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LONG TIME NUMERICAL SOLUTION OF THE NAVIER–STOKES EQUATIONS BASED ON A SEQUENTIAL REGULARIZATION FORMULATION*

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Abstract. The sequential regularization method is a reformulation of the unsteady Navier–Stokes equations from the viewpoint of constrained dynamical systems or the approximate Helmholtz–Hodge projection. In this paper we study the long time behavior of the sequential regularization formulation. We give a uniform-in-time estimate between the solution of the reformulated system and that of the Navier–Stokes equations. We also conduct an error analysis for the temporal discrete system and show that the error bound is independent of time. A couple of long time flow examples are computed to demonstrate this method.

Key words. Navier–Stokes equations, sequential regularization, iterative penalty method, long time solution, constrained dynamical system, approximate projection

AMS subject classifications. 65M12, 76D05

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1. Introduction.

Consider nonstationary Navier–Stokes equations with a non-slip boundary condition

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{f}, \tag{1.1}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{1.2}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \;\; \mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{1.3}$$

in a smooth bounded domain $\Omega \in \mathbb{R}^2$ over a long time. In estimating the error of numerical methods the Gronwall inequality is used in dealing with the nonlinear convection term and the pressure p term. Then a factor $\exp(Ct)$ $(C > 0)$ usually appears in the error bound. This is a difficulty in that usual analysis techniques do not work in studying long time approximations of the Navier–Stokes equations. Another computational difficulty of the Navier–Stokes equations is to maintain the incompressibility constraint $\mathrm{div}\mathbf{u} = 0$ over a long time.

Various techniques have been developed to overcome some of the above difficulties. The dissipative term $\nu\Delta\mathbf{u}$ is usually used to control the nonlinear convection term and the pressure term. Then the Reynolds number has to be restricted to be quite low in order to have enough dissipation. The projection method may be a widely used method in dealing with the incompressibility constraint. However, in the case of a no-slip boundary only normal velocity is enforced in the projection formulation.

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The tangential velocity is not enforced in the formulation and may cause a slip error in a long time simulation. Another method is to write the Navier–Stokes equations to a first-order system and then derive a new formulation based on the least squares minimization of its residual. However, quite a few new variables are introduced in the formulation which complicate the problem and increase the computational cost. Other formulations are available in dealing with the above difficulties, for example, a class of pseudocompressibility methods, which include the penalty method, the artificial compressibility method, etc. (cf. [13, 2, 7, 12, 18]). But the small parameter associated with these methods introduces extra stiffness into the system, and the incompressibility constraint is not guaranteed over a long time. In [1, 7, 12], a sequential regularization method (SRM) is introduced and analyzed to maintain the incompressibility constraint and to solve the unsteady Navier–Stokes equations. In this paper we would like to study the long time behavior of the method because, in this method, many of the above issues are resolved or improved. The incompressibility constraint is uniformly under control in time, while there is no extra stiffness introduced due to the approximate Hodge projection (to be explained in next section) or the built-in Baumgarte stabilization [1, 7] for the incompressibility constraint. There is no slip error since the no-slip boundary condition is fully enforced. The pressure term is eliminated/replaced by a damping term, and then there is no need to control it in the analysis.

The method or formulation reads: given an initial guess p_0 and a positive constant α, for $s = 1, 2, 3, \dots$, find the solution $(\mathbf{u}_s, p_s)$ of the following system:

$$(\mathbf{u}_s)_t + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s = \nu\Delta\mathbf{u}_s - \nabla p_s + \mathbf{f}, \tag{1.4}$$

$$(\mathrm{div}\mathbf{u}_s)_t + \mathrm{div}\alpha\mathbf{u}_s = \epsilon(p_{s-1} - p_s), \tag{1.5}$$

$$\mathbf{u}_s|_{\partial\Omega} = \mathbf{0}, \ \ \mathbf{u}_s|_{t=0} = \mathbf{u}_0. \tag{1.6}$$

