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ABSTRACT

In this paper, we consider the mean field limit of Brownian particles with Coulomb repulsion in 3D space using compactness. Using a symmetrization technique, we are able to control the singularity and prove that the limit measure almost surely is a weak solution to the limiting nonlinear Fokker-Planck equation. Moreover, by proving that the energy almost surely is bounded by the initial energy, we improve the regularity of the weak solutions. By a natural assumption, we also establish the weak-strong uniqueness principle, which is closely related to the propagation of chaos.

I. INTRODUCTION

There are many phenomena in natural and social sciences that are related to interacting particles.[1–4] An effective method for studying these large and complex systems where small individuals interact with each other is the mean field approximation.[5–8] In this approximation, the effect of surrounding particles is approximated by a consistent averaged force field so that we have a one body problem. The mean field approximation naturally applies to the kinetic theory where the macroscopic properties of gases are studied.[9–12] A desired property in the mean field limit is the so-called "propagation of chaos."[10,13–15] Roughly speaking, starting with a chaotic initial configuration where the particles are from independent copies of the initial state, the statistical correlation between finite groups of particles vanishes at a later (fixed) time as the number of particles goes to infinity. In other words, the particles reduce to independent copies of nonlinear Markov processes.

In this paper, we are interested in the mean field limit of Brownian particles with Coulomb interaction in three dimensional space. More precisely, we consider the N particle system

$$dX_t^{i,N} = -\frac{1}{N}\nabla_{x_i}\mathcal{H}(X_t^{1,N},\dots,X_t^{N,N})\,dt + \sqrt{2}\,dB_i^N,\ i = 1,\dots,N, \tag{1.1}$$

where

$$\mathcal{H}(x_1,\dots,x_N) = \sum_{i,j:i<j} g(x_i - x_j) = \frac{1}{2}\sum_{i,j:i\neq j} g(x_i - x_j), \tag{1.2}$$

with g being the interaction potential. This system can be regarded as the overdamped limit of Langevin equations[16] so that dX's are from the friction terms, while $\frac{1}{N}\nabla_{x_i}\mathcal{H}$ can be thought as the interacting forces. Hence, in the remaining part of the paper, we will call this the "interacting forces," though it may have other interpretations in some applications. The scaling $1/N$ appears because we desire to have a

total mass or charge to be $O(1)$ so that there is a mean field limit as $N \to \infty$. The Brownian motion is not scaled since the strength is determined by the temperature instead of the mass of the particle. Of course, if one desires to consider other scaling regimes, there may be factors depending on N for both terms. The initial values $\{X_0^{i,N}\}$ are independent and identically distributed (iid) from some given density ρ_0. Also, $\{B_i^N\}_{i=1}^N$ are independent d-dimensional standard Brownian motions. Hence, the N-particle system is totally determined by $(X_0^{1,N}, \dots, X_0^{N,N}, B_1^N, \dots, B_N^N)$. It is standard by the Kolmogorov extension theorem[17,18] that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that all the random variables $\{(X_0^{1,N}, \dots, X_0^{N,N}, B_1^N, \dots, B_N^N)\}_{N=1}^\infty$ are on this probability space and they are all independent. Clearly, if we identify B_i^N for different N's, the law of $X_t^{i,N}$ is unchanged. Hence, we will drop the index N for the Brownian motions from now on. Moreover, we use $\mathbb{E}$ to mean the expectation under $\mathbb{P}$. If the interaction potential is given by

$$g(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & d = 2, \\ C_d |x|^{-(d-2)}, & d \geq 3, \end{cases} \tag{1.3}$$

where

$$C_d = \frac{1}{d(d-2)\pi} \left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \right)^{\frac{2}{d}} \tag{1.4}$$

is chosen such that $-\Delta g = \delta_0$ holds in the distributional sense, then the interaction is called the Coulomb repulsive interaction. Moreover, we define the Coulomb repulsive force as

$$F(x) = -\nabla g(x) = (d-2) C_d \frac{x}{|x|^d}. \tag{1.5}$$

We will particularly focus on the $d = 3$ case, but some discussions are performed for general d.

Our goal is to show that as $N \to \infty$, the empirical measure

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}} \tag{1.6}$$

[as a random variable taking values in $\mathbf{P}(C([0,T]; \mathbb{R}^d))$] converges in law to a random measure μ, whose density is almost surely a solution of the nonlinear Fokker-Planck equation

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla (g * \rho)), \quad \rho_t|_{t=0} = \rho_0, \tag{1.7}$$

provided that the initial empirical measure converges weakly to the initial data ρ_0. The meaning of "solution" here is the weak solution, which will be clarified later (Definition 3.1). In fact, we have the following result.

Theorem (Informal version of Theorem 3.1). *When $d = 3$, under suitable conditions of the initial data ρ_0, any limit point μ of the empirical measure μ^N under the topology of convergence in law [as random variables in $\mathbf{P}(C([0,T]; \mathbb{R}^d))$] has a density ρ a.s., and ρ is a weak solution to (1.7) a.s. in the sense of Definition 3.1.*

If the solution ρ is proved to be unique so that the limit measure μ is deterministic, then we have the propagation of chaos (see Ref. 15, Proposition 2.2). However, the regularity of the weak solution in Definition 3.1 is limited and it is very challenging to show the uniqueness of the weak solutions under these conditions even though the initial data ρ_0 are very good. On the other hand, it is standard to show that if $\rho_0 \geq 0$, $\rho_0 \in L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$ with some $m > d/2$, the equation has a unique global strong solution $\rho \in C([0,T]; L^1(\mathbb{R}^d)) \cap C([0,T]; H^m(\mathbb{R}^d))$ with $\rho \geq 0$. Moreover, $\|\rho\|_1 = \|\rho_0\|_1$ and $\rho \in C^\infty((0,\infty), H^s)$ for all $s \geq 0$. See Appendix A (Propositions A.2 and A.3) for reference. Hence, one desires to improve the regularity of the weak solutions so that one can eventually show that the "weak solution" by the limit measure is the same as the strong solution, which is one common way to establish the propagation of chaos. This is known to be the "weak-strong uniqueness principle."[19–22]

As a second main result of this work, we show that the energy almost surely is bounded by the initial energy so that we can improve the regularity of the limiting weak solutions (see Propositions 3.2 and 3.3). Together with this, the extra assumption that the weak solution $\rho \in L^2_{loc}([0,T]; L^2(\mathbb{R}^3))$ can imply the weak-strong uniqueness principle.

Theorem (Informal version of Propositions 3.2 and 4.1). *With suitable assumptions on the initial data ρ_0, for general dimension $d \geq 3$, for any limit point μ of μ^N, the energy is bounded by the initial energy almost surely,*

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y) \mu_t(dx) \mu_t(dy) \leq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y) \rho_0(x) \rho_0(y)\, dxdy.$$

Consequently, the density satisfies $\rho \in L^\infty(0,T; H^{-1})$ and $\nabla(g * \rho) \in L^\infty(0,T; L^2)$ almost surely. Finally, for $d = 3$, if one further assumes such a weak solution $\rho \in L^2_{\text{loc}}(0,T; L^2(\mathbb{R}^3))$, then it must be the unique strong solution.

We will explain in Sec. IV that the $L^2_{\text{loc}}(0,T;L^2(\mathbb{R}^3))$ assumption makes perfect physical sense due to the energy dissipation. However, rigorous justification is not easy. Combining these two results, we obtain a condition for the propagation of chaos.

Theorem (Informal version of Theorem 4.1). *Consider $d = 3$. With suitable assumptions on the initial data ρ_0, if the density ρ for any limit point μ of the empirical measures μ^N satisfies $\mathbb{E}\int_0^T \int_{\mathbb{R}^3} \rho^2\, dxdt < \infty$, then there is propagation of chaos.*

The tool we use to establish the above results is the compactness method based on entropy and Fisher information estimates. In Ref. 23, the propagation of chaos result for the $d = 2$ case is proved using compactness through a self-consistent martingale problem. The proof needs to control singularity with $(d-1)$-th order using the Fisher information from Ref. 24 (see Lemma 2.5 for a slightly generalized form). However, the proof there cannot be applied directly to $d \geq 3$ cases. What we do is to use certain symmetrization to reduce the singularity in the third term of (1.7) from $d-1$ to $d-2$. Using this trick and the estimates of Fisher information, we show that the limit measure almost surely is a weak solution to the limiting nonlinear Fokker-Planck equation (1.7) for $d = 3$ (Theorem 3.1). This is the first main result in this work. As already mentioned, the weak-strong uniqueness principle is only established by assuming that the density of the limit measure almost surely is in $L^2_{\text{loc}}(0,T;L^2(\mathbb{R}^3))$ (see Proposition 4.1). Although physically significant, the justification seems hard. As will be remarked in Sec. III, the compactness method based on Fisher information seems not to work for $d \geq 4$ cases, and new tools should be developed to tackle this problem.

Let us mention some related references, which by no means are exhaustive. In Refs. 12, 21, and 25, the mean field limit problems for particle systems without Brownian motions with various interaction kernels have been established. In particular, in Ref. 12, Serfaty and Duerinckx established the results for particles with Coulomb interaction even for $d \geq 3$. When Brownian motions are present, we have stochastic systems.[23,24,26–30] In Ref. 27, propagation of chaos was proved uniformly in time when the interaction kernel is regular enough and a confining potential is present. In Ref. 23, the propagation of chaos for 2D Coulomb interaction was proved using nonlinear martingale problems. In Ref. 29, the propagation of chaos for $W^{-1,\infty}$ kernels has been established, and this includes the kernels considered in Refs. 23 and 24. By estimating the relative entropy, they found the convergence rate of propagation of chaos for some models. However, the 3D Coulomb kernel is not included in their model, so their method does not apply.

The rest of the paper is organized as follows. In Sec. II, we review and prove some basic results for Fisher information of probability measures and N particle systems. In particular, the empirical measures of the N particle systems are tight so that any subsequence has a further converging subsequence to some limiting measure. Also, there are uniform estimates of the Fisher information. In Sec. III, using a symmetrization technique together with the Fisher information estimate, we show that the limit measure almost surely is a weak solution to the nonlinear Fokker-Planck equation (1.7). In Sec. IV, we establish the weak-strong uniqueness principle based on the assumption $\rho \in L^2_{\text{loc}}(0,T;L^2(\mathbb{R}^3))$ and remark on the propagation of chaos. In Appendices A and B, we provide the notes for strong solutions and missing proofs for reference.

II. SETUP AND EXISTING RESULTS

In this section, we first recall some basic properties of Fisher information and extend the estimates in Ref. 24 to high-dimensional cases. Then, we give an alternative proof for the well-posedness of the system (1.1). Finally, we present the results of tightness of the empirical measures in Ref. 23.

A. Entropy and Fisher information of probability measures

We begin with the definition of Fisher information. For any probability measure $f \in \mathbf{P}((\mathbb{R}^d)^k)$, we recall that the entropy and Fisher information are defined, respectively, by

$$H(f) := \begin{cases} \int_{\mathbb{R}^{kd}} \rho \log \rho\, dx, & \text{if } f = \rho\, dx, \\ +\infty & \text{otherwise,} \end{cases} \quad I(f) := \begin{cases} \int_{\mathbb{R}^{kd}} \frac{|\nabla \rho|^2}{\rho} dx, & \text{if } f = \rho\, dx, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

We also introduce the normalized entropy and Fisher information for $f \in \mathbf{P}((\mathbb{R}^d)^k)$,

$$H_k(f) := \frac{1}{k} H(f),\ I_k(f) := \frac{1}{k} I(f). \tag{2.2}$$

The normalized version is introduced so that $H_k(f^{\otimes k}) = H_1(f)$ and $I_k(f^{\otimes k}) = I_1(f)$ for $f \in \mathbf{P}(\mathbb{R}^d)$, which is convenient for the mean field limit discussion. We remark that the notations we use here are different from those in Ref. 31, where they use H to mean the normalized version while H_j is the un-normalized version. In the following discussion, we sometimes use $I_k(\rho)$ and $H_k(\rho)$ to represent $I_k(f)$ and $H_k(f)$ [or $I(\rho)$ and $H(\rho)$ to represent $I(f)$ and $H(f)$].

We denote the set of all symmetric probability measures on $(\mathbb{R}^d)^k$ by $\mathbf{P}_{sym}((\mathbb{R}^d)^k)$. By "symmetric," we mean the measure stays unchanged under the pushforward corresponding to any permutation of the k copies of $\mathbb{R}^d$. If the distribution of some k particle system is symmetric for all time, then the system is said to be exchangeable. Recall that for the joint probability distribution $F \in \mathbf{P}(\mathcal{X}^k)$ of k random variables taking values in $\mathcal{X}$, the marginal distribution of $X^{i_1}, \dots, X^{i_r}$ with $\{i_1, \dots, i_r\} \subset \{1, \dots, N\}$ is defined by

$$F_{i_1,\cdots,i_r} := \int_{\mathcal{X}^{k-r}} F \, dx^{i_{r+1}} \dots dx^{i_k}, \tag{2.3}$$

where $\{i_{r+1}, \dots, i_k\} = \{1, \dots, N\} \backslash \{i_1, \dots, i_r\}$. If F is symmetric, the marginal distributions are the same for different choices of $i_1, \dots, i_r$, and this will be called the r-marginal distribution later, denoted by $F^{(r)}$.

Below, we list out some standard properties of the entropy and Fisher information.

Lemma 2.1 (Ref. 31, Lemma 3.3). *We have the following superadditivity of entropy:*

1. *Suppose all the one marginal distributions of $F \in \mathbf{P}((\mathbb{R}^d)^k)$ are the same, denoted by $f \in \mathbf{P}(\mathbb{R}^d)$. If $H_1(f) < \infty$, then $H_k(F) \geq H_1(f)$. The equality holds if and only if $F = f^{\otimes k}$.*
2. *Consider $F \in \mathbf{P}((\mathbb{R}^d)^k)$. Then, the un-normalized entropies satisfy*

$$H(F) \geq H(F_{i_1,\cdots,i_r}) + H(F_{i_{r+1},\cdots,i_k}). \tag{2.4}$$

The equality holds if and only if $F = F_{i_1,\dots,i_r} \otimes F_{i_{r+1},\dots,i_k}$, where $r \in \{1, \dots, k-1\}$.

Proof. By Jensen's inequality, we have $\int_E g \log \frac{h}{g} \, dx \leq 0$ for any probability densities g, h on a Polish space E. Hence,

$$\int_E g \log g \, dx \geq \int_E g \log h \, dx. \tag{2.5}$$

The equality holds if and only if $g = h$, a.e. Now, we take $E = (\mathbb{R}^d)^k$. The first part of the claim follows by taking $g = F$ and $h = f^{\otimes k}$ in (2.5). Also, the second part follows by taking $g = F$ and $h = F_{i_1,\dots,i_r} \otimes F_{i_{r+1},\dots,i_k}$. □

Lemma 2.2 (Ref. 32, Theorem 3). *Suppose $F \in \mathbf{P}((\mathbb{R}^d)^k)$. Then, the non-normalized Fisher information satisfies*

$$I(F) \geq I(F_{i_1,\dots,i_r}) + I(F_{i_{r+1},\dots,i_k}). \tag{2.6}$$

The equality holds if and only if $F = F_{i_1,\dots,i_r} \otimes F_{i_{r+1},\dots,i_k}$.

From Lemmas 2.1 and 2.2, for $f \in \mathbf{P}_{sym}((\mathbb{R}^d)^k)$ with jth marginal distribution $f^{(j)}$, where $k = qj + r$, $q, r \in \mathbb{Z}, q \geq 0, 0 \leq r \leq j-1$, one then has

$$kI_k(f) \geq qjI_j(f^{(j)}) + rI_r(f^{(r)}), \quad kH_k(f) \geq qjH_j(f^{(j)}) + rH_r(f^{(r)}). \tag{2.7}$$

Moreover, Ref. 31, Lemma 3.7 shows that

$$I_j(f^{(j)}) \leq I_k(f), \tag{2.8}$$

i.e., for symmetric probability measures, the normalized Fisher information for marginal distributions $f^{(j)}$ can always be bounded by $I_k(f)$.

Since the entropy might be negative, we do not have $H_j(f^{(j)}) \leq H_k(f)$ and $H_j(f^j) \leq (k/(qj))H_k(f)$. To resolve this, we note the following lemma, which gives a lower bound for the entropy by moments of f.

Lemma 2.3 (Ref. 31, Lemma 3.1). *For any $p, \lambda > 0$, there exists a constant $C_{p,\lambda} \in \mathbb{R}$ such that for any $k \geq 1$, $F \in \mathbf{P}((\mathbb{R}^d)^k)$ with*

$$M_p(F) = \int_{(\mathbb{R}^d)^k} \frac{1}{k} \sum_{i=1}^{k} (|x_i|^2 + 1)^{\frac{p}{2}} F(dx) < \infty,$$

one has

$$H_k(F) \geq -C_{p,\lambda} - \lambda M_p(F). \tag{2.9}$$

Combining Eqs. (2.7) and (2.9), one gets a control of $H_j(f^j)$ in terms of $H_k(f)$ as follows:

$$H_j(f^{(j)}) \le \frac{k}{qj} H_k(f) + \frac{r}{qj}(\lambda M_p(f^{(r)}) + C_{p,\lambda}). \tag{2.10}$$

Next, we extend the estimates in Ref. 24 to high-dimensional cases.

Lemma 2.4. Let $d \ge 3$. For any probability density f in $\mathbb{R}^d$ with finite Fisher information $I(f)$, one has

$$\forall q \in \left[1, \frac{d}{d-1}\right], \|\nabla f\|_{L^q(\mathbb{R}^d)} \le C_{q,d} I(f)^{\frac{d+1}{2} - \frac{d}{2q}}, \tag{2.11}$$

$$\forall p \in \left[1, \frac{d}{d-2}\right], \|f\|_{L^p(\mathbb{R}^d)} \le C_{p,d} I(f)^{\frac{d}{2}(1-\frac{1}{p})}. \tag{2.12}$$

If $d = 2$, then (2.11) holds for $q \in [1,2)$, while (2.12) holds for $p \in [1, +\infty)$.

Proof. We start from (2.11). By Hölder's inequality,

$$\|\nabla f\|_q^q \le \left(\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} dx\right)^{\frac{q}{2}} \|f\|_{\frac{q}{2-q}}^{\frac{q}{2}}. \tag{2.13}$$

For $1 \le q \le \frac{d}{d-1}$, we use the interpolation along with Sobolev's inequality,

$$\|f\|_{\frac{q}{2-q}} \le \|f\|_1^{1-\theta} \|f\|_{\frac{dq}{d-q}}^{\theta} \le C_{q,d} \|\nabla f\|_q^{\theta}, \tag{2.14}$$

where θ is given by $\frac{2-q}{q} = \frac{d-q}{dq}\theta + (1-\theta)$. Note that f is a probability density. Plugging (2.13) into (2.14), we get (2.11).

Now, for $1 \le p \le \frac{d}{d-2}$, we can find some $1 \le r \le \frac{d}{d-1}$ satisfying $p = \frac{r}{2-r}$. Then, by (2.14) and (2.11), we can easily obtain (2.12). □

The following lemma is a slight generalization of those in Ref. 31 to higher dimension, which is important to control some singular integrals using Fisher information.

Lemma 2.5. Suppose (X_1, X_2) is a random variable with density F in $\mathbb{R}^d \times \mathbb{R}^d$. Assume that F has finite Fisher information $I(F)$.

1. *For any $0 < \gamma < 2$ and $\frac{\gamma}{d} < \beta \le \frac{2}{d}$, there exists $C_{\gamma,\beta}$ such that*

$$\mathbb{E}[|X_1 - X_2|^{-\gamma}] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{F(x_1, x_2)}{|x_1 - x_2|^{\gamma}} dx_1 dx_2 \le C_{\gamma,\beta}\left(I(F)^{\frac{\beta d}{2}} + 1\right). \tag{2.15}$$

Moreover, for any $\varepsilon > 0$, the following estimate holds:

$$\int_{|x-y|<\varepsilon} \frac{F(x,y)}{|x-y|^{\gamma}} dxdy \le C_{\gamma,\beta} \varepsilon^{d\beta-\gamma} I(F)^{\frac{d\beta}{2}}. \tag{2.16}$$

2. *For $d \ge 3$ and $\gamma = 2$, it also holds that*

$$\mathbb{E}[|X_1 - X_2|^{-2}] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{F(x_1, x_2)}{|x_1 - x_2|^2} dx_1 dx_2 \le \frac{C}{(d-2)^2} I(F). \tag{2.17}$$

Proof. Set $Y_1 = \frac{X_1 - X_2}{\sqrt{2}}$, $Y_2 = \frac{X_1 + X_2}{\sqrt{2}}$ and denote the joint distribution of (Y_1, Y_2) by $\tilde{F}(y_1, y_2)$. Then, $I(F) = I(\tilde{F})$. Denote the density of Y_1 by $\tilde{f}$. From the superadditivity property of Fisher information (Lemma 2.2), we see that $I(\tilde{f}) \le I(\tilde{F}) = 2I_2(F)$.

