including the famous Scharfetter-Gummel scheme ((SG) scheme) [45], widely used for silicon diode models) are often adopted. In particular, let $B : \mathbb{R} \to \mathbb{R}^+$ satisfy: (i) B is Lipschitz continuous; (ii) $B(0) = 1$, $B(w) > 0$ for all $s \in \mathbb{R}$; (iii) $B(w) - B(-w) = -w$

$\forall w \in \mathbb{R}$. The flux for

$$\partial_t \rho + \partial_x(s(x)\rho) = \partial_x(D(x)\partial_x\rho) \tag{1.11}$$

is then given by

$$J_{j+1/2} = \frac{D_{j+1/2}}{h}\left[B\left(-\frac{s_{j+1/2}h}{D_{j+1/2}}\right)\rho_j - B\left(\frac{s_{j+1/2}h}{D_{j+1/2}}\right)\rho_{j+1}\right]. \tag{1.12}$$

With this flux expression, one can also write out the master equation as in (1.9) (see Section 3 for more details). Again, if the discrete equation is well-posed, one can similarly expect that the B schemes are nonnegativity-preserving and ℓ^1 nonexpansive.

For summary, these discretizations in space give nonnegativity preserving and ℓ^1 nonexpansive schemes (at least in the formal way since the well-posedness needs further investigation). These properties make them useful in numerical analysis. For example, the ℓ^1 nonexpansion will imply the uniqueness of solutions. If the scheme is also TVD as the upwind schemes for hyperbolic conservation laws, the existence of weak solutions to the PDEs can be established on $\mathbb{R}\times[0,T]$ by taking a convergent subsequence of the numerical solutions. (As we will see in sections 3 and 4, these properties hold and the mass is conserved for the problems we consider.)

The convergence on $\mathbb{R}\times[0,T]$, however, is not enough if we care about the long time asymptotic behaviors. Our observation is that when the equation is linear, the master equation (1.9) can be regarded as the forward equation of a jump process (time continuous Markov chain) [32]. In this case, we can normalize ρ to the probability measure of the chain on $\mathbb{Z}$. Since jump processes are well-studied [32, 38] in the community of probability, we can then use tools from probability to study the large time behaviors so that it is possible to show that these schemes can capture the correct physics.

In fact, analyzing the discrete schemes in the viewpoint of time continuous Markov chains and probability has been widely adopted in the literature [11–13, 16, 40, 46]. In [11, 12, 46], the upwind discretization was considered for linear transport equations in $\mathbb{R}^d$. Using the fluctuations of the Markov chains, the 1/2 order accuracy of upwind scheme for nonsmooth initial data was recovered in a nice probabilistic way. The authors of [13, 40] focused on the finite-dimensional Markov chains and discretization of Fokker-Planck equations in bounded domains. Using the viewpoints of gradient flows, they were able to establish certain discrete log-Sobolev inequalities (the relation between relative entropy and Onsager matrix) and show the convergence of these schemes. In [16], the discretization of the special Fokker-Planck equation $\partial_t F + v\partial_x F - \partial_v(\partial_v + v)F = 0$ was considered and the exponential convergence was established. This diffusion is degenerate in x direction so some discrete hypercoercivity was used. For other related references, one can also refer, for example, to [14, 15, 20, 30]. The Donsker invariance principle [14, 15] claims that a certain rescaled random walk converges to the standard Brownian motion on time interval $[0, 1]$ in distribution. In [20, 30], Markov chains have been used to approximate diffusion processes and the weak convergence of the scheme on fixed time interval has been proved.

In this work, we investigate the large time behaviors of the typical numerical schemes of conservation laws mentioned above using jump processes. We are able to establish the discrete Poincaré inequality under some assumptions inspired by theories in [32, 38], using the discrete Hardy's inequality. Then, we prove the exponential convergence to equilibrium states and the uniform $O(h)$ error. In particular, with Assumption 2.1, i.e., $b(x) \cdot x \le -r|x|^2$ for $|x|$ large enough and σ^2 to be uniformly bounded below and above, we show the following.

Theorem (Informal version of Theorems 5.2 and 3.2). *For the upwind schemes and a class of B schemes, the numerical solutions $\rho^h(t)$ converge exponentially fast to some stationary solution with rate κ_1 independent of h (for sufficiently small h)*

$$\left\|\rho^h(t) - \frac{1}{h}\|\rho^h(0)\|_{\ell^1}\pi^h\right\|_{\ell^1} \le C\exp(-\kappa_1 t),$$

and $\rho^h(t)$ approximates the solution of the Fokker-Planck equation (1.1) *with uniform $O(h)$ error, hence also approximating the equilibrium solution.*

For dynamics on the torus, we are also able to establish the results as follows.

Theorem (Informal version of Theorems 6.1 and 6.2). *For the upwind schemes and the B-schemes, one has similar results for the numerical solutions on the torus even though the detailed balance does not hold.*

These then verify that the mentioned schemes can capture the correct physics even on unbounded domains. We remark that the existing results regarding exponential convergence for discrete schemes of conservation laws are often on bounded domains (see [7, 13, 19, 40]). In fact, the large time behavior of B-schemes for advection-diffusion equations on bounded domains has been studied recently in [7] already. Compared with [11, 12, 46], we focus on the large time behaviors of schemes, and compared with [7, 13, 40, 46], we focus on equations in unbounded domains. We hope our work will bring more understanding to the numerical schemes of parabolic conservation laws and inspire understanding for schemes of hyperbolic conservation laws.

The rest of the paper is organized as follows. In section 2, we give a brief introduction to SDEs and the associated Fokker-Planck equation. We also have a review of results regarding the stationary distribution and ergodicity. In section 3, we move on to the discrete schemes for the Fokker-Planck equations on $\mathbb{R}$ and show the uniform error estimates. In section 4, we prove some elementary properties of the jump process for the upwind schemes and B-schemes. In particular, we show some basic properties of the discrete backward equation of the Markov jump process and show that the numerical solution converges to a stationary solution in the case of weak confinement. In section 5, we focus on the strong confinement and study the asymptotic behaviors of the numerical schemes. We show the uniform geometric convergence to the steady states using a discrete Poincaré inequality on the whole space. We then prove the $O(h)$ accuracy for the stationary solution, proving the unproved claim (Theorem 3.1) in section 3. Further, in section 6, we establish the results on the torus for which detailed balance may not hold. Finally, in section 7, we propose a Monte Carlo method to numerically solve the numerical schemes in a probabilistic way.

2. Preliminaries: Basic facts of SDEs

We have mentioned that the linear conservation law with positive diffusion is the Fokker-Planck equation for an SDE. This is the focus of this paper, so we will have a brief review of SDEs for general dimension d in this section.

2.1. **Basic setup of SDEs.** The time homogeneous SDEs driven by Wiener process in the Itô sense are given by [41]:

$$dX = b(X)\,dt + \sigma(X)\,dW. \tag{2.1}$$

Here, $X = X(t)$ is the unknown process, and the functions b and σ are called the drift and diffusion coefficients, respectively. W is the standard Wiener process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When b and σ are Lipschitz continuous and have linear growth at infinity, (2.1) has global strong solutions [41, sect. 5.2] for $L^2(\mathbb{P})$ initial data. The conditions imposed $b(\cdot)$ in [41, sect. 5.2] is too strong for many applications. In fact, it is also known that locally Lipschitz and confinement conditions can also imply the existence and uniqueness of solutions (For example, in [33, Theorem 2.3.5], it is shown that $\max(x\cdot b(x), |\sigma|^2) \le C_1 + C_2|x|^2$ is enough for the well-posedness, which allows b like $-(1+|x|^2)^p x$.)

The most frequently used confinement condition in this work is the following.

Assumption 2.1. Suppose b and σ are smooth. The function b satisfies

$$b(x)\cdot x \le -r|x|^2 \tag{2.2}$$

when $|x| > R$ for some R. Also, σ satisfies $\|\sigma\|_\infty < \infty$ and $\Lambda = \sigma\sigma^T \ge S_1 I > 0$.

Besides this, we sometimes weaken the conditions as follows.

Assumption 2.2. Suppose b and σ are smooth. The function b satisfies

$$\lim_{|x|\to\infty} \frac{-b(x)\cdot x}{|x|} = \infty. \tag{2.3}$$

Also, σ satisfies $\|\sigma\|_\infty < \infty$ and $\Lambda = \sigma\sigma^T \ge S_1 I > 0$.

We will use $\mathbb{E}$ to represent the expectation under $\mathbb{P}$. The notation $\mathbb{E}_x$ indicates that the expectation is conditioned on $X(0) = x$. Let μ_t be the law of $X(t)$, which is a measure in $\mathbb{R}^d$. Then we have

$$\mathbb{E}f(X(t)) = \langle \mu_t, f\rangle = \int_{\mathbb{R}^d} f\,d\mu_t. \tag{2.4}$$

For smooth bounded function $f(x)$, define

$$u(x,t) = \mathbb{E}_x f(X(t)). \tag{2.5}$$

By Itô's calculus [41], u satisfies

$$\partial_t u(x,t) = \mathbb{E}_x \mathcal{L} f(X(t)), \tag{2.6}$$

where $\mathcal{L}$ is the generator of the process

$$\mathcal{L} := b\cdot\nabla + \frac{1}{2}\Lambda_{ij}\partial_{ij}, \tag{2.7}$$

where we used Einstein summation convention (i.e., $\Lambda_{ij}\partial_{ij} \equiv \sum_{i,j=1}^d \Lambda_{ij}\partial_{x_i x_j}$) and

$$\Lambda = \sigma\sigma^T. \tag{2.8}$$

This is a special case of Dynkin's formula. The density of the law of $X(t)$ starting x, denoted by $p(t,x,y)$, is called the Green's function. When Λ is positive definite, $p(t,x,y)$ is a smooth function for $t > 0$. Equation (2.6) implies that $p(t,x,y)$ satisfies the forward Kolmogorov equation, or the Fokker-Planck equation for $t > 0$:

$$\partial_t p = -\nabla_y\cdot(b(y)p) + \frac{1}{2}\partial_{y_i y_j}(\Lambda_{ij}(y)p) := \mathcal{L}_y^* p, \tag{2.9}$$

where the subindex y means that the derivatives are taken on y variable. By the well-posedness of (2.1), we have under the confinement conditions that

$$\int_{\mathbb{R}^d} p(t,x,y)\,dy = 1\ \forall x \in \mathbb{R}^d, t>0. \tag{2.10}$$

Clearly, for general starting probability measure μ_0, the law of $X(t)$ also satisfies (2.9) in the distributional sense:

$$\frac{d}{dt}\langle \mu_t, f\rangle = \langle \mu_t, \mathcal{L}f\rangle$$

for any smooth bounded f, which is clearly a generalization of (2.6). Moreover, let $v:(x,t)\mapsto v(x,t)$ solve the backward Kolmogorov equation

$$\partial_t v = \mathcal{L}v = b\cdot\nabla v + \frac{1}{2}\Lambda_{ij}\partial_{ij}v \tag{2.11}$$

with initial condition $v(x,0)=f(x)$. Let $X(t)$ be the process satisfying (2.1) with initial condition $X(0)=x$. We check that $\mathcal{M}(s)=v(X(s),t-s)$ is a martingale and therefore

$$v(x,t)=\mathbb{E}\mathcal{M}(0)=\mathbb{E}\mathcal{M}(t)=\mathbb{E}v(X(t),0)=\mathbb{E}f(X(t))=u(x,t). \tag{2.12}$$

This means that (2.5) solves the backward Kolmogorov equation. Combining with (2.6), we can infer that the Green's function satisfies $\mathcal{L}_y^* p(t,x,y)=\mathcal{L}_x p(t,x,y)$, or

$$\begin{aligned} &-\nabla_y\cdot(b(y)p(t,x,y))+\frac{1}{2}\partial_{y_iy_j}(\Lambda_{ij}(y)p(t,x,y)) \\ &\qquad = b(x)\cdot\nabla_x p(t,x,y)+\frac{1}{2}\Lambda_{ij}(x)\partial_{x_ix_j}p(t,x,y). \end{aligned} \tag{2.13}$$

2.2. **Stationary solutions and ergodicity.** Under Assumption 2.1, using Itô's formula and test function $f(x)=\exp(c|x|^2)$, one can show that

$$\mathbb{E}_x\exp(c|X_t|^2)\le \exp(c|x|^2)e^{-rt}+C \tag{2.14}$$

for some positive constants c,r,C. This implies that the process has certain recurrent properties so that the SDE (2.1) has a unique stationary distribution π [28, sect. 4.4-4.7]. Moreover, π has a density with respect to Lebesgue measure [28, Lemma 4.16]. Below, we may abuse the notation a little bit and use $\pi(\cdot)$ to mean this density for convenience. The Green's function $p(t,x,y)$ converges to $\pi(y)$ pointwise as $t\to\infty$ for all $x\in\mathbb{R}^d$ [28, Lemma 4.17]. Clearly, $\pi(y)$ has finite moment of any order by (2.14). Since $\pi(y)$ is a solution to the parabolic equation (2.9) with the diffusion coefficient matrix positive definite, $\pi(y)$ is smooth and $\pi(y)>0$.

Often people study the ergodicity of SDEs in the L^p spaces. We will use $L^p(\mathbb{R}^d)$ to represent the L^p spaces associated with the Lebesgue measure while $L^p(\nu)$ to mean the L^p spaces associated with the measure ν. If ν has a density w, we also write $L^p(\nu)$ as $L^p(w)$. The most frequently used weight is $w=\pi$. Let $p(\cdot,t)$ be the density of μ_t. We often define

$$q(x,t):=\frac{p(x,t)}{\pi(x)}\ge 0,$$

and study the convergence of $q(\cdot,t)$ to 1 in $L^p(\pi)$ spaces.

Note that Λ_{ij} is symmetric and

$$-\nabla\cdot(b\pi)+\frac{1}{2}\partial_{ij}(\Lambda_{ij}\pi)=0. \tag{2.15}$$

We have

$$\partial_t q = \Big(\frac{1}{\pi}\nabla\cdot(\Lambda\pi) - b\Big)\cdot\nabla q + \frac{1}{2}\Lambda_{ij}\partial_{ij}q. \tag{2.16}$$

If the detailed balance condition

$$b = \frac{1}{2\pi}\nabla\cdot(\Lambda\pi) \tag{2.17}$$

holds (for example, $\Lambda = 2DI$ and $b = -\nabla V$), which clearly indicates (2.15), then we have the useful identity

$$\mathcal{L}^*(f\pi) = \pi\mathcal{L}f + f\mathcal{L}^*\pi = \pi\mathcal{L}f. \tag{2.18}$$

Then (2.16) can be rewritten as

$$\partial_t q = b\cdot\nabla q + \frac{1}{2}\Lambda_{ij}\partial_{ij}q, \tag{2.19}$$

which is the backward equation (2.11). In this case, the semigroup $e^{t\mathcal{L}}$ is symmetric in $L^2(\pi)$ and $e^{t\mathcal{L}^*}$ is symmetric in $L^2(1/\pi)$ by (2.18). Hence, it is convenient to investigate $u(\cdot,t)\to\langle\pi,f\rangle$ and $q(\cdot,t)\to 1$ in $L^2(\pi)$ using (2.11). If the detailed balance is not satisfied, the modified generator

$$\tilde{\mathcal{L}} = \Big(\frac{1}{\pi}\nabla\cdot(\Lambda\pi) - b\Big)\cdot\nabla + \frac{1}{2}\Lambda_{ij}\partial_{ij} =: \tilde{b}\cdot\nabla + \frac{1}{2}\Lambda_{ij}\partial_{ij} \tag{2.20}$$

corresponds to another SDE

$$dY = \tilde{b}\,dt + \sigma dY, \tag{2.21}$$

which has the same stationary distribution π, or $\tilde{\mathcal{L}}^*\pi = 0$. Suppose the law of $X(0)$ has a density $p^0(y)$. It follows from (2.21) that

$$q(x,t) = \mathbb{E}\Big(\frac{p^0(Y(t))}{\pi(Y(t))}|Y(0)=x\Big). \tag{2.22}$$

Hence, though the semigroups generated by $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are not symmetric in $L^2(\pi)$, one can still consider the convergence of $u(\cdot,t)\to\langle\pi,f\rangle$ and $q(\cdot,t)\to 1$ in $L^2(\pi)$ using Kolmogorov backward equations.