The method is developed from the viewpoint of constrained dynamical systems (cf. [5, 7]). It is a combination of the penalty method (cf. [14, 15]) and the Baumgarte time stabilization (cf. [1, 7]) for the constraint. It may be seen as an extension of the augmented Lagrangian method or the iterative penalty method for unsteady Navier–Stokes equations (cf. [3]). It may also be seen as an approximate Helmholtz–Hodge projection, which will be explained later. Unlike the penalty method the parameter ϵ is not necessarily very small, and thus the reformulated system is more stable or less stiff (see [7, 12] or the convergence estimate (3.10) later). It approximates the incompressibility constraint better than the penalty method. Roughly speaking we expect from (1.5) and (3.10) that the divergence of the velocity $\mathbf{u}_s$ is of $O(\epsilon^s)$ and the bound may be independent of the time. From the regularity point of view, the method is a more natural formulation than the penalty formulation for the unsteady Navier–Stokes equations (see [12] for more details). This method also decouples the velocity and the pressure. We can eliminate p_s from the system, solve an equation only with unknown $\mathbf{u}_s$, and then recover p_s from (1.5). When we eliminate p_s from system (1.4)–(1.6), we obtain an equation only with the unknown $\mathbf{u}_s$. Let us omit the iteration index s and rewrite $\mathbf{f} - \nabla p_{s-1}$ as $\mathbf{f}$; the equation can be written in the following form:

$$\mathbf{u}_t - \frac{1}{\epsilon}\nabla\mathrm{div}(\mathbf{u}_t + \alpha\mathbf{u}) - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}, \tag{1.7}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \ \ \mathbf{u}|_{t=0} = \mathbf{a}. \tag{1.8}$$

In [12], we consider (1.4)–(1.6) in a finite time interval $[0, T]$ and obtain the existence and regularity of the solution as well as the error estimates of the SRM.

For some flow problems, we need to compute the solution over a very long time, for instance, the steady state solution of driven cavity flow. It is necessary to study the long time behavior of this formulation and extend the results in [12] over a long time.

This paper will be organized as follows. We will relate the SRM to an approximate Helmholtz–Hodge projection and give another explanation/understanding of the method in section 2. Some preliminaries such as notations and assumptions will be given there as well. In sections 3 and 4, we establish energy inequalities and error estimates for the continuous equations and the temporal discrete equations, respectively. A couple of long time flow examples will be computed in section 5.

2. Approximate Helmholtz–Hodge projection and the SRM. The SRM can be explained as an approximate Helmholtz–Hodge projection. Let $\mathcal{P} : L^2(\Omega, \mathbb{R}^N) \to (\nabla H^1(\Omega))^{\perp}$ denote the Helmholtz–Hodge projection onto vector fields that are divergence-free and have zero normal component on the boundary. In principle we may define $\mathcal{P}\mathbf{u} = \mathbf{u} - \nabla\phi$, where ϕ satisfies $\Delta\phi = \mathrm{div}\mathbf{u}$, $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{u}\cdot\mathbf{n}$, and $\int_\Omega \phi d\mathbf{x} = 0$. One may apply $\mathcal{P}$ to both sides of (1.1) and obtain

$$\mathbf{u}_t + \mathcal{P}(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}) = \nu\mathcal{P}\Delta\mathbf{u}. \tag{2.1}$$

The dissipation in (2.1) appears degenerate due to the fact that $\mathcal{P}$ annihilates gradients. So the analysis of (2.1) is usually restricted to a divergence-free space. However, for the stability of numerical computation, some additional damping effects to the divergence need to be added [9]. In this formulation, solutions formally satisfy $\partial_t(\mathrm{div}\mathbf{u}) = 0$. Consequently the divergence-free condition (1.2) needs to be imposed only on the initial data. However, when a numerical perturbation applies, weak instability to the divergence-free condition may occur. In addition, the projection $\mathcal{P}$ may not be easily implemented numerically. To overcome these difficulties with the projection, we can do the following.

We introduce an approximate/desingularized projection operator $\mathcal{P}_\epsilon$ by $\mathcal{P}_\epsilon\mathbf{u} := \mathbf{u} - \nabla\phi_\epsilon$, where ϕ_ϵ satisfies $\Delta\phi_\epsilon - \epsilon\phi_\epsilon = \mathrm{div}\mathbf{u}$ and $\frac{\partial\phi_\epsilon}{\partial\mathbf{n}} = \mathbf{u}\cdot\mathbf{n}$. We can then verify

$$\left(I - \frac{\nabla\mathrm{div}}{\epsilon}\right)\mathcal{P}_\epsilon\mathbf{u} = \mathbf{u}, \quad \mathcal{P}_\epsilon\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{2.2}$$

Thus we can denote $\mathcal{P}_\epsilon^{-1} = (I - \frac{\nabla\mathrm{div}}{\epsilon})$, which is a local operator asking no additional boundary conditions. Now, replacing $\mathcal{P}$ by $\mathcal{P}_\epsilon$ and taking the inverse operator yields an approximate Navier–Stokes equation:

$$\mathbf{u}_t + \mathcal{P}_\epsilon(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}) = \nu\mathcal{P}_\epsilon\Delta\mathbf{u}$$

or

$$\left(I - \frac{\nabla\mathrm{div}}{\epsilon}\right)\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}.$$

It is then the SRM (1.7) if adding a damping term $-\frac{\alpha}{\epsilon}\nabla\mathrm{div}\mathbf{u}$ to the left-hand side of above equation (accordingly to adding $-\mathcal{P}\frac{\alpha}{\epsilon}\nabla\mathrm{div}\mathbf{u} = 0$ to (2.1)). It is (1.7), which will be discussed in this paper. From (1.5) the divergence of $\mathbf{u}$ will not drift too much away from zero as the time goes. Therefore, this formulation has better stability.