1. By simple computation,

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{F(x_1, x_2)}{|x_1 - x_2|^{\gamma}} dx_1 dx_2 &= 2^{\frac{\gamma}{2}} \int_{\mathbb{R}^d} \frac{\tilde{f}(y)}{|y|^{\gamma}} dy \\ &\le 2^{\frac{\gamma}{2}} \left(\int_{|y|>1} \tilde{f}(y) dy + \int_{|y|\le 1} \frac{\tilde{f}(y)}{|y|^{\gamma}} dy\right). \end{aligned} \tag{2.18}$$

The first term does not exceed 1, while for the second term one applies Hölder's inequality and (2.12),

$$\int_{|y|\le 1} \frac{\tilde{f}(y)}{|y|^{\gamma}} dy \le \left(\int_{|y|\le 1} |y|^{-\frac{\gamma}{\beta}} dy\right)^{\beta} \|\tilde{f}\|_{\frac{1}{1-\beta}} \le C_{\gamma,\beta} I(\tilde{f})^{\frac{\beta d}{2}} \le 2C_{\gamma,\beta} I_2(F)^{\frac{d\beta}{2}}. \tag{2.19}$$

Note that the restriction $\frac{\gamma}{\beta} < d$ comes from the integrability of $|y|^s$, while $\beta \le \frac{2}{d}$ comes from (2.12). Therefore, (2.15) holds. For (2.16), one has

$$\begin{aligned}\int_{|x-y|<\varepsilon} \frac{F(x,y)}{|x-y|^\gamma} dxdy &= \int_{|y|\le \frac{\varepsilon}{\sqrt{2}}} \frac{\tilde f(y)}{|y|^\gamma} dy \\ &\le \left(\int_{|y|\le \frac{\varepsilon}{\sqrt{2}}} |y|^{-\frac{\gamma}{\beta}} dy\right)^\beta \|\tilde f\|_{\frac{1}{1-\beta}} \le C_{\gamma,\beta}\varepsilon^{d\beta-\gamma} I(\tilde f)^{\frac{d\beta}{2}} \le 2C_{\gamma,\beta}\varepsilon^{d\beta-\gamma} I_2(F)^{\frac{d\beta}{2}},\end{aligned} \tag{2.20}$$

which implies (2.16).

2. For $d \ge 3$ and $\gamma = 2$, we note

$$\int_{\mathbb{R}^d} \frac{1}{|y|^2}\tilde f(y)\, dy = -\frac{1}{d-2}\int_{\mathbb{R}^d} \frac{y}{|y|^2}\cdot \nabla \tilde f(y)\, dy \le \frac{1}{d-2}\left(\delta \int_{\mathbb{R}^d} \frac{1}{|y|^2}\tilde f(y)\, dy + \frac{1}{4\delta}\int_{\mathbb{R}^d} \frac{|\nabla \tilde f(y)|^2}{\tilde f(y)}\, dy\right).$$

One can choose $\delta = \frac{d-2}{2}$ and obtain

$$\int_{\mathbb{R}^d} \frac{1}{|y|^2}\tilde f(y)\, dy \le \frac{1}{(d-2)^2} I(\tilde f).$$

The integration by parts can be easily justified by approximating $\tilde f$ with compactly supported smooth functions. The claim therefore follows. □

B. The *N*-particle system

In this part, we study the N-particle system (1.1) and provide some estimates on the entropy and energy. Most of the results have been established in Ref. 23, but we will give alternative proofs here for the convenience of the readers. These results will be used further in the proof for the propagation of chaos result in Sec. IV.

Assume that the dimension $d \ge 3$ and N is set to be fixed throughout this part. We will also use X_t^i to represent $X_t^{i,N}$ in this section for convenience. The joint distribution of the particles $(X_t^1, \dots, X_t^N)$ is denoted by f_t^N. The important quantities associated with the system include entropy and energy. Recall the entropy defined in (2.2) and we also define the energy given by [recall (1.6) for the empirical measure μ^N]

$$\mathscr{E}_N(t) := \frac{1}{2}\iint_{\mathcal{D}^c} g(x-y)\mu^N(dx)\mu^N(dy)(t) = \frac{1}{2N^2}\sum_{i\ne j} g(X_t^i - X_t^j), \tag{2.21}$$

where $\mathcal{D}$ represents the diagonal $\{(x,y) : x = y\}$. For convenience, we define

$$h_0 = (-\Delta)^{-1}\rho_0 = \int_{\mathbb{R}^d} g(x-y)\rho_0(y)\, dy. \tag{2.22}$$

The continuous system has an initial energy,

$$\mathcal{E}(\rho_0) := \frac{1}{2}\iint_{(\mathbb{R}^d)^2} g(x-y)\rho_0(x)\rho_0(y)dxdy = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla h_0(x)|^2\, dx = \frac{1}{2}\|\rho_0\|_{H^{-1}}^2 \le C\|\rho_0\|_{\frac{2d}{d+2}}^2, \tag{2.23}$$

by the Hardy-Littlewood-Sobolev inequality. Moreover, it holds that

$$\mathbb{E}\mathscr{E}_N(0) = \frac{N-1}{N}\mathcal{E}(\rho_0).$$

We first of all state the results about the well-posedness of the system (1.1).

Theorem 2.1. *For any $d \ge 3$ and $N \ge 2$, consider a sequence of independent d-dimensional Brownian motions $\{(B_t^i)_{t\ge 0}\}_{i=1}^N$ and the independent and identically distributed* (i.i.d.) *initial data $\{X_0^i\}_{i=1}^N$ with a common distribution f_0 that has a density ρ_0 satisfying $H_1(\rho_0) < +\infty$, $\mathcal{E}(\rho_0) < +\infty$, and $\rho_0 \in L^1(\mathbb{R}^d, (1+|x|^2)dx)$. Then, there exists a unique global strong solution to (1.1) with $X_t^i \ne X_t^j$ a.s. for all $t > 0$ and $i \ne j$.*

The proof for the noncollision result and energy estimate is based on the mollification approximation. Recall that the potential $g(x) = C_d|x|^{2-d}$ is the solution to $-\Delta g = \delta$, and the mollification we use is given by

$$g_\varepsilon = J_\varepsilon * g, \quad F_\varepsilon(x) = -\nabla g_\varepsilon(x), \tag{2.24}$$

where $J_\varepsilon = \frac{1}{\varepsilon^d} J(\frac{x}{\varepsilon})$, for some fixed $J(x) \in C^2(\mathbb{R}^d)$ which is non-negative, radial, with supp $J(x) \subset B(0,1)$ and $\int_{\mathbb{R}^d} J(x)\,dx = 1$. This mollification has the following standard properties, for which we omit the proofs.

Lemma 2.6 (Ref. 23, Lemma 2.1).

(i) $g_\varepsilon(x) = g(x), F_\varepsilon(x) = F(x)$ *whenever* $|x| \geq \varepsilon$;
(ii) $F_\varepsilon(0) = 0$, $\nabla \cdot F_\varepsilon(x) = J_\varepsilon(x)$;
(iii) $|F_\varepsilon(x)| \leq \min\{\frac{C_d|x|}{\varepsilon^d}, |F(x)|\}$.

We first consider the following regularized system for (1.1):

$$dX_t^{i,\varepsilon} = \frac{1}{N}\sum_{j=1,j\neq i}^{N} F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})dt + \sqrt{2}dB_t^i, \quad X_t^{i,\varepsilon}|_{t=0} = X_0^i. \tag{2.25}$$

We try to use system (2.25) to approximate (1.1). Since $F_\varepsilon \in C_b^2(\mathbb{R}^d)$, (2.25) is well-defined and has a unique strong solution. We start with the *a priori* estimates of the entropy and energy for this regularized system.

Lemma 2.7. Let $\{X_t^{i,\varepsilon}\}_{i=1}^N$ *be the unique strong solution to* (2.25) *with joint distribution* $(f_t^{N,\varepsilon})_{t\geq 0}$ *and density* $(\rho_t^{N,\varepsilon})_{t\geq 0}$. *Then, we have the following relation for energy:*

$$\begin{aligned}\langle \rho_t^{N,\varepsilon}, E^{N,\varepsilon}\rangle + \int_0^t \langle \rho_s^{N,\varepsilon}, |F_1^{N,\varepsilon}|^2\rangle ds + \frac{N-1}{N}\int_0^t \langle \rho_s^{N,\varepsilon}, J^\varepsilon(x_1 - x_2)\rangle\, ds \\ = \frac{N-1}{2N}\iint_{(\mathbb{R}^d)^2} g^\varepsilon(x-y)\rho_0(x)\rho_0(y)\,dxdy \leq C_1\mathcal{E}(\rho_0),\end{aligned} \tag{2.26}$$

and we have uniform estimates for entropy and second moment as follows:

$$\begin{aligned}&H_N(f_t^{N,\varepsilon}) + \int_0^t I_N(f_s^{N,\varepsilon})ds + \frac{N-1}{N}\int_0^t \langle \rho_s^{N,\varepsilon}, J^\varepsilon(x_1-x_2)\rangle\, ds = H_N(f_0^N) = H_1(\rho_0), \\ &\mathbb{E}[|X_t^{i,\varepsilon}|^2] \leq 3\mathbb{E}[|X_0^1|^2] + Ct\mathcal{E}(\rho_0) + 6td.\end{aligned} \tag{2.27}$$

Here,

$$E^{N,\varepsilon}(x) = \frac{1}{2N^2}\sum_{i,j=1,i\neq j}^{N} g^\varepsilon(x_i - x_j), \quad F_1^{N,\varepsilon}(x) = \frac{1}{N}\sum_{j=2}^{N} F_\varepsilon(x_j - x_1). \tag{2.28}$$

Sketch of the proof. Since the force field is bounded and smooth and the initial density $\rho_0^{N,\varepsilon}$ is continuous, $\rho_t^{N,\varepsilon}$ is a classical non-negative solution to the Fokker-Planck equation,

$$\partial_t \rho_t^{N,\varepsilon} = \frac{1}{2}\nabla\cdot(\rho_t^{N,\varepsilon}\nabla g^{N,\varepsilon}) + \Delta\rho_t^{N,\varepsilon}, \tag{2.29}$$

where

$$g^{N,\varepsilon}(x) = \frac{1}{N}\sum_{i,j:i\neq j} g^\varepsilon(x_i - x_j) \Rightarrow \nabla_{x_1} g^{N,\varepsilon}(x) = -2F_1^{N,\varepsilon}.$$

For (2.26), one starts with (2.29) to obtain

$$\frac{d}{dt}\langle \rho_t^{N,\varepsilon}, g^{N,\varepsilon}\rangle = -\left\langle \rho_t^{N,\varepsilon}, \frac{1}{2}|\nabla g^{N,\varepsilon}|^2\right\rangle - \left\langle \rho_t^{N,\varepsilon}, \frac{2}{N}\sum_{i,j=1,i\neq j}^{N} J_\varepsilon(x_i - x_j)\right\rangle. \tag{2.30}$$

By exchangeability,

$$\langle \rho^{N,\varepsilon}, |\nabla g^{N,\varepsilon}|^2\rangle = N\langle \rho^{N,\varepsilon}, |\nabla_{x_1} g^{N,\varepsilon}|^2\rangle = 4N\langle \rho^{N,\varepsilon}, |F_1^{N,\varepsilon}|^2\rangle.$$

By dividing both sides by $2N$ and integrating on time, the equality in (2.26) follows. Moreover,

$$\frac{1}{2N}\langle \rho_0^{N,\varepsilon}, g^{N,\varepsilon}\rangle = \frac{N-1}{2N}\iint_{(\mathbb{R}^d)^2} g^{\varepsilon}(x-y)\rho_0(x)\rho_0(y)\,dxdy \le \frac{1}{2}\int_{\mathbb{R}^d}\nabla h_0(x)\cdot\nabla h_0^{\varepsilon}(x)\,dx \le C\|\nabla h_0\|_2^2.$$

This holds because $\|\nabla h_0^{\varepsilon}\|_2 \le \|\nabla h_0\|_2$ by Young's convolutional inequality.

Now, by simple computations and integration by parts,

$$\frac{d}{dt}H_N(f_t^{N,\varepsilon}) = -I_N(f_t^{N,\varepsilon}) - \frac{1}{N^2}\int_{\mathbb{R}^{Nd}}\sum_{i,j=1,i\neq j}^{N} J_{\varepsilon}(x_i-x_j)\rho_t^{N,\varepsilon}dx, \tag{2.31}$$

which gives the entropy relation in (2.27) since J_{ε} is non-negative.

For the moment estimate in (2.27), since $\mathbf{X}_t^{N,\varepsilon}$ is the solution to (2.25), one can deduce that

$$|X_t^{i,\varepsilon}|^2 \le 3|X_0^i|^2 + \frac{3t}{N^2}\int_0^T\left|\sum_{j=1,j\neq i}^{N} F_{\varepsilon}(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\right|^2 ds + 6|B_t^i|^2. \tag{2.32}$$

Taking expectation of (2.32), and noting exchangeability, one has

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{j=1,j\neq i}^{N} F_{\varepsilon}(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\right|^2\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left|\sum_{j=1,j\neq i}^{N} F_{\varepsilon}(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\right|^2\right]\\ &= N^2\langle \rho_s^{N,\varepsilon}, |F_1^{N,\varepsilon}|^2\rangle.\end{aligned}$$

Then, the moment estimate in (2.27) follows from (2.26) directly. □

Proof for Theorem 2.1. First, we restrict ourselves to a finite time period $[0,T]$. In order to show that the particles in (1.1) a.s. never collide, we consider system (2.25) along with the stopping time

$$\tau_{\varepsilon} = \inf\{t\ge 0 | \min_{i\neq j}|X_t^{i,\varepsilon}-X_t^{j,\varepsilon}| \le \varepsilon\}.$$

Since (1.1) and (2.25) take the same initial value, and by the fact that $F_{\varepsilon}(x) = F(x)$ whenever $|x|\ge\varepsilon$, for any $\varepsilon_1 > \varepsilon$ and $\omega\in\Omega$, if we define

$$\hat{X}_s^{\varepsilon}(\omega) = \mathbf{1}_{\tau_{\varepsilon_1}(\omega)\le t}X_s^{\varepsilon}(\omega) + \mathbf{1}_{\tau_{\varepsilon_1}(\omega)>t}X_s^{\varepsilon_1}(\omega), \tag{2.33}$$

then (2.33) is also a solution for system (2.25) on $[0,t]$ since $F_{\varepsilon}(x) = F_{\varepsilon_1}(x)$ when $|x|\ge\varepsilon_1$. Therefore, from the uniqueness of the solution (since F_{ε} is Lipschitz over $\mathbb{R}^d$), we see that

$$\mathbb{P}\left(X_s^{\varepsilon_1}\mathbf{1}_{\tau_{\varepsilon_1}>t} = X_s^{\varepsilon}\mathbf{1}_{\tau_{\varepsilon_1}>t}, \forall 0\le s\le t\right) = 1. \tag{2.34}$$

Now, we consider the set

$$A_{\varepsilon,\varepsilon_1} := \bigcap_{t\in\mathbb{Q}}\left\{X_s^{\varepsilon_1}\mathbf{1}_{\tau_{\varepsilon_1}>t} = X_s^{\varepsilon}\mathbf{1}_{\tau_{\varepsilon_1}>t}, \forall 0\le s\le t\right\},$$

where $\mathbb{Q}$ is the set of rational numbers. By (2.34), $\mathbb{P}(A_{\varepsilon,\varepsilon_1}) = 1$. For $\omega\in A_{\varepsilon,\varepsilon_1}$, if $\tau_{\varepsilon}(\omega) < \tau_{\varepsilon_1}(\omega)$, then there exists a rational number $t\in\mathbb{Q}$ such that $\tau_{\varepsilon}(\omega) < t < \tau_{\varepsilon_1}(\omega)$, and then by the definition of $A_{\varepsilon,\varepsilon_1}$, we see that $X_s^{\varepsilon_1} = X_s^{\varepsilon}$ for $0\le s\le t$, which contradicts with the assumption $\tau_{\varepsilon}(\omega) < t$. Therefore, we have proved that when $\varepsilon_1 > \varepsilon$, $\tau_{\varepsilon}\ge\tau_{\varepsilon_1}$ for a.s. $\omega\in\Omega$.

We take $\varepsilon_n = \frac{1}{2^n}$. Consider

$$A := \bigcap_{n>m\ge 1} A_{\varepsilon_n,\varepsilon_m}.$$

From the discussion above, $\{\tau_{\varepsilon_n}\}$ is nondecreasing as $n\to\infty$ for $\omega\in A$ and $\mathbb{P}(A) = 1$.

Define

$$A_0 := \{\tau_{\varepsilon_n}\uparrow +\infty\}.$$

If we can show that

$$\mathbb{P}(A_0) = 1, \tag{2.35}$$

then for $\omega \in A_0 \cap A$, there exists an $M(\omega)$ such that $\tau_{\varepsilon_n}(\omega) > T$ when $n \geq M(\omega)$. This implies that $X_t^{\varepsilon_n}(\omega) = X_t^{\varepsilon_{M(\omega)}}(\omega)$ for $n \geq M(\omega)$ and $0 \leq t \leq T$. Therefore, we can define

$$\tilde{X}_t(\omega) = X_t^{\varepsilon_{M(\omega)}}(\omega) \tag{2.36}$$

whenever $\omega \in A_0 \cap A$, and for other $\omega \in \Omega$, we just put $\tilde{X}_t(\omega) = X_0(\omega)$. Then, $\tilde{X}_t$ satisfies (1.1) when $0 \leq t \leq T$ a.s. $\omega \in \Omega$, which gives the existence of the solution.

For the uniqueness, suppose that X_t is another solution that solves (1.1). Consider the stopping time

$$\sigma_\varepsilon = \inf\{t \geq 0 | \min_{i \neq j} |X_t^i - X_t^j| \leq \varepsilon\}.$$

Similar to (2.33),

$$\hat{X}_s^{\ \varepsilon}(\omega) = \mathbf{1}_{\sigma_\varepsilon(\omega) \leq t} X_s^\varepsilon(\omega) + \mathbf{1}_{\sigma_\varepsilon(\omega) > t} X_s(\omega) \tag{2.37}$$

gives a solution for (2.25), from which by using the uniqueness it is not hard to see the set

$$A_1 := \bigcap_{t \in \mathbb{Q}} \{X_s \mathbf{1}_{\sigma_{\varepsilon_n} > t} = X_s^\varepsilon \mathbf{1}_{\sigma_{\varepsilon_n} > t}, \forall 0 \leq s \leq t, n \geq 1\}$$

satisfies $\mathbb{P}(A_1) = 1$. Now, for $\omega \in A_1$, if $\sigma_\varepsilon < \tau_\varepsilon$ for some $\varepsilon = \varepsilon_n$, since for fixed ω, X_t and X_t^ε are continuous in t, from the definition of the stopping time, we see

$$\min_{i \neq j} |X_{\sigma_\varepsilon}^i - X_{\sigma_\varepsilon}^j| = \varepsilon, \min_{i \neq j} |X_{\sigma_\varepsilon}^{i,\varepsilon} - X_{\sigma_\varepsilon}^j| > \varepsilon;$$

by continuity, there exists a $t \in \mathbb{Q}$ such that

$$\min_{i \neq j} |X_t^i - X_t^j| - \min_{i \neq j} |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}| < 0, \ \ t < \sigma_\varepsilon,$$

which contradicts with the definition of A_1. This gives the fact that $\sigma_{\varepsilon_n}(\omega) \geq \tau_{\varepsilon_n}(\omega)$ as long as $\omega \in A_1$. Now, if (2.35) holds, then $\mathbb{P}(A_0 \cap A \cap A_1) = 1$, and for $\omega \in A_0 \cap A \cap A_1$, we have $X_t(\omega) = X_t^{\varepsilon_{M(\omega)}}(\omega) = \tilde{X}_t(\omega)$, which concludes the proof for uniqueness.