It is well known that Assumption 2.1 implies geometric ergodicity (i.e., convergence to a unique invariant measure with exponential rate) regarding the convergence of $u(\cdot,t)$ to $\langle\pi,f\rangle$ or μ_t to π using coupling argument for SDEs. In particular, we have the V-uniform geometric ergodicity for $u(\cdot,t)\to\langle\pi,f\rangle$ ([35, 37]) or exponential convergence of $\mu_t\to\pi$ in Wasserstein space ([17, 18]). Besides the coupling argument, one may prove the exponential convergence of $u(\cdot,t)$ to $\langle\pi,f\rangle$ in $L^p(\pi)$ spaces using spectral gap and Perron-Frobenius-type theorems (see [37, Chap. 20]; [1, 21, 22, 27, 39] for example). The V-uniform ergodicity and ergodicity in $L^p(\pi)$ do not necessarily imply each other, unless extra conditions are imposed [37, Chap. 20].

The geometric convergence of $q(\cdot,t)$ to 1 (equivalent to the convergence of $u(\cdot,t)\to\langle\pi,f\rangle$ for the modified SDE (2.21)) in $L^p(\pi)$ spaces can also be obtained directly using the Fokker-Planck equation and some functional inequalities (Poincaré inequality, or log-Sobolev inequality, etc.) [3]. These functional inequalities will imply spectral gaps of the semigroups. Let us explain this briefly. Take a smooth function φ and recall (2.16). We find

$$\frac{d}{dt}\varphi(q) = \tilde{\mathcal{L}}(\varphi(q)) - \frac{1}{2}\varphi''(q)\Lambda_{ij}\partial_i q\partial_j q.$$

Multiplying π and taking integral (recall $\tilde{\mathcal{L}}^*(\pi)=0$), we have the energy-dissipation relation

$$\frac{d}{dt}\mathcal{F} := \frac{d}{dt}\int_{\mathbb{R}} \varphi(q)\pi\,dx = -\frac{1}{2}\int_{\mathbb{R}} \varphi''(q)\Lambda_{ij}\partial_i q\partial_j q\,\pi dx =: -\mathcal{D}. \tag{2.23}$$

If φ is the quadratic function and the Poincaré inequality associated with π can be established, the geometric convergence of $q(\cdot,t)$ to 1 in $L^2(\pi)$ follows. This clearly implies that the geometric convergence of $q(\cdot,t)$ to 1 in $L^1(\pi)$ and hence the geometric convergence of $p(\cdot,t)$ to π in $L^1(\mathbb{R}^d)$ norm (total variation norm). Alternatively, one may take $\varphi(q)=q\log q-q+1$ and then $\mathcal{F}$ becomes the relative entropy or Kullback–Leibler (KL) divergence. If the log-Sobolev inequality holds, one can then establish the geometric convergence of the relative entropy and thus in total variation norm by Pinsker's inequality. The advantage of log-Sobolev inequality is that the constant is dimension free. For the case $b=-\nabla V$ and $\sigma=\sqrt{2D}I$, these results are well-known and one can refer to the review by Markowich and Villani [34].

For $d=1$, we have the following straightforward observation, which is needed for the error analysis of the discrete schemes.

Lemma 2.1. *Let $d=1$. If b and σ satisfy Assumption 2.1, then for any index $n>0$, there exist positive constants $C_n>0$, $\nu_n>0$ such that*

$$\Big|\frac{d^n}{dx^n}\pi(x)\Big| \le C_n\exp(-\nu_n|x|^2). \tag{2.24}$$

To see this, we note that the detailed balance condition $-b\pi+\frac{1}{2}\partial_x(\sigma^2\pi)=0$ holds. We can then solve $\sigma^2\pi$ and therefore π. Using the formula, Lemma 2.1 follows directly. The details are omitted. For $d>1$, in the case $b=-\nabla V$ and $\sigma=\sqrt{2D}I$, the claim is also trivial since $\pi\propto\exp(-V/D)$. For general dimension and general b,σ, we believe Lemma 2.1 is still true due to (2.14) (one may replace the test function $\exp(c|x|^2)$ with the derivatives $x\exp(c|x|^2)$ to get the estimates for derivative of π). For interested readers, one may refer to [43] for the pointwise estimates at infinity and to, for example, [2,4,43] for the theories of elliptic equations in unbounded domains.

3. Several schemes for Fokker-Planck equations on $\mathbb{R}$

In this section, we focus on the discretization of one dimensional Fokker-Planck equations, and view the discrete equations as jump processes. We can rewrite the 1D Fokker-Planck equation into the conservative form as

$$\partial_t\rho = -\partial_x((b-\sigma\sigma')\rho)+\frac{1}{2}\partial_x(\sigma^2\partial_x\rho). \tag{3.1}$$

We assume

$$\rho(x,0)=\rho^0(x)\ge 0. \tag{3.2}$$

Clearly, $f(x,\rho)=(b-\sigma\sigma')\rho =: s(x)\rho$. In this case, we have the corresponding decomposition

$$b-\sigma\sigma' =: s^+ - s^-,\ s^{\pm}\ge 0. \tag{3.3}$$

Recall that we use spatial step h to discretize the space and $x_j = jh$. Moreover, we use $R_g : C(\mathbb{R}) \to \mathbb{R}^{\mathbb{Z}}$ to mean the restriction onto the grid:

$$R_g\varphi = (\varphi(x_j)). \tag{3.4}$$

Now, consider the schemes for conservation laws discussed in the introduction. Since the Fokker-Planck equation is a parabolic conservation law, we consider both the upwind scheme (1.5) for general conservation laws and the B-schemes for parabolic conservation laws.

Direct discretization of the conservation form (3.1) using the upwind idea (1.5) yields

$$\begin{aligned}\frac{d}{dt}\rho_j(t) = &-\left(\frac{s_j^+\rho_j - s_{j-1}^+\rho_{j-1}}{h} - \frac{s_{j+1}^-\rho_{j+1} - s_j^-\rho_j}{h}\right)\\ &+ \frac{1}{2h^2}\left(\sigma_{j+1/2}^2\rho_{j+1} - (\sigma_{j+1/2}^2 + \sigma_{j-1/2}^2)\rho_j + \sigma_{j-1/2}^2\rho_{j-1}\right).\end{aligned} \tag{3.5}$$

The rates (1.8) for the master equation (1.9) are independent of ρ:

$$\alpha_j = \frac{s_j^+}{h} + \frac{1}{2h^2}\sigma_{j+1/2}^2,\ \beta_j = \frac{s_j^-}{h} + \frac{1}{2h^2}\sigma_{j-1/2}^2. \tag{3.6}$$

Similarly, consider the B-schemes with the flux

$$J_{j+1/2} = \frac{D_{j+1/2}}{h}\left[B\left(-\frac{s_{j+1/2}h}{D_{j+1/2}}\right)\rho_j - B\left(\frac{s_{j+1/2}h}{D_{j+1/2}}\right)\rho_{j+1}\right],\ D_{j+1/2} = \frac{1}{2}\sigma_{j+1/2}^2. \tag{3.7}$$

We have

$$\begin{aligned}\frac{d}{dt}\rho_j = &\frac{1}{h^2}D_{j+1/2}B\left(\frac{s_{j+1/2}h}{D_{j+1/2}}\right)\rho_{j+1} + \frac{1}{h^2}D_{j-1/2}B\left(-\frac{s_{j-1/2}h}{D_{j-1/2}}\right)\rho_{j-1}\\ &- \frac{1}{h^2}\left[D_{j+1/2}B\left(-\frac{s_{j+1/2}h}{D_{j+1/2}}\right) + D_{j-1/2}B\left(\frac{s_{j-1/2}h}{D_{j-1/2}}\right)\right]\rho_j.\end{aligned} \tag{3.8}$$

Consequently,

$$\alpha_j = \frac{D_{j+1/2}}{h^2}B\left(-\frac{s_{j+1/2}h}{D_{j+1/2}}\right) > 0,\ \beta_j = \frac{D_{j-1/2}}{h^2}B\left(\frac{s_{j-1/2}h}{D_{j-1/2}}\right) > 0. \tag{3.9}$$

If $B(w) = 1 + w^- = 1 + (-w)^+$, the flux is given by

$$J_{j+1/2} = \frac{D_{j+1/2}}{h}(\rho_j - \rho_{j+1}) + s_{j+1/2}^+\rho_j - s_{j+1/2}^-\rho_{j+1}.$$

This is also an upwind scheme. The difference from (3.5) is that we used a shifted $s(\cdot)$ function. Clearly, this upwind scheme is also consistent. In the case $B(w) = \frac{w}{e^w-1}$, the scheme is an SG scheme. The flux is then given by

$$J_{j+1/2} = s_{j+1/2}\frac{\rho_j - e^{-s_{j+1/2}h/D_{j+1/2}}\rho_{j+1}}{1 - e^{-s_{j+1/2}h/D_{j+1/2}}}.$$

As $D_{j+1/2} \to 0$, the SG scheme will degenerate to the upwind scheme without diffusion.

In this work, for technical reasons regarding the discrete Poincaré inequality, we only consider B-schemes that satisfy the following.

Assumption 3.1. The function B satisfies

$$0 < \inf_{w \geq 0} B(w) \leq \sup_{w \geq 0} B(w) < +\infty. \tag{3.10}$$

The function $B(w) = 1 + w^- = 1 + (-w)^+$ satisfies Assumption 3.1, while the SG scheme does not. However, we emphasize that we can modify the B function for large w so that the modified SG scheme satisfies the assumption. The modification near $+\infty$ does not alter the behaviors of the schemes too much as the local behavior of B near 0 matters. If $\lim_{w\to+\infty} B(w)$ has a limit, as $D_{j+1/2} \to 0$, the modified SG scheme still tends to the upwind scheme without diffusion. The point of Assumption 3.1 is that as $s_{j+1/2} \to +\infty$, the α_j rate does not vanish so that the diffusive behavior at $+\infty$ still exists.

The implication of Assumption 3.1 due to $B(w) - B(-w) = -w$ is that

$$\lim_{s\to+\infty} \frac{w}{B(-w)} = 1, \lim_{s\to+\infty} \frac{B(w)}{B(-w)} = 0. \tag{3.11}$$

Below, we discuss these schemes uniformly in the viewpoint of jump processes. Denote the sequence

$$\rho^h(t) := (\rho_j(t))_{j\in\mathbb{Z}}. \tag{3.12}$$

We assume $\rho^0(\cdot) \in L^1(\mathbb{R})$ and $\rho^h(0)$ is constructed such that

$$\|\rho^h(0) - R_g\rho^0\|_{\ell^1} \leq Ch, \ |\|\rho^h(0)\|_{\ell^1} - \|\rho^0\|_{L^1(\mathbb{R})}| \leq Ch. \tag{3.13}$$

Recall that ℓ^1 and $L^1(\mathbb{R})$ spaces are introduced in equation (1.10) and section 2.2, respectively. Let $p^0(x) = \rho^0/\|\rho^0\|_{L^1(\mathbb{R})}$. With Assumption 2.2, the SDE (2.1) is not explosive by [33, Theorem 2.3.5]. Hence, $p(x,t)$, the density of the law of $X(t)$, exists and is unique with $\int_{\mathbb{R}} p(x,t)\,dx = 1$. It is the solution of (3.1) with initial condition $p(x,0) = p^0(x)$, and thus $\rho(x,t) = p(x,t)\|\rho^0\|_{L^1(\mathbb{R})}$.

Since the discrete equation is also linear, we can normalize

$$p_j(t) := h\frac{\rho_j(t)}{\|\rho^h(0)\|_{\ell^1}} \geq 0 \tag{3.14}$$

so that $p_j(0) \geq 0$ and $\sum_j p_j(0) = 1$. For convenience, we define the sequence

$$p^h(t) := (p_j(t))_{j\in\mathbb{Z}}. \tag{3.15}$$

Remark 3.1. Note that $\rho^h(t)$ is the numerical approximation of $\rho(\cdot,t)$, but $p^h(t)$ is not the numerical approximation of the continuous probability density $p(\cdot,t)$ directly. Instead, $h^{-1}p^h(t)$ approximates the probability density $p(\cdot,t)$ and the reason we use this convention shall be clear soon.

The upwind scheme (3.5) and the B-schemes (3.8) ensure that α_j, β_j are nonnegative. Hence, the equation for $p^h(t)$,

$$\frac{d}{dt}p_j = \alpha_{j-1}p_{j-1} + \beta_{j+1}p_{j+1} - (\alpha_j + \beta_j)p_j =: (\mathcal{L}_h^* p^h)_j, \tag{3.16}$$

can be regarded as the forward equation (discrete Fokker-Planck equation) of a jump process or time continuous Markov chain $Z(t)$ [32]. α_j is the rate of jumping from site j to site $j+1$ while β_j is the rate of jumping from j to $j-1$.

Here, $\mathcal{L}_h^* : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is defined for any sequence, but the equation may not have solutions for arbitrarily given initial data. Later in section 4, we will see that under Assumption 2.2 the chain is nonexplosive and equation (3.16) is well-posed for ℓ^1

initial data. Moreover, $p_j(0) \ge 0$ and $\sum_j p_j(0) = 1$ imply that and that $p_j(t) \ge 0$, $\sum_j p_j(t) = 1$. Then $p_j(t)$ is the probability of appearing at site j and this is why we use the normalization in (3.14).

For convenience, we define the semigroup as

$$e^{t\mathcal{L}_h^*} p^h(0) := p^h(t). \tag{3.17}$$

With the well-posedness facts and the discussion in the introduction (section 1), we can deduce easily the following, and we omit the proofs.

Lemma 3.1. *The semigroup $e^{t\mathcal{L}_h^*}$ for upwind scheme* (3.5) *or* (3.16) *is ℓ^1 non-expansive and nonnegativity preserving. Moreover, the scheme* (3.5) *is TVD for $\rho_0^h \in \ell^1$ (i.e., $\sum_j |\rho_j(t) - \rho_{j-1}(t)|$ is nonincreasing).*

Remark 3.2. The jump process interpretation for general d is straightforward like in [11]. The focus of [11] is to establish the convergence accuracy using the viewpoint of jump processes, so the analysis can be generalized to multi-dimensions. For our purpose, theoretic study of the large time behavior is nontrivial for multi-dimensional space, especially for nonuniform meshes in unbounded domains. One issue is that we may lose the detailed balance for discrete schemes and the uniform functional inequalities for these cases are hard to prove, lacking also compactness. We expect that the functional inequalities for bounded domains are doable, possibly using the ideas in [13, 40].

3.1. **Stationary solutions.** Consider a stationary solution π^h to (3.16) or (1.6). We then find

$$J_{j+1/2}^h = J = const.$$

We take $j \to \infty$ and find $J = 0$. Hence, we have

$$\alpha_j \pi_j^h = \beta_{j+1} \pi_{j+1}^h. \tag{3.18}$$

This is the detailed balance condition.

Lemma 3.2. *Suppose the weak confinement Assumption* 2.2 *holds. Then, there is a unique stationary distribution π^h for the jump process $Z(t)$ corresponding to* (3.16).