The divergence of the projected solution is uniformly controlled due to the following identity (obtained from (2.2)):

$$\|\mathcal{P}_\epsilon\mathbf{u}\|^2 + \frac{2}{\epsilon}\|\mathrm{div}\mathcal{P}_\epsilon\mathbf{u}\|^2 + \frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathcal{P}_\epsilon\mathbf{u}\|^2 = \|\mathbf{u}\|^2.$$

Here and in what follows we used notations $\|\cdot\|$ and $\|\cdot\|_m$ as the norms for L^2 and H^m, respectively. Unlike the penalty method, there is no additional time step restriction. From the definition of $\mathcal{P}$ and $\mathcal{P}_\epsilon$ earlier, we can have

$$\|(\mathcal{P}-\mathcal{P}_\epsilon)\mathbf{u}\| \le C\epsilon\|\mathbf{u}\|,$$

which justifies the approximation to the Navier–Stokes equation as well. The approximate Helmholtz projection can also be viewed as a regularization to the Helmholtz projection. The Fourier symbol for $\mathcal{P}$ is represented by $\hat{\mathcal{P}} = I - \frac{\vec{\xi}\vec{\xi}^\top}{\|\vec{\xi}\|^2}$, while the symbol for $\mathcal{P}_\epsilon$ is a desingularized one:

$$\hat{\mathcal{P}}_\epsilon = I - \frac{\vec{\xi}\vec{\xi}^\top}{\|\vec{\xi}\|^2+\epsilon}.$$

Simple computation shows that the symbol of its inverse is given by

$$\hat{\mathcal{P}}_\epsilon^{-1} = I + \frac{\vec{\xi}\vec{\xi}^\top}{\epsilon},$$

which again agrees with $\mathcal{P}_\epsilon^{-1} = I - \frac{\nabla\mathrm{div}}{\epsilon}$.

It is known that the solution of the Navier–Stokes equation is regular up to an arbitrary time interval in a two-dimensional domain (cf. [16, 17]). More precisely, consider two conditions:

$$\|\mathbf{u}_0\|_1 \le N_1, \quad \int_0^\infty \|\mathbf{f}(\cdot,t)\|^2 dt \le N_1, \quad \mathrm{div}\mathbf{u}_0 = 0, \tag{2.3}$$

$$\|\mathbf{u}_0\|_2 \le N_2, \quad \sup_{0<t<\infty} \|\mathbf{f}(\cdot,t)\| \le N_2, \quad \int_0^\infty \|\mathbf{f}_t\|^2 dt \le N_2. \tag{2.4}$$

If condition (2.3) holds, then there exists a constant M_1 which depends on N_1 such that

$$\sup_{0<t<\infty} \|\mathbf{u}\|_1^2 + \int_0^\infty \left(\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 + \|\mathbf{u}_t\|^2\right) dt \le M_1. \tag{2.5}$$

If conditions (2.3) and (2.4) hold, then there exists a constant M_2 which depends on N_1 and N_2 such that

$$\sup_{0<t<\infty} (\|\mathbf{u}\|_2 + \|\mathbf{u}_t\| + \|\nabla p\|) \le M_2. \tag{2.6}$$

For the sequential regularized reformulation, we have similar results for (1.7)–(1.8); i.e., if we assume that (2.3) holds, then there exists a constant M_1 such that

$$\sup_{0<t<\infty} \|\mathbf{u}\|_1^2 + \int_0^\infty \left(\|\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}\|^2 + \|\mathbf{u}\|_2^2\right) ds \le M_1. \tag{2.7}$$

If (2.3) and (2.4) hold, then there exists a constant M_2 such that

$$\sup_{0<t<\infty} \left(\frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}\|^2 + \|\mathbf{u}\|_2^2 + \|\mathbf{u}_t\|^2\right) \le M_2. \tag{2.8}$$

Since the pressure can be recovered by $\mathrm{div}\mathbf{u}_t + \alpha\mathrm{div}\mathbf{u}$, the same bound for pressure follows. The key part in the proof is Lemma 3.2, which is similar to Lemma 4.1 in [12] but holds uniformly in time. Based on these results, we can extend the error estimation in [12] to any time interval. Therefore, this sequential regularization formulation is suitable for long time computations. The numerical results in the last section will also demonstrate this claim.