Now, we show (2.35). Since τ_{ε_n} is a.s. nondecreasing, to show $\tau_{\varepsilon_n} \uparrow +\infty$, a.s., it suffices to show that for any fixed T,

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon \leq T) = 0. \tag{2.38}$$

We consider the un-normalized energy

$$\Phi_t^{\varepsilon,N} := \frac{1}{N} \sum_{i,j=1, i \neq j}^N g^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}).$$

Then, we have the following basic fact:

$$\{\tau_\varepsilon \leq T\} \subset \left\{ \sup_{t \in [0,T]} \Phi_{t \wedge \tau_\varepsilon}^{\varepsilon,N} \geq \Phi_{\tau_\varepsilon}^{\varepsilon,N} \right\}. \tag{2.39}$$

Since $g^\varepsilon \in C_b^2(\mathbb{R}^d)$, by Itô's formula and the fact that $-\Delta g^\varepsilon(x) = J_\varepsilon(x) = 0$ on $|x| \geq \varepsilon$, we get

$$\Phi_t^{\varepsilon,N} = \Phi_0^{\varepsilon,N} - \frac{2}{N^2} \int_0^t \sum_{i=1}^N \left| \sum_{j=1, j \neq i}^N F^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) \right|^2 ds - \frac{2}{N} \sum_{i,j=1, i \neq j}^N \int_0^t J_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) ds - M_t^\varepsilon, \tag{2.40}$$

where

$$M_t^\varepsilon = \frac{\sqrt{2}}{N}\sum_{i,j=1,i\neq j}^{N}\int_0^t F_\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\cdot(dB_s^i-dB_s^j) = \frac{2\sqrt{2}}{N}\sum_{i,j=1,i\neq j}^{N}\int_0^t F_\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\cdot dB_s^i.$$

Since F_ε is bounded, M_t^ε is a martingale with

$$\begin{aligned}\mathbb{E}[(M_t^\varepsilon)^2] &= \frac{8}{N^2}\int_0^t\sum_{i=1}^{N}\mathbb{E}\left[\left|\Big(\sum_{j=1,j\neq i}^{N}F_\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\Big)\right|^2\right]ds\\ &= \int_0^t\langle\rho_s^{N,\varepsilon},2|\nabla g^{N,\varepsilon}|^2\rangle ds\\ &= 8N\int_0^t\langle\rho^{N,\varepsilon},|F_1^{N,\varepsilon}|^2\rangle\,ds \le CN\mathcal{E}(\rho_0).\end{aligned} \tag{2.41}$$

The last inequality comes from (2.26). Combining (2.39) and (2.40), from the positivity of g^ε (since $d\ge 3$), we have

$$\{\tau_\varepsilon\le T\}\subset\left\{(\Phi_0^{\varepsilon,N}-\inf_{0\le t\le T}M_t^\varepsilon)>\frac{1}{N}g(\varepsilon)\right\}, \tag{2.42}$$

$$\subset\{\Phi_0^{\varepsilon,N}>R\}\cup\left\{\sup_{0\le t\le T}(-M_t^\varepsilon)>\frac{1}{N}g(\varepsilon)-R\right\}, \tag{2.43}$$

for any $R>0$. Here, $g(\varepsilon)=C_d\varepsilon^{2-d}$. We notice that

$$\mathbb{E}[\Phi_0^{\varepsilon,N}] = (N-1)\int_{\mathbb{R}^{2d}}\rho_0(x)\rho_0(y)g^\varepsilon(x-y)dxdy \le C(N-1)\mathcal{E}(\rho_0). \tag{2.44}$$

Therefore, Markov's inequality gives

$$\mathbb{P}(\Phi_0^{\varepsilon,N}>R)\le\frac{1}{R}(N-1)C\mathcal{E}(\rho_0). \tag{2.45}$$

For the second term, we apply Doob's inequality for martingales (p. 203, Theorem 7.31 in Ref. 33),

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T}(-M_t^\varepsilon)>\frac{1}{N}g(\varepsilon)-R\right) &\le \frac{N}{g(\varepsilon)-NR}\left(\mathbb{E}[|\sup_{0\le t\le T}(-M_t^\varepsilon)|^2]\right)^{\frac{1}{2}}\\ &\le \frac{4N}{g(\varepsilon)-NR}(\mathbb{E}[|M_T^\varepsilon|^2])^{\frac{1}{2}} \le \frac{C(N,d,\mathcal{E}(\rho_0))}{g(\varepsilon)-NR},\end{aligned} \tag{2.46}$$

where we used (2.41). Combining (2.42), (2.45), and (2.46),

$$\mathbb{P}(\tau_\varepsilon\le T)\le C(N)\left(\frac{1}{R}+\frac{1}{g(\varepsilon)-NR}\right). \tag{2.47}$$

We take $R=g(\varepsilon)^{\frac{1}{2}}$, and the conclusion follows from the fact that $g(\varepsilon)=C(d)\varepsilon^{2-d}\to\infty$ as $\varepsilon\to 0$.

Finally, we conclude the global existence and uniqueness. For $k\ge 1$, suppose $X_t^{(k)}$ is the a.s. unique solution to (1.1) on the time interval $t\in[0,k]$. From the previous local existence and uniqueness proof, we find that the set

$$S_0=\bigcap_{\ell\ge 1}\left\{X_t^{(k)}=X_t^{(\ell)},\ \forall k\ge\ell, t\in[0,\ell]\right\}$$

has probability 1. Therefore, if we define

$$\tilde{X}_t(\omega)=X_t^{([t+1])}(\omega)\mathbf{1}_{S_0}(\omega)+X_0(\omega)\mathbf{1}_{S_0^c}(\omega),$$

then $\tilde{X}_t$ satisfies (1.1) for all $t>0$ a.s. (here, $[t]$ rounds t to the nearest integer). Meanwhile, if X_t is another global solution for $t>0$, then by local uniqueness we know that for any $k\ge 1$, $X_t=X_t^{(k)}$, $\forall 0\le t\le k$, a.s. This implies that $X_t=\tilde{X}_t$, which gives the global uniqueness. □

Next, we state some useful estimates for the N-particle system (1.1).

Proposition 2.1. Suppose $\{X_t\}_{t\geq 0}$ is the solution for (1.1) with joint distribution f_t^N. Then, f_t^N has a density function ρ_t^N. Moreover, we have the following estimates:

$$H_N(f_t^N) + \int_0^t I_N(f_s^N)ds \leq H_1(f_0)(= H_N(f_0^N)), \tag{2.48}$$

$$\mathbb{E}\mathscr{E}_N(t) + \mathbb{E}\int_0^t \frac{1}{N}\sum_i \left|\frac{1}{N}\sum_{j:j\neq i} F(X_j - X_i)\right|^2 ds \leq \frac{N-1}{N}\mathcal{E}(\rho_0), \tag{2.49}$$

$$\mathbb{E}[|X_t^i|^2] \leq 3\mathbb{E}[|X_0^1|^2] + Ct\mathcal{E}(\rho_0) + 6td. \tag{2.50}$$

Proof. Throughout this proof, C denotes the constant that depends on N, ρ_0, d and so on, but not on ε. Note that for any density $\rho \in L^1(\mathbb{R}^{Nd})$, $|\rho \log^- \rho| < 1$. For $\alpha < 1$,

$$\int_{|x|\geq 1} |\rho(x)\log^- \rho(x)|\, dx \leq C\int_{|x|\geq 1} |\rho|^\alpha dx \leq C\left(\int_{\mathbb{R}^{Nd}} |x|^2\rho dx\right)^\alpha \left(\int_{|x|\geq 1} |x|^{-\frac{2\alpha}{1-\alpha}} dx\right)^{1-\alpha}. \tag{2.51}$$

If we take $\frac{Nd}{Nd+2} < \alpha < 1$, then from the uniform estimate (2.27), we deduce

$$\int_{\mathbb{R}^{Nd}} |\rho_t^{N,\varepsilon}(x)\log \rho_t^{N,\varepsilon}(x)|dx \leq C, \tag{2.52}$$

which means that $\rho_t^{N,\varepsilon}$ is uniformly integrable in $L^1(\mathbb{R}^{Nd})$. Consider the sequence $\{\rho_t^{N,\varepsilon_n} : \varepsilon_n = \frac{1}{2^n}\}$. Denote by $B_r = \{|x| \leq r\}$ the ball in $\mathbb{R}^{Nd}$ centered at the origin with radius r. By the Dunford-Pettis theorem (p. 412, Theorem 12 in Ref. 34), there exists a subsequence $\{\rho_{t,(1)}^{N,\varepsilon_n}\}$ converging weakly in $L^1(B_1)$ to some $\rho_{t,(1)}^N$. This subsequence has a further subsequence converging weakly in $L^1(B_2)$ to some $\rho_{t,(2)}^N$. From the uniqueness of the weak limit, we see that $\rho_{t,(2)}^N = \rho_{t,(1)}^N$ a.e. on B_1. Proceeding this process and taking the diagonal sequence, there exists a subsequence (without relabeling) and a $\rho_t^N \in L^1_{\text{loc}}(\mathbb{R}^{Nd})$ such that

$$\rho_t^{N,\varepsilon_n} \rightharpoonup \rho_t^N \text{ in } L^1(B_k), \forall k \geq 1. \tag{2.53}$$

(2.53) also gives $\|\rho_t^N\|_{L^1(\mathbb{R}^{Nd})} \leq 1$. Now, the moment estimate in (2.27) gives the tightness of $\rho_t^{N,\varepsilon}$, i.e., $\int_{|x|\geq M}\rho^{N,\varepsilon}(x)dx$ goes to 0 as $M \to \infty$ uniformly in ε; by (2.53), it is not hard to observe that for any $\phi \in L^\infty(\mathbb{R}^{Nd})$, $\langle \phi, \rho_t^{N,\varepsilon_n}\rangle \to \langle \phi, \rho_t^N\rangle$, i.e.,

$$\rho_t^{N,\varepsilon_n} \rightharpoonup \rho_t^N \text{ in } L^1(\mathbb{R}^{Nd}). \tag{2.54}$$

In particular, $\|\rho_t^N\|_{L^1(\mathbb{R}^{Nd})} = 1$. From the Proof of Theorem 2.1, we see that $X_t^{\varepsilon_n} \to X_t$, a.s.; therefore, for any $\varphi \in C_b(\mathbb{R}^{Nd})$, we have

$$\langle \varphi, \rho_t^N\rangle = \lim_{n\to\infty}\langle \varphi, \rho_t^{N,\varepsilon_n}\rangle = \lim_{n\to\infty}\mathbb{E}[\varphi(X_t^{\varepsilon_n})] = \mathbb{E}[\varphi(X_t)] = \langle \varphi, f_t^N\rangle. \tag{2.55}$$

This gives the fact that f_t^N has density ρ_t^N. Note that

$$\langle \rho_t^{N,\varepsilon}, E^{N,\varepsilon}\rangle + \int_0^t \langle \rho_s^{N,\varepsilon}, |F_1^{N,\varepsilon}|^2\rangle ds = \mathbb{E}[E^{N,\varepsilon}(\mathbf{X}_t^{N,\varepsilon})] + \int_0^t \mathbb{E}[|F_1^{N,\varepsilon}(\mathbf{X}_s^{N,\varepsilon})|^2]ds \tag{2.56}$$

and that for a.s. $\omega \in \Omega$, $\mathbf{X}_t^{N,\varepsilon_n}(\omega) = \mathbf{X}_t^N(\omega)$ when n is big enough, from Fatou's lemma, the exchangeability, and (2.26), we obtain (2.49). Similarly, (2.50) holds. Now, combining the entropy estimate in (2.27) and the fact that the functionals H and I are both lower semicontinuous with respect to weak convergence (Theorems 5.4 and 5.7 in Ref. 31), we see that

$$H_N(f_t^N) + \int_0^t I_N(f_s^N)ds \leq \liminf_{n\to\infty}\left(H_N(f_t^{N,\varepsilon_n}) + \int_0^t I_N(f_s^{N,\varepsilon_n})ds\right) \leq H_1(\rho_0), \tag{2.57}$$

which gives (2.48). □

C. The weak convergence of the empirical measures

In this part, we recall the results in Ref. 23 for the weak convergence of the empirical measures.

Proposition 2.2 (Ref. 23, Lemma 3.1). *For any $N \geq 2$ and $d \geq 3$, let $\{(X_t^{i,N})_{0\leq t\leq T}\}_{i=1}^N$ be the unique strong solution to (1.1) with the iid initial data $\{X_0^{i,N}\}_{i=1}^N$. Suppose the common density $\rho_0(x) \in L^{\frac{2d}{d+2}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|^2)dx)$ and $H_1(\rho_0) < \infty$. Recall the empirical measure μ^N defined in (1.6).*

(i) *The sequence $\{\mathcal{L}(X^{1,N})\}$ is tight in $\mathbf{P}(C([0,T];\mathbb{R}^d))$.*
(ii) *The sequence $\{\mathcal{L}(\mu^N)\}$ is tight in $\mathbf{P}(\mathbf{P}(C([0,T];\mathbb{R}^d)))$.*

Here, $\mathcal{L}(X^{1,N})$ is the law of $X^{1,N}$, i.e., $\mathcal{L}(X^{1,N})(A) = \mathbb{P}(X^{1,N} \in A)$ for $A \subset C([0,T];\mathbb{R}^d)$ that is Borel measurable, similar for $\mathcal{L}(\mu^N)$. For the convenience of the readers, we provide a concise proof in Appendix B.

We consider the projection $\pi_t : (C([0,T];\mathbb{R}^d), \mathcal{B}) \to \mathbb{R}^d$, where $\mathcal{B}$ is the standard Borel σ-algebra in $C([0,T];\mathbb{R}^d)$,

$$\pi_t(X) = X(t).$$

Then, for some measure $\nu \in \mathbf{P}(C([0,T];\mathbb{R}^d))$, we define the time marginal ν_t as the pushforward of ν under π_t,

$$\nu_t := (\pi_t)_\# \nu \tag{2.58}$$

or

$$\nu_t(E) = \nu(\pi_t^{-1}(E)),\ \forall E \in \mathbb{R}^d, \text{Borel measurable}.$$

Consequently, we have

$$\left(t \mapsto \nu_t\right) \in C([0,T];\mathbf{P}(\mathbb{R}^d)),$$

where $\mathbf{P}(\mathbb{R}^d)$ is equipped with the topology of weak convergence.

We easily conclude the following by change of measures, i.e., for $T : (\mathcal{X}, \nu) \to (\mathcal{Y}, \tilde{\nu})$ with $\tilde{\nu}(A) = \nu(T^{-1}(A))$, one has $\int_{\mathcal{Y}} f d\tilde{\nu} = \int_{\mathcal{X}} f \circ T d\nu$.

Lemma 2.8. Suppose $\nu \in \mathbf{P}(C([0,T];\mathbb{R}^d))$ with time marginal $\nu_t \in \mathbf{P}(\mathbb{R}^d)$, and ψ is a Borel measurable function on $\mathbb{R}^d$. Then, for $0 \leq t \leq T$, the equation

$$\int_{C([0,T];\mathbb{R}^d)} \psi(X_t)\nu(dX) = \int_{\mathbb{R}^d} \psi(x)\nu_t(dx) \tag{2.59}$$

holds if either side is integrable. Similarly, for the product space $C([0,T];\mathbb{R}^d) \times C([0,T];\mathbb{R}^d)$ and Borel measurable function ψ on $\mathbb{R}^d \times \mathbb{R}^d$,

$$\iint_{C([0,T];\mathbb{R}^d)^2} \psi(X_t, Y_t)\nu(dX)\nu(dY) = \int_{C([0,T];\mathbb{R}^d)} \int_{\mathbb{R}^d} \psi(x, Y_t)\nu_t(dx)\nu(dY) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \psi(x,y)\nu_t(dx)\nu_t(dy) \tag{2.60}$$

if either side of (2.60) is integrable.

Recall that a sequence of random variables Z_n taking values in some Polish space $\mathcal{X}$ converges in law to Z meaning that $\mathbb{E}\varphi(Z_n) \to \mathbb{E}\varphi(Z)$ for any $\varphi \in C_b(\mathcal{X})$ (i.e., bounded continuous functions). The following lemma gives the consequence of the tightness in Proposition 2.2.

Lemma 2.9. 1. *There is a subsequence of the empirical measures, $\mu^N \in \mathbf{P}(C([0,T];\mathbb{R}^d))$ (without relabeling), and a random measure $\mu : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbf{P}(C([0,T];\mathbb{R}^d))$ such that*

$$\mu^N \to \mu \text{ in law as } N \to \infty. \tag{2.61}$$

[Or $\mathcal{L}(\mu^N)$ converges weakly to $\mathcal{L}(\mu)$ in $\mathbf{P}(\mathbf{P}(C([0,T];\mathbb{R}^d)))$.]

2. *For the subsequence in 1, μ_t^N, as $\mathbf{P}(\mathbb{R}^d)$ valued random measures, converge in law to μ_t. In other words, $\mathcal{L}(\mu_t^N)$ converges weakly to $\mathcal{L}(\mu_t)$ in $\mathbf{P}(\mathbf{P}(\mathbb{R}^d))$.*

Proof. The first claim follows from the tightness of $\{\mathcal{L}(\mu^N)\}$ in $\mathbf{P}(\mathbf{P}(C([0,T];\mathbb{R}^d)))$ by Prokhorov's theorem.

For the second, we first note that a sequence $\nu^n \in \mathbf{P}(C([0,T];\mathbb{R}^d))$ converging weakly to $\nu \in \mathbf{P}(C([0,T];\mathbb{R}^d))$ will imply that $(\nu^n)_t \in \mathbf{P}(\mathbb{R}^d)$ converges weakly to $\nu_t \in \mathbf{P}(\mathbb{R}^d)$. In fact, for any function $\phi \in C_b(\mathbb{R}^d)$, we have $\int_{\mathbb{R}^d}\phi(x)d(\nu_n)_t = \int_{C([0,T];\mathbb{R}^d)}\phi(X_t)d\nu_n$. Note that $X \to \phi(X_t)$ is a continuous functional on $C([0,T];\mathbb{R}^d)$ and thus

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi(x)d(\nu_n)_t = \int_{C([0,T];\mathbb{R}^d)}\phi(X_t)d\nu = \int_{\mathbb{R}^d}\phi(x)d\nu_t.$$

Now, consider a continuous functional $\Gamma : \mathbf{P}(\mathbb{R}^d) \to \mathbb{R}$. We define $\Gamma_1 : \mathbf{P}(C([0,T];\mathbb{R}^d)) \to \mathbb{R}$ as

$$\Gamma_1(\nu) := \Gamma(\nu_t).$$

According to what has been justified, Γ_1 is a continuous functional on $\mathbf{P}(C([0,T];\mathbb{R}^d))$. Consequently,

$$\mathbb{E}\Gamma_1(\mu^N) \to \mathbb{E}\Gamma_1(\mu) \Rightarrow \mathbb{E}\Gamma(\mu_t^N) \to \mathbb{E}\Gamma(\mu_t).$$

This then verifies the second claim. □

The following lemma gives another property which will be useful to us.

Lemma 2.10. Let $\mathcal{X}$ be a Polish space. Suppose μ^N, μ are random measures on $\mathcal{X}$ (i.e., $\mathbf{P}(\mathcal{X})$-valued random variables) such that μ^N converge to μ in law. For any $\psi \in C_b(\mathcal{X}\times\mathcal{X})$, if we define a functional $\mathcal{K}_\psi : \mathbf{P}(\mathcal{X}) \to \mathbb{R}$ with

$$\mathcal{K}_\psi(\nu) = \int_{\mathcal{X}^2}\psi(X,Y)\nu(dX)\nu(dY),$$

then $\mathcal{K}_\psi(\mu^N) \to \mathcal{K}_\psi(\mu)$ in law as $N \to \infty$.

Proof. We consider the metric on $\mathbf{P}(\mathcal{X})$ induced by weak convergence. By p. 23, Theorem 2.8 in Ref. 35, $\nu^N \to \nu$ in $\mathbf{P}(\mathcal{X})$ implies that $\mathcal{K}_\psi(\nu^N) \to \mathcal{K}_\psi(\nu)$; therefore, for any $\phi \in C_b(\mathbb{R})$, $\phi \circ \mathcal{K}_\psi$ is a bounded continuous functional on $\mathbf{P}(\mathcal{X})$, and then

$$\mathbb{E}\big[\phi\big(\mathcal{K}_\psi(\mu^N)\big)\big] \to \mathbb{E}[\phi(\mathcal{K}_\psi(\mu))],\ N \to \infty,$$

which gives the last claim. □

We note the following facts regarding the marginal distributions (see Ref. 31, Lemma 5.6; Ref. 24, Theorem 4.1; Ref. 23, Lemma 3.2). The results are modified for our purpose here, and we sketch a quick proof for reference.

Proposition 2.3. Under the assumption of Proposition 2.2, we denote by $(f_t^N)_{t\ge0}$ the joint distribution of $\{(X_t^{i,N})_{t\ge0}\}_{i=1}^N$ and $f_t^{(j),N}$ the j-th marginal of f_t^N for any $j \ge 1$.

(i) *For any j that is a positive integer, we have*

$$\sup_{t\in[0,T],N}\int_{\mathbb{R}^{dj}}|x|^2 f_t^{(j),N}(dx) < \infty,\quad \sup_{t\in[0,T],N} H_j(f_t^{(j),N}) < \infty,\ \sup_N\int_0^T I_j(f_t^{(j),N})dt < \infty. \tag{2.62}$$

(ii) *$f_t^{(j),N}$ has a density $\rho_t^{(j),N}$. Consider $\rho^{(j),N} = (\rho_t^{(j),N}) \in L^1([0,T]\times\mathbb{R}^{dj})$. It has a subsequence $\rho^{(j),N}$ (without relabeling) weakly converging to $\rho^{(j)}$ in $L^1([0,T]\times\mathbb{R}^{dj})$ as $N\to\infty$, and also for a.e. $t\in[0,T]$, $f_j^{(j),N} = \rho_t^{(j),N}dx$ converges weakly to $\rho_t^{(j)}dx$ as probability measures. Besides,*

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^{dj}}|x|^2\rho_t^{(j)}dx < \infty,\quad \sup_{t\in[0,T]}\int_{\mathbb{R}^{dj}}\rho_t^{(j)}|\log\rho_t^{(j)}|\,dx < \infty,\quad \int_0^t I_j(\rho_t^{(j)})ds < \infty. \tag{2.63}$$

Moreover, let μ be the limit (random) measure of any further subsequence of μ^N, and let μ_t be the time marginal of μ. Then, for a.e. $t\in[0,T]$, it holds that

$$\int_{\mathbb{R}^{dj}}\rho_t^{(j)}\varphi\,dx = \mathbb{E}(\langle\mu_t^{\otimes j},\varphi\rangle),\ \ \forall\varphi\in C_b(\mathbb{R}^{dj}).$$

(iii) *The entropy and Fisher information of the limit random measure μ satisfy that for all $t\in[0,T]$,*

$$\mathbb{E}(H_1(\mu_t)) = \sup_{j\geq 1} H_j(\rho_t^{(j)}) \leq \liminf_{N\to\infty} H_N(f_t^N),$$
$$\mathbb{E}(I_1(\mu_t)) = \sup_{j\geq 1} I_j(\rho_t^{(j)}) \leq \liminf_{N\to\infty} I_N(f_t^N). \tag{2.64}$$

(iv) *We have the following estimates for the Fisher information:*

$$\mathbb{E}\int_0^T I_1(\mu_t)dt < C. \tag{2.65}$$

Consequently, for a.s. ω, $\mu(\omega)$ has a density $(\rho_t(\omega))_{t\in[0,T]}$. At time $t = 0$, $\rho_t(\omega) = \rho_0$ for a.s. ω.