Proof. Using (3.18), we find

$$\pi_j^h = \pi_0^h \prod_{k=1}^{j} \frac{\alpha_{k-1}}{\beta_k}.$$

With the condition, $b(x_j) < 0$ and $|b(x_j)| \to \infty$ for $j \to \infty$. For the usual upwind scheme with rates (3.6), and for the B schemes due to (3.11), we have $\lim_{j\to\infty} \frac{\alpha_{j-1}}{\beta_j} \to 0$. Hence, π_j^h decays with at least geometric rate. This means $\sum_{j\ge 0} \pi_j^h < \infty$. Similarly, $\sum_{j<0} \pi_j^h < \infty$ also holds. Hence, $\sum_j \pi_j^h < \infty$ and we can normalize it to a probability distribution so that π_0^h is determined uniquely. $\square$

Similar to p^h, π^h does not approximate the density $\pi(\cdot)$ of stationary distribution of (2.1). Instead, $h^{-1}\pi^h$ approximates $\pi(\cdot)$. In fact, in section 5.2, we will prove the following, which says that $h^{-1}\pi^h$ approximates $\pi(\cdot)$ with error h:

Theorem 3.1. *Suppose Assumption* 2.1 *holds with* $S_1 \le \sigma^2 \le S_2$. *Let* π^h *be the stationary distribution of the jump process* $Z(t)$ *for* (3.16) *corresponding to the upwind scheme* (3.5) *or the* B*-schemes* (3.8) *satisfying Assumption* 3.1, *and let* $\pi(\cdot)$ *be the stationary solution of* (1.1) *with total mass* 1 (*or density of the stationary distribution of* (2.1)). *Then there exist* $h_0 > 0$ *and* $C > 0$ *such that* (*recall* (3.4) *for* R_g)

$$\left\| R_g \pi - \frac{1}{h}\pi^h \right\|_{\ell^1} \le Ch \ \forall h \le h_0. \tag{3.19}$$

On bounded domain, usual techniques for the finite difference method of elliptic equations can be used to prove such types of results. The difference is that now the domain is unbounded. The proof relies on the spectral gap of the operator. See section 5.2 for more details.

Now, as an example, let us apply the upwind scheme (3.5) to the Ornstein-Uhlenbeck (OU) process with $b(x) = -x,\ \sigma = 1$. Then,

$$\begin{cases} \alpha_j = \frac{1}{2h^2},\ \beta_j = j + \frac{1}{2h^2},\ j \ge 0, \\ \alpha_j = |j| + \frac{1}{2h^2},\ \beta_j = \frac{1}{2h^2},\ j < 0. \end{cases}$$

Using the fact $\frac{\pi_{j+1}^h}{\pi_j^h} = \frac{\alpha_j}{\beta_{j+1}}$, we find that π_j^h is even. Hence, we only need to focus on $j \ge 0$. Clearly, for $j \ge 1$,

$$\pi_j^h = A_h \prod_{k=1}^{j} \frac{1}{1 + 2kh^2} =: A_h v_j, \tag{3.20}$$

where $A_h = \pi_0^h$. Let $w := \sqrt{2\pi} R_g \pi$ (whether "π" means the circular ratio or stationary distribution should be clear), or

$$\pi(x_j) = \frac{1}{\sqrt{2\pi}} \exp(-(jh)^2) = \frac{1}{\sqrt{2\pi}} w_j. \tag{3.21}$$

As $j \to \infty$, the leading behavior of v_j is like

$$v_j = \exp\Big(-\sum_{k=1}^{j} \ln(1 + 2kh^2) \Big) \sim \exp(-C_h j \ln j)$$

which decays slower than w_j.

Clearly, v_j is decreasing and

$$\sum_{k \ge j+1} h v_k \le v_j h \sum_{m=1}^{\infty} \frac{1}{(1 + 2jh^2)^m} = \frac{v_j}{2jh}.$$

Hence, we find

$$\Big| \sum_{j \in \mathbb{Z}} h v_j - \sum_{j \in \mathbb{Z}} h w_j \Big| \le \left| \sum_{|j| \le M} h |v_j - w_j| \right| + \frac{2v_M}{2Mh} + \frac{C\pi(x_M)}{Mh}. \tag{3.22}$$

Moreover, since $-x \le -\ln(1+x) \le -x + \frac{1}{2}x^2$, using $\sum_{k=1}^{j} k^2 \le j^3$, we find

$$w_j \exp(-jh^2) \le v_j \le w_j \exp(-jh^2 + 2j^3 h^4).$$

It follows that $|v_j - w_j| \le w_j C \max(jh^2, 2j^3h^4)$ for $j^3h^4 \le 1$ and $jh^2 \le 1$. Since there exists C independent of h such that

$$\sum_{j\in\mathbb{Z}} hw_j \max(jh, 2j^3h^3) < C,$$

we find (3.22) can be controlled by

$$\Big|\sum_{j\in\mathbb{Z}} hv_j - \sum_{j\in\mathbb{Z}} hw_j\Big| \le Ch + \Big(\frac{2v_M}{2Mh} + \frac{C\pi(x_M)}{Mh}\Big)|_{M=h^{-4/3}} \le C_1 h.$$

Hence, $|h^{-1}A_h - \frac{1}{\sqrt{2\pi}}| \le C_2 h$. Consequently, $h^{-1}\pi^h - R_g\pi$ is controlled by h both in ℓ^∞ and in ℓ^1.

Remark 3.3. This OU process considered here is the homogeneous Fokker-Planck equation in [16]. The same discrete equilibrium formula is obtained there. Moreover, they also prove a discrete Poincaré inequality regarding this discrete equilibrium with the Poincaré constant to be 1. The proof in [16] needs the concrete property of the equilibrium state. The discrete Poincaré inequality we establish in section 5 is more general and the proof is different.

3.2. **Uniform error estimates.** Note $\tilde{b}(x) = b(x)$ (since for $d = 1$ the detailed balance condition is satisfied always). We now use the equation for q to investigate the uniform approximation of upwind scheme to the Fokker-Planck equation.

In [49, sect. 3.1], the following exponential decay has been proved.

Proposition 3.1. *Suppose that Assumption* 2.1 *holds and that the derivatives of b and σ are bounded. Then for any index $n > 0$, there exist a polynomial p_n and $\gamma_n > 0$ such that*

$$\Big|\frac{\partial^n}{\partial x^n}(q(x,t) - 1)\Big| \le p_n(x)\exp(-\gamma_n t). \tag{3.23}$$

Proposition 3.1, Lemma 2.1, and Theorem 3.1 imply the following.

Theorem 3.2. *Suppose that Assumption* 2.1 *holds and that the derivatives of b and σ are bounded. Then, for any $n \ge 0$, there exist $C_n > 0$ and $\tilde{\gamma}_n > 0$ such that*

$$\Big|\frac{\partial^n}{\partial x^n}(p(x,t) - \pi(x))\Big| \le C_n \exp(-\nu_n|x|^2)\exp(-\tilde{\gamma}_n t). \tag{3.24}$$

Suppose π^h is the stationary solution for (3.16) *with $\sum_j \pi_j^h = 1$ and recall R_g* (3.4). *Then*

$$\sup_{t\ge 0}\sum_{j\in\mathbb{Z}} |p(x_j,t)h - p_j(t)| \le Ch + 2\Big\|R_g\pi - \frac{1}{h}\pi^h\Big\|_{\ell^1} \le Ch. \tag{3.25}$$

Hence, the upwind scheme (3.5) *and the B-schemes* (3.8) *satisfying Assumption* 3.1 *can solve the Fokker-Planck equation* (1.1) *with $O(h)$ error uniformly in time:*

$$\sup_{t\ge 0}\|R_g\rho(\cdot,t) - \rho^h(t)\|_{\ell^1} \le Ch. \tag{3.26}$$

Proof. Note that $p - \pi = \pi(q-1)$. Lemma 2.1 and Proposition 3.1 imply (3.24).

We insert $\psi := p - \pi$ into the discrete Fokker-Planck equation (3.16) and by the standard Taylor expansion in numerical analysis scheme, we have

$$\frac{d}{dt}\psi(x_j,t) = \mathcal{L}_h^*\psi(x_j,t) + g(x_j,t)h,$$

where $\|g(x_j,t)\|_{\ell^1} \le C\exp(-\gamma t)$ holds uniformly for small h by (3.24). Then, we have $p(x_j,t)-\pi(x_j) = e^{t\mathcal{L}_h^*}(p(x_j,0)-\pi(x_j)) + h\int_0^t e^{(t-s)\mathcal{L}_h^*} g\,ds$. Since the $p_j(t) = (e^{t\mathcal{L}_h^*}p^h(0))_j$ and $\pi^h = e^{t\mathcal{L}_h^*}\pi^h$, we have

$$\begin{aligned} &p(x_j,t)-\pi(x_j)-\frac{1}{h}(p_j(t)-\pi_j^h) \\ &\quad = e^{t\mathcal{L}_h^*}\Big(p(x_j,0)-\pi(x_j)-\frac{1}{h}(p_j(0)-\pi_j^h)\Big) + h\int_0^t e^{(t-s)\mathcal{L}_h^*} g(x_j,s)\,ds. \end{aligned}$$

Since $e^{t\mathcal{L}_h^*}$ is ℓ^1 nonexpansive by Lemma 3.1, we have

$$\begin{aligned} \Big\|R_g p(\cdot,t)-\frac{1}{h}p^h(t)\Big\|_{\ell^1} &\le 2\Big\|R_g\pi - \frac{1}{h}\pi^h\Big\|_{\ell^1} \\ &\quad + \Big\|R_g p(\cdot,0)-\frac{1}{h}p^h(0)\Big\|_{\ell^1} + hC\int_0^t \exp(-\gamma z)dz. \end{aligned}$$

The first claim (3.25) follows by noticing (3.13) and Theorem 3.1. The claim (3.26) follows from the relation between ρ^h and p^h given in (3.15). □

4. Properties of the jump process

We will investigate the forward and backward equations associated with the jump process $Z(t)$ corresponding to (3.16). It is convenient to introduce the Green's function

$$p_t(i,j) := \mathbb{P}(Z(t)=j|Z(0)=i) \ge 0. \tag{4.1}$$

Following [32, Chapter 2], we introduce the Q matrix as

$$Q(i,j) = \frac{d}{dt}p_t(i,j)|_{t=0}. \tag{4.2}$$

4.1. **Forward and backward equations.** The Green's function $p_t(i,j)$ is a solution to (3.16) with the initial distribution $p_0(i,j)=\delta_{ij}$ (possibly not unique without Assumption 2.2). The equation for the Green's function is

$$\frac{d}{dt}p_t(i,j) = \alpha_{j-1}p_t(i,j-1) + \beta_{j+1}p_t(i,j+1) - (\alpha_j+\beta_j)p_t(i,j). \tag{4.3}$$

It follows that

$$Q(j,j) = -(\alpha_j+\beta_j),\ Q(j,j-1)=\beta_j,\ Q(j,j+1)=\alpha_j. \tag{4.4}$$

Recall the definition of irreducibility.

Definition 4.1 ([32, Definition 2.47])**.** A Markov chain is irreducible if $p_t(i,j)>0$ for all i,j and $t>0$

The following observation follows from positivity of α_j and β_j [32], for which we omit the proofs.

Lemma 4.1. *The jumping process $Z(t)$ corresponding to* (3.16) *is irreducible.*

Then, by [32, Corollary 2.58] and Lemma 3.2, if Assumption 2.2 holds, the chain is recurrent.

The backward equation corresponding to the forward equation (3.16) reads

$$\frac{d}{dt}u_i(t) = \sum_{j\in\mathbb{Z}} Q(i,j)u_j(t) = \beta_i u_{i-1} - (\alpha_i+\beta_i)u_i + \alpha_i u_{i+1} =: (\mathcal{L}_h u)_i. \tag{4.5}$$

Clearly, $\mathcal{L}_h : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is the dual operator of $\mathcal{L}_h^*$. In fact, letting

$$\langle u, v\rangle_h := \sum_{j\in\mathbb{Z}} h u_j v_j, \tag{4.6}$$

we have

$$\langle \mathcal{L}_h g, f\rangle_h = \langle g, \mathcal{L}_h^* f\rangle_h$$

for any test sequence f that has finite nonzero entries. (Note that sequences with finite nonzero entries are dense in ℓ^p with $p < \infty$, so this is general enough.) Let $u(t) = (u_j(t))_{j\in\mathbb{Z}}$ be the solution of (4.5). The semigroup defined by

$$e^{t\mathcal{L}_h} u(0) := u(t) \tag{4.7}$$

is the dual of $e^{t\mathcal{L}_h^*}$.

It is well known that besides the forward equation (4.3), the Green's function also satisfies the backward equation (see [32, Theorem 2.14]):

(4.8)
$$\frac{d}{dt} p_t(i,j) = \sum_{k\in\mathbb{Z}} Q(i,k) p_t(k,j) = \beta_i p_t(i-1,j) - (\alpha_i + \beta_i) p_t(i,j) + \alpha_i p_t(i+1,j).$$

Formally, $P = e^{tQ}$ and we have $Qe^{tQ} = e^{tQ}Q$. This fact is an analogy to the continuous case (2.13). Since the chain is irreducible and recurrent, by [32, Corollary 2.34], the total probability is conserved $\sum_j p_t(i,j) = 1$ for all i (i.e., no probability leaks to infinity). By [32, Theorem 2.26] and [32, Exercise 2.38], the backward equation (4.8) has a unique bounded solution in ℓ^∞ given any initial data $u(0) \in \ell^\infty$. Correspondingly, for general initial data $p^h(0) \in \ell^1$, the solution is a linear combination of $p_t(i,j)$. Hence, the forward equation is also well-posed, nonnegativity preserving and it preserves sum

$$\sum_{j\in\mathbb{Z}} p_j(t) = \sum_{j\in\mathbb{Z}} p_j(0). \tag{4.9}$$

Hence $e^{t\mathcal{L}_h}$ maps ℓ^∞ to ℓ^∞ and the semigroup $e^{t\mathcal{L}_h^*}$ given in (3.17) maps ℓ^1 to ℓ^1.

Note that though the Green's function $p_t(i,j)$ satisfies the backward equation, the probability distribution $p_i(t)$ for general initial data does not. Instead, the lemma below shows that $\sum_i p_t(j,i) u_i(0)$ satisfies the backward equation. Before we state the results, we introduce the weighted ℓ^p spaces here, which are analogies of the weighted $L^p(w)$ spaces in section 2.2. Given w with $w_j \ge 0$, we define $\ell^p(w)$ as

$$\ell^p(w) := \left\{ q : \|q\|_{\ell^p(w)} := \left(\sum_{j\in\mathbb{Z}} w_j |q_j|^p\right)^{1/p} < \infty \right\}. \tag{4.10}$$

Proposition 4.1. *Let $S(t) := e^{t\mathcal{L}_h}$. Then,*

(1) *For any $u(0) \in \ell^\infty$, it holds that*

$$(S(t)u(0))_j = \sum_{i\in\mathbb{Z}} p_t(j,i) u_i(0). \tag{4.11}$$

(2) *The semigroup $S(t)$ is TVD, i.e., if $u(0) \in \ell^1$, then $\sum_j |u_j(t) - u_{j-1}(t)|$ is nonincreasing.*
(3) *$S(t)$ is symmetric in $\ell^2(\pi^h)$ for any $t \ge 0$.*
(4) *$S(t)$ is nonexpansive in $\ell^p(\pi^h)$ for any $p \in [1, \infty]$.*

Proof. (1). Let $v_j(t) = \sum_i p_t(j,i)u_i(0)$. Using the Fubini theorem, we find that $v \in \ell^\infty$. Moreover, since $p_t(j,\cdot) \in \ell^1$ for all j and $t \geq 0$, we find by (4.8),

$$\frac{d}{dt}v_j(t) = \sum_{i\in\mathbb{Z}} \Big(\beta_j p_t(j-1,i) + \alpha_j p_t(j+1,i) - (\alpha_j+\beta_j)p_t(j,i)\Big)u_i(0)$$
$$= \beta_j v_{j-1}(t) + \alpha_j v_{j+1}(t) - (\alpha_j+\beta_j)v_j(t).$$

Hence, $v = u$ by the uniqueness of the bounded solution.