Proof. (i) The second moment estimate follows directly from Eq. (2.50). Equations (2.8) and (2.48) imply that $\int_0^T I_j(f_t^{(j),N})\,dt \leq \int_0^T I_N(f_t^{(j),N})\,dt \leq H_1(f_0) - H_N(f_t^N)$. By the second moment estimates and Lemma 2.3 with $p = 2, \lambda = 1$, we see that

$$H_1(\rho_T^{(1),N}) \geq -C_{p,\lambda} - M_2(\rho_T^{(1),N}) \geq -C, \tag{2.66}$$

where C depends only on ρ_0, T and d. By Lemma 2.1, we have $H_N(\rho_T^N) \geq H_1(\rho_T^{(1),N}) \geq -C$. We thus have

$$\sup_N \int_0^T I_j(f_t^{(j),N})\,dt \leq \sup_N \int_0^T I_N(f_t^N)\,dt \leq C(j,T,d) < \infty. \tag{2.67}$$

We note that $H_N(f_t^N)$ is uniformly bounded. Then, by (2.7), we have (note that entropy can be negative)

$$H_j(f_t^{(j),N}) \leq \left(1 + \frac{N-mj}{mj}\right) H_N(f_t^N) - \frac{N-mj}{mj} H_{N-mj}\left(f_t^{(N-mj),N}\right), \tag{2.68}$$

where m is an integer chosen so that $N - mj \in [0, j)$. A simple application of (2.9) with second moment gives the uniform bound for $H_j(f_t^{(j),N})$.

(ii) By the uniform second moment estimate, $\rho^{(j),N}$ is tight in $L^1([0,T]\times\mathbb{R}^{dj})$. Moreover, we have $\int_{\mathbb{R}^{dj}} \rho^{(j),N} |\log \rho^{(j),N}|\,dx$ to be uniformly bounded by the uniform estimates of H_j and a similar calculation for (2.52). Hence, $\rho^{(j),N}$ is uniformly integrable on $[0,T]\times\mathbb{R}^{dj}$. Although the Dunford-Pettis theorem is stated for finite measures, combined with the tightness, the uniform integrability implies that $\rho^{(j),N}$ is weakly compact in $L^1([0,T]\times\mathbb{R}^{dj})$. Hence, we can find a subsequence $\rho^{(j),N}$ converging weakly to $\rho^{(j)}$ in $L^1([0,T]\times\mathbb{R}^{dj})$.

The second moment mapping $\nu \mapsto \int_{\mathbb{R}^{dj}} |x|^2 \nu(dx)$ is lower-semicontinuous with respect to the topology of weak convergence, which can be seen by approximating $|x|$ with $|x| \wedge m$. After taking sup in t, it is still lower semicontinuous. It has been proved in Ref. 24 (Lemma 4.2) that H_j and I_j are lower semicontinuous. Taking supremum in t or taking integral of non-negative lower semicontinuous functionals still yields lower semicontinuous functionals. Hence, taking $N \to \infty$ in (2.62), we get the corresponding estimates for $\rho_t^{(j)}$. The second moment and entropy estimates then yield $\sup_{t\in[0,T]} \int_{\mathbb{R}^{dj}} \rho_t^{(j)} |\log \rho_t^{(j)}|\,dx < \infty$ similarly as we did in (2.52).

We now take $\phi \in C[0,T]$ and $\varphi \in C_b(\mathbb{R}^{dj})$. Then, $\Gamma : C([0,T];\mathbb{R}^d)^j \to \mathbb{R}$ defined by $(X^1,\dots,X^j) \to \int_0^T \phi(t)\varphi(X_t^1,\dots,X_t^j)\,dt$ is a bounded continuous functional. A slight generalization of Lemma 2.10 with $\mathcal{X} = C([0,T];\mathbb{R}^d)$ shows that

$$\mathbb{E}\langle (\mu^N)^{\otimes j}, \Gamma\rangle \to \mathbb{E}\langle \mu^{\otimes j}, \Gamma\rangle = \int_0^T \phi(t)\mathbb{E}\langle \mu_t^{\otimes j}, \varphi\rangle\,dt,$$

where the last term is obtained by Fubini and the definition of μ_t. Let

$$\nu_t := \mathcal{L}(\mu_t) \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d)).$$

Define

$$\nu_t^j := \int_{\mathbf{P}(\mathbb{R}^d)} g^{\otimes j} \nu_t(dg) \in \mathbf{P}(\mathbb{R}^{dj}).$$

By this definition, we have for any $\varphi \in C_b(\mathbb{R}^{dj})$ that

$$\langle \nu_t^j, \varphi\rangle = \int_{\mathbf{P}(\mathbb{R}^d)} \int_{\mathbb{R}^{dj}} \varphi(x) g^{\otimes j}(dx)\nu_t(dg) = \mathbb{E}(\langle \mu_t^{\otimes j}, \varphi\rangle).$$

This means that $\mathbb{E}\langle (\mu^N)^{\otimes j}, \Gamma\rangle \to \int_0^T \phi(t)\langle \nu_t^j, \varphi\rangle\,dt$.

On the other hand, by definition and the Fubini theorem,

$$\mathbb{E}\langle(\mu^N)^{\otimes j},\Gamma\rangle = \int_0^T \phi(t)\mathbb{E}\Big(\frac{1}{N^j}\sum\varphi(X_t^{i_1,N},\dots,X_t^{i_j,N})\Big)dt = \frac{N!}{N^j(N-j)!}\int_0^T\phi(t)\times$$
$$\mathbb{E}\varphi(X_t^{1,N},\dots,X_t^{j,N})dt + \frac{1}{N^j}\sum_{\text{some } i_k's \text{ are equal}}\int_0^T\phi(t)\mathbb{E}\varphi(X_t^{i_1,N},\dots,X_t^{i_j,N})dt. \tag{2.69}$$

A simple estimate shows that the second term goes to zero as $N\to\infty$, while the first term converges to $\int_0^T\phi(t)\int_{\mathbb{R}^{dj}}\varphi\rho_t^{(j)}\,dxdt$ by the results we have just proved.

Since $\phi(t)$ is arbitrary, for a fixed φ, we have for a.e. t that

$$\int_{\mathbb{R}^{dj}}\varphi\rho_t^{(j)}\,dx = \langle\nu_t^j,\varphi\rangle. \tag{2.70}$$

Moreover, since C_c^∞ is separable, we know for a.e. t and all $\varphi\in C_c$ that (2.70) holds. Using the uniform second moment bounds of $\rho_t^{(j)}$ and ν_t^j (the proof of second moment for ν_t^j can be obtained similarly as for $\rho_t^{(j)}$), we know that they are tight. We thus can pass from C_c to C_b for these t. Hence, $\rho_t^{(j)}$ is in fact the density of ν_t^j for a.e. $t\in[0,T]$.

Now, a slight generalization of Lemma 2.10 with $\mathcal{X}=\mathbb{R}^{dj}$ shows that for $\varphi\in C_b(\mathbb{R}^{dj})$,

$$\mathbb{E}\langle(\mu_t^N)^{\otimes j},\varphi\rangle\to\mathbb{E}\langle(\mu_t)^{\otimes j},\varphi\rangle = \langle\nu_t^j,\varphi\rangle$$

since μ_t^N converges in law to μ_t by Lemma 2.9. A similar computation of (2.69) shows that we in fact have

$$\lim_{N\to\infty}\int_{\mathbb{R}^{dj}}\varphi f_t^{(j),N}(dx) = \langle\nu_t^j,\varphi\rangle.$$

This in fact means $f_t^{(j),N} = \rho_t^{(j),N}\,dx \rightharpoonup \nu_t^j$ for all t. Thus, for a.e. t, $\rho_t^{(j),N}\,dx\rightharpoonup\rho_t^{(j)}\,dx$ as probability measures.

(iii) In Ref. 24, Lemma 4.2, it is proved that the functional I_j is convex, proper, and lower semicontinuous. Then, Ref. 31, Lemma 5.6 shows that

$$\mathbb{E}(I_1(\mu_t)) = \int_{\mathbf{P}(\mathbb{R}^d)}I_1(g)\nu_t(dg) = \sup_{j\ge1}I_j(\nu_t^j).$$

On the one side, the convexity gives

$$\int_{\mathbf{P}(\mathbb{R}^d)}I_1(g)\nu_t(dg) = \int_{\mathbf{P}(\mathbb{R}^d)}\frac{1}{j}I(g^{\otimes j})\nu_t(dg)\ge\frac{1}{j}I\Big(\int_{\mathbf{P}(\mathbb{R}^d)}g^{\otimes j}\nu_t(dg)\Big) = I_j(\nu_t^j).$$

On the other side, it is more tricky. One uses a type of affine property for the functional $\nu\mapsto\sup_{j\ge1}I_j(\nu)$, and we refer the readers to Ref. 31, Theorems 5.4 and 5.7.

Then, using (2.7), it is clear that $\lim_{N\to\infty}I_j(\rho^{(j),N})\le\lim_{N\to\infty}I_N(f_t^N)$. The lower semicontinuity then implies $I_j(\nu_t^j)\le\lim_{N\to\infty}I_j(\rho^{(j),N})$.

For the entropy, it is shown in Ref. 24 (Lemma 4.2) that H_j is convex, lower semicontinuous and a certain affine property. $\mathbb{E}(H_1(\mu_t)) = \int_{\mathbf{P}(\mathbb{R}^d)}H_1(g)\nu_t(dg) = \sup_{j\ge1}H_j(\nu_t^j)$ holds.

Since the entropy could be negative, we should use the fact that the second moment of $\rho^{(j),N}$ is uniformly bounded and (2.9) to control the entropy of the marginal distributions $\rho^{(j),N}$. We apply (2.9) in (2.68) and have $H_j(\rho^{(j),N})\le(1+\frac{N-mj}{mj})H_N(f_t^N)+\frac{N-mj}{mj}\big(M_2(\rho^{(N-mj),N})+C_{j,2}\big)$. It then follows that $\lim_{N\to\infty}H_j(\rho^{(j),N})\le\lim_{N\to\infty}H_N(f_t^N)$ still holds. The lower semicontinuity then gives the desired result.

(iv) By (2.67), we obtain (2.65). Now, since $\int_0^TI_1(\mu_s)\,ds<\infty$ a.s., the definition of Fisher information [Eq. (2.1)] implies that for such ω, $(\mu_s)_{s\in[0,T]}$ has density for a.e. $s\in[0,T]$. The claim for $t=0$ is a simple consequence of law of large numbers. □

III. THE LIMIT MEASURE ALMOST SURELY IS A WEAK SOLUTION

Now, we define the weak solution of (1.7) in the following sense:

Definition 3.1. We say $\rho\in L^\infty(0,T;L^1(\mathbb{R}^d))$ is a weak solution to (1.7) if

- *$\rho\,dx\in C([0,T];C_b(\mathbb{R}^d)')$ and $\rho\nabla h\in L^1(0,T;L^1(\mathbb{R}^d))$, where $h=g^*\rho$.*

- *For all* $t \in [0,T]$,

$$\langle \rho_t, \phi\rangle - \langle \rho_0, \phi\rangle - \int_0^t \int_{\mathbb{R}^d} \nabla\phi(x)\cdot\nabla h(x)\rho_s(x)\,dxds - \int_0^t \langle \rho_s, \Delta\phi\rangle ds = 0 \tag{3.1}$$

for any $\phi \in C_c^2(\mathbb{R}^d)$.

We first of all prove the following important result for $d = 3$.

Proposition 3.1. Let $d = 3$ *and* $\{(X_t^{i,N})\}_{i=1}^N$ *be the unique solution to* (1.1) *with the iid initial data* $\{X_0^{i,N}\}_{i=1}^N$. *Suppose that the common density* ρ_0 *satisfies* $H(\rho_0) < \infty$, $m_2(\rho_0) < \infty$, *and* $\mathcal{E}(\rho_0) < \infty$. *Assume that the random measure* μ *on* $C([0,T];\mathbb{R}^3)$ *is a limit point of* μ^N *under the topology induced by convergence in law. Then,* μ *has a density* $(\rho_s)_{s\in[0,T]}$ *a.s. as we have seen, and for fixed* $\phi \in C_b^2(\mathbb{R}^3)$ *and* $t \in [0,T]$, ρ *satisfies the following integral equation almost surely:*

$$\langle \rho_t, \phi\rangle - \langle \rho_0, \phi\rangle - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} (\nabla\phi(x) - \nabla\phi(y))\cdot F(x-y)\rho_s(x)\rho_s(y)\,dxdy\,ds - \int_0^t \langle \rho_s, \Delta\phi\rangle ds = 0. \tag{3.2}$$

Proof. We divide our proof into the following steps.

Step 1 The integral (3.2) involves the singularity; therefore, we need to show that it is well-defined. Since $\phi \in C_b^2(\mathbb{R}^3)$, we only need to show that the third term is integrable for a.s. $\omega \in \Omega$. By Tonelli's theorem, it suffices to show that

$$\mathbb{E}\left[\frac{1}{2}\int_0^t \int_{\mathbb{R}^{2d}} |(\nabla\phi(x) - \nabla\phi(y))\cdot F(x-y)|\mu_s(dx)\mu_s(dy)ds\right] < \infty. \tag{3.3}$$

Since $\phi \in C_b^2(\mathbb{R}^3)$, by Lemma 2.5, we take $d = 3$ and $\frac{1}{3} < \beta \le \frac{2}{3}$, and there exists a constant C depending only on ϕ, T, and β such that

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |(\nabla\phi(x) - \nabla\phi(y))\cdot F(x-y)|\mu_s(dx)\mu_s(dy)ds\right] \\
&\le 2\|\nabla^2\phi\|_\infty \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_s(x)\rho_s(y)}{|x-y|}dxdyds\right|\right] \\
&\le C\mathbb{E}\left[\int_0^T (I_2(\rho_s^{\otimes 2})^{\frac{3\beta}{2}} + 1)ds\right] = C\mathbb{E}\left[\int_0^T I_1(\rho_s)^{\frac{3\beta}{2}} ds\right] + CT.
\end{aligned} \tag{3.4}$$

Since $\beta < \frac{2}{3}$, using Hölder's inequality, there exists a constant $C = C(\phi, T, \beta)$ such that

$$\mathbb{E}\left[\int_0^T I_1(\rho_s)^{\frac{3\beta}{2}} ds\right] \le C\left(\int_0^T I(\rho_s)ds\right)^{\frac{3\beta}{2}}. \tag{3.5}$$

Combining (2.65), (3.4), and (3.5) together, we obtain (3.3), which means that the integral (3.2) is well-defined.

Now from Lemma 2.8 and (2.60), we can rewrite the integral (3.2) as

$$\begin{aligned}
&\langle \mu_t, \phi\rangle - \langle \rho_0, \phi\rangle - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} (\nabla\phi(x) - \nabla\phi(y))\cdot F(x-y)\mu_s(dx)\mu_s(dy)ds - \int_0^t \langle \mu_s, \Delta\phi\rangle ds \\
&= -\frac{1}{2}\int_0^t \iint_{(C([0,T];\mathbb{R}^3))^2} (\nabla\phi(X_s) - \nabla\phi(Y_s))\cdot F(X_s - Y_s)\mu(dX)\mu(dY)ds \\
&\quad + \int_{C([0,T];\mathbb{R}^3)} (\phi(X_t) - \phi(X_0))\mu(dX) - \int_{C([0,T];\mathbb{R}^3)} \int_0^t \Delta\phi(X_s)ds\mu(dX).
\end{aligned} \tag{3.6}$$

For $X, Y \in C([0,T];\mathbb{R}^3)$, we define the functional

$$\psi(X,Y) = \phi(X_t) - \phi(X_0) - \frac{1}{2}\int_0^t (\nabla\phi(X_s) - \nabla\phi(Y_s))\cdot F(X_s - Y_s)ds - \int_0^t \Delta\phi(X_s)ds, \tag{3.7}$$

and similarly, $\psi_\varepsilon(X,Y)$ is the functional with F being replaced by F_ε. We also define functional $\mathcal{K}_\psi$ and $\mathcal{K}_{\psi_\varepsilon}$ on $\mathbf{P}(C([0,T];\mathbb{R}^d))$ by

$$\begin{aligned}
\mathcal{K}_\psi(\nu) &= \int_{(C([0,T];\mathbb{R}^3))^2} \psi(X,Y)\nu(dX)\nu(dY), \\
\mathcal{K}_{\psi_\varepsilon}(\nu) &= \int_{(C([0,T];\mathbb{R}^3))^2} \psi_\varepsilon(X,Y)\nu(dX)\nu(dY).
\end{aligned} \tag{3.8}$$

If ν is a random measure, i.e., a (measurable) mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbf{P}(C([0,T];\mathbb{R}^3))$, then $\mathcal{K}_\psi(\nu)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Since we can change the order of integration in light of (3.3), from definition (3.8), we see that for a.s. $\omega \in \Omega$, the left side of (3.2) is actually equal to $\mathcal{K}_\psi(\mu(\omega))$. Therefore, it suffices to show that $\mathbb{E}[|\mathcal{K}_\psi(\mu)|] = 0$.

For $\varepsilon > 0$, one certainly has

$$\mathbb{E}[|\mathcal{K}_\psi(\mu)|] \le \mathbb{E}[|\mathcal{K}_\psi(\mu) - \mathcal{K}_{\psi_\varepsilon}(\mu)|] + \mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu)|]. \tag{3.9}$$

In the following steps, we show that each term of (3.9) goes to 0 as $\varepsilon \to 0$.

Step 2 Now, we investigate the first term of (3.9). For fixed $\omega \in \Omega$, μ is a probability measure on $C([0,T],\mathbb{R}^3)$; thus, we can apply Lemma 2.8 and obtain

$$\begin{aligned}
&\mathbb{E}[|\mathcal{K}_\psi(\mu) - \mathcal{K}_{\psi_\varepsilon}(\mu)|] \\
&= \frac{1}{2}\mathbb{E}\left[\left|\int_0^t \int_{C\times C} [\nabla\phi(X_s) - \nabla\phi(Y_s)] \cdot [F_\varepsilon(X_s - Y_s) - F(X_s - Y_s)]\mu(dX)\mu(dY)ds\right|\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left|\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} [\nabla\phi(x) - \nabla\phi(y)] \cdot [F_\varepsilon(x-y) - F(x-y)]\mu_s(dx)\mu_s(dy)ds\right|\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left|\int_0^t \iint_{|x-y|<\varepsilon} [\nabla\phi(x) - \nabla\phi(y)] \cdot [F_\varepsilon(x-y) - F(x-y)]\rho_s(x)\rho_s(y)dxdyds\right|\right] \\
&\le C_d\mathbb{E}\|\nabla^2\phi\|_\infty \int_0^t \iint_{|x-y|<\varepsilon} \frac{\rho_s(x)\rho_s(y)}{|x-y|}dxdyds.
\end{aligned} \tag{3.10}$$

In the equation, we used the fact that $|F_\varepsilon(x)| \le |F(x)|$ and $F_\varepsilon(x) = F(x)$ when $|x| \ge \varepsilon$. Now, we apply (2.16) in Lemma 2.5 by taking $\gamma = 1$ and obtain

$$\begin{aligned}
\mathbb{E}\int_0^t \iint_{|x-y|<\varepsilon} \frac{\rho_s(x)\rho_s(y)}{|x-y|}dxdyds &\le C_\beta\varepsilon^{3\beta-1}\mathbb{E}\int_0^t I(\rho_s^{\otimes 2})^{\frac{3\beta}{2}}ds \\
&\le C(\beta,T)\varepsilon^{3\beta-1}\left(\mathbb{E}\int_0^t I(\rho_s)ds\right)^{\frac{3\beta}{2}},
\end{aligned} \tag{3.11}$$

where $\frac{1}{3} < \beta < \frac{2}{3}$. Therefore, by (2.65), there exists $C = C(\phi, T, \beta, \rho_0)$ such that

$$\mathbb{E}[|\mathcal{K}_\psi(\mu) - \mathcal{K}_{\psi_\varepsilon}(\mu)|] \le C\varepsilon^{3\beta-1}. \tag{3.12}$$

Step 3 For the second term of (3.9), since ψ_ε is bounded and continuous on $C([0,T];\mathbb{R}^3) \times C([0,T];\mathbb{R}^3)$ and $\mu^N \to \mu$ in law, applying Lemma 2.10 with $\mathcal{X} = C([0,T];\mathbb{R}^3)$, the random variable $\mathcal{K}_{\psi_\varepsilon}(\mu^N)$ converges to $\mathcal{K}_{\psi_\varepsilon}(\mu)$ in law for fixed ε. Since $\mathcal{K}_{\psi_\varepsilon}(\mu^N)$ and $\mathcal{K}_{\psi_\varepsilon}(\mu)$ are bounded by $\|\psi_\varepsilon\|_{L^\infty}$, we can take $\phi(x) = |x| \wedge \|\psi_\varepsilon\|_{L^\infty}$ as the test function and conclude

$$\lim_{N\to\infty} \mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu^N)|] = \mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu)|]. \tag{3.13}$$

Now, we investigate $\mathbb{E}\left[|\mathcal{K}_{\psi_\varepsilon}(\mu^N)|\right]$. By definition, it holds that

$$\mathcal{K}_{\psi_\varepsilon}(\mu^N)\frac{1}{N^2}\sum_{i,j=1}^N \psi_\varepsilon(X_t^{i,N}, X_t^{j,N}) = \frac{1}{N^2}\sum_{i,j=1}^N \left[\phi(X_t^{i,N}) - \phi(X_0^{i,N}) - \frac{1}{2}\int_0^t (\nabla\phi(X_s^{i,N}) - \nabla\phi(X_s^{j,N})) \cdot F_\varepsilon(X_s^{i,N} - X_s^{j,N})ds - \int_0^t \Delta\phi(X_s^{i,N})ds\right]. \tag{3.14}$$