(2). The backward equation (4.5) can be rearranged into $\frac{d}{dt}u_j = \alpha_j(u_{j+1}-u_j) - \beta_j(u_j - u_{j-1})$. It follows that

$$\frac{d}{dt}(u_{j+1}-u_j) = \alpha_{j+1}(u_{j+2}-u_{j+1}) - (\alpha_j+\beta_{j+1})(u_{j+1}-u_j) + \beta_j(u_j-u_{j-1}).$$

This is a forward equation for the sequence $\{u_{j+1}-u_j\}$ and the rates are given so that the equation is well-posed. Note that $\{u_j(0)-u_{j-1}(0)\} \in \ell^1$ since $u(0) \in \ell^1$. Since well-posed forward equations are ℓ^1 nonexpansions, $S(t)$ is TVD. Intuitively, we can multiply $\sigma_j := \mathrm{sgn}(u_{j+1}-u_j)$ on both sides of the equations and use $\sigma_j(u_{j+2}-u_{j+1}) \leq |u_{j+2}-u_{j+1}|$, $\sigma_j(u_j-u_{j-1}) \leq |u_j-u_{j-1}|$ to obtain

$$\frac{d}{dt}|u_{j+1}-u_j| \leq \alpha_{j+1}|u_{j+2}-u_{j+1}| - (\alpha_j+\beta_{j+1})|u_{j+1}-u_j| + \beta_j|u_j-u_{j-1}|.$$

(3). We denote $S := S(1)$ and $p(i,j) := p_1(i,j)$. Clearly, we only have to show that S is symmetric by the semigroup property. Using the detailed balance, we have:

$$\sum_j \pi_j^h f_j (Sg)_j = \sum_j \sum_i f_j g_i \pi_j^h p(j,i) = \sum_{ij} \pi_i^h p(i,j) f_j g_i = \sum_i \pi_i^h g_i (Sf)_i.$$

(4). Let $(u_j^i(t))$, $i = 1,2$ be two solutions and define $\tilde{u}_j = u_j^1 - u_j^2$. Then $(\tilde{u}_j)$ is also a solution and for any convex function φ it holds that

$$\frac{d}{dt}\varphi(\tilde{u}_j) = \mathcal{L}_h\varphi(\tilde{u})_j + \alpha_j(\varphi(\tilde{u}_j) + \varphi'(\tilde{u}_j)(\tilde{u}_{j+1}-\tilde{u}_j) - \varphi(\tilde{u}_{j+1}))$$
$$+ \beta_j(\varphi(\tilde{u}_j) + \varphi'(\tilde{u}_j)(\tilde{u}_{j-1}-\tilde{u}_j) - \varphi(\tilde{u}_{j-1})) \leq \mathcal{L}_h\varphi(\tilde{u})_j. \tag{4.12}$$

If φ is not differentiable at $\tilde{u}_j$, $\varphi'(\tilde{u}_j)$ is understood as one element in the subdifferential. Multiplying π_j^h and applying the detailed balance (3.18), we have $\frac{d}{dt}\pi_j^h\varphi(\tilde{u}_j) \leq \mathcal{L}_h^*(\pi\varphi(\tilde{u}))_j$. Taking sum on j yields that $\frac{d}{dt}\sum_j \pi_j^h\varphi(\tilde{u}_j) \leq 0$. Choosing $\varphi(z) = |z|^p$ which is convex, we have the claims for $p \in [1,\infty)$.

For $p = \infty$, we multiply $\sigma_j := \mathrm{sgn}(\tilde{u}_j)$ on both sides of the equation and obtain

$$\frac{d}{dt}|\tilde{u}_j| \leq \mathcal{L}_h|\tilde{u}|_j.$$

This implies that $\|\tilde{u}\|_{\ell^\infty}$ is nonincreasing. □

An important observation is that the discrete scheme always satisfies the detailed balance. If we define

$$q^h(t) := (q_j(t))_{j\in\mathbb{Z}},\ q_j(t) = \frac{p_j(t)}{\pi_j^h}, \tag{4.13}$$

then q^h satisfies the backward equation using the detailed balance condition (3.18):

$$\frac{d}{dt}q_j = \beta_j q_{j-1} + \alpha_j q_{j+1} - (\alpha_j+\beta_j)q_j. \tag{4.14}$$

With this interpretation, the relation (4.11) can be checked directly:

$$q_j(t)=\frac{1}{\pi_j^h}\sum_{i\in\mathbb{Z}}p_i(0)p_t(i,j)=\sum_{i\in\mathbb{Z}}p_t(i,j)\frac{p_i(0)}{\pi_j^h}.$$

Using the detailed balance (3.18), we have $\pi_i^h p_t(i,j)=p_t(j,i)\pi_j^h$. Hence,

$$q_j(t)=\sum_{i\in\mathbb{Z}}p_t(j,i)q_i(0). \tag{4.15}$$

4.2. **Convergence for the weak confinement.** The theory for irreducible time continuous Markov chain with countable state space is well-developed. See [32, Chapter 2]. We now use these theories to establish some basic properties of the jump processes and the numerical schemes we consider. We have the following.

Proposition 4.2. *Suppose Assumption* 2.2 *holds. The jump process* $Z(t)$ *for* (3.16) *satisfies*

$$p_t(i,j)\to\pi_j^h,\ t\to\infty \text{ for all } i,j.$$

Moreover, if we assume $p_j(0)=\frac{h\rho_j(0)}{\|\rho^h(0)\|_{\ell^1}}\le C\pi_j^h$ *for all* $j\in\mathbb{Z}$*, we then have*

$$\sum_{j\in\mathbb{Z}}|p_j(t)-\pi_j^h|\to 0,\ t\to\infty. \tag{4.16}$$

Consequently, for the upwind scheme (3.5) *and the* B*-schemes* (3.8) *satisfying Assumption* 3.1*,*

$$\left\|\rho^h(t)-\frac{1}{h}\pi^h\|\rho^h(0)\|\right\|_{\ell^1}\to 0. \tag{4.17}$$

Proof. By [32, Theorem 2.88, Theorem 2.66], we have for all i,j that $p_t(i,j)\to\pi_j^h$ as $t\to\infty$. Now, in general, we have

$$p_j(t)=\sum_{i\in\mathbb{Z}}p_i(0)p_t(i,j).$$

Since $|p_t(i,j)|\le 1$, the dominant convergence theorem implies that

$$p_j(t)\to\pi_j^h,\ t\to\infty\ \forall j\in\mathbb{Z}.$$

Equation (4.14) has the maximal principle following the last claim in Proposition 4.1:

$$|q_j(t)-\theta|\le\max_{j\in\mathbb{Z}}|q_j(0)-\theta|\ \forall\theta\in\mathbb{R}.$$

In particular, we take $\theta=1$. By the assumption, we have $|q_j(0)|\le C$ and thus $|q_j(t)-1|\le C_1\ \forall t\ge 0$. Since $p_j(t)\to\pi_j^h$, we have $q_j(t)\to 1\forall j$. Dominant convergence theorem then yields

$$\sum_{j\in\mathbb{Z}}\pi_j^h|q_j(t)-1|=\sum_{j\in\mathbb{Z}}|p_j(t)-\pi_j^h|\to 0,\ t\to\infty.$$

Using the relation between p^h and ρ^h, we find

$$\left\|\rho^h(t)-\frac{1}{h}\pi^h\|\rho^h(0)\|_{\ell^1}\right\|_{\ell^1}\to 0,\ t\to\infty.$$

□

The above proof makes use of the boundedness of $p_t(i,j)$ heavily. This clearly has no correspondence in the continuous case as $h \to 0$. Naturally, one may wonder whether we have the convergence uniform in $h \to 0$. We will investigate this in the next section.

5. Large time behaviors for strong confinement

In section 4.2, we have seen that the distribution of the jump process converges to the stationary solution under the weak confinement assumption. However, we do not have any rate for the convergence. Under the strong confinement (Assumption 2.1), we know that the convergence of the distribution for SDE (2.1) in $L^1(\mathbb{R})$ norm is exponential, which is obtained by using relative entropy and log-Sobolev inequality [34]. Naturally, we desire that under Assumption 2.1 the jump process (3.16) has uniform geometric ergodicity under ℓ^1 norm.

The convergence of $p^h(t)$ to π^h in total variation norm (or $h^{-1}p^h(t) \to h^{-1}\pi^h$ in ℓ^1) is equivalent to convergence of $q^h(t)$ to 1 in $\ell^1(\pi^h)$. Hence, we can consider the geometric convergence of $q^h(t)$ to 1 in $\ell^p(\pi^h)$ $(p \ge 1)$, which is closely related to spectral gaps of the semigroup $\{e^{t\mathcal{L}_h}\}$. This is a typical Perron-Frobenius-type question. Besides the traditional compactness requirement of the semigroup $\{e^{t\mathcal{L}_h}\}$ in $\ell^p(\pi^h)$, some sufficient conditions for the Perron-Frobenius-type theorems include the hypercontractivity and uniform integrability [21, 51, 52]. The classical result of Gross [23] tells us that the hypercontractivity is equivalent to log-Sobolev inequality. Proving such types of results for finite-dimensional Markov chains can be found, for example, in [13, 40]. For infinite discrete states, one may prove the discrete log-Sobolev inequality using the results in [5, 8] and similar strategy in section 5.1. It occurs to us that showing the discrete Poincaré inequality seems more convenient, which uses a quadratic Lyapunov function compared with the relative entropy for log-Sobolev inequalities.

In subsection 5.1, we use the quadratic function as the Lyapunov function and derive the discrete Poincaré inequality. In subsection 5.2, we establish the uniform geometric ergodicity.

5.1. **A discrete Poincaré inequality.**

Slightly different from equation (4.12), we note the following for a smooth function φ:

$$\frac{d}{dt}\varphi(q_j) = \mathcal{L}_h(\varphi'(q)q)_j + \beta_j q_{j-1}\Big(\varphi'(q_j) - \varphi'(q_{j-1})\Big) + \alpha_j q_{j+1}\Big(\varphi'(q_j) - \varphi'(q_{j+1})\Big). \tag{5.1}$$

By the detailed balance condition (3.18), this gives for convex function φ that

$$\frac{d}{dt}\sum_{j\in\mathbb{Z}} \pi_j^h \varphi(q_j) = -\sum_{j\in\mathbb{Z}} \alpha_j \pi_j^h (q_j - q_{j+1})(\varphi'(q_j) - \varphi'(q_{j+1})) \le 0. \tag{5.2}$$

This is the energy dissipation relation. If $\varphi(q) = q\log q - q + 1$, $\sum_j \pi_j^h \varphi(q_j)$ gives the relative entropy. What we find useful is the quadratic function $\varphi(q) = \frac{1}{2}(q - \sum_k \pi_k^h q_k)^2$. Then, we have

$$\frac{d}{dt}\mathcal{F}_h = -\mathcal{D}_h \tag{5.3}$$

with

$$\mathcal{F}_h := \frac{1}{2}\sum_{j\in\mathbb{Z}}\pi_j^h\left(q_j - \sum_{k\in\mathbb{Z}}\pi_k^h q_k\right)^2,\ \mathcal{D}_h := \sum_{j\in\mathbb{Z}}\alpha_j\pi_j^h(q_j-q_{j+1})^2. \tag{5.4}$$

Now we need to control $\mathcal{F}_h$ using $\mathcal{D}_h$. This type of control is achieved by Poincaré inequality. Below is a lemma modified from [38, Proposition 1] or [51, Lemma 1.3.10], which is a discrete Hardy inequality. For the convenience of the readers, we also attach the proof in Appendix A. See also [8] for relevant discussions.

Lemma 5.1. *Let θ be a nonnegative sequence with $\sum_j \theta_j < \infty$ and let μ be a positive sequence on $\mathbb{Z}$. Set*

$$A := \sup_f\left\{\max\left(\sum_{j\ge 0}\theta_j\Big(\sum_{k=0}^{j} f_k\Big)^2, \sum_{j\le -1}\theta_j\Big(\sum_{k=j}^{-1} f_k\Big)^2\right) : \sum_{j\in\mathbb{Z}}\mu_j f_j^2 = 1\right\} \tag{5.5}$$

and

$$B := \max\left(\sup_{j\ge 0}\Big(\sum_{k=0}^{j}\mu_k^{-1}\Big)\sum_{k\ge j}\theta_k\ ,\ \sup_{j<0}\Big(\sum_{k=j}^{-1}\mu_k^{-1}\Big)\sum_{k\le j}\theta_k\right). \tag{5.6}$$

Then it holds that $B \le A \le 4B$.

Using Lemma 5.1 and the approach in [51, sect. 1.3.3], it is straightforward to find the following.

Lemma 5.2. *Let α and β be the rates in* (3.16) *for the jump process $Z(t)$. Define*

$$\kappa := \inf_f\left\{\sum_{j\in\mathbb{Z}}\alpha_j\pi_j^h(f_{j+1}-f_j)^2 : \sum_{j\in\mathbb{Z}}\pi_j^h f_j^2 = 1, \sum_{j\in\mathbb{Z}}\pi_j^h f_j = 0\right\}. \tag{5.7}$$

Then we have

$$\kappa^{-1} \le 8\max\left(\sup_{j\ge 0}\Big(\sum_{k=0}^{j}(\alpha_k\pi_k^h)^{-1}\Big)\sum_{k\ge j+1}\pi_k^h,\ \sup_{j\le 0}\Big(\sum_{k=j}^{0}(\beta_k\pi_k^h)^{-1}\Big)\sum_{k\le j-1}\pi_k^h\right). \tag{5.8}$$

Proof. Consider θ, μ, A, and B in Lemma 5.1. Let

$$A_1 := \sup_g\left\{\sum_{j\ge 0}\theta_j\Big(\sum_{k=0}^{j} g_k\Big)^2 + \sum_{j\le -1}\theta_j\Big(\sum_{k=j}^{-1} g_k\Big)^2 : \sum_{j\in\mathbb{Z}}\mu_j g_j^2 = 1\right\}.$$

Then we have $A \le A_1 \le 2A$.

Clearly, for any sequence g we can define a sequence f such that

$$f_0 = 0,\ g_k = f_{k+1} - f_k$$

and this is a one-to-one correspondence. Then, we can rewrite A_1 in terms of f as

$$A_1 = \sup_f\left\{\sum_{j\ge 0}\theta_j f_{j+1}^2 + \sum_{j\le -1}\theta_j f_j^2 : \sum_{j\in\mathbb{Z}}\mu_j(f_{j+1}-f_j)^2 = 1,\ f_0 = 0\right\}. \tag{5.9}$$

It is clear that

$$A_1 = \sup_f \left\{ \frac{\sum_{j\ge 0}\theta_j(f_{j+1}-f_0)^2 + \sum_{j\le -1}\theta_j(f_j-f_0)^2}{\sum_{j\in\mathbb{Z}}\mu_j(f_{j+1}-f_j)^2} : f\not\equiv const,\ \sum_{j\in\mathbb{Z}}\mu_j(f_{j+1}-f_j)^2<\infty \right\}. \tag{5.10}$$

Now we define $\theta_j = \pi^h_{j+1}$ for $j\ge 0$ and $\theta_j=\pi^h_j$ for $j\le -1$, and let $\mu_j=\alpha_j\pi^h_j$. Then, A_1 under this particular choice of θ and μ is

$$A_1 = \sup_f \left\{ \frac{\sum_{j\in\mathbb{Z}}\pi^h_j(f_j-f_0)^2}{\sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2} : f\not\equiv const,\ \sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2<\infty \right\}. \tag{5.11}$$

It is then straightforward to find

$$A_1^{-1} = \inf_f \left\{ \sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2 : \sum_{j\in\mathbb{Z}}\pi^h_j(f_j-f_0)^2 = 1 \right\}. \tag{5.12}$$

In fact, if all sequences with $\sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2<\infty$, $f\not\equiv const$ satisfy $\sum_{j\in\mathbb{Z}}\pi^h_j(f_j-f_0)^2<\infty$, then (5.12) is clear. If there exists f such that

$$\sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2<\infty$$

but $\sum_{j\in\mathbb{Z}}\pi^h_j(f_j-f_0)^2=\infty$, then $A_1=\infty$. If this case happens, we can then take $\tilde{f}^N = A_N(f_i 1_{|i|\le N})_{i\in\mathbb{Z}}$ with A_N picked so that $\sum_j \pi^h_j(\tilde{f}^N_j-\tilde{f}^N_0)^2=1$. Then, $A_N\to 0$ and the infimum in (5.12) over $\tilde{f}_N$ is zero. Hence, (5.12) holds.

Using (5.12), we have

$$\kappa = \inf_f \left\{ \sum_{j\in\mathbb{Z}}\alpha_j\pi^h_j(f_{j+1}-f_j)^2 : \sum_{j\in\mathbb{Z}}\pi^h_j\left(f_j-\sum_k f_k\pi^h_k\right)^2 = 1 \right\} \ge A_1^{-1}.$$

This is because for $f\in\ell^2(\pi^h)$, the constant c that minimizes $\inf_c\sum_{j\in\ell^2(\pi^h)}\pi^h_j(f_j-c)^2$ is the mean $c=\sum_k f_k\pi^h_k$. Hence, we conclude by Lemma 5.1 that

$$\kappa \ge \frac{1}{2}A^{-1} \ge \frac{1}{8}B^{-1}.$$

Using the detailed balance $\alpha_k\pi^h_k = \beta_{k+1}\pi^h_{k+1}$ for $k\le -1$, we have

$$B = \max\left\{ \sup_{j\ge 0}\left(\sum_{k=0}^{j}(\alpha_k\pi^h_k)^{-1}\right)\sum_{k\ge j+1}\pi^h_k,\ \sup_{j\le 0}\left(\sum_{k=j}^{0}(\beta_k\pi^h_k)^{-1}\right)\sum_{k\le j-1}\pi^h_k \right\}.$$

The claim then follows. □

Lemma 5.3. *Suppose $S_1\le\sigma^2\le S_2$ for $S_2>S_1>0$ and b is a smooth function. Then, fixing $R>0$, we can find $C(R)>0$ and $h_0>0$ such that*

$$\max_{0\le j\le [R/h]+1}\pi^h_j \le C(R)\min_{0\le j\le[R/h]+1}\pi^h_j \quad \forall h\le h_0 \tag{5.13}$$

and that

$$\max_{-[R/h]-1\le j\le 0}\pi_j^h \le C(R)\min_{-[R/h]-1\le j\le 0}\pi_j^h \ \forall h\le h_0. \tag{5.14}$$

Proof. We only prove the claim for $0\le j\le [R/h]+1$. The other case is similar.

For the upwind scheme (3.5):

$$\pi_j^h=\pi_0^h\prod_{k=1}^j\frac{\alpha_{k-1}}{\beta_k}=\pi_0^h\prod_{k=1}^j\frac{s_{k-1}^+/h+\sigma_{k-1/2}^2/(2h^2)}{s_k^-/h+\sigma_{k-1/2}^2/(2h^2)}. \tag{5.15}$$

Hence, for h small enough, we have

$$\pi_0^h\prod_{k=1}^j\frac{1}{1+2h|s(x_k)|/S_1}\le \pi_j^h\le \pi_0^h\prod_{k=1}^j\Big(1+2h\frac{|s(x_{k-1})|}{S_1}\Big). \tag{5.16}$$

Using (5.16), we find

$$\frac{\max_{0\le j\le [R/h]+1}\pi_j^h}{\min_{0\le j\le [R/h]+1}\pi_j^h}\le \prod_{k=1}^{[R/h]+1}\Big(1+2h\frac{|s(x_{k-1})|}{S_1}\Big)\prod_{k=1}^{[R/h]+1}\Big(1+2h\frac{|s(x_k)|}{S_1}\Big).$$

Note that $\prod_{k=1}^{[R/h]+1}\Big(1+2h\frac{|s(x_k)|}{S_1}\Big)\le \exp(\frac{2}{S_1}\sum_{k=1}^{[R/h]+1}h|s(x_k)|)$. The inside of the right hand side is the Riemann sum for the integral $\frac{2}{S_1}\int_0^{R+h}|s(x)|\,dx$. Hence, the right hand side is bounded by a number depending on R when h is small enough. Similarly, $\prod_{k=1}^{[R/h]+1}(1+2h\frac{|s(x_{k-1})|}{S_1})\le C_1(R)$.

For the B-shemes (3.8), we note

$$\frac{B(-s)}{B(s)}=1+\frac{s}{B(s)}=\frac{1}{1-\frac{s}{B(-s)}}. \tag{5.17}$$

When h is small enough, $B\left(\frac{s_{j-1/2}h}{D_{j-1/2}}\right)\ge \frac{1}{2}$ and thus by (5.17),

$$\frac{1}{1+h\frac{|s(x_{k-1/2})|}{S_1}}\le\frac{\alpha_{k-1}}{\beta_k}\le 1+h\frac{|s(x_{k-1/2})|}{S_1}.$$

The arguments are similar. □

Now, we are able to conclude the discrete Poincaré inequality.

Theorem 5.1. *Suppose Assumption* 2.1 *holds with* $S_1\le\sigma^2\le S_2$*. Let* π^h *be the stationary distribution of the jump process* $Z(t)$ *corresponding to the upwind scheme* (3.5) *or the B-schemes* (3.8) *satisfying Assumption* 3.1*. Then the discrete Poincaré inequality holds for measure* π^h *when* h *is small enough. In other words, there exist* $h_0>0$ *and* $\kappa_1>0$ *independent of* h *so that for any* $f\in\ell^2(\pi^h)$*, we have*

$$\kappa_1\left(\sum_{j\in\mathbb{Z}}\pi_j^h f_j^2-(\sum_{k\in\mathbb{Z}}\pi_k^h f_k)^2\right)\le\sum_{j\in\mathbb{Z}}\alpha_j\pi_j^h(f_{j+1}-f_j)^2, \tag{5.18}$$

where α_j *is the rate in* (3.6).

Proof. Recall that

$$B_1 := \max\left\{\sup_{j\geq 0}\left(\sum_{k=0}^{j}(\alpha_k\pi_k^h)^{-1}\right)\sum_{k\geq j+1}\pi_k^h,\ \sup_{j\leq 0}\left(\sum_{k=j}^{0}(\beta_k\pi_k^h)^{-1}\right)\sum_{k\leq j-1}\pi_k^h\right\} =: (I_+,\ I_-),$$

Below, we consider I_+ only because the discussion for I_- is just parallel.

We can find $R>0$ such that $s(x)=b(x)-\sigma(x)\sigma'(x)<-r|x|$ for $x>R$. Let us recall that

$$\pi_j^h=\pi_0^h\prod_{k=1}^{j}\frac{\alpha_{k-1}}{\beta_k}.$$

For $j\geq [R/h]+1=:j^*$,

$$\pi_{j+n}^h=\pi_j^h\prod_{i=1}^{n}\frac{\alpha_{j+i-1}}{\beta_{j+i}}=\pi_j^h\prod_{i=1}^{n}\frac{\sigma_{i+j-1/2}^2}{\sigma_{i+j-1/2}^2+2hs_{i+j}^-}\leq\pi_j^h\prod_{i=1}^{n}\frac{1}{1+2hs_{i+j}^-/S_2},\ n\geq 1.$$

Hence, we have

$$\sum_{k\geq j+1}\pi_k^h\leq\pi_j^h\sum_{k\geq j+1}\frac{1}{(1+2rh^2(j+1)/S_2)^{k-j}}=\frac{S_2}{2r}\frac{\pi_j^h}{(j+1)h^2}, \tag{5.19}$$

where we have used $s_{i+j}^-\geq r(j+1)h$ for $i\geq 1$.

Let $K:=\frac{S_2}{2r}$. If $0\leq j\leq [R/h]=j^*-1$, we have by (5.19) that

$$\sum_{k\geq j+1}\pi_k^h\leq (j^*-j)\max_{0\leq k\leq j^*}\pi_k^h+K\frac{\pi_{j^*}^h}{(j^*+1)h^2}.$$

Consequently, by Lemma 5.3,

$$h^2(j+1)\left(\max_{0\leq k\leq j}(\pi_k^h)^{-1}\right)\sum_{k\geq j+1}\pi_k^h\leq ((R+h)^2+K)C(R),$$

and the right hand side is uniformly bounded for $h\leq h_0$.

If $j\geq j^*$, using (5.19) again, we have

$$h^2(j+1)\left(\max_{0\leq k\leq j}(\pi_k^h)^{-1}\right)\sum_{k\geq j+1}\pi_k^h\leq K\left(\min_{0\leq k\leq j}\pi_k^h\right)^{-1}\pi_j^h\leq KC(R).$$

The last inequality holds because

$$\min_{0\leq k\leq j}\pi_k^h=\min\left(\min_{0\leq k\leq j^*}\pi_k^h,\pi_j^h\right).$$

Clearly, $\pi_j^h\leq\pi_{j^*}^h$. If $\pi_j^h\geq\min_{0\leq k\leq j^*}\pi_k^h$, then

$$(\min_{0\leq k\leq j}\pi_k^h)^{-1}\pi_j^h\leq(\min_{0\leq k\leq j^*}\pi_k^h)^{-1}\pi_{j^*}^h\leq C(R)$$

by Lemma 5.3. Otherwise, $(\min_{0\leq k\leq j}\pi_k^h)^{-1}\pi_j^h=1$. Hence, I_+ is bounded.

We now consider the B-schemes satisfying Assumption 3.1. Using (3.9), we find

$$\frac{\alpha_{k-1}}{\beta_k}=\frac{B(w_k)}{B(-w_k)}=\frac{1}{1+\frac{w_k}{B(w_k)}},$$

with

$$w_k = -\frac{s_{k-1/2}h}{D_{k-1/2}}.$$

For $k \ge j^*$, $B(w_j)$ has both upper and lower bound. Also, the rate α_j is bounded below by $\frac{C_1}{h^2}$ for all $j \ge 0$ due to Assumption 3.1 (when $0 \le j \le j^*$, $|w_j|$ is bounded independent of h so $B(w_j)$ is also bounded).The argument is similar as above for the upwind scheme (3.5).

Overall, B_1 is bounded by a constant M depending on R, r, S_1, S_2, and h_0. Then, by Lemma 5.2, we have

$$\kappa \ge \frac{1}{8B_1} \ge \frac{1}{8M}.$$

Taking $\kappa_1 = 1/(8M)$ finishes the proof. □

5.2. **Uniform ergodicity.** Recall that ℓ^1 and $\ell^p(w)$ are defined in equations (1.10) and (4.10), respectively. Using Theorem 5.1, we are able to conclude the following.

Theorem 5.2. *Suppose Assumption* 2.1 *holds with $S_1 \le \sigma^2 \le S_2$. Consider the jump process $Z(t)$ corresponding to* (3.16) *and q defined by* (4.13). *Then for the upwind scheme* (3.5) *or the B-schemes* (3.8) *satisfying Assumption* 3.1*,*

$$\left\| q^h(t) - \sum_{j\in\mathbb{Z}} \pi_j^h q_j \right\|_{\ell^2(\pi^h)} = \|q^h(t) - 1\|_{\ell^2(\pi^h)} \le \|q^h(0) - 1\|_{\ell^2(\pi^h)} e^{-\kappa_1 t}. \tag{5.20}$$

Consequently, $p^h(t)$ converges to π^h exponentially fast in the total variation norm:

$$\sum_{j\in\mathbb{Z}} |p_j(t) - \pi_j^h| \le C\exp(-\kappa_1 t) \ \forall t > 0 \tag{5.21}$$

and

$$\left\| \rho^h(t) - \frac{1}{h}\|\rho^h(0)\|_{\ell^1}\pi^h \right\|_{\ell^1} \le C\exp(-\kappa_1 t). \tag{5.22}$$

Proof. Recall the definition of $\mathcal{F}_h$ and $\mathcal{D}_h$ in (5.4). Then, by Theorem 5.1, we have

$$\frac{d}{dt}\mathcal{F}_h = -\mathcal{D}_h \le -2\kappa_1\mathcal{F}_h.$$

Noticing $\sum_j \pi_j^h q_j = \sum_j p_j = 1$ and $\mathcal{F}_h = \|q - \sum_j \pi_j^h q_j\|_{\ell^2(\pi^h)}^2$, the first claim follows.

By Hölder's inequality, it holds that

$$\sum_{j\in\mathbb{Z}} |p_j(t) - \pi_j^h| = \|q^h(t) - 1\|_{\ell^1(\pi^h)} \le \|q^h(t) - 1\|_{\ell^2(\pi^h)} \le C\exp(-\kappa_1 t).$$

Since

$$\rho_j(t) = \frac{1}{h}\|\rho^h(0)\|_{\ell^1} p_j(t),$$

we then have

$$\left\| \rho^h(t) - \frac{1}{h}\|\rho^h(0)\|_{\ell^1}\pi^h \right\|_{\ell^1} \le \|\rho^h(0)\|_{\ell^1} \sum_j |p_j(t) - \pi_j^h| \le C\exp(-\kappa_1 t).$$

□

Using the second claim of Theorem 5.2, we conclude the following property of the semigroup $e^{t\mathcal{L}_h^*}$.

Corollary 5.1. *Suppose that $v \in \ell^1$ and $\sum_j hv_j = 0$. Then,*

$$\left\| e^{t\mathcal{L}_h^*} v \right\|_{\ell^1} \le C \exp(-\kappa_1 t). \tag{5.23}$$

Proof. Let $v^+ = \{v_j \vee 0\}$ and $v^- = \{-v_j \wedge 0\}$ so that $v = v^+ - v^-$. Let

$$p^1(t) := e^{tL_h^*} \frac{hv^+}{\|v^+\|_{\ell^1}},\ p^2(t) := e^{tL_h^*} \frac{hv^-}{\|v^-\|_{\ell^1}}.$$

By Theorem 5.2, we have

$$\sum_{j\in\mathbb{Z}} |p_j^i(t) - \pi_j^h| \le C_i \exp(-\kappa_1 t),\ i = 1, 2,$$

for some constants C_i.

Note that $\sum_j hv_j = 0$ implies $\|v^+\|_{\ell^1} = \|v^-\|_{\ell^1} = \frac{1}{2}\|v\|_{\ell^1}$. We have

$$\|e^{t\mathcal{L}_h^*} v\|_{\ell^1} = \sum_{j\in\mathbb{Z}} \Big| \|v^+\|_{\ell^1} p_j^1(t) - \|v^-\|_{\ell^1} p_j^2(t) \Big| = \frac{1}{2}\|v\|_{\ell^1} \sum_{j\in\mathbb{Z}} |p_j^1(t) - p_2^j(t)| \le C \exp(-\kappa_1 t).$$

□

Corollary 5.1 tells us that $e^{t\mathcal{L}_h^*}$ has a spectral gap in ℓ^1. For any $v \in \ell^1$, we define the projection onto the space spanned by π^h as

$$Pv := \Big(\sum_{j\in\mathbb{Z}} hv_j \Big) \left(\frac{1}{h} \pi^h \right). \tag{5.24}$$

Clearly, Pv is invariant under $e^{t\mathcal{L}_h^*}$. Corollary 5.1 implies that if v has no component in the direction of π^h, then $e^{t\mathcal{L}_h^*} v$ decays exponentially fast.

Now, we are able to conclude Theorem 3.1, i.e., bounding the error for approximating $\pi(x_j)$ using π_j^h. Note that for $j \in \mathbb{Z}$

$$\mathcal{L}_h^* \left(\pi(x_j) - \frac{1}{h}\pi_j^h \right) = \mathcal{L}_h^*(\pi(x_j)) = \tau_j h, \tag{5.25}$$

where $|\tau_j| \le C$ and $\sum_j h|\tau_j| \le C$ by direct Taylor expansion and Lemma 2.1. Intuitively, $P(R_g\pi - \frac{1}{h}\pi^h) = O(h)$, and $\mathcal{L}_h^*$ has a spectral gap in ℓ^1. Hence, we may possibly invert $\mathcal{L}_h^*$ and obtain

$$\left\| R_g \pi - \frac{1}{h}\pi^h \right\|_{\ell^1} \le Ch.$$

This understanding is not quite a rigorous proof. Below, we provide a rigorous proof.

Proof of Theorem 3.1. We have the following identity for operators from ℓ^1 to ℓ^1:

$$I = e^{t\mathcal{L}_h^*} + \int_0^t e^{(t-s)\mathcal{L}_h^*} \mathcal{L}_h^* \, ds. \tag{5.26}$$

In fact, for any $v \in \ell^1$ that does not depend on time, we set $f = \mathcal{L}_h^* v$. Then, $\frac{d}{dt}v + \mathcal{L}_h^* v = f$ implies that $v(t) = e^{t\mathcal{L}_h^*} v(0) + \int_0^t \exp((t-s)\mathcal{L}_h^*) f(s)\, ds$. Since we have assumed $v(t) \equiv v$, the identity is proved.