Now, we apply the Itô's formula to $\phi \in C_b^2(\mathbb{R}^3)$ and obtain

$$\sum_{i=1}^N (\phi(X_t^{i,N}) - \phi(X_0^{i,N})) = \frac{1}{N}\sum_{i=1}^N \sum_{k=1,k\ne i}^N \int_0^t \nabla\phi(X_s^{i,N}) \cdot F(X_s^{i,N} - X_s^{k,N})ds + \sum_{i=1}^N \int_0^t \Delta\phi(X_s^{i,N})ds + \sqrt{2}\sum_{i=1}^N \int_0^t \nabla\phi(X_s^{i,N}) \cdot dB_s^i. \tag{3.15}$$

Note that by symmetry

$$\sum_{j=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1,k\neq i}^{N}\int_0^t\nabla\phi(X_s^{i,N})\cdot F(X_s^{i,N}-X_s^{k,N})ds=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\int_0^t(\nabla\phi(X_s^{i,N})-\nabla\phi(X_s^{j,N}))\cdot F(X_s^{i,N}-X_s^{j,N})ds. \tag{3.16}$$

Therefore, one has

$$\mathcal{K}_{\psi_\varepsilon}(\mu^N)\frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\int_0^t\Big[(\nabla\phi(X_s^{i,N})-\nabla\phi(X_s^{j,N}))\cdot(F(X_s^{i,N}-X_s^{j,N})-F_\varepsilon(X_s^{i,N}-X_s^{j,N}))\Big]ds+\frac{\sqrt{2}}{N}\sum_{i=1}^{N}\int_0^t\nabla\phi(X_s^{i,N})\cdot dB_s^i \tag{3.17}$$

and thus

$$\mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu^N)|]\le\frac{N-1}{2N}\int_0^t\iint_{\mathbb{R}^3\times\mathbb{R}^3}\rho_s^{(2),N}(x,y)\Big|(\nabla\phi(x)-\nabla\phi(y))\cdot(F(x-y)-F_\varepsilon(x-y))\Big|dxdyds+\mathbb{E}\left[\left|\frac{\sqrt{2}}{N}\sum_{i=1}^{N}\int_0^t\nabla\phi(X_s^{i,N})\cdot dB_s^i\right|\right]. \tag{3.18}$$

Again note that $F_\varepsilon(x)=F(x)$ when $|x|\ge\varepsilon$, and one has

$$\int_0^t\iint_{\mathbb{R}^3\times\mathbb{R}^3}\rho_s^{(2),N}(x,y)|(\nabla\phi(x)-\nabla\phi(y))\cdot(F(x-y)-F_\varepsilon(x-y))|dxdyds\le 2C_d\|\nabla^2\phi\|_\infty\int_0^t\iint_{|x-y|<\varepsilon}\frac{\rho_s^{(2),N}(x,y)}{|x-y|}dxdyds. \tag{3.19}$$

Applying Lemma 2.5 once more with $\frac{1}{3}<\beta<\frac{2}{3}$ and $\gamma=1$, one obtains

$$\int_0^t\iint_{|x-y|<\varepsilon}\frac{\rho_s^{(2),N}(x,y)}{|x-y|}dxdyds\le C_\beta\varepsilon^{3\beta-1}\int_0^t I(\rho_s^{(2),N})^{\frac{3\beta}{2}}ds\le C(\beta,T)\varepsilon^{3\beta-1}\left(\int_0^t I(\rho_s^{(2),N}(x,y))ds\right)^{\frac{3\beta}{2}}\le C\varepsilon^{3\beta-1}. \tag{3.20}$$

Here, the constant $C=C(\phi,T,\beta,\rho_0)$ comes from (2.62) for $j=2$.

For the second term of (3.18), from the independence of the Brownian motions $\{B_t^i\}_{i=1}^N$, one can easily calculate its second moment,

$$\mathbb{E}\left[\left|\frac{\sqrt{2}}{N}\sum_{i=1}^{N}\int_0^t\nabla\phi(X_s^{i,N})\cdot dB_s^i\right|^2\right]=\frac{2}{N^2}\sum_{i=1}^{N}\int_0^t\mathbb{E}[|\nabla\phi(X_s^{i,N})|^2]ds\le\frac{2}{N}T\|\nabla\phi\|_\infty^2, \tag{3.21}$$

which implies

$$\mathbb{E}\left[\left|\frac{\sqrt{2}}{N}\sum_{i=1}^{N}\int_0^t\nabla\phi(X_s^{i,N})\cdot dB_s^i\right|\right]\le\frac{C}{\sqrt{N}}. \tag{3.22}$$

Plugging (3.20) and (3.22) into (3.18), one finds that for any $\varepsilon,N>0$,

$$\mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu^N)|]\le C\left(\frac{1}{\sqrt{N}}+\varepsilon^{3\beta-1}\right). \tag{3.23}$$

Step 4 Finally, we combine the estimates above together.

Plugging (3.23) into (3.13), one finds that

$$\mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu)|]=\lim_{N\to\infty}\mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu^N)|]\le C\varepsilon^{3\beta-1}. \tag{3.24}$$

By (3.12),

$$\mathbb{E}[|\mathcal{K}_\psi(\mu)-\mathcal{K}_{\psi_\varepsilon}(\mu)|] \le C\varepsilon^{3\beta-1}. \tag{3.25}$$

Finally by (3.24) and (3.25),

$$\mathbb{E}[|\mathcal{K}_\psi(\mu)|] \le \liminf_{\varepsilon\to 0}(\mathbb{E}[|\mathcal{K}_\psi(\mu)-\mathcal{K}_{\psi_\varepsilon}(\mu)|] + \mathbb{E}[|\mathcal{K}_{\psi_\varepsilon}(\mu)|]) = 0. \tag{3.26}$$

This is the desired conclusion. □

Note that in (3.2), we have symmetrized Eq. (3.1). This symmetrization technique reduces the singularity from $|x|^{-3}$ to $|x|^{-2}$ so that Lemma 2.5 can be applied to control the singularity. This is one of the important observations in this work. The bottleneck for general d is that the singularity allowed in Lemma 2.5 is only $(0, 2)$ for all d. In fact, for $d \ge 4$ cases, Proposition 3.2 below (the proof does not rely on d) actually implies that (3.3) still holds. Therefore, step 1 and step 2 of the proof are still valid with (3.12) replaced by $\lim_{\varepsilon\to 0}\mathbb{E}[|\mathcal{K}_\psi(\mu)-\mathcal{K}_{\psi_\varepsilon}(\mu)|] = 0$. However, the difficulty arises from the N particle system (3.20), where the Fisher Information no longer provides the uniform estimate and we know nothing about

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\int_0^t \iint_{|x-y|<\varepsilon} \frac{\rho_s^{(2),N}(x,y)}{|x-y|^{d-2}}\,dxdyds.$$

Recalling the Proof of Lemma 2.5, if we can find better uniform L^p estimates for the density of $X_t^{i,N} - X_t^{j,N}$, then one might be able to pass the limit for $d \ge 4$ cases. Hence, we think that for general $d \ge 4$, the entropy way does not work unless new estimates are found.

We now give some L^p estimates for the density ρ of the limit measure μ. For convenience, we will then reserve h as

$$h = (-\Delta)^{-1}\rho = g * \rho. \tag{3.27}$$

Lemma 3.1. Let $d = 3$. Let ρ be the density of the (random) limit measure μ. Then, for a.s. $\omega \in \Omega$, we have the following claims:

$$\rho \in L^{\frac{2q}{4q-3}}(0,T;\dot{W}^{1,q}) \cap L^{\frac{2p}{3(p-1)}}(0,T;L^p),\ p \in [1,3], q \in [1,3/2],$$
$$\nabla h \in L^{2p_1/(2p_1-3)}(0,T;L^{p_1}),\ p_1 \in (3/2,\infty).$$

Consequently, $\rho\nabla h \in L^1(0,T;L^1(\mathbb{R}^3))$ a.s.

The claims for ρ follow from Eq. (2.65) and Lemma 2.4. The claims of ∇h are due to the Hardy-Littlewood-Sobolev inequality since $\nabla h = \nabla g * \rho$. We skip the details.

By Proposition 3.1 and Lemma 3.1, now we are able to prove that the density of the limit measure μ is a.s. a weak solution for Eq. (1.7).

Theorem 3.1. *Suppose that $d = 3$ and ρ_0, μ_N, μ, ρ satisfy the assumptions in Proposition 3.1, i.e., the common density ρ_0 satisfies $H(\rho_0) < \infty$, $m_2(\rho_0) < \infty$, and $\mathcal{E}(\rho_0) < \infty$, while the random measure μ on $C([0,T];\mathbb{R}^3)$ is a limit point of μ^N under the topology induced by convergence in law with a.s. density ρ. Then, for a.s. $\omega \in \Omega$, ρ is a weak solution to the nonlinear Fokker-Planck equation (1.7).*

Proof. First, we fix ϕ and show that (3.2) holds for all $t \in [0,T]$ and a.s. $\omega \in \Omega$. In fact, by Proposition 3.1 and (3.4), the following set has probability 1:

$$A = \left\{\omega \in \Omega \middle| \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |(\nabla\phi(x) - \nabla\phi(y)) \cdot F(x-y)|\mu_s(dx)\mu_s(dy)ds < \infty, \text{(3.2) holds for } t \in [0,T] \cap \mathbb{Q}\right\}. \tag{3.28}$$

For any probability measure $\mu \in \mathbb{P}(C([0,T];\mathbb{R}^3)))$, $\psi \in C_b(\mathbb{R}^3)$, and $t_n \to t$, we may apply the dominant convergence theorem,

$$\int_{C([0,T];\mathbb{R}^3)} \psi(X_{t_n})\mu(dX) \to \int_{C([0,T];\mathbb{R}^3)} \psi(X_t)\mu(dX), \tag{3.29}$$

which gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\psi(x)\mu_{t_n}(dx) = \int_{\mathbb{R}^3}\psi(x)\mu_t(dx) \tag{3.30}$$

by Lemma 2.8. From (3.30), we see that both $\langle\Delta\phi,\mu_t\rangle$ and $\langle\phi,\mu_t\rangle$ are continuous functions on $[0,T]$. The continuity then implies that for $\omega \in A$, (3.2) holds for all $t \in [0,T]$.

Now, we show that for a.s. $\omega \in \Omega$, $\rho(\omega)$ satisfies (3.2) both for all $t \in [0,T]$ and all $\phi \in C_c^2(\mathbb{R}^3)$. In fact, since $C_c^2(\mathbb{R}^3)$ is separable (note that $C_b^2(\mathbb{R}^3)$ is not separable), there is a countable dense set $\{\phi_n\}$. Then, for a.s. $\omega \in \Omega$, $\mu(\omega)$ satisfies (3.2) for all $t \in [0,T]$ and $\phi = \phi_n$. Now, in light of (3.4), for a.s. $\omega \in \Omega$, the left side of (3.2) can be viewed as a bounded linear functional on $C_c^2(\mathbb{R}^3)$. The conclusion then follows from the density of $\{\phi_n\}$.

Finally, $\rho \nabla h \in L^1(0,T;L^1(\mathbb{R}^3))$ from Lemma 3.1, and we can then change the symmetric integral equation (3.2) into the usual one (3.1). □

The weak solution defined above has the minimal regularity requirement. In fact, the system we consider could give more information and we can improve the regularity. We first of all have the following important claim about the energy.

Proposition 3.2. Consider a general dimension $d \geq 3$. Suppose μ is any limit point of μ^N which a.s. has density as we have seen. Then, for a.s. $\omega \in \Omega$, the energy

$$\mathscr{E}(t,\omega) := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y)\mu_t(dx)\mu_t(dy) \tag{3.31}$$

is bounded by the initial energy

$$\sup_{t\in[0,T]} \mathscr{E}(t,\omega) \leq \mathcal{E}(\rho_0) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y)\rho_0(x)\rho_0(y)\,dxdy.$$

Proof. From (2.40), it holds that

$$\sup_{0\leq t\leq T} \frac{1}{N^2} \sum_{i,j=1,i\neq j}^{N} g^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}) \leq \frac{1}{N^2} \sum_{i,j=1,i\neq j}^{N} g^\varepsilon(X_0^i - X_0^j) + \frac{1}{N} \sup_{0\leq t\leq T}(-M_t^\varepsilon). \tag{3.32}$$

Since for fixed N, $g^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}) = g(X_t^i - X_t^j)$ for all $t \in [0,T]$ outside a set A_ε whose probability goes to zero as $\varepsilon \to 0$ by the noncollision result, we then have almost surely that

$$\sup_{0\leq t\leq T} \frac{1}{N^2} \sum_{i,j=1,i\neq j}^{N} g(X_t^i - X_t^j) = \lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} \frac{1}{N^2} \sum_{i,j=1,i\neq j}^{N} g^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}).$$

Fatou's lemma gives us that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T} \frac{1}{N^2} \sum_{i,j:i\neq j} g(X_t^i - X_t^j) - 2\mathcal{E}(\rho_0)\right]^+ \\ &\leq \liminf_{\varepsilon\to 0} \mathbb{E}\left(\frac{1}{N^2} \sum_{i,j:i\neq j} g^\varepsilon(X_0^i - X_0^j) - 2\mathcal{E}(\rho_0)\right)^+ + \liminf_{\varepsilon\to 0} \frac{1}{N}\mathbb{E}\left[\sup_{0\leq t\leq T}(-M_t^\varepsilon)\right]^+. \end{aligned} \tag{3.33}$$

Doob's L^p inequality for martingale (p. 203, Theorem 7.31 in Ref. 33) and (2.41) imply that

$$\frac{1}{N}\left\|\sup_{0\leq t\leq T}(-M_t^\varepsilon)\right\|_{L^2(\mathbb{P})} \leq \frac{1}{N}\left(\mathbb{E}\sup_{0\leq t\leq T}(-M_t^\varepsilon)^2\right)^{1/2} \leq \frac{2}{N}\mathbb{E}[(M_T^\varepsilon)^2]^{\frac{1}{2}} \leq \frac{C_{d,\rho_0}}{\sqrt{N}}. \tag{3.34}$$

Hence, the last term goes to zero as $N \to \infty$. Moreover, at $t = 0$, the joint distribution of $(X_0^i, X_0^j, X_0^m, X_0^n)$ is simply $\rho_0^{\otimes 4}$ if they are all distinct. In the square of $\sum_{i,j:i\neq j} g^\varepsilon(X_0^i - X_0^j)$, the number of terms where some X_0^i's are repeated is $O(N^3)$. Hence, most terms are those where the four X_0^i's are distinct. Using this fact and direct computation, we find

$$\begin{aligned} &\lim_{N\to\infty} \liminf_{\varepsilon\to 0} \mathbb{E}\left(\frac{1}{N^2} \sum_{i,j:i\neq j} g^\varepsilon(X_0^i - X_0^j) - 2\mathcal{E}(\rho_0)\right)^+ \\ &\qquad \leq \lim_{N\to\infty} \liminf_{\varepsilon\to 0} \left(\mathbb{E}\left|\frac{1}{N^2} \sum_{i,j:i\neq j} g^\varepsilon(X_0^i - X_0^j) - 2\mathcal{E}(\rho_0)\right|^2\right)^{\frac{1}{2}} = 0. \end{aligned} \tag{3.35}$$

Consequently,

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \frac{1}{N^2} \sum_{i,j:i\neq j} g(X_t^i - X_t^j) - 2\mathcal{E}(\rho_0)\right]^+ = 0.$$

Now, for $v \in \mathbf{P}(C[0,T],\mathbb{R}^d)$, we define

$$Q_t(v) := \iint_{\mathcal{D}^c} g(x-y)v_t(dx)v_t(dy), \tag{3.36}$$

where v_t is defined in (2.58).

We also define

$$Q(v) := \sup_{0\le t\le T} Q_t(v) = \sup_{0\le t\le T} \iint_{\mathcal{D}^c} g(x-y)v_t(dx)v_t(dy). \tag{3.37}$$

We claim that if we consider the topology induced by weak convergence on $\mathbf{P}(C[0,T],\mathbb{R}^d)$, then $Q(v)$ is a lower semicontinuous functional on $\mathbf{P}(C[0,T],\mathbb{R}^d)$. In fact, we can define

$$Q_t^{(m)}(v) := \iint_{(C([0,T];\mathbb{R}^d))^2} g^{(m)}(X(t)-Y(t))v(dX)v(dY),$$

where $g^{(m)}(x) = g(|x|)$ if $|x| \ge 1/m$ and $g^{(m)}(x) = m|x|g(1/m)$ if $|x| < 1/m$. Since $g^{(m)}$ is a continuous bounded function, by Ref. 35, Theorem 2.8 (p. 23), $Q^{(m)}$ is a continuous functional. Moreover, by the monotone convergence theorem, $\forall v \in \mathbf{P}(C[0,T],\mathbb{R}^d)$, $Q_t(v) = \sup_m Q^{(m)}(v)$. Hence, Q_t is lower semicontinuous, and thus, $Q = \sup_t Q_t$ is lower semicontinuous.

From the previous proof, a subsequence of $\{\mu_N\}$ converges in law to some random measure μ. Since $\mathbf{P}(C[0,T],\mathbb{R}^d)$ is now a Polish space, from Ref. 36, p. 415, Theorem 11.7.2, there exists some probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random measures $\tilde\mu_N,\tilde\mu : (\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}}) \to \mathbf{P}(C[0,T],\mathbb{R}^d)$ such that $\tilde\mu_N \to \tilde\mu$ a.s., and $\tilde\mu_N,\tilde\mu$ has the same law as μ_N,μ. By the Fatou Lemma and the lower semicontinuity, we have

$$\mathbb{E}[(Q(\mu)-2\mathcal{E}(\rho_0))^+] = \tilde{\mathbb{E}}[(Q(\tilde\mu)-2\mathcal{E}(\rho_0))^+] \le \liminf_{N\to\infty} \tilde{\mathbb{E}}[(Q(\tilde\mu_N)-2\mathcal{E}(\rho_0))^+] = \liminf_{N\to\infty} \mathbb{E}[(Q(\mu_N)-2\mathcal{E}(\rho_0))^+] = 0. \tag{3.38}$$

Moreover, since μ has density almost surely, then almost surely it holds that

$$\sup_{0\le t\le T} \iint_{\mathbb{R}^d\times\mathbb{R}^d} g(x-y)\mu_t(dx)\mu_t(dy) \le 2\mathcal{E}(\rho_0). \tag{3.39}$$

□

With the above estimate, $\rho \in L^\infty(0,T;H^{-1})$ and $\nabla h \in L^\infty(0,T;L^2)$. Then, we have the following improved weak solution, and we provide the proof in Appendix B.

Proposition 3.3. Let $d = 3$. Suppose $\mu(.)$ is a time-dependent probability measure, which has a density ρ. Assume that ρ is a weak solution to (1.7) in the sense of Definition 3.1. If moreover $\int_0^T I(\mu_t)\,dt < \infty$ and $\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} g(x-y)\mu_t(dx)\mu_t(dy) \le 2\mathcal{E}(\rho_0)$, then

1. *$\rho \in L^{3r/(5r-6)}(0,T;L^r)$ for $r \in [3/2,3]$; $\nabla h \in L^{q/(q-2)}(0,T;L^q)$ for $q \ge 2$; consequently, $\rho\nabla h$ is in $L^{3p/(8p-6)}(L^p)$ for $1 \le p \le \frac{6}{5}$ (recall $h = g * \rho$);*
2. *in $L^{6/5}((0,T),W^{-1,12/11})$, it holds that*

$$\partial_t\rho = \nabla\cdot(\rho\nabla h) + \Delta\rho. \tag{3.40}$$

Moreover, ρ is a mild solution in $L^{4/3}(0,T;L^{3/2}(\mathbb{R}^3))$ so that

$$\rho(t) = e^{t\Delta}\rho_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot(\rho\nabla h)\,ds. \tag{3.41}$$

Note that the mild solution form here does not necessarily give the continuity of $\rho(t)$ at $t = 0$ because we do not know whether the second term goes to 0 as $t \to 0^+$.

IV. A COMMENT ABOUT PROPAGATION OF CHAOS IN 3D

We have established the fact that the limit measure is almost surely a weak solution to the nonlinear Fokker-Planck equation (1.7). An important question in the mean field limit research is whether we have propagation of chaos. In other words, we expect that the j-marginal tends to the tensor product of the limit law ρ. First, we recall the following standard equivalent notions of propagation of chaos which can be found in the lecture of Sznitman (Proposition 2.2 in Ref. 15).

Definition 4.1. *Let $\mathcal{X}$ be a Polish space and f be a probability measure on $\mathcal{X}$. A sequence of symmetric probability measures f^N on $\mathcal{X}^N$ is said to be $f-$chaotic, if one of the three following equivalent conditions is satisfied:*

(i) *The sequence of second marginals $f^{(2),N} \rightharpoonup f \otimes f$ as $N \to \infty$.*
(ii) *For all $j \geq 1$, the sequence of j-th marginals $f^{(j),N} \rightharpoonup f^{\otimes j}$ as $N \to \infty$;*
(iii) *Let $(X^{1,N}, \dots, X^{N,N}) \in \mathcal{X}^N$ be drawn randomly according to f^N. The empirical (random) measure $\mu^N = \frac{1}{N}\sum_i \delta_{X^{i,N}}$ converges in law to the constant probability measure f as $N \to \infty$.*

Note that since $\mathcal{X}$ is a Polish space, there exists a metric d_0 on $\mathbf{P}(\mathcal{X})$ such that for $\nu^N, \nu \in \mathbf{P}(\mathcal{X})$, $\nu^N \to \nu$ in law if and only if $d_0(\nu^N, \nu) \to 0$ as $N \to \infty$. Therefore, as f is constant, (iii) is equivalent to μ^N converging to f in probability.