Now, we act on the identity on $E_j = \pi(x_j) - \frac{1}{h}\pi_j^h$. Using equation (5.25), we have

$$E = e^{t\mathcal{L}_h^*} E + h \int_0^t e^{(t-s)\mathcal{L}_h^*} \tau \, ds,$$

where $\|\tau\|_{\ell^1} \le C$. Since τ is in the range of $\mathcal{L}_h^*$, we therefore have (recall (4.6))

$$\sum_{j\in\mathbb{Z}} h\tau_j = \langle 1, \tau_j\rangle_h = \langle \mathcal{L}_h 1, E\rangle_h = 0,$$

by approximating E with sequences that have finite nonzero entries. Moreover, we define

$$\bar{\pi}_j = \frac{1}{h}\int_{x_j-h/2}^{x_j+h/2} \pi(y)\,dy,$$

and have $\|\bar{\pi} - R_g\pi\|_{\ell^1} \le C_1 h$. Applying Corollary 5.1, we have

$$\|E\|_{\ell^1} \le \|e^{t\mathcal{L}_h^*}(\bar{\pi} - R_g\pi)\|_{\ell^1} + \lim_{t\to\infty}\|e^{t\mathcal{L}_h^*}(\bar{\pi} - h^{-1}\pi^h)\|_{\ell^1} + h\int_0^\infty Ce^{-(t-s)\kappa_1 t}\,ds.$$

The second term is zero by Corollary 5.1 and the result follows. □

6. Bounded domain with periodic boundary condition

If the domain is bounded with periodic boundary condition or we consider the problems on the torus with length L

$$\mathbb{T} = \mathbb{R}/(L\mathbb{Z}), \tag{6.1}$$

many of the proofs above can be significantly simplified. However, the proofs in this section also differ from the above arguments in the sense that there is no detailed balance. Hence, this section may give inspiration to general schemes of conservation laws in higher dimensions.

The Wiener process W is the standard Wiener process in $\mathbb{R}$ wrapped into $\mathbb{T}$. Hence, the generator and the Kolmogorov equations are unchanged. For SDEs on the torus, one may refer to [10, 36]. We will assume generally the following.

Assumption 6.1. Assume b, σ are smooth functions on $\mathbb{T}$ and $\sigma^2 \ge S_1 > 0$.

By [36, section 2], Assumption 6.1 implies that the SDE has a unique stationary measure with smooth density. In fact, for $d = 1$, we can verify this directly. Letting $v(x) = \pi(x)\sigma^2(x)$ and $b_1(x) = b(x)/\sigma^2(x)$, the equation $\mathcal{L}^*\pi = 0$ implies that

$$v(x) = \exp\Big(-\int_0^x b_1(y)\,dy\Big)\left(v(0) + C\int_0^x \exp(\int_0^z b_1(y)\,dy)\,dz\right). \tag{6.2}$$

Using $v(L) = v(0)$, we find

$$v(0) + C\int_0^L \exp\Big(\int_0^z b_1(y)\,dy\Big)\,dz = v(0)\exp\Big(\int_0^L b_1(y)\,dy\Big) > 0, \tag{6.3}$$

which determines C uniquely. Since $\int_0^x \exp(\int_0^z b_1(y)\,dy)\,dz \le \int_0^L \exp(\int_0^z b_1(y)\,dy)\,dz$, $v(x) > 0$ for all $x \in [0, L]$. Hence, we can normalize so that $\int_0^L \pi(x)\,dx = 1$.

Note that for the Fokker-Planck equation on the torus, the corresponding jump process may not be reversible (the stationary distribution does not have detailed balance). The function $q(x,t) = p(x,t)/\pi(x)$ satisfies (2.16) and the modified SDE is given by

$$dY = \left(\frac{1}{\pi}\partial_x(\sigma^2\pi) - b\right)dt + \sigma\,dW. \tag{6.4}$$

As before, π is also the stationary solution to the modified SDE, and (2.22) still holds. With this observation, we have the following.

Lemma 6.1. *Suppose Assumption* 6.1 *holds. Then, let* $u(x,t) = \mathbb{E}_x\varphi(X)$ *for the SDE* (2.1) *or* $u(x,t) = \mathbb{E}_x\varphi(Y)$ *for the modified SDE* (6.4) *where* $\varphi \in C^\infty(\mathbb{T})$. *Then for any integer* $k > 0$ *we have for some* $\lambda_k > 0$ *that*

$$\|u - \langle \pi, \varphi\rangle\|_{C^k(\mathbb{T})} \le C\exp(-\lambda_k t). \tag{6.5}$$

Consequently, for any index n, *we can find* $\gamma_n > 0$ *such that*

$$\sup_{x\in\mathbb{T}}\left|\frac{\partial^n}{\partial x^n}(\rho(x,t) - \pi(x))\right| \le C_n\exp(-\gamma_n t). \tag{6.6}$$

The proof of Lemma 6.1 follows closely [49, section 6.1.2], and we put it in Appendix B for convenience. This fact is also used in [10, H3].

For the discretization, we pick a positive integer N and define

$$h = \frac{L}{N},\ x_j = jh,\ 0 \le j \le N-1. \tag{6.7}$$

If j falls out of $[0,N]$, we wrap it back into $[0,N]$ using periodicity. (For example, $j = N+2$ will be understood as $j = 2$.) We again consider the upwind scheme (3.5) and the B-schemes (3.8). However, we emphasize that the Assumption 3.1 for the B-schemes is no longer needed in this section.

Lemma 6.2. *Equation* (3.16) *has on* $\mathbb{T}$ *a unique stationary solution up to multiplicative constants. Besides, the one with* $\sum_j \pi_j^h = 1$ *satisfies* $\pi_j^h > 0$ *for all* j. *Moreover, we have for any sequence* f *that*

$$-\sum_{j=0}^{N-1}\pi_j^h f_j\mathcal{L}_h f_j = \sum_{j=0}^{N-1}\frac{\beta_{j+1}\pi_{j+1}^h + \alpha_j\pi_j^h}{2}(f_{j+1} - f_j)^2, \tag{6.8}$$

where $\mathcal{L}_h$ *is the generator of the jump process* $Z(t)$ *for* (3.16) *on* $\mathbb{T}$.

Proof. Note that the jump process $Z(t)$ is irreducible and aperiodic with finite states. The existence of a unique stationary distribution follows from the standard theory of Markov chains. See [32], for example. This stationary distribution (denoted as π^h) is clearly a positive solution of $\mathcal{L}_h^* f = 0$ with $\sum_j \pi_j^h = 1$. We fix this π^h now, and show that all solutions are multiples of π^h.

Direct computation shows that for any $j = 0, \dots, N-1$

$$f_j\mathcal{L}_h f_j = \frac{1}{2}(\mathcal{L}_h f^2)_j - \frac{\beta_j}{2}(f_{j-1}^2 - f_j)^2 - \frac{\alpha_j}{2}(f_j - f_{j+1})^2.$$

Multiplying π_j^h and taking the sum on j yield (6.8).

According to (6.8), we find that $\mathcal{L}_h f = 0$ only has constant solutions. This means that the right eigenspace of $\mathcal{L}_h$ corresponding to eigenvalue 0 is one dimensional. Hence, the left eigenspace of $\mathcal{L}_h$ for eigenvalue 0 is also one dimensional. This means that $\mathcal{L}_h^* f = 0$ has a unique solution up to multiplying constants $\square$

The stationary solution has the following property.

Lemma 6.3. *There exists a constant* C *independent of* h *such that for sufficiently small* h

$$\max_{0\le j\le N-1}\pi_j^h \le C\min_{0\le j\le N-1}\pi_j^h. \tag{6.9}$$

Proof. We introduce the variable

$$z_j := \pi_j^h/\pi(x_j),\ j = 0, \dots, N-1. \tag{6.10}$$

Since $\pi(\cdot)$ is bounded from below and above, we only need to investigate z_j.

The discussion for the upwind scheme (3.5) and the B-schemes (3.8) are similar. We only take (3.5) as the example.

Consider first the equation for π_j^h.

$$-\left(\frac{s_j^+\pi(x_j)z_j - s_{j-1}^+\pi(x_{j-1})z_{j-1}}{h} - \frac{s_{j+1}^-\pi(x_{j+1})z_{j+1} - s_j^-\pi(x_j)z_j}{h}\right) \tag{6.11}$$
$$+\frac{1}{2h^2}(\sigma_{j+1/2}^2\pi(x_{j+1})z_{j+1} - (\sigma_{j+1/2}^2 + \sigma_{j-1/2}^2)\pi(x_j)z_j + \sigma_{j-1/2}^2\pi(x_{j-1})z_{j-1}) = 0.$$

Since $\pi(x)$ is a solution to $\mathcal{L}^*\pi = 0$, there exists $h_0 > 0$ such that for all $h \le h_0$,

$$\mathcal{L}_h^*\pi(x_j) = \tau_j h\ \forall 0 \le j \le N-1, \tag{6.12}$$

where $\|\tau_j\|_{\ell^\infty} \le C_1$ uniformly for $h \le h_0$. Subtracting (6.11) with $z_j\mathcal{L}_h^*\pi(x_j)$ and using (6.12), we have

$$T_h z_j := -\left(s_{j-1}^+\pi(x_{j-1})\frac{z_j - z_{j-1}}{h} - s_{j+1}^-\pi(x_{j+1})\frac{z_{j+1} - z_j}{h}\right) \tag{6.13}$$
$$+\frac{1}{2h^2}\Big(\sigma_{j+1/2}^2\pi(x_{j+1})(z_{j+1} - z_j) - \sigma_{j-1/2}^2\pi(x_{j-1})(z_j - z_{j-1})\Big) = -z_j\tau_j h.$$

Expanding $\pi(x_{j\pm1})$ in $\sigma_{j\pm1/2}^2\pi(x_{j\pm1})$ terms around $x_{j\pm1/2}$, it is not hard to see T_h is a first order consistent difference scheme for the modified backward operator

$$\tilde{\mathcal{L}}\,q = \frac{1}{2}\partial_x(\pi\sigma^2\partial_x q) - \Big(\frac{1}{2}\sigma^2\partial_x\pi - s\pi\Big)\partial_x q, \tag{6.14}$$

which is clearly the same as the one in (2.20).

The crucial observation is that both T_h and $\tilde{\mathcal{L}}$ with Dirichlet boundary conditions have maximum principles. This allows us to prove the stability of T_h. Let us now investigate this in detail. Assume z_j attains the maximum value at j^*. Without loss of generality, we can assume $j^* = 0$. Then, define for $j = 0, \dots, N-1$ that

$$\zeta_j := \frac{z_j}{z_0} - 1. \tag{6.15}$$

We find then

$$T_h\zeta_j = -\frac{z_j}{\|z\|_{\ell^\infty}}\tau_j h \text{ for } j = 1, \dots, N-1,$$
$$\zeta_0 = \zeta_N = 0.$$

Consider the equation

$$\tilde{\mathcal{L}}\,\phi(x) = 1,\ \phi(0) = \phi(L) = 0.$$

By the maximum principle, $\phi(x) < 0$ for $x \in (0, L)$. Since T_h is a consistent scheme for $\tilde{\mathcal{L}}$, for sufficiently small h, we have

$$T_h\phi(x_j) \ge 1/2,\ j = 1, \dots, N-1.$$

Letting $\xi_j := 2\|\tau\|_\infty\phi(x_j)h - \zeta_j$, we have for $j = 1, \dots, N-1$,

$$T_h(\xi)_j \ge 0$$

with $\xi_0 = \xi_N = 0$. This means $\xi_j \le 0$ by maximum principle and hence

$$\zeta_j \ge 2\|\tau\|_\infty \phi(x_j) h.$$

Similarly, replacing ζ with $-\zeta$, we have $\zeta_j \le -2\|\tau\|_\infty \phi(x_j)h$. This means

$$\max_{0\le j\le N-1} |\zeta_j| = \max_{0\le j\le N-1} \left|\frac{z_j}{z_0} - 1\right| \le 2\|\tau\|_\infty \|\phi\|_\infty h. \tag{6.16}$$

Hence, for all $j = 0, \dots, N-1$,

$$\frac{z_j}{z_0} \ge 1 - 2\|\tau\|_\infty \|\phi\|_\infty h \ge \frac{1}{2}, \tag{6.17}$$

when h is sufficiently small. The claim (6.9) follows since π is bounded from above and below by positive numbers. $\square$

Now, we prove the uniform consistency, which is an analogy of Theorems 3.1 and 3.2.

Theorem 6.1. *Consider the upwind scheme* (3.5) *or the B-schemes* (3.8)*, and the jump process $Z(t)$ corresponding to* (3.16) *on $\mathbb{T}$. Suppose Assumption* 6.1 *holds. Then,*

(i) *The stationary distribution of* (3.16) *satisfies that*

$$\max_{0\le j\le N-1} \left|\frac{1}{h}\pi^h - \pi(x_j)\right| \le Ch. \tag{6.18}$$

(ii) *The following uniform error estimate holds for* (3.5)*:*

$$\sup_{t\ge 0} \|R_g \rho(\cdot, t) - \rho^h(t)\|_{\ell^1} \le Ch.$$

The first claim is essentially proven in the proof of Lemma 6.3. There, we have seen that $|z_j/\|z\|_\infty - 1| \le Ch$. Since $|\sum_j h\pi(x_j) - 1| \le C_1 h$ and $\sum_j z_j \pi(x_j) = 1$, we then conclude that $|h^{-1}\|z\|_{\ell^\infty} - 1| \le C_2 h$. The second claim can be proved in the same way as in the proof of Theorem 3.2.

We now move on to the convergence to equilibrium. Using Lemma 6.3 and that the torus is a bounded domain, the following version of discrete Poincaré inequality (analogy of Theorem 5.1) can be proved in a straightforward way (one can refer to [19, Proposition 4.6] for a similar discussion).

Lemma 6.4. *Suppose Assumption* 6.1 *holds. Then there exists $h_0 > 0$ and $\kappa_1 > 0$, so that for any sequence f, we have*

$$\kappa_1 \sum_{j=0}^{N-1} \pi_j^h \left(f_j - \sum_{i=0}^{N-1} \pi_i^h f_i\right)^2 \le \sum_{j=0}^{N-2} \frac{\beta_{j+1}\pi_{j+1}^h + \alpha_j \pi_j^h}{2} (f_{j+1} - f_j)^2. \tag{6.19}$$

Proof. Since $f_j - f_0 = \sum_{k=1}^{j} (f_k - f_{k-1})$, we have

$$\begin{aligned}
\sum_{j=0}^{N-1} \pi_j^h (f_j - f_0)^2 &\le \sum_{j=1}^{N-1} \pi_j^h j \sum_{k=1}^{j} (f_k - f_{k-1})^2 \\
&= \sum_{k=1}^{N-1} \frac{\beta_k \pi_k^h + \alpha_{k-1}\pi_{k-1}^h}{2} (f_k - f_{k-1})^2 \sum_{j\ge k} \frac{2j\pi_j^h}{\beta_k \pi_k^h + \alpha_{k-1}\pi_{k-1}^h}.
\end{aligned}$$

The claim follows from the fact that when h is sufficiently small

$$\begin{aligned}\sum_{k\le j\le N-1}\frac{2j\pi_j^h}{\beta_k\pi_k^h+\alpha_{k-1}\pi_{k-1}^h} &\le \frac{2N^2}{\min_{j,k}(\beta_k\pi_k^h/\pi_j^h+\alpha_{k-1}\pi_{k-1}^h/\pi_j^h)}\\ &\le \frac{2CN^2}{\min_k(\beta_{k+1}+\alpha_k)}\\ &\le \frac{2C}{S_1}N^2h^2,\end{aligned}$$

where we have applied Lemma 6.3 to obtain $\min_j \pi_k^h/\pi_j^h \ge \frac{1}{C}$ and $\min_j \pi_{k-1}^h/\pi_j^h \ge \frac{1}{C}$ for any k. Since $Nh = L$ and $\sum_j \pi_j^h\left(f_j-\sum_i \pi_i^h f_i\right)^2 \le \sum_j \pi_j^h(f_j-f_0)^2$, the claim follows. $\square$

The chain in general is not reversible. In fact, for the stationary solutions, we have

$$J_{j+1/2} = J = const.$$

If $J=0$, then we must have $\prod_{j=0}^{N-1}\alpha_j = \prod_{j=0}^{N-1}\beta_j$, which may not be true. Hence, in general $J\neq 0$ and the process is not reversible. Defining

$$\tilde{\beta}_j := \frac{\alpha_{j-1}\pi_{j-1}^h}{\pi_j^h},\ \tilde{\alpha}_j := \frac{\beta_{j+1}\pi_{j+1}^h}{\pi_j^h},\ j=0,\dots,N-1 \tag{6.20}$$

we have

$$\alpha_j+\beta_j = \tilde{\alpha}_j+\tilde{\beta}_j,\ j=0,\dots,N-1.$$

Hence, using (3.16), we can write the equation for $q^h = p^h/\pi^h$ (p^h and q^h are similarly defined as in (3.15) and (4.13)) as

$$\frac{d}{dt}q_j = \tilde{\beta}_j q_{j-1}+\tilde{\alpha}_j q_{j+1}-(\tilde{\alpha}_j+\tilde{\beta}_j)q_j =: (\tilde{\mathcal{L}}_h q^h)_j,\ j=0,\dots,N-1. \tag{6.21}$$

It is easily verified that π^h is also a stationary solution of $\tilde{\mathcal{L}}_h^*$, the dual operator of $\tilde{\mathcal{L}}_h$:

$$(\tilde{\mathcal{L}}_h^*\pi^h)_j = \tilde{\alpha}_{j-1}\pi_{j-1}^h-(\tilde{\alpha}_j+\tilde{\beta}_j)\pi_j^h+\tilde{\beta}_{j+1}\pi_{j+1}^h = \beta_j\pi_j^h-(\alpha_j+\beta_j)\pi_j^h+\alpha_j\pi_j^h = 0. \tag{6.22}$$

With the preparation, we easily conclude the following, similar to Theorem 5.2.