The key point of propagation of chaos is therefore to establish a strong-weak uniqueness principle for the solutions so that $\rho_t(\omega)$ is a.s. deterministic. The definition of weak solution in Definition 3.1 is too weak, and it is very hard to prove the uniqueness. We need to put more constraints to make it unique. In fact, we have the strong-weak uniqueness principle by assuming $\rho \in L^2_{loc}((0,T); L^2(\mathbb{R}^3))$.

Proposition 4.1. *Let the initial density $\rho_0 \in H^m(\mathbb{R}^3)$ with $m > 3/2$. Suppose that $\mu(.)$ is a time-dependent probability measure, which has a density ρ. Assume that ρ is a weak solution to (1.7) in the sense of Definition 3.1. If moreover $\int_0^T I(\mu_t)\, dt < \infty$, $\rho \in L^2_{\text{loc}}((0,T); L^2(\mathbb{R}^3))$, and*

$$\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} g(x-y)\mu_t(dx)\mu_t(dy) \leq 2\mathcal{E}(\rho_0) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} g(x-y)\rho_0(x)\rho_0(y)\, dxdy,$$

then ρ_t is the unique strong solution of (1.7).

The Proof of this proposition, though important, is tedious, and we attach it in Appendix B. In fact, we do not have good enough *a priori* L^p estimates, so the usual hypercontractivity method for Keller-Segel equations (for instance, see Refs. 37 and 38) will not work. What we use is an energy method appeared in Ref. 12.

Recall that the energy equality (2.26) tells us that

$$\sup_{N,\varepsilon} \int_0^T \iint_{\mathbb{R}^3\times\mathbb{R}^3} J_\varepsilon(x_1 - x_2)\mu^{(2)}_{N,\varepsilon}(x_1, x_2)\, dx_1 dx_2 dt < \infty. \tag{4.1}$$

Formally, if we take $\varepsilon \to 0$, we would have

$$\int_0^T \tilde{\rho}^N(0,t)\, dt \leq C,$$

where $\tilde{\rho}^N$ is the density for $X_1^N - X_2^N$. As $N \to \infty$, it is expected that

$$\int_0^T \int_{\mathbb{R}^3} \mu^{(2)}(x,x)\, dxdt \leq C_1.$$

This should be

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^3} \rho^2\, dxdt \leq C_1,$$

which is desired. However, rigorously justifying these limits needs some uniform convergence, and this seems hard. We will keep on working on the weak-strong uniqueness principle.

One may be tempted to send $N \to \infty$ first in

$$\sup_{N,\varepsilon} \int_0^T \iint_{\mathbb{R}^3\times\mathbb{R}^3} J^\varepsilon(x_1 - x_2)\mu^{(2)}_{N,\varepsilon}(x_1, x_2)\, dx_1 dx_2 dt < C.$$

The mollified system has the propagation of chaos, and the limit measure is unique, which is the strong solution ρ_ε to the nonlinear Fokker-Planck equation with F being replaced by F_ε. Although ρ_ε has a uniform $L^2(0,T;L^2)$ bound, one can show that ρ_ε converges to the strong solution of the nonlinear Fokker-Planck equation constructed in Appendix A, instead of the limit measure ρ. Hence, this does not work.

Remark 4.1. Note that for the $d = 2$ case, the assumption $\rho \in L^2(0,T;L^2(\mathbb{R}^2))$ is a direct corollary from (2.12) in Lemma 2.4 by taking $p = d = 2$, and one can check that the Proof of Proposition 4.1 is valid for $d = 2$. Also, for Proposition 3.1, the self-consistent martingale problem proved in Sec. 4 of Ref. 23 implies the conclusion. Hence, combining these two results, one obtains the propagation of chaos result for $d = 2$ easily.

Remark 4.2. In fact, in the energy estimate, one also expects that the first negative term will give us

$$\int_0^T \int_{\mathbb{R}^3} \rho |\nabla h|^2 \, dxdt < C. \tag{4.2}$$

If this is true, many proofs can be simplified. For example, we will then have $\nabla\rho \cdot \nabla h = \nabla\sqrt{\rho}(\sqrt{\rho}\nabla h) \in L^1(0,T;L^1)$. Then, using the mild solution form (3.41) and the non-negativity of ρ^2, one finds $\rho \in L^2(0,T;L^2)$. However, (4.2) seems difficult to justify.

With Theorem 3.1 and Proposition 4.1, we conclude the following.

Theorem 4.1. *For $d = 3$, let $\{(X_t^{i,N})\}_{i=1}^N$ be the unique solution to (1.1) with iid initial data $\{(X_0^{i,N})\}_{i=1}^N$. Suppose the common initial density $\rho_0 \in L^1(\mathbb{R}^3) \cap H^m(\mathbb{R}^3)$ for $m > 3/2$, with $m_2(\rho_0) < \infty$, $H_1(\rho_0) < \infty$. Suppose that any limit point of the empirical measure μ^N defined in (1.6) satisfies*

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^3} \rho^2 \, dxdt < \infty.$$

Then, μ^N goes in probability to a deterministic measure $\overline{\mu} := (\rho_t \, dx)_{t\in[0,T]}$ in $\mathbf{P}(C([0,T];\mathbb{R}^3))$ as $N \to \infty$, where ρ_t is the unique strong solution to (1.7) with initial value ρ_0.

Proof. We consider the metric d_0 on $\mathbf{P}(C([0,T],\mathbb{R}^3))$ induced by weak convergence. From Proposition 2.2, we know that $\mathcal{L}(\mu^N)$ is tight in $\mathbf{P}(\mathbf{P}(C([0,T];\mathbb{R}^3)))$. Therefore, for any subsequence of μ^N, there exists a further subsequence $\{\mu^{N_k}\}$ converging in law to some random measure μ: $(\Omega,\mathcal{F},\mathbb{P}) \to C([0,T];\mathbb{R}^3)$. Then, by Theorem 3.1, for a.s. $\omega \in \Omega$, the limiting point μ has a density ρ, which is a weak solution to (1.7). By the assumption and Proposition 4.1, the weak solution to (1.7) is unique. Therefore, if we denote $\overline{\mu}$ the (deterministic) random measure with density $\overline{\rho}$, which is the strong solution to (1.7), then for any $0 \le t \le T$ and $\omega \in \Omega$, $\mu = \overline{\mu}$ for a.s. $\omega \in \Omega$. Since the subsequence $\{\mu^{N_k}\}$ converge in law to μ and μ is a.s. equal to the deterministic probability measure $\overline{\mu}$, we see that μ^{N_k} converge in probability to $\overline{\mu}$. In other words, any subsequence of $\{\mu^N\}$ has a further subsequence $\{\mu^{N_k}\}$ converging in probability to $\overline{\mu}$. Hence, $\{\mu^N\}$ converges in probability to the deterministic probability measure $\overline{\mu}$ in $\mathbf{P}(C([0,T];\mathbb{R}^3))$. □

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APPENDIX A: NOTES ON THE NONLINEAR FOKKER-PLANCK EQUATION

In this part, we investigate some properties of the nonlinear Fokker-Planck equation (1.7). We will show the local existence and uniqueness of strong solution for (1.7), given the initial data are small in some space H^s, and then, we will discuss some potential methods for the uniqueness of the weak solution.

First, we state a useful lemma in Ref. 39, which is some type of Banach fixed point theorem.

Lemma A.1. Let $(X, \|\cdot\|_X)$ be a Banach space and $H : X \times X \to X$ be a bounded bilinear form satisfying $\|H(x_1,x_2)\|_X \le \eta\|x_1\|_X\|x_2\|_X$ for all $x_1, x_2 \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $f \in X$ is such that $\|f\|_X < \varepsilon$, the equation $x = f + H(x,x)$ has a solution in X such that $\|x\|_X \le 2\varepsilon$. This solution is unique in the ball $\overline{B}(0,2\varepsilon)$.

In light of Duhamel's principle, we define the mild solution of (1.7) in the following sense.

Definition A.1. Let X be a Banach space over space and time. We call $\rho \in X$ a mild solution to (1.7) with initial data ρ_0 if ρ satisfies the following equation in X:

$$\rho(x,t) = e^{t\Delta}\rho_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(\rho(s,\cdot)\nabla(g * \rho(s,\cdot))\Big)ds. \tag{A1}$$

Now, we have the following local existence and uniqueness of mild solution:

Proposition A.1. Suppose $m > \frac{d}{2}$, $d \ge 3$, and the initial data $\rho_0 \in L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$. Then, there exists a $T > 0$ such that Eq. (1.7) admits a unique mild solution ρ in $C([0,T];L^1(\mathbb{R}^d)) \cap C([0,T];H^m(\mathbb{R}^d))$. If we define T_b to be the largest time of existence, i.e.,

$$T_b = \sup\{T > 0 |\ (1.7) \text{ has a mild solution in } C([0,T];L^1(\mathbb{R}^d)) \cap C([0,T];H^m(\mathbb{R}^d))\},$$

then $T_b < \infty$ *implies that* $\limsup_{t\to T_b^-}(\|\rho_t\|_{H^m} + \|\rho_t\|_{L^1}) = +\infty$. *Moreover, the integral of the mild solution is preserved, i.e.,*

$$\int_{\mathbb{R}^d} \rho(x,t)dx = \int_{\mathbb{R}^d} \rho_0(x)dx. \tag{A2}$$

Proof. We will apply Lemma A.1 to prove this result. We set

$$X := C([0,T];L^1(\mathbb{R}^d)) \cap C([0,T];H^m(\mathbb{R}^d))$$

with norm $\|u\|_X := \|u\|_{C([0,T];L^1)} + \|u\|_{C([0,T];H^m)}$ and define the bilinear form H on $X \times X$ by

$$H(u,v) = \int_0^t e^{(t-s)\Delta}\big(\nabla\cdot(u(s,\cdot)\nabla(g * v(s,\cdot)))\, ds.$$

We also denote

$$\|f\| := \|f\|_{L^1} + \|f\|_{H^m}$$

for $f \in L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$.

First, since H^m is an algebra as long as $m > \frac{d}{2}$, for $f_1, f_2 \in L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$, we have

$$\|f_1\nabla(g * f_2)\|_{H^m} \le C_m\|f_1\|_{H^m}\|\nabla(g * f_2)\|_{H^m}. \tag{A3}$$

Note that

$$\begin{aligned}\|\nabla(g * f_2)\|_{H^m}^2 &= \int_{\mathbb{R}^d} |\widehat{\nabla(g * f_2)}(\xi)|^2(1+|\xi|^2)^m d\xi \\ &= \int_{|\xi|\le 1} \frac{|\widehat{f_2}|^2(1+|\xi|^2)^m}{|\xi|^2} d\xi + \int_{|\xi|>1} \frac{|\widehat{f_2}|^2(1+|\xi|^2)^m}{|\xi|^2} d\xi \\ &\le C_{m,d}\|\hat{f_2}\|_\infty^2 + \|f_2\|_{H^m}^2.\end{aligned} \tag{A4}$$

Combining (A3) and (A4) and the fact that $\|\widehat{f_2}\|_\infty \le \|f_2\|_{L^1}$, one finds that

$$\|f_1\nabla(g * f_2)\|_{H^m} \le C_{m,d}\|f_1\|\|f_2\|. \tag{A5}$$

For $0 \le \alpha < 1$, one also has the following for $f \in H^m$:

$$\begin{aligned}\|e^{t\Delta}\nabla f\|_{H^{m+\alpha}}^2 &= \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{m+\alpha}|\xi|^2 e^{-2|\xi|^2 t} d\xi \\ &\le \|f\|_{H^m}^2 \sup_{\xi\in\mathbb{R}^d}(|\xi|^2(1+|\xi|^2)^\alpha e^{-2|\xi|^2 t}) \\ &\le C\|f\|_{H^m}^2(t^{-1} + t^{-1-\alpha}).\end{aligned} \tag{A6}$$

Hence, for $u, v \in X$, one has that

$$\sup_{t\in[0,T]} \|H(u,v)\|_{H^{m+\alpha}} \le C(T^{1/2} + T^{(1-\alpha)/2})\|u\|_X\|v\|_X. \tag{A7}$$

The heat kernel $P(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$ satisfies $\|\nabla P(\cdot,t)\|_{L^1} = \alpha_d t^{-\frac{1}{2}}$, where α_d is a constant. Note that $e^{(t-s)\Delta}\nabla\cdot(u\nabla(g * v)) = \int_{\mathbb{R}^d} \nabla P(x-y, t-s)\cdot u\nabla(g * v)(y,s)\, dy$. One thus has

$$\begin{aligned}\sup_{t\in[0,T]} \|H(u,v)\|_{L^1} &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|u\nabla(g * v)\|_1(s)\, ds \\ &\le C\|u\|_X\|v\|_X\int_0^T (T-s)^{-1/2}\, ds.\end{aligned} \tag{A8}$$

Note that we used $\|u(s)\nabla(g * v)(s)\|_1 \le \|u(s)\|_2\|\nabla g * v(s)\|_2 \le C\|u(s)\|_{H^m}\|v(s)\|$ by setting $m = 0$ in (A4).

We now check that $H(u,v)\in X$. By (A7) and (A8), it is easy to verify that $H(u,v)$ is continuous at $t=0$ in the $H^{m+\alpha}$ and L^1 norm. We now fix $t>0$. Pick $\delta_1\in(0,t)$ and set $w=u\nabla(g*v)$. We calculate for $|\delta|$ small enough (note that δ can be negative) that

$$\begin{aligned}\|H(u,v)(t+\delta)-H(u,v)(t)\|_{H^{m+\alpha}}\le&\int_0^{t-\delta_1}\|(e^{(t+\delta-s)\Delta}-e^{(t-s)\Delta})\nabla\cdot w\|_{H^{m+\alpha}}ds\\&+\int_{t-\delta_1}^{t}\|e^{(t-s)\Delta}\nabla\cdot w(s)\|_{H^{m+\alpha}}ds+\int_{t-\delta_1}^{t+\delta}\|e^{(t+\delta-s)\Delta}\nabla\cdot w(s)\|_{H^{m+\alpha}}ds.\end{aligned}\tag{A9}$$

Using (A6), the last two terms of (A9) are bounded by $C(|\delta|+\delta_1)^{\min(\frac12,\frac{1-\alpha}{2})}\|u\|_X\|v\|_X$. The first term of (A9) is similarly estimated as in (A6),

$$\begin{aligned}&\int_0^{t-\delta_1}\|(e^{(t+\delta-s)\Delta}-e^{(t-s)\Delta})\nabla\cdot w(s)\|_{H^{m+\alpha}}ds\\&\le C\|u\|_X\|v\|_X\int_0^{t-\delta_1}\|(1+|\xi|^{2\alpha})(e^{-|\xi|^2(t+\delta-s)}-e^{-|\xi|^2(t-s)})\cdot\xi\|_\infty^{1/2}ds\\&\le C\|u\|_X\|v\|_X\int_{\delta_1}^{t}\|\xi e^{-|\xi|^2(s-\delta_1/2)}(1+|\xi|^{2\alpha})(e^{-\delta_1|\xi|^2/2}-e^{-|\xi|^2(\delta_1/2+\delta)})\|_\infty^{1/2}ds.\end{aligned}\tag{A10}$$

By discussing the domains for $|\xi|\ge L$ and $|\xi|\le L$, one can easily find that as $\delta\to0$, the $\|\cdot\|_\infty$ norm goes to zero. Hence, $H(u,v)$ is continuous at t under the $H^{m+\alpha}$ norm. So, we have actually verified that $H(u,v)\in C([0,T];H^{m+\alpha}(\mathbb{R}^d))$ where $0\le\alpha<1$.

Similar to (A9), we have

$$\begin{aligned}\|H(u,v)(t+\delta)-H(u,v)(t)\|_{L^1}\le&\int_0^{t-\delta_1}\|(e^{(t+\delta-s)\Delta}-e^{(t-s)\Delta})\nabla\cdot w(s)\|_{L^1}ds\\+\int_{t-\delta_1}^{t}\|e^{(t-s)\Delta}\nabla\cdot w(s)\|_{L^1}ds&+\int_{t-\delta_1}^{t+\delta}\|e^{(t+\delta-s)\Delta}\nabla\cdot w(s)\|_{L^1}ds.\end{aligned}\tag{A11}$$

Similarly as in (A8), the last two terms of (A11) are controlled by $4C_d\sqrt{|\delta|+\delta_1}\|u\|_X\|v\|_X$. For the first term, we similarly write

$$\begin{aligned}&\int_0^{t-\delta_1}\|(e^{(t+\delta-s)\Delta}-e^{(t-s)\Delta})\nabla\cdot w(s)\|_{L^1}ds\\&\le C\|u\|_X\|v\|_X\int_0^{t-\delta_1}\|\nabla P(\cdot,t+\delta-s)-\nabla P(\cdot,t-s)\|_{L^1}.\end{aligned}\tag{A12}$$

Since $\nabla P\in C([\delta_1,T],L^1(\mathbb{R}^d))$ and thus uniformly continuous in time on $[\delta_1,T]$. This term goes to zero as $\delta\to0$. Hence, $H(u,v)\in C([0,T];L^1(\mathbb{R}^d))$. We thus have $H(u,v)\in X$ with

$$\|H(u,v)\|_X\le C_{d,m}\sqrt{T}\|u\|_X\|v\|_X.\tag{A13}$$

Now, we apply Lemma A.1 by taking $f=e^{t\Delta}\rho_0$. Since

$$\|e^{t\Delta}\rho_0\|_{L^1}=\|P(\cdot,t)*\rho_0\|_{L^1}\le\|\rho_0\|_{L^1},$$

$$\|e^{t\Delta}\rho_0\|_{H^m}=\|e^{-|\xi|^2t}(1+|\xi|^2)^{\frac{m}{2}}\widehat{\rho_0}\|_{L^2},$$

we find that $\|f\|_X\le\|\rho_0\|_{L^1}+\|\rho_0\|_{H^m}=\|\rho_0\|$. Therefore, by Lemma A.1, Eq. (1.7) admits a unique mild solution $\rho\in C([0,T];L^1(\mathbb{R}^d))\cap C([0,T];H^m(\mathbb{R}^d))$, where $T=\frac{1}{16C_{d,m}^2}\|f\|_X^{-2}$. Moreover, $\|\rho\|_X\le2\|f\|_X\le2\|\rho_0\|$.

Moreover, we claim that the mild solution is also unique on $[0,T_b)$, not just on $[0,T]$. In fact, for two mild solutions $\rho_i(t)$, $i=1,2$. Define $I=\{t:\rho_1(s)=\rho_2(s),\text{ for all }s\le[0,t)\}$. Clearly, I is an interval and $[0,T]\subset I$. By viewing $\rho_1(t),t\in I$ as the new initial data and applying Lemma A.1 again, we find that ρ is unique on some interval $[t,t+\varepsilon(t)]$ with $\varepsilon(t)>0$. Hence, I is an open subinterval of $[0,T_b)$ with the topology inherited from $\mathbb{R}$. Moreover, by the continuity of $\rho_i(t)$, I is also closed. Hence, $I=[0,T_b)$.

If the blow-up criterion does not hold, there exists $M>0$ such that $\sup_{t\in[0,T_b)}\|\rho(t)\|_X\le M$. Set $t_1:=\frac{1}{16C_{d,m}^2}M^{-2}$. Equation (1.7) with initial data $\rho(T_b-t_1/2)$ has a mild solution $\tilde\rho$ in $C([0,t_1];L^1(\mathbb{R}^d))\cap C([0,t_1];H^m(\mathbb{R}^d))$. If we define $\bar\rho(t)=\rho(t)$ for $0\le t\le T_b-t_1/2$ and $\bar\rho(t)=\tilde\rho(t-(T_b-t_1/2))$ for $t\in[T_b-t_1/2,T_b+t_1/2]$, then $\bar\rho$ is a mild solution on $[0,T_b+t_1/2]$, which contradicts with the definition of T_b.

Finally, we have

$$\int_{\mathbb{R}^d}\rho(x,t)dx=\int_{\mathbb{R}^d}\left(e^{t\Delta}\rho_0(x)+\int_0^t e^{(t-s)\Delta}\nabla\cdot(\rho(s,\cdot)\nabla(g*\rho(s,\cdot))ds\right)dx.\tag{A14}$$

Since we have shown in (A8) that the right side is in L^1, we can freely change the order of the integral and the integral preservation follows. □

We now show that the mild solution is a strong solution. We say $\rho \in C([0,T];L^1(\mathbb{R}^d)) \cap C([0,T];H^m(\mathbb{R}^d))$ is a strong solution if (i) ρ is a weak solution that satisfies the equation in the distributional sense and (ii) both $\partial_t\rho$ and $\nabla\cdot(\rho\nabla(g*\rho)) + \Delta\rho$ are locally integrable functions on $(0,T)\times\mathbb{R}^d$ so that the equation holds.

Proposition A.2. Let $\rho_0 \in L^1(\mathbb{R}^d)\cap H^m(\mathbb{R}^d)$ with $m > \frac{d}{2}$. Then, the mild solution ρ is a strong solution belonging to $C^\infty((0,T_b),H^{m'}(\mathbb{R}^d))$ for any $m' \geq m$. Moreover, the strong solution is unique.

Proof. We take $T \in (0,T_b)$. From the proof of previous proposition, for $0 \leq \alpha < 1$,

$$H(\rho,\rho) \in C([0,T];H^{m+\alpha}(\mathbb{R}^d)).$$

Meanwhile, since $\rho_0 \in H^m(\mathbb{R}^d)$, it is easy to verify that $e^{t\Delta}\rho_0 \in C((0,T];H^{m'})$ for any $m' > 0$. Therefore, we see that ρ is in $C((0,T];H^{m+\alpha}(\mathbb{R}^d))$.

Now, for any $0 < t_1 < T$, we take $\alpha = \frac{1}{2}$ with the new initial value $\rho_0^{(1)} = \rho_{\frac{t_1}{2}}$. Then, $\rho^{(1)}(t) := \rho(t-\frac{t_1}{2})$ is a mild solution of (1.7) in $C([0,T-\frac{t_1}{2}];H^{m+\alpha}(\mathbb{R}^d)) \cap C([0,T-\frac{t_1}{2}];L^1(\mathbb{R}^d))$. Therefore, the previous argument implies that $\rho^{(1)} \in C((0,T-\frac{t_1}{2}];H^{m+2\alpha}(\mathbb{R}^d))$. Then, we can take the new initial value $\rho_0^{(2)} = u^{(1)}_{\frac{t_1}{4}}$ along with $\rho^{(2)}(t) = u^{(1)}(t-\frac{t_1}{4})$. Iterating this process for $2(m'-m)+2$ times, we find that $\rho \in C([t_1,T],H^{m'}(\mathbb{R}^d))$.

Take $t_1 > 0$. Let $\bar\rho(t) = \rho(t+t_1)$. Then, $\bar\rho$ satisfies

$$\bar\rho(t) = e^{t\Delta}\bar\rho(0) + \int_0^t e^{(t-s)\Delta}\nabla\cdot(\bar\rho\nabla(g*\bar\rho)).$$

We have $w := \nabla\cdot(\bar\rho\nabla(g*\bar\rho)) \in C([0,T-t_1];H^m(\mathbb{R}^d))$ for any $m > 0$. It then follows

$$\Delta\bar\rho(t) = \Delta e^{t\Delta}\bar\rho(0) + \int_0^t e^{(t-s)\Delta}\Delta w_s\,ds. \tag{A15}$$

By the property for heat equation with L^2 initial data, we have $e^{t\Delta}u - u = \int_0^t \Delta e^{\tau\Delta}u\,d\tau$ if $u \in L^2$. Hence,

$$\begin{aligned}\int_0^t(\Delta\bar\rho(\tau)+w(\tau))d\tau &= \int_0^t \Delta e^{\tau\Delta}\bar\rho_0 d\tau + \int_0^t\int_0^\tau e^{(\tau-s)\Delta}\Delta w_s\,ds d\tau + \int_0^t w(\tau)d\tau\\ &= (e^{t\Delta}\bar\rho_0 - \bar\rho_0) + \int_0^t\int_s^t e^{(\tau-s)\Delta}\Delta w_s\,d\tau ds + \int_0^t w(\tau)\,d\tau\\ &= \bar\rho(t) - \bar\rho_0.\end{aligned} \tag{A16}$$

We exchanged the order of integral since $e^{(\tau-s)\Delta}\Delta w_s$ is bounded under the L^2 norm. This identity first of all implies that $\bar\rho$ is a weak solution since $\rho \in C([0,T];L^2)$. Moreover, it also implies that $\bar\rho \in C^\infty([t_1,T])$ under any H^m norm. Hence, taking derivative on time, we find that $\bar\rho$ is a strong solution. Since t_1 is arbitrary, the claim follows.

The strong solution is a mild solution on $[0,T]$. The uniqueness then follows trivially by the uniqueness of mild solutions. □

We are more interested in the non-negative initial data due to the problem we consider.

Proposition A.3. Besides the conditions in Proposition A.1, if we also have $\rho_0 \geq 0$, then

1. *for all t in the integral of existence, we have $\rho(x,t) \geq 0$ and*
2. *the strong solution exists globally, i.e., $T_b = \infty$.*

Proof. 1. The proof of non-negativity follows in a similar way as in Ref. 40. Here, we sketch the proof briefly.

We fix an arbitrary $T \in (0,T_b)$ and let ρ be the mild solution on $[0,T]$. We consider the approximated problem

$$\rho_n(t) = e^{t\Delta}\rho_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot(\rho_n^+(s)\nabla(g^{\varepsilon_n}*\rho)),$$

where $\varepsilon_n = 1/n$. Since $\nabla(g^{\varepsilon_n}*\rho)$ is a smooth function with the derivatives bounded, $\rho_n \in C([0,T],H^1)\cap C([0,T],L^1)\cap C^\infty((0,T),H^1)$. Then, for $t > 0$, it holds in H^{-1} that

$$\partial_t \rho_n = \nabla \cdot \left(\rho_n^+(s)\nabla(g^{\varepsilon_n} * \rho) + \Delta\rho_n.\right.$$

Multiply $\rho_n^- = -\min(\rho_n, 0)$ on both sides and integrate. The right hand side is equal to $\|\nabla\rho_n^-\|_2^2$. Consider the left hand side. Since $\partial_t\rho_n = \lim_{h\to 0^+}\frac{\rho_n(t)-\rho_n(t-h)}{h}$ converges in L^2, we have

$$\langle\rho_n^-, \partial_t\rho_n\rangle = \lim_{h\to 0^+}\left\langle\rho_n^-, \frac{\rho_n(t)-\rho_n(t-h)}{h}\right\rangle \le -\frac{1}{2}\lim_{h\to 0^+}\frac{\|\rho_n^-(t)\|_2^2 - \|\rho_n^-(t-h)\|^2}{h}.$$

Since $|\rho_n^-(t_2) - \rho_n^-(t_1)| \le |\rho_n(t_2) - \rho_n(t_1)|$ holds pointwise and therefore in L^2, we find that $t \to \|\rho_n^-\|_2$ is in $C[0,T]\cap C^1(0,T)$ and

$$\partial_t\|\rho_n^-\|_2^2 \le 0.$$

This implies that $\rho_n^- = 0$ and thus $\rho_n^+ = \rho_n$. As $n\to\infty$, we can show that $\rho_n \to \rho$ in $C([0,T], L^2)$, which further implies that $\rho \ge 0$ on $[0,T]$.

2. It suffices to show that the solution does not blow up in the L^1 and H^m norm in a finite time. By the integral preservation and positivity preservation, $\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$. Hence, we only need to consider the H^m norm.

Note $\rho_0 \in L^p$ for all $p\in[1,\infty)$. Using the facts that $\rho \in H^{m'}$ for any $m' > 0$ and that ρ is smooth in time for $t > 0$, we can multiply the equation with $p\rho^{p-1}$ and integrate to have, for $t > 0$,

$$\begin{aligned}\partial_t\|\rho\|_p^p &= -p(p-1)\langle\nabla\rho, \rho^{p-2}\nabla\rho\rangle - \langle\rho^2, p\rho^{p-1}\rangle + \langle\nabla(g*\rho), \nabla\rho^p\rangle\\ &= -(p-1)\int_{\mathbb{R}^d}(p\rho^{p-2}|\nabla\rho|^2 + \rho^{p+1})\,dx. \end{aligned}\tag{A17}$$

Using the non-negativity of ρ, we find that $\|\rho\|_p \le C_p$ is uniformly bounded.

Now, we consider $n = [m] + i$ (where $i = 1, 2$ so that n is even) and use the data at some $t_1 > 0$ as the initial data. Set $\|f\|_{\dot H^s} = \|(-\Delta)^{\frac{s}{2}}f\|_{L^2} = \|\xi|^s\widehat{f}\|_{L^2}$, then $\|\rho\|_{H^n}$ can be controlled by $\|\rho\|_{\dot H^n}$ and $\|\rho\|_{L^2}$. Therefore, we only need to show that $\|\rho\|_{\dot H^n}$ does not blow up, which clearly will indicate that the original $\|\cdot\|_{H^m}$ norm does not blow up. Multiplying (1.7) by $(-\Delta)^n\rho$ and integrating, we have, for $t > t_1$,

$$\frac{1}{2}\partial_t\|\rho\|_{\dot H^n}^2 = -\|\rho\|_{\dot H^{n+1}}^2 - \langle\rho^2, (-\Delta)^n\rho\rangle + \langle\nabla\rho\cdot\nabla(g*\rho), (-\Delta)^n\rho\rangle.\tag{A18}$$

For the second term of (A18), after integrating by parts for n times, we obtain

$$-\langle\rho^2, (-\Delta)^n\rho\rangle = -\langle(-\Delta)^{n/2}\rho, (-\Delta)^{n/2}(\rho^2)\rangle.\tag{A19}$$

Expanding $(-\Delta)^{n/2}(\rho^2)$ out, this contains terms of the form $C_\ell D^\ell\rho D^{n-\ell}\rho$, where $\ell = 0, 1, \dots, n$, and D denotes any partial derivative. For the $\ell = 0, n$ terms, we use the non-negativity of ρ and find $-\int_{\mathbb{R}^d}\rho|(-\Delta)^{n/2}\rho|^2\,dx \le 0$. Consider that $1 \le \ell \le n-1$. By the Gagliardo-Nirenberg inequality,

$$\|D^j f\|_{L^p} \le C\|f\|_{L^r}^{1-\alpha}\|f\|_{\dot H^{n+1}}^{\alpha},\ \alpha = \frac{j - d/p + d/r}{(n+1) - d/2 + d/r}.\tag{A20}$$

Setting $p_\ell = \frac{2n}{\ell}, q_\ell = \frac{2n}{n-\ell}$ and applying the Hölder inequality and (A20), we find that for $1 \le \ell \le n-1$,

$$|\langle D^\ell\rho D^{n-\ell}\rho, D^n\rho\rangle| \le \|D^\ell\rho\|_{L^{p_\ell}}\|D^{n-\ell}\rho\|_{L^{q_\ell}}\|\rho\|_{\dot H^n} \le C\|\rho\|_{\dot H^{n+1}}^{\frac{n-d/2+2d/r}{(n+1)-d/2+d/r}}\|\rho\|_{\dot H^n}.\tag{A21}$$

Here, we have used the fact that $\|\rho_t\|_{L^r} \le \|\rho_0\|_{L^r}$. Here, p_ℓ and q_ℓ are chosen so that the corresponding $\alpha \in (0,1)$. We pick $r > d$, and then, the power of $\|\rho\|_{\dot H^{m+1}}$ is less than 1.

For the third term of (A18), we similarly have

$$\langle\nabla\rho\cdot\nabla(g*\rho), (-\Delta)^n\rho\rangle = \langle(-\Delta)^{\frac{n}{2}}(\nabla\rho\cdot\nabla(g*\rho)), (-\Delta)^{\frac{n}{2}}\rho\rangle.\tag{A22}$$

Expanding out, we have terms of the form $(\nabla D^{n-\ell}\rho)D^\ell\nabla(g*\rho)$. The $\ell = 0$ term contributes to

$$\int_{\mathbb{R}^d}\nabla((-\Delta)^{n/2}\rho)\cdot\nabla(g*\rho)(-\Delta)^{\frac{n}{2}}\rho\,dx = -\frac{1}{2}\int_{\mathbb{R}^d}\rho|(-\Delta)^{\frac{n}{2}}\rho|^2\,dx \le 0.$$

When $\ell \ge 1$, by the singular integral theory, we have

$$\|D^\ell \nabla(g*\rho)\|_{L^{p_\ell}} = \|D\nabla(g*D^{\ell-1}\rho)\|_{L^{p_\ell}} \le \|D^{\ell-1}\rho\|_{L^{p_\ell}},\ 1<p_\ell<\infty. \tag{A23}$$

Due to this reason, we find that when $\ell = 1$, the pairing is controlled by

$$\begin{aligned}\langle(\nabla D^{n-\ell}\rho)D^\ell\nabla(g*\rho),(-\Delta)^{\frac{n}{2}}\rho\rangle &\le \|\nabla D^{n-1}\rho\|_{2+\delta}\|D\nabla(g*\rho)\|_{2(2+\delta)/\delta}\|\rho\|_{\dot H^n}\\ &\le C\|\nabla D^{n-1}\rho\|_{2+\delta}\|\rho\|_{\dot H^n}\end{aligned}.$$

By the Galiardo-Nirenberg inequality again, $\|\nabla D^{n-1}\rho\|_{2+\delta} \le C\|\rho\|^{\alpha}_{\dot H^{n+1}}\|\rho\|^{1-\alpha}_{\dot H^n}$, where $\alpha = \frac{d\delta}{2(2+\delta)}$. For $\ell > 1$, using (A23), the pairing $\langle(\nabla D^{n-\ell}\rho)D^\ell\nabla(g*\rho),(-\Delta)^{\frac{n}{2}}\rho\rangle$ is similarly controlled as in (A21). Hence, we finally have, for $t > t_1$,

$$\frac{1}{2}\partial_t\|\rho\|^2_{\dot H^n} \le -\|\rho\|^2_{\dot H^{n+1}} + C\|\rho\|^{2-\alpha}_{\dot H^n}\|\rho\|^{\alpha}_{\dot H^{n+1}} + C\|\rho\|_{\dot H^n}\|\rho\|^{\nu}_{\dot H^{n+1}}, \tag{A24}$$

where $\alpha \in (0,1)$ and $\nu \in (0,1)$. This gives that $\|\rho\|_{\dot H^n}$ never blows up in finite time for $t > t_1$, which further implies that $\|\rho\|_{H^m}$ does not blow up. □

APPENDIX B: THE MISSING PROOFS

Proof of Proposition 2.2. Note that for any N, $0 \le s < t \le T$, one has

$$X^{1,N}_t - X^{1,N}_s = \frac{1}{N}\int_s^t \sum_{j\neq 1}^N F(X^{1,N}_r - X^{j,N}_r)\,dr + \sqrt{2}(B^1_t - B^1_s).$$

This then motivates us to define

$$Z_N := \sup_{s,t:s<t}\frac{\sqrt{2}|B^1_t - B^1_s|}{(t-s)^{1/2}},\quad U_N := \frac{1}{N}\left(\int_0^T\left(\sum_{j\neq 1}^N F(X^1_t - X^j_t)\right)^2 dt\right)^{1/2}.$$

Clearly,

$$|X^{1,N}_t - X^{1,N}_s| \le (t-s)^{1/2}(Z_N + U_N).$$

Moreover, Z_N's have the same distribution for all N, and

$$Z_N < \infty, a.s.$$

Consequently,

$$\lim_{R_1\to\infty}\sup_{N\ge 2}\mathbb{P}(|Z_N| > R_1) = 0.$$

Using the energy estimate (2.49), we have

$$\mathbb{E}U_N^2 \le \mathcal{E}(\rho_0).$$

Moreover, $\mathbb{E}|X^{1,N}_0|^2 < \infty$.

We define

$$K := \{X \in C([0,T];\mathbb{R}^d), |X_0| \le A, |X_t - X_s| \le R(t-s)^{1/2}, \forall 0 \le s < t \le T\}.$$

Clearly, K is a compact set in $C([0,T],\mathbb{R}^d)$ by the Arzela-Ascoli theorem.

Moreover,

$$\begin{aligned}\sup_{N\ge 2}\mathbb{P}(X^{1,N}_t \notin K) &\le \sup_{N\ge 2}\mathbb{P}(|X^{1,N}_0| > A) + \sup_{N\ge 2}\mathbb{P}(Z_N + U_N > R)\\ &\le \sup_{N\ge 2}\mathbb{P}(|X^{1,N}_0| > A) + \sup_{N\ge 2}(\mathbb{P}(Z_N > R/2) + \mathbb{P}(U_N > R/2)).\end{aligned}$$

Using the uniform bound on the moments of $X^{1,N}_0$, U_N, we find that for any $\varepsilon > 0$, there exist $A > 0, R > 0$ such that

$$\sup_{N\geq 2}\mathbb{P}(X_t^{1,N}\notin K)<\varepsilon,$$

which concludes the tightness of the law of $X^{1,N}$.

The tightness of $\mathcal{L}(\mu^N)$ follows from (i) and the exchangeability of the system. See Ref. 15, Proposition 2.2. □

Proof of Proposition 3.3. Recall that $h=g*\rho$. By the assumption, one has

$$\rho\in L^\infty((0,T);H^{-1}),\ \ \nabla h\in L^\infty((0,T);L^2).$$

Since $\int_{\mathbb{R}^3}\rho^{3/2}\,dx\leq\|\rho\|_{H^{-1}}\|\sqrt{\rho}\|_{\dot{H}^1}$, we thus have

$$\|\rho\|_{L^3((0,T);L^{3/2})}\leq C\int_0^T\|\nabla\sqrt{\rho}\|_{L^2}^2\,dt=C\int_0^T I(\rho)\,dt.$$

Interpolating this with $\rho\in L^1(0,T;L^3)$, we have $\rho\in L^{3r/(5r-6)}(0,T;L^r)$ for $r\in[3/2,3]$.

Moreover, $\nabla^2 h\in L^1(0,T;L^3)$ since $\rho\in L^1(0,T;L^3)$; interpolating with $\nabla h\in L^\infty(0,T;L^2)$ by the Gagliardo-Nirenberg inequality, we find that

$$\nabla h\in L^{q/(q-2)}((0,T);L^q),q\geq 2.$$

Hence, $\rho\nabla h\in L^{3p/(8p-6)}(0,T;L^p)$ for $p\in[1,6/5]$. In particular, we have

$$\rho\nabla h\in L^{6/5}((0,T),L^{12/11}).$$

Since Δ is bounded from $W^{1,q}$ to $W^{-1,q}$, by Lemma 3.1, we have $\Delta\rho\in L^{8/5}((0,T),W^{-1,12/11})$.

Recall that

$$\langle\rho_t,\phi\rangle-\langle\rho_0,\phi\rangle-\int_0^t\int_{\mathbb{R}^3}\nabla\phi(x)\cdot\nabla h(x)\rho_s(x)\,dxds-\int_0^t\langle\rho_s,\Delta\phi\rangle ds=0 \tag{B1}$$

for any $\phi\in C_c^2(\mathbb{R}^3)$ and $t\in(0,T]$. Using the regularity, we find that this holds for all $\phi\in C_b^2(\mathbb{R}^3)$. In fact, we can take smooth truncation of $\phi_n=\phi\chi_n$, where $\chi_n=\chi(x/n)$ and $\chi=1$ in $B(0,1)$. Then, $\nabla(\phi\chi_n)\cdot\nabla h\to\nabla\phi\cdot\nabla h$ in $L^2(0,T;L^4)$ and $\langle\rho_s,D^\alpha(\phi_n)\rangle\to\langle\rho_s,D^\alpha\phi\rangle$, where $|\alpha|\leq 2$. The latter holds because $\rho_s\in L^1$ and $D^\alpha(\phi-\phi_n)$ is bounded and nonzero only outside $B(0,n)$. That $\int_0^t\langle\rho_s,\Delta\phi_n\rangle\,ds\to\int_0^t\langle\rho_s,\Delta\phi\rangle\,ds$ holds by the dominate convergence theorem.

Then, we claim that for any $\phi\in C^1([0,T],C_b^2(\mathbb{R}^3))$ and $t\in(0,T]$, it holds that

$$\begin{aligned}\langle\rho_t,\phi(x,t)\rangle-\langle\rho_0,\phi(x,0)\rangle-\int_0^t\int_{\mathbb{R}^3}\rho(x,s)\partial_t\phi\,dxds\\+\int_0^t\int_{\mathbb{R}^3}\nabla\phi(x,s)\cdot\nabla h(x)\rho(x,s)\,dxds-\int_0^t\langle\rho(x,s),\Delta\phi(\cdot,s)\rangle ds=0.\end{aligned} \tag{B2}$$

In fact, we can take $t=t_1$ and $t=t_2$ in (B1) and take the difference to obtain $\langle\rho_{t_2},\varphi\rangle-\langle\rho_{t_1},\varphi\rangle-\int_{t_1}^{t_2}\int\ldots-\int_{t_1}^{t_2}\ldots=0$, where the omitted content is clear. Then, we can take $\varphi=\phi(.,t_n)$ so that we have kind of Riemann sum. The regularity ensures that the Riemann sum converges to the desired integral form.