Theorem 6.2. *Consider the upwind scheme* (3.5) *or the B-schemes* (3.8)*, and the equivalent discrete Fokker-Planck equation* (3.16) *on the torus. Suppose Assumption* 6.1 *holds. Then, we have $\|q^h(t)-1\|_{\ell^2(\pi^h)} \le \|q^h(0)-1\|_{\ell^2(\pi^h)}e^{-\kappa_1 t}$. Consequently, $p^h(t)$ converges to π^h exponentially fast in total variation norm $\sum_j |p_j(t)-\pi_j^h| \le C\exp(-\kappa_1 t)$, and thus $\|\rho^h(t)-\frac{1}{h}\|\rho^h(0)\|_{\ell^1}\pi^h\|_{\ell^1} \le C\exp(-\kappa_1 t)$.*

Proof. Let φ be a smooth function defined on $\mathbb{T}$. Applying (6.21) and using a similar calculation as in equation (4.12), we have

$$\begin{aligned}\frac{d}{dt}\sum_{j=0}^{N-1}\pi_j^h\varphi(q_j) &= \sum_{j=0}^{N-1}\pi_j^h\tilde{\alpha}_j(\varphi(q_j)+\varphi'(q_j)(q_{j+1}-q_j)-\varphi(q_{j+1}))\\ &\quad+\sum_{j=0}^{N-1}\pi_j^h\tilde{\beta}_j\left(\varphi(q_j)+\varphi'(q_j)(q_{j-1}-q_j)-\varphi(q_{j-1})\right).\end{aligned} \tag{6.23}$$

If we take $\varphi(q_j) = \frac{1}{2}(q_j - \sum_i \pi_i^h q_i)^2$, we then have by (6.23) and (6.20) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1}\pi_j^h\left(q_j - \sum_i \pi_i^h q_i\right)^2 &= -\sum_{j=0}^{N-1}\frac{\tilde{\alpha_j}\pi_j^h + \tilde{\beta}_{j+1}\pi_{j+1}^h}{2}(q_{j+1}-q_j)^2 \\
&= -\sum_{j=0}^{N-1}\frac{\beta_{j+1}\pi_{j+1}^h + \alpha_j\pi_j^h}{2}(q_{j+1}-q_j)^2 \\
&\le -\sum_{j=0}^{N-2}\frac{\beta_{j+1}\pi_{j+1}^h + \alpha_j\pi_j^h}{2}(q_{j+1}-q_j)^2.
\end{aligned}
\tag{6.24}
$$

Using Lemma 6.4, the remaining proof is similar to the proof of Theorem 5.2, and we omit it. □

7. A Monte Carlo method

In this section, we propose some Monte Carlo methods [44] to approximate the upwind scheme (3.5) or the B-schemes (3.8). One idea is to construct a jump process $\{Z_n^{\Delta t}\}$ with transition probability $\tilde{P} = I + \Delta t\, Q$ using forward Euler scheme in time. In other words, the probability distribution satisfies

$$p^{n+1} = (I + \Delta t\, Q)p^n, \tag{7.1}$$

where p^n refers to the probability distribution at the nth step. There are two drawbacks. First, the forward Euler introduces numerical errors in time discretization; secondly $I + \Delta t\, Q$ may have negative entries for any Δt. One can also consider the backward Euler scheme where the transition probability is $(I - \Delta t\, Q)^{-1}$. The disadvantage of this matrix is that it is usually full and inconvenient for the full space $\mathbb{R}$. Another idea is to use the continuous time random walk. The process waits for a random time that satisfies an exponential distribution at a site and then performs a jump. This idea can avoid using the time discretization to recover (3.5). If we consider the schemes on $\mathbb{R}$, we need the exponential distribution for the waiting time to depend on the site j, and a corresponding Monte Carlo method can be developed. For the jump process $Z(t)$ on the torus, we can choose the exponential distribution independent of the sites. Then the number of jumps is a Poisson process and this motivates another Monte Carlo algorithm. For convenience, we focus on the problems on the torus only and explain this Monte Carlo algorithm in detail.

Lemma 7.1 ([32, Example 2.5])**.** *Let P be a transition matrix. Let $\mathcal{N}(t)$ be a Poisson process of intensity λ. If $Z_1(t)$ is the process that takes transitions at jumps of $\mathcal{N}(t)$ according to P, then $Z_1(t)$ is a continuous time jump process with Q matrix to be*

$$Q = \lambda(P - I). \tag{7.2}$$

Recall that Q matrix is defined in (4.2) so that $p_t(i,j) = \mathbb{P}(Z_1(t) = j | Z_1(0) = i)$ satisfies

$$\frac{d}{dt}p_t(i,j) = \sum_k Q(i,k)p_t(k,j) = \sum_k p_t(i,k)Q(k,j).$$

Lemma 7.1 follows easily from the fact $Z_1(t)$ is Markovian and that

$$p_t(i,j) = \mathbb{E}P^{\mathcal{N}(t)}(i,j) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}P^n(i,j). \tag{7.3}$$

Here, P^n is defined inductively by $P^{m+1}(i,j) = \sum_k P^m(i,k)P(k,j)$ with $P^1 = P$. If $Q(i,j)$ is bounded, we can take λ large enough so that

$$P = I + \lambda^{-1}Q \tag{7.4}$$

has nonnegative entries. For problems on the torus, we can do this and then $Z_1(t)$ is a realization of $Z(t)$. This then gives the following Monte Carlo method:

(1) Fix $T > 0$. Pick $\lambda \ge \max(\alpha + \beta)$ with α, β in (3.6) or (3.9). Pick M for the number of samples.
(2) For $m = 1 : M$:
   - Sample $\mathcal{N} \sim Poisson(\lambda T)$, and $j_0 \sim p_j(0)$.
   - Sample $Y_{\mathcal{N}}$ according to the j_0th row of $P^{\mathcal{N}}$. (In other words, we have a discrete time Markov chain $\{Y_n\}_{n=1}^{\mathcal{N}}$ with $Y_0 = j_0$ and transition matrix P in (7.4), or $P(j,j) = 1 - \lambda^{-1}(\alpha_j + \beta_j)$, $P(j, j-1) = \lambda^{-1}\beta_j$, and $P(j, j+1) = \lambda^{-1}\alpha_j$.)
(3) Let $\tilde{p}$ be the empirical distribution of $Y_{\mathcal{N}}$ (with M values of $Y_{\mathcal{N}}$). Then, $\tilde{\rho}(x_j, T) = h^{-1}\|\rho_0^h\|_{\ell^1}\tilde{p}_j$ is the numerical solution.

As is well known, the Monte Carlo method converges with the error bound $\sqrt{\mathrm{var}(Z(t))/M}$ [44]. While the variance is bounded here in time according to the uniform ergodicity, the convergence is uniformly in the rate $1/\sqrt{M}$.

Remark 7.1. Since $\mathbb{E}\mathcal{N} = \lambda T$, λ^{-1} is like the time step. Hence, $\lambda^{-1}\max(\alpha+\beta) \le 1$ is like the CFL condition (for parabolic equations).

Note that we may use fast algorithms to pre-compute P^n to save time. Consider the following SDE on $\mathbb{T}$ with $L = 2\pi$ and

$$b(x) = \cos(x)\exp(\sin(x)),\ \sigma(x) = \exp\Big(\frac{1}{2}\sin(x)\Big).$$

It follows that

$$s(x) = b(x) - \sigma(x)\sigma'(x) = \frac{1}{2}\cos x \exp(\sin x),\ \pi(x) \propto \exp(\sin(x)).$$

By the symbol "π" in this example, whether we mean the circular ratio or the stationary solution should be clear in the context.

Now, we take $\rho(x,0) = \frac{1}{2\pi}$ so that $\lim_{t\to\infty}\rho(x,t) = \pi(x)$. The initial distribution for j_0 is therefore the uniform distribution. Figure 1 shows the computed $\tilde{\rho}$ for the upwind scheme (3.5) at $t = 1, 4, 10, 12$, where we take the number of grid points $N = 2^6$, $h = 2\pi/N$, $\lambda = \max(\alpha_j + \beta) + 10 \approx 291.7$ and the number of samples $M = 10^6$. We find that the numerical solution of the Monte Carlo method for the jump process indeed converges to a stationary solution fast. Moreover, the stationary solution of the numerical solution is close to the stationary distribution of the SDE. This example therefore verifies our theory and the Monte Carlo method.

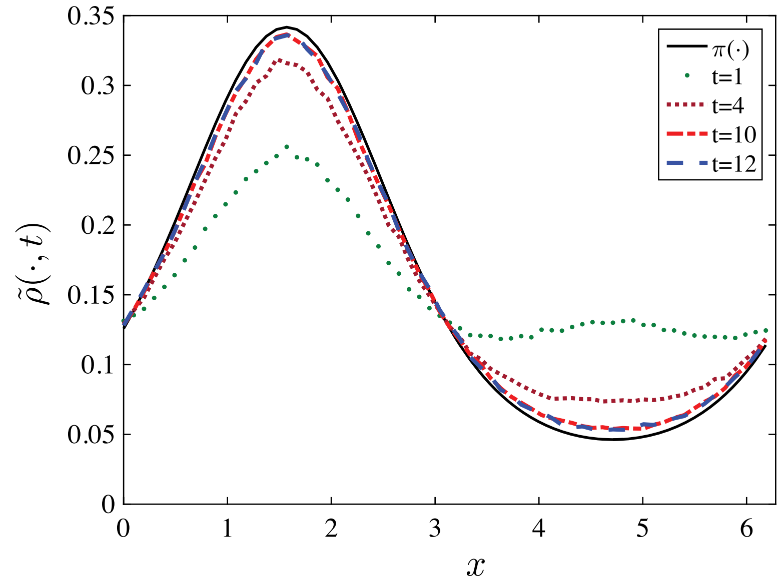

FIGURE 1. Monte Carlo simulation of the jump process corresponding to the upwind scheme (3.5). Number of grids $N = 2^6$, $\lambda \approx 291.7$ and number of samples $M = 10^6$. The solid black line shows the exact stationary solution $\pi(\cdot)$. Others show the computed numerical solution at $t = 1$(green dots), $t = 4$ (brown dotted line), $t = 10$ (red dash-dotted line), and $t = 12$ (blue dashed line). The stationary solution of the numerical solution is close to the stationary distribution of the SDE.

APPENDIX A. PROOF OF LEMMA 5.1

Proof of Lemma 5.1. Recall that θ is a nonnegative sequence with $\sum_j \theta_j < \infty$ and μ is a positive sequence on $\mathbb{Z}$. We first pick $f_i = \mu_i^{-1} 1_{[0,M]}(i)$. By the definition of A, we have

$$A \sum_{k=0}^{M} \mu_k^{-1} = A \sum_{k=-\infty}^{\infty} \mu_k f_k^2 \geq \sum_{j \geq 0} \theta_j \left(\sum_{k=0}^{j} f_k \right)^2 \geq \sum_{j \geq M} \left(\sum_{k=0}^{M} \mu_k^{-1} \right)^2 \theta_j.$$

Similarly, if we pick $f_i = \mu_i^{-1} 1_{[-M,-1]}(i)$, we have

$$A \sum_{k=-M}^{-1} \mu_k^{-1} = A \sum_{k=-\infty}^{\infty} \mu_k f_k^2 \geq \sum_{j \leq -1} \theta_j \left(\sum_{k=j}^{-1} f_k \right)^2 \geq \sum_{j \leq -M} \left(\sum_{k=-M}^{-1} \mu_k^{-1} \right)^2 \theta_j.$$

This verifies that $A \geq B$.

On the other hand, let us assume $\sum_j \mu_j f_j^2 = 1$. Note the basic inequality

$$\text{(A.1)} \qquad \frac{b-a}{2\sqrt{b}} \leq \sqrt{b} - \sqrt{a}, \;\; a \geq 0, b > 0.$$

Now let $\gamma_j := \sum_{k=0}^{j} \mu_k^{-1}$. Applying (A.1) and noting $\gamma_0 = \mu_0^{-1}$, we obtain

$$\sum_{k=0}^{j} \frac{\mu_k^{-1}}{\sqrt{\gamma_k}} = \frac{\mu_0^{-1}}{\sqrt{\gamma_0}} + \sum_{k=1}^{j} \frac{\gamma_k - \gamma_{k-1}}{\sqrt{\gamma_k}} \le \sqrt{\gamma_0} + 2\sqrt{\gamma_j} - 2\sqrt{\gamma_0} \le 2\sqrt{\gamma_j}. \tag{A.2}$$

Similarly,

$$\sum_{j\ge k} \frac{\theta_j}{\sqrt{\sum_{i\ge j}\theta_i}} = \sum_{j\ge k} \frac{\sum_{i\ge j}\theta_i - \sum_{i\ge j+1}\theta_i}{\sqrt{\sum_{i\ge j}\theta_i}} \le 2\sqrt{\sum_{i\ge k}\theta_i}. \tag{A.3}$$

Consequently, we find

$$\begin{aligned}
\sum_{j\ge 0}\theta_j \left(\sum_{k=0}^{j} f_k\right)^2 &\le \sum_{j\ge 0}\theta_j \left(\sum_{k=0}^{j} f_k^2 \mu_k \sqrt{\gamma_k}\right)\left(\sum_{k=0}^{j}\frac{\mu_k^{-1}}{\sqrt{\gamma_k}}\right) \\
&\le 2\sum_{j\ge 0}\theta_j\sqrt{\gamma_j}\sum_{k=0}^{j} f_k^2\mu_k\sqrt{\gamma_k} \\
&\le 2\sqrt{B}\sum_{j\ge 0}\frac{\theta_j}{\sqrt{\sum_{i\ge j}\theta_i}}\sum_{k=0}^{j} f_k^2\mu_k\sqrt{\gamma_k} \\
&= 2\sqrt{B}\sum_{k\ge 0} f_k^2\mu_k\sqrt{\gamma_k}\sum_{j\ge k}\frac{\theta_j}{\sqrt{\sum_{i\ge j}\theta_i}} \\
&\le 4B\sum_{k\ge 0} f_k^2\mu_k \le 4B.
\end{aligned}$$