For $\phi\in C^1([0,T];C_c^2(\mathbb{R}^3))$, we then have

$$\int_0^t\langle\partial_t\rho,\phi\rangle=\int_0^t\int_{\mathbb{R}^3}\nabla\phi(x,s)\cdot\nabla h(x)\rho(x,s)\,dxds+\int_0^t\langle\nabla\rho,\nabla\phi\rangle\,ds,$$

where $\partial_t\rho$ is the distributional derivative of ρ. Clearly, the right hand side is a bounded functional for $\phi\in L^6(0,T;W^{1,12})$. By a possible mollification procedure, we find

$$\partial_t\rho=\nabla\cdot(\rho\nabla h)+\Delta\rho,\ \text{in}\ L^{6/5}((0,T),W^{-1,12/11}). \tag{B3}$$

In fact, this weak solution is also a mild solution. To see this, we mollify ρ as

$$\rho^{\varepsilon,\delta}=J_1^\delta(t)*\rho*J_2^\varepsilon(x).$$

Here, J_1 is the mollification in time, while J_2 is in space. Then, on $t\in(\delta,T-\delta)$, we have

$$\partial_t\rho^{\varepsilon,\delta}=J_1^\delta*J_2^\varepsilon*\partial_t\rho=\nabla\cdot(J_1^\delta*\rho\nabla h*J_2^\varepsilon)+\Delta\rho^{\varepsilon,\delta}.$$

Since all functions are smooth and bounded, with derivatives bounded, we have for $\delta < t_1 < t < T - \delta$ that

$$\rho^{\varepsilon,\delta}(t) = e^{(t-t_1)\Delta}\rho^{\varepsilon,\delta}(t_1) + \int_{t_1}^{t} e^{(t-s)\Delta}\nabla\cdot(J_1^{\delta} * \rho\nabla h * J_2^{\varepsilon})\,ds.$$

Now, we claim that

$$\rho^{\varepsilon}(t) := J_2^{\varepsilon} * \rho(t)$$

is a bounded continuous function on $[0,T]\times\mathbb{R}^3$. In fact, we let $\phi_x(y) = J_2^{\varepsilon}(x-y) \in C_b$. Then, we can define

$$p_{x,t}(X) = \phi_x(X(t)).$$

This is a continuous bounded functional on $C([0,T];\mathbb{R}^3)$. Then, $p_{x,t} \to p_{(x,t_0)}$ pointwise as $t \to t_0$ and are bounded functionals. Hence, we have by the dominate convergence theorem

$$\int_{C([0,T];\mathbb{R}^3)} p_{(x,t)}(X)d\mu \to \int_{C([0,T];\mathbb{R}^3)} p_{(x,t_0)}(X)d\mu.$$

This means

$$\rho^{\varepsilon}(x,t) \to \rho^{\varepsilon}(x,t_0),\ \forall x \in \mathbb{R}^3.$$

Hence, $\rho^{\varepsilon}(x,t)$ is continuous and bounded. Since $\rho^{\varepsilon} \in L^1 \cap L^{\infty}$, then taking $\delta \to 0$, we find

$$\rho^{\varepsilon,\delta}(t) \to \rho^{\varepsilon}(t), \text{in } L^1 \cap L^{3/2}, t \in (0,T).$$

Note that

$$\|\nabla P\|_r \le C_r t^{3/(2r)-2},\ r \in [1,\infty].$$

Picking $r = 4/3$, applying Ref. 41, Theorem 4 with $\rho\nabla h \in L^{6/5}(0,T;L^{12/11})$, we find that $\int_{t_1}^{t}\nabla e^{(t-s)\Delta}\cdot(\rho\nabla h)\,ds \in L^{4/3}((0,T);L^{3/2})$. Taking $\delta \to 0$, we have in $L^{4/3}((0,T);L^{3/2})$ that

$$\int_{t_1}^{t} e^{(t-s)\Delta}\nabla\cdot(J_1^{\delta} * \rho\nabla h * J_2^{\varepsilon})\,ds \to \int_{t_1}^{t} e^{(t-s)\Delta}\nabla\cdot(J_2^{\varepsilon} * (\rho\nabla h))\,ds.$$

Then, we have in $L^{4/3}((0,T);L^{3/2})$ that

$$\rho^{\varepsilon}(t) = e^{(t-t_1)\Delta}\rho^{\varepsilon}(t_1) + \int_{t_1}^{t} e^{(t-s)\Delta}\nabla\cdot(J_2^{\varepsilon} * (\rho\nabla h))\,ds.$$

Now, for any $t > 0$, we take $t_1 \to 0$. By dominate convergence, we have

$$\rho^{\varepsilon}(t) = e^{t\Delta}\rho^{\varepsilon}(0) + \int_{0}^{t} e^{(t-s)\Delta}\nabla\cdot(J_2^{\varepsilon} * (\rho\nabla h))\,ds.$$

Eventually, we take $\varepsilon \to 0$ and we have in $L^{4/3}((0,T);L^{3/2})$ that

$$\rho(t) = e^{t\Delta}\rho_0 + \int_{0}^{t} e^{(t-s)\Delta}\nabla\cdot(\rho\nabla h)\,ds. \tag{B4}$$

□

Proof of Proposition 4.1. **Step 1** The L^p bound for $t > 0$.
We fix $\delta > 0$ and then

$$\rho \in L^2([\delta,T],L^2(\mathbb{R}^3)).$$

Mollifying the equation for ρ in Proposition 3.3, we have

$$\partial_t\rho^{\varepsilon} - \nabla h\cdot\nabla\rho^{\varepsilon} - \Delta\rho^{\varepsilon} = J^{\varepsilon} * (\nabla\cdot(\rho\nabla h)) - \nabla h\cdot\nabla\rho^{\varepsilon} =: r^{\varepsilon}. \tag{B5}$$

Note that we do not have $\nabla\rho\cdot\nabla h \in L^1(\delta,T;L^1)$, so $\nabla\rho\cdot\nabla h$ may not be a distribution. This is why we cannot have such a term in the equation. Recall that $\rho \in L^2(\delta,T;L^2(\mathbb{R}^3))$ and $\nabla^2 h \in L^2(\delta,T;L^2(\mathbb{R}^3))$ by singular integral theory. We then use the proof of Lemma II.1 in Ref. 42 and conclude

$$r^\varepsilon \to -\rho^2, \text{ in } L^1(\delta, T; L^1_{loc}(\mathbb{R}^3)).$$

(There is a small typo in the Proof of Lemma II.1 in Ref. 42, where ε^{-N} is lost in the expressions on p. 517.) In fact,

$$\begin{aligned} J^\varepsilon * (\nabla\cdot(\rho\nabla h)) - \nabla h\cdot\nabla\rho^\varepsilon &= -\int_{\mathbb{R}^3}\nabla_y(J^\varepsilon(x-y))\cdot\rho(y)\nabla h(y)\,dy - \nabla h\cdot\nabla\rho^\varepsilon \\ &= \int_{\mathbb{R}^3}(\nabla J^\varepsilon)(x-y)\cdot(\nabla h(y)-\nabla h(x))\rho(y)\,dy. \end{aligned}$$

By our construction, $\|\nabla J^\varepsilon\|_\infty \le C\varepsilon^{-4}$. Consequently,

$$\left\|\int_{\mathbb{R}^3}(\nabla J^\varepsilon)(x-y)\cdot(\nabla h(y)-\nabla h(x))\rho(y)\,dy\right\|_{L^1(B(0,R))} \le C\int_{B(0,R)}dx\int_{|y-x|\le C\varepsilon}\varepsilon^{-3}\rho(y)\frac{|\nabla h(y)-\nabla h(x)|}{\varepsilon}\,dy.$$

Let $D_1 = \{(x,y): x\in B(0,R), |y-x|\le C\varepsilon\}$. We then have the above controlled by

$$\left(\iint_{D_1}\rho^2(y)dxdy\right)^{1/2}\varepsilon^{-3}\left(\iint_{D_1}\left(\frac{|\nabla h(y)-\nabla h(x)|}{\varepsilon}\right)^2 dxdy\right)^{1/2}.$$

The first term is controlled by $C\varepsilon^{3/2}\|\rho\|_{L^2(B(0,R+1))}$. The second term is controlled by

$$\varepsilon^{-3}\left(\int_{B(0,R)}dx\int_{|y-x|\le C\varepsilon}dy\left(\frac{|\nabla h(y)-\nabla h(x)|}{\varepsilon}\right)^2\right)^{1/2} \le \varepsilon^{-3/2}\left(\int_{B(0,R)}dx\int_{|z|\le C}dz\left(\int_0^1|\nabla^2 h(x+t\varepsilon z)|\right)^2\right)^{1/2}.$$

Since

$$\left[\int_{B(0,R)}dx\left(\int_0^1|\nabla^2 h(x+t\varepsilon z)|\right)^2\right]^{1/2} \le \int_0^1\|\nabla^2 h(\cdot+t\varepsilon z)\|_{L^2(B(0,R))}\,dt \le \|\nabla^2 h\|_{L^2(B(0,R+1))},$$

we have the second term controlled by $\varepsilon^{-3/2}C\|\nabla^2 h\|_{L^2(B(0,R+1))}$. Hence,

$$\|r^\varepsilon(t)\|_{L^1(B(0,R))} \le C\|\rho(t)\|_{L^2(B(0,R+1))}\|\nabla^2 h(t)\|_{L^2(B(0,R+1))}.$$

This bound is uniform in ε. With the time dimension added in, the corresponding norms are similarly controlled. By a density argument, we can then approximate ρ and $\nabla^2 h$ with smooth functions in their respective spaces. For smooth functions, the limit is clearly $\rho\Delta h = -\rho^2$.

Recall that $\rho\,dx \in C([0,T]; C_b(\mathbb{R}^3)')$; we have $\rho^\varepsilon \in C([0,T]; L^1(\mathbb{R}^3))$. Using basically the same argument as in Step 1 of the Proof of Lemma 2.5 in Ref. 43, we obtain that ρ^ε is a Cauchy sequence in $L^1(\delta, T; L^1_{\text{loc}})$ as $\varepsilon\to 0$. In fact, for convex function $\beta\in C^1(\mathbb{R}^3)$, we have the following chain rule:

$$\partial_t\beta(\rho^\varepsilon) = \nabla\beta(\rho^\varepsilon)\cdot\nabla h + \beta'(\rho^\varepsilon)r^\varepsilon + \Delta\beta(\rho^\varepsilon) - \beta''(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2. \tag{B6}$$

Equations (B5) and (B6) will hold if we replace ρ^ε with $\rho^{\varepsilon_1}-\rho^{\varepsilon_2}$ and r^ε with $r^{\varepsilon_1}-r^{\varepsilon_2}$ since (B5) is linear. In particular, we choose $\beta(s) = s^2/2$ for $|s|\le A$ and $\beta(s) = A|s| - A^2/2$ for $|s|\ge A$. This will give $\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0}\sup_{0\le t\le T}\int_{\mathbb{R}^3}\beta(\rho^{\varepsilon_1}-\rho^{\varepsilon_2})\chi(x)\,dx = 0$ for any $\chi\in C_c^\infty$. [There is only one difference from Ref. 43: to justify $\int_{\mathbb{R}^3}\beta(\rho^{\varepsilon_1}-\rho^{\varepsilon_2})\nabla h\cdot\nabla\chi\to 0$, we use $\nabla h\in L^\infty(L^2)$ and $\rho^{\varepsilon_1}-\rho^{\varepsilon_2}\to 0$ in $L^2(\delta,T;L^2)$.]

Consequently, we have $\rho\in C([\delta,T]; L^1_{loc}(\mathbb{R}^3))$. Moreover, using the uniform of estimates $\int_{\mathbb{R}^3}\rho|x|^2\,dx$ so that $\{\rho(t)\}_{t\in[\delta,T]}$ is tight, we have

$$\rho\in C([\delta,T]; L^1(\mathbb{R}^3)). \tag{B7}$$

Next, following the proof of Lemma 2.5 in Ref. 43, taking some non-negative test function $\chi\in C_c^\infty(\mathbb{R}^3)$ and integrating (B6) over $[\delta_1, t_1]$ where $\delta\le\delta_1\le t_1$, we find

$$\begin{aligned} \int_{\mathbb{R}^3}\beta(\rho^\varepsilon_{t_1})\chi dx + \int_{\delta_1}^{t_1}\int_{\mathbb{R}^3}\beta''(\rho^\varepsilon_s)|\nabla\rho^\varepsilon_s|^2\chi dxds &= \int_{\mathbb{R}^3}\beta(\rho^\varepsilon_{\delta_1})\chi dx \\ &+ \int_{\delta_1}^{t_1}\int_{\mathbb{R}^3}\{\beta(\rho^\varepsilon_s)\Delta\chi + (\beta'(\rho^\varepsilon_s)r^\varepsilon + \rho_s\beta(\rho^\varepsilon_s))\chi - \beta(\rho^\varepsilon_s)\nabla h\cdot\nabla\chi\}dxds. \end{aligned} \tag{B8}$$

We first consider a function $\beta \in C^2(0,\infty)$ that satisfies (i) convex, linear outside a compact set (i.e., β'' is continuous with compact support); (ii) for any $|u| \le L$, there is $C(L)$ such that $|\beta(u)| \le C(L)|u|$. Taking the limit $\varepsilon \to 0$ first and then $\chi_R(x) = \chi(\frac{x}{R})$ with $R \to \infty$, we obtain that

$$\int_{\mathbb{R}^3} \beta(\rho_{t_1})dx + \int_{\delta_1}^{t_1}\int_{\mathbb{R}^3} \beta''(\rho_s)|\nabla \rho_s|^2 dxds \le \int_{\mathbb{R}^3} \beta(\rho_{\delta_1})dx + \int_{\delta_1}^{t_1}\int_{\mathbb{R}^3} (-\beta'(\rho_s)\rho_s^2 + \rho_s\beta(\rho_s))^+ dxds. \tag{B9}$$

The left hand side is obtained by Fatou's lemma since $\beta''(s) \ge 0$. The convergence of the first term on the right can be obtained by decomposing $\beta(u) = (\beta(u) - A|s|) + A|s|$. $(\beta(u) - A|s|)$ is treated by the dominate convergence theorem, while $A|s|$ is treated by the L^1 convergence of mollification. Other terms are dealt with by the regularity and the convergence of $r^\varepsilon \to -\rho^2$ in $L^1(\delta, T; L^1_{loc}(\mathbb{R}^3))$.

For general convex function $\beta \in C^2(0,\infty)$ satisfying (i) $0 \le \beta(u) \le C(1 + u|\log(u)|)$, $u\beta(u) - u^2\beta'(u) \le C(1+u^2)$; (ii) for any $|u| \le L$, there is $C(L)$ such that $|\beta(u)| \le C(L)|u|$, we may choose a sequence of smooth convex functions β_R with linear growth at infinity to approximate β and obtain (B9). In fact, for such β, $\int_{\delta_1}^{t_1}\int_{\mathbb{R}^3}(-\beta'(\rho_s)\rho_s^2 + \rho_s\beta(\rho_s))^+ dxds$ is integrable by decomposing the integrals into domains for $\rho \le 1$ and $\rho \ge 1$.

Now, mimicking the Proof of Lemma 2.7 in Ref. 43, we take for $p \ge 2$

$$\beta(u) := \frac{u^p}{p}\mathbf{1}_{u\le K} + \left(\frac{K^{p-1}}{\log K}(u\log u - u) - \frac{1}{q}K^p + \frac{K^p}{\log K}\right)\mathbf{1}_{u\ge K},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then, β is convex, non-negative, and $\beta'(x) \ge 0$. Moreover, $u\beta(u) - u^2\beta'(u) \le 0$ if K is large enough. By plugging β into (B9) and sending $K \to \infty$, we find that

$$\frac{1}{p}\|\rho_{t_1}\|_p^p + \frac{4(p-1)}{p^2}\int_{\delta_1}^{t_1}\int_{\mathbb{R}^3}|\nabla\rho_s^{p/2}|^2 dxds \le \frac{1}{p}\|\rho_{\delta_1}\|_p^p.$$

Note that we know $\rho \in L^1(0,T;L^3(\mathbb{R}^3))$ and that $0 < \delta \le \delta_1$ are arbitrary, and we then find that

$$\rho \in L^\infty_{\text{loc}}((0,T);L^p(\mathbb{R}^3)),\ p \in (1,3].$$

Remark B.1. If we instead have $\rho \in L^2(0,T;L^2(\mathbb{R}^3))$, we will have $\rho \in L^\infty(0,T;L^p)$ for any $p \in [1,\infty)$.

Step 2 Weak–strong uniqueness.

For $t > 0$, we have $\frac{d}{dt}\|\rho\|_2^2 + 2\|\nabla\rho\|_2^2 \le 0$. Since we do not have $\rho \in L^\infty(0,T;L^{3/2})$, it is very hard to obtain the usual hypercontractivity,

$$\sup_{0<t\le T} t^{p-\frac{3}{2}}\|\rho\|_p^p \le C.$$

Using $\rho \in L^\infty(0,T;L^1(\mathbb{R}^d))$, we only have

$$\frac{d}{dt}\|\rho\|_2^2 + \|\rho\|_2^{10/3} \le 0,$$

which yields $\sup_{0<t\le T} t^{3/2}\|\rho\|_2^2 \le C$. This is not enough to prove the uniqueness as in the standard Keller-Segel equations (see Ref. 37). Hence, we must use other methods to prove the uniqueness. Here, we use the strategy in Ref. 12, where weak–strong uniqueness was shown by using the Coulomb energy. (If we know $\rho \in L^2(0,T;L^2)$, then $\rho \in L^\infty(0,T;L^p)$ for any $p \in [1,\infty)$, and the usual method will work.)

Using $\rho \in L^\infty_{\text{loc}}((0,T);L^p(\mathbb{R}^3))$, $p \in (1,3]$ and converting the semigroup equation for ρ into the semigroup form for ∇h, we see $\nabla h \in AC(t_0,T;L^2)$ for any $t_0 > 0$. Here, "AC" means absolutely continuous. It is then clear that

$$\partial_t \nabla h = \nabla g * \nabla(\rho\nabla h) + \Delta\nabla h. \tag{B10}$$

Consider the strong solution ρ_2. $\nabla h_2 \in C^\infty$ is clear,

$$\nabla h_2 = \nabla g * \rho_2 = \nabla g|_{|x|\le 1} * \rho_2 + \nabla g|_{|x|\ge 1} * \rho_2.$$

We have

$$|\nabla h_2| \le \|\nabla g|_{|x|\le 1}\|_1\|\rho_2\|_\infty + \|\nabla g_{|x|\ge 1}\|_\infty\|\rho_2\|_1.$$

Hence, ∇h_2 is bounded. The derivatives of ∇h_2 are clearly bounded. Moreover, by the Hardy-Littlewood-Sobolev inequality, $\nabla h_2 \in L^\infty(0,T;L^q(\mathbb{R}^3))$ for all $q > 3/2$. Hence, $\rho_2 \nabla h_2 \in L^\infty(L^p)$ for any $p \geq 1$. We also have the equation

$$\partial_t \nabla h_2 = \nabla g * \nabla(\rho_2 \nabla h_2) + \Delta \nabla h_2.$$

Note that the singular integral theory tells us that $\nabla g * \nabla(\rho_2 \nabla h_2) \in L^\infty(0,T;L^q)$, $q > 1$.

For $t > 0$, we have

$$\partial_t(\nabla h - \nabla h_2) = \nabla g * \nabla(\rho(\nabla h - \nabla h_2)) + \nabla g * \nabla((\rho - \rho_2)\nabla h_2) + \Delta(\nabla h - \nabla h_2).$$

For $t > t_0 > 0$, $\nabla h - \nabla h_2 \in L^\infty[t_0, T, L^6]$, and then, we can pair the equation with $\nabla h - \nabla h_2$.

Then,

$$\frac{1}{2}\partial_t \|\nabla h - \nabla h_2\|_2^2 = -\langle \nabla h - \nabla h_2, \rho(\nabla h - \nabla h_2)\rangle - \langle \nabla h - \nabla h_2, (\rho - \rho_2)\nabla h_2\rangle + \langle \nabla h - \nabla h_2, \Delta(\nabla h - \nabla h_2)\rangle =: I_1 + I_2 + I_3.$$

Here, we have used the fact

$$\langle \nabla\phi, \nabla g * \nabla \cdot v\rangle = -\langle g * \Delta\phi, \nabla \cdot v\rangle = \langle \nabla(g * \Delta\phi), v\rangle = -\langle \nabla\phi, v\rangle.$$

Clearly, $I_1 \leq 0$ and $I_3 \leq 0$,

$$\begin{aligned} I_2 &= -\int_{\mathbb{R}^3} \operatorname{div}(\nabla(h - h_2) \otimes \nabla(h - h_2) - \frac{1}{2}|\nabla(h - h_2)|^2 I)\nabla h_2 \, dx \\ &= \int_{\mathbb{R}^3} (\nabla(h - h_2) \otimes \nabla(h - h_2) - \frac{1}{2}|\nabla(h - h_2)|^2 I)\nabla^2 h_2 \, dx \leq C \int_{\mathbb{R}^3} |\nabla(h - h_2)|^2 \, dx. \end{aligned}$$

Note that it is exactly at this point we need ρ_2 to be the strong solution.

Using Grönwall, we have, for $t \in [t_0, T]$,

$$\|\nabla h(t) - \nabla h_2(t)\|_2^2 \leq C(T)\|\nabla h(t_0) - \nabla h_2(t_0)\|_2^2.$$

Finally,

$$\|\nabla h(t_0) - \nabla h_2(t_0)\|_2^2 = 2\mathscr{E}(t_0) - 2\int_{\mathbb{R}^3} \nabla h(t_0) \cdot \nabla h_2(t_0) \, dx + 2\mathscr{E}_2(t_0),$$

where

$$\mathscr{E}(t) := \frac{1}{2}\iint_{\mathbb{R}^3 \times \mathbb{R}^3} g(x - y)\rho(x,t)\rho(y,t) \, dxdy = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla h(x,t)|^2 \, dx,$$

and $\mathscr{E}_2$ is similarly defined for ρ_2 and h_2. The second term is equal to $2\int_{\mathbb{R}^3} h_2(t_0)\rho(t_0) \, dx$, which is continuous in t_0. $\mathscr{E}_2(t)$ is also continuous. Hence, we then have

$$\lim_{t_0 \to 0} \frac{1}{2}\|\nabla h(t_0) - \nabla h_2(t_0)\|_2^2 = \lim_{t_0 \to 0} \mathscr{E}(t) - 2\mathcal{E}(\rho_0) + \mathcal{E}(\rho_0) \leq \mathcal{E}(\rho_0) - 2\mathcal{E}(\rho_0) + \mathcal{E}(\rho_0)$$

by the condition. This means $\rho - \rho_2 = 0$ in $L^\infty(0,T;H^{-1})$. Since they are both in $L^\infty(0,T;L^1)$, they must be equal almost everywhere. □

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