The first inequality is due to Hölder inequality. The second inequality is due to (A.2). The third inequality is due to (recall the definition of γ_j and definition of B)

$$\sqrt{\gamma_j}\sqrt{\sum_{i\ge j}\theta_i} \le \sqrt{B}.$$

The second to last inequality is due to (A.3)

$$\sqrt{\gamma_k}\sum_{j\ge k}\frac{\theta_j}{\sqrt{\sum_{i\ge j}\theta_i}} \le 2\sqrt{\gamma_k}\sqrt{\sum_{i\ge k}\theta_i} \le 2\sqrt{B}.$$

Similarly, defining $\gamma_j = \sum_{k=j}^{-1}\mu_k^{-1}$, one can control

$$\sum_{j\le -1}\theta_j\left(\sum_{k=j}^{-1} f_k\right)^2 \le 4B.$$

Hence, $A \le 4B$. □

Appendix B. Proof of Lemma 6.1

Proof of Lemma 6.1. Recall the notation

$$\langle \pi, f\rangle = \int_{\mathbb{T}} f(x)\pi(x)\,dx.$$

Without loss of generality, we assume $\langle \pi, \varphi\rangle = 0$ and consider the equation of u for SDE (2.1) (the proof for the modified SDE (6.4) is just the same):

$$\partial_t u = \mathcal{L}u = b\,\partial_x u + \frac{1}{2}\Lambda \partial_{xx} u. \tag{B.1}$$

We see $\langle \pi, u\rangle = 0$ for all $t > 0$. Multiplying $2u$, we have

$$\partial_t |u|^2 = \mathcal{L}|u|^2 - \Lambda |\partial_x u|^2.$$

Multiplying π and integrating yields

$$\frac{d}{dt}\int_{\mathbb{T}} \pi(x)|u|^2(x)\,dx = -\int_{\mathbb{T}} \pi\Lambda|\partial_x u|^2\,dx \le -\lambda \int_{\mathbb{T}} \pi|u|^2\,dx. \tag{B.2}$$

The inequality follows from Poincaré inequality since $\langle \pi, u\rangle = 0$. We then obtain the exponential decay of $\langle \pi, |u|^2\rangle$:

$$\langle \pi, |u|^2\rangle = \int_{\mathbb{T}} |u|^2\pi\,dx \le \langle \pi, \varphi^2\rangle \exp(-\lambda t).$$

Consequently, multiplying $e^{(\lambda-\delta)t}$ in (B.2) for $\delta > 0$ small and taking integral,

$$\int_0^\infty e^{(\lambda-\delta)t}\int_{\mathbb{T}} \pi\Lambda|\partial_x u|^2\,dx = -\int_0^\infty e^{(\lambda-\delta)t}\frac{d}{dt}\int_{\mathbb{T}} \pi|u|^2\,dxdt \le C$$

means that $\int_0^\infty e^{(\lambda-\delta)t}\langle \pi, |\partial_x u|^2\rangle\,dt < \infty$.

Now, we perform induction. For convenience, we will use D to mean either $\frac{d}{dx}$ or $\frac{\partial}{\partial x}$. Assume that we have proved that for all $m \le n-1$

$$\langle \pi, |D^m u|^2\rangle \le C_m \exp(-\gamma_m t) \tag{B.3}$$

and that for all $m \le n$

$$\int_0^\infty e^{\tilde{\lambda}_m t}\langle \pi, |D^m u|^2\rangle\,dt < \infty. \tag{B.4}$$

We show (B.3)-(B.4) hold for $m \le n$ and $m \le n+1$, respectively. Taking the nth order derivative of (B.1), we have

$$\partial_t D^n u = \mathcal{L}D^n u + g_{n,0}(x)D^{n+1}u + g_{n,1}(x)D^n u + \sum_{m\le n-1} g_{n,n-m+1}D^m u,$$

where $g_{n,m}(x)$ are smooth functions involving b, σ and their derivatives. Multiplying $2\pi D^n u$ and taking the integral, we have

$$\begin{aligned}\partial_t\langle \pi, |D^n u|^2\rangle \le& -\int_{\mathbb{T}} \Lambda|D^{n+1}u|^2\pi\,dx + C\int_{\mathbb{T}} |D^{n+1}uD^n u|\pi\,dx \\ &+ C\langle \pi, |D^n u|^2\rangle + \sum_{m\le n-1} C_m\langle \pi, |D^m uD^n u|\rangle.\end{aligned} \tag{B.5}$$

Since $\int_{\mathbb{T}} |D^{n+1}uD^n u|\pi\,dx \le \nu\langle \pi, |D^{n+1}u|^2\rangle + \frac{1}{4\nu}\langle \pi, |D^n u|^2\rangle$, the $D^{n+1}u$ term is controlled by the first term on the right hand side. Multiplying on both sides with $e^{\tilde{\lambda}_n t}$ and taking the integral from 0 to t, one can get the results (B.3), (B.4) for $m = n$ and $m = n+1$, respectively. This then finishes the induction.

Now (B.3)-(B.4) hold for all $m \ge 0$. Since π is bounded from below, we find that $\|u - \langle \pi, \varphi\rangle\|^2_{H^k(\mathbb{T})} \le C_n \exp(-\gamma_n t)$. The claims for the decay of $\|u - \langle \pi, \varphi\rangle\|_{C^k}$ follow from Sobolev embedding.

We see $p(x,t) = q(x,t)\pi(x)$ where q satisfies the backward equation for the modified SDE (6.4). The first part of this lemma says that $\|q(\cdot,t)-1\|_{C^k} \le C\exp(-\gamma_k t)$. Since π is smooth on $\mathbb{T}$, we then have $\|\rho(\cdot,t)-\pi\|_{C^k} = \|\pi(q(\cdot,t)-1)\|_{C^k}$ decays to zero exponentially fast. $\square$

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References

[1] S. Aida, *Uniform positivity improving property, Sobolev inequalities, and spectral gaps*, J. Funct. Anal. **158** (1998), no. 1, 152–185, DOI 10.1006/jfan.1998.3286. MR1641566
[2] L. A. Bagirov, *Elliptic equations in an unbounded domain* (Russian), Mat. Sb. (N.S.) **86(128)** (1971), 121–139. MR0296750
[3] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Vol. 348, Springer Science & Business Media, 2013.
[4] H. Berestycki, L. Caffarelli, and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 69–94 (1998). Dedicated to Ennio De Giorgi. MR1655510
[5] S. G. Bobkov and F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal. **163** (1999), no. 1, 1–28, DOI 10.1006/jfan.1998.3326. MR1682772
[6] C. Chainais-Hillairet and J. Droniou, *Finite-volume schemes for noncoercive elliptic problems with Neumann boundary conditions*, IMA J. Numer. Anal. **31** (2011), no. 1, 61–85, DOI 10.1093/imanum/drp009. MR2755937
[7] C. Chainais-Hillairet and M. Herda, *Large-time behavior of a family of finite volume schemes for boundary-driven convection-diffusion equations*, IMA J. Numer. Anal. **00** (2019), 1–32.
[8] M. F. Chen, *Variational formulas of Poincaré-type inequalities for birth-death processes*, Acta Math. Sin. (Engl. Ser.) **19** (2003), no. 4, 625–644, DOI 10.1007/s10114-003-0308-9. MR2023358
[9] M. G. Crandall and A. Majda, *Monotone difference approximations for scalar conservation laws*, Math. Comp. **34** (1980), no. 149, 1–21, DOI 10.2307/2006218. MR551288
[10] A. Debussche and E. Faou, *Weak backward error analysis for SDEs*, SIAM J. Numer. Anal. **50** (2012), no. 3, 1735–1752, DOI 10.1137/110831544. MR2970763
[11] F. Delarue and F. Lagoutière, *Probabilistic analysis of the upwind scheme for transport equations*, Arch. Ration. Mech. Anal. **199** (2011), no. 1, 229–268, DOI 10.1007/s00205-010-0322-x. MR2754342
[12] F. Delarue, F. Lagoutière, and N. Vauchelet, *Convergence order of upwind type schemes for transport equations with discontinuous coefficients*, J. Math. Pures Appl. (9) **108** (2017), no. 6, 918–951, DOI 10.1016/j.matpur.2017.05.012. MR3723162
[13] K. Disser and M. Liero, *On gradient structures for Markov chains and the passage to Wasserstein gradient flows*, Netw. Heterog. Media **10** (2015), no. 2, 233–253, DOI 10.3934/nhm.2015.10.233. MR3356496
[14] M. D. Donsker, *An invariance principle for certain probability limit theorems*, Mem. Amer. Math. Soc., **No. 6** (1951), 12. MR0040613
[15] M. D. Donsker, *Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems*, Ann. Math. Statistics **23** (1952), 277–281. MR47288
[16] G. Dujardin, F. Hérau, and P. Lafitte, *Coercivity, hypocoercivity, exponential time decay and simulations for discrete Fokker-Planck equations*, arXiv preprint arXiv:1802.02173 (2018).
[17] A. Eberle, *Reflection coupling and Wasserstein contractivity without convexity* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **349** (2011), no. 19-20, 1101–1104, DOI 10.1016/j.crma.2011.09.003. MR2843007

[18] A. Eberle, *Reflection couplings and contraction rates for diffusions*, Probab. Theory Related Fields **166** (2016), no. 3-4, 851–886, DOI 10.1007/s00440-015-0673-1. MR3568041
[19] F. Filbet and M. Herda, *A finite volume scheme for boundary-driven convection-diffusion equations with relative entropy structure*, Numer. Math. **137** (2017), no. 3, 535–577, DOI 10.1007/s00211-017-0885-7. MR3712285
[20] A. Gerardi, F. Marchetti, and A. M. Rosa, *Simulation of diffusions with boundary conditions*, Systems & Control Letters **4** (1984), 253–261.
[21] F. Gong and L. Wu, *Spectral gap of positive operators and applications* (English, with English and French summaries), J. Math. Pures Appl. (9) **85** (2006), no. 2, 151–191, DOI 10.1016/j.matpur.2004.11.004. MR2199011
[22] L. Gross, *Existence and uniqueness of physical ground states*, J. Functional Analysis **10** (1972), 52–109, DOI 10.1016/0022-1236(72)90057-2. MR0339722
[23] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083, DOI 10.2307/2373688. MR420249
[24] A. Harten, *On a class of high resolution total-variation-stable finite-difference schemes*, SIAM J. Numer. Anal. **21** (1984), no. 1, 1–23, DOI 10.1137/0721001. With an appendix by Peter D. Lax. MR731210
[25] A. Harten and P. D. Lax, *A random choice finite difference scheme for hyperbolic conservation laws*, SIAM J. Numer. Anal. **18** (1981), no. 2, 289–315, DOI 10.1137/0718021. MR612144
[26] A. Harten, P. D. Lax, and B. van Leer, *On upstream differencing and Godunov-type schemes for hyperbolic conservation laws*, SIAM Rev. **25** (1983), no. 1, 35–61, DOI 10.1137/1025002. MR693713
[27] M. Hino, *Exponential decay of positivity preserving semigroups on L^p*, Osaka J. Math. **37** (2000), no. 3, 603–624. MR1789439
[28] R. Khasminskii, *Stochastic stability of differential equations*, 2nd ed., Stochastic Modelling and Applied Probability, vol. 66, Springer, Heidelberg, 2012. With contributions by G. N. Milstein and M. B. Nevelson. MR2894052
[29] S. N. Kružkov, *First order quasilinear equations in several independent variables*, Mathematics of the USSR-Sbornik **10** (1970), no. 2, 217.
[30] H. J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Stochastic Modelling and Applied Probability 2nd ed., Applications of Mathematics (New York), vol. 24, Springer-Verlag, New York, 2001. MR1800098
[31] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*, Steady-state and time-dependent problems, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. MR2378550
[32] T. M. Liggett, *Continuous Time Markov Processes*: *An introduction*, Graduate Studies in Mathematics, vol. 113, American Mathematical Society, Providence, RI, 2010. MR2574430
[33] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Series in Mathematics & Applications, Horwood Publishing Limited, Chichester, 1997. MR1475218
[34] P. A. Markowich and C. Villani, *On the trend to equilibrium for the Fokker-Planck equation: an interplay between physics and functional analysis*, Mat. Contemp. **19** (2000), 1–29. VI Workshop on Partial Differential Equations, Part II (Rio de Janeiro, 1999). MR1812873
[35] J. C. Mattingly, A. M. Stuart, and D. J. Higham, *Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise*, Stochastic Process. Appl. **101** (2002), no. 2, 185–232, DOI 10.1016/S0304-4149(02)00150-3. MR1931266
[36] J. C. Mattingly, A. M. Stuart, and M. V. Tretyakov, *Convergence of numerical time-averaging and stationary measures via Poisson equations*, SIAM J. Numer. Anal. **48** (2010), no. 2, 552–577, DOI 10.1137/090770527. MR2669996
[37] S. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, With a prologue by Peter W. Glynn 2nd ed., Cambridge University Press, Cambridge, 2009. MR2509253
[38] L. Miclo, *An example of application of discrete Hardy's inequalities*, Markov Process. Related Fields **5** (1999), no. 3, 319–330. MR1710983
[39] L. Miclo, *On hyperboundedness and spectrum of Markov operators*, Invent. Math. **200** (2015), no. 1, 311–343, DOI 10.1007/s00222-014-0538-8. MR3323580
[40] A. Mielke, *Geodesic convexity of the relative entropy in reversible Markov chains*, Calc. Var. Partial Differential Equations **48** (2013), no. 1-2, 1–31, DOI 10.1007/s00526-012-0538-8. MR3090532

[41] B. Øksendal, *Stochastic Differential Equations: An introduction with applications*, 6th ed., Universitext, Springer-Verlag, Berlin, 2003. MR2001996

[42] L. Pareschi and T. Rey, *Residual equilibrium schemes for time dependent partial differential equations*, Comput. & Fluids **156** (2017), 329–342, DOI 10.1016/j.compfluid.2017.07.013. MR3693223

[43] P. J. Rabier and C. A. Stuart, *Exponential decay of the solutions of quasilinear second-order equations and Pohozaev identities*, J. Differential Equations **165** (2000), no. 1, 199–234, DOI 10.1006/jdeq.1999.3749. MR1771794

[44] R. Y. Rubinstein and D. P. Kroese, *Simulation and the Monte Carlo Method*, Wiley Series in Probability and Statistics, John Wiley & Sons, Inc., Hoboken, NJ, 2017. Third edition [of MR0624270]. MR3617204

[45] D. Scharfetter and H. Gummel, *Large-signal analysis of a silicon read diode oscillator*, IEEE Transactions on electron devices **16** (1969), no. 1, 64–77.

[46] A. Schlichting and C. Seis, *Convergence rates for upwind schemes with rough coefficients*, SIAM J. Numer. Anal. **55** (2017), no. 2, 812–840, DOI 10.1137/16M1084882. MR3631392

[47] D. Serre, *Systems of Conservation Laws. 2: Geometric structures, oscillations, and initial-boundary value problems; Translated from the 1996 French original by I. N. Sneddon*, Cambridge University Press, Cambridge, 2000. MR1775057

[48] C.-W. Shu, *Total-variation-diminishing time discretizations*, SIAM J. Sci. Statist. Comput. **9** (1988), no. 6, 1073–1084, DOI 10.1137/0909073. MR963855

[49] D. Talay, *Second-order discretization schemes of stochastic differential systems for the computation of the invariant law*, Stochastics: An International Journal of Probability and Stochastic Processes **29** (1990), no. 1, 13–36.

[50] C. Wang and J.-G. Liu, *Positivity property of second-order flux-splitting schemes for the compressible Euler equations*, Discrete Contin. Dyn. Syst. Ser. B **3** (2003), no. 2, 201–228, DOI 10.3934/dcdsb.2003.3.201. MR1973414

[51] F. Wang, *Functional Inequalities, Markov Semigroups and Spectral Theory*, Science Press, Beijing, 2005.

[52] L. Wu, *Uniformly integrable operators and large deviations for Markov processes*, J. Funct. Anal. **172** (2000), no. 2, 301–376, DOI 10.1006/jfan.1999.3544. MR1753178

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