Next we introduce a few operators for the treatment of the nonlinear convection term. Let

$$B(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v}, \qquad \bar{B}(\mathbf{u},\mathbf{v}) = B(\mathbf{u},\mathbf{v}) + \frac{1}{2}(\mathrm{div}\mathbf{u})\mathbf{v},$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = (B(\mathbf{u},\mathbf{v}),\mathbf{w}), \quad \bar{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = (\bar{B}(\mathbf{u},\mathbf{v}),\mathbf{w}).$$

One can easily check

$$\bar{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\{b(\mathbf{u},\mathbf{v},\mathbf{w}) - b(\mathbf{u},\mathbf{w},\mathbf{v})\} \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\Omega).$$

Therefore,

$$\bar{b}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

For the trilinear form b (or $\bar{b}$), we can prove the following inequality by combination of integration of parts, the Hölder inequality, and the Sobolev inequality (see, for instance, [17])

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C \begin{cases} \|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \\ \|\mathbf{u}\|_2\|\mathbf{v}\|_1\|\mathbf{w}\| \\ \|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|. \end{cases} \tag{2.9}$$

Moreover, if $\Omega \in R^2$, we have the following:

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{u}\|_1^{\frac{1}{2}}\|\mathbf{v}\|_1^{\frac{1}{2}}\|\mathbf{v}\|_2^{\frac{1}{2}}\|\mathbf{w}\| \quad \forall \mathbf{v} \in \mathbf{H}^2 \cap \mathbf{H}_0^1, \mathbf{u} \in \mathbf{H}_0^1, \mathbf{w} \in \mathbf{L}^2, \tag{2.10}$$

where the trilinear form b can be replaced by $\bar{b}$. In what follows we will replace B by $\bar{B}$. Hence (1.7)–(1.8) become

$$\mathbf{u}_t - \frac{1}{\epsilon}\nabla\mathrm{div}(\mathbf{u}_t + \alpha\mathbf{u}) - \nu\Delta\mathbf{u} + \bar{B}(\mathbf{u},\mathbf{u}) = \mathbf{f}, \tag{2.11}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{2.12}$$

We remark here that we can also replace B by $\omega \times \mathbf{u}$ based on the identity $(\mathbf{u}\cdot\nabla)\mathbf{u} = \omega \times \mathbf{u} + \nabla\frac{|\mathbf{u}|^2}{2}$, where ω is the vorticity $\nabla \times \mathbf{u}$ and the second term on the right-hand side may be absorbed into the pressure (see [10, 19]). Our analysis may be carried out for this form as well.

3. A priori estimations. We first assume that the solution of (2.11)–(2.12) exists and then establish an a priori estimation. We will give a basic energy estimation $\sup_{0<t<\infty}\|\mathbf{u}\|^2 + \int_0^\infty \|\mathbf{u}\|_1^2 dt$ and a strong solution estimation $\sup_{0<t<\infty}\|\mathbf{u}\|_1^2 + \int_0^\infty \|\mathbf{u}\|_2^2 dt$ in Theorem 3.3 and then obtain a higher-order regularity estimation $\sup_{0<t<\infty}\|\mathbf{u}\|_2^2$ in Theorem 3.4.

Since we consider an arbitrary long time interval $[0,T]$, we need to improve the solution estimates in [12] such that the choice of ϵ and the bound of solution do not depend on the time length T. The following two Lemmas 3.1 and 3.2 play an essential role in obtaining a time-independent estimation.

LEMMA 3.1. *Let X be a Banach space, let $f : (0,\infty) \to X$ be a vector value function which satisfies $\int_0^\infty \|f(t)\|^2 dt < +\infty$, and let α be a positive constant; then we have*

$$\int_0^T \left[\int_0^t \|f(s)\| e^{\alpha(s-t)} ds\right]^2 dt \le \frac{1}{\alpha^2}\int_0^T \|f(s)\|^2 ds, \tag{3.1}$$

$$\sup_{0<t<T} \left[\int_0^t \|f(s)\| e^{\alpha(s-t)} ds\right]^2 \le \frac{1}{\alpha^2}\sup_{0<t<T} \|f(t)\|^2. \tag{3.2}$$

Proof. We apply the Cauchy–Schwarz inequality and change the integration order to obtain

$$\begin{aligned}
\int_0^T \left[\int_0^t \|f(s)\| e^{\alpha(s-t)} ds\right]^2 dt &= \int_0^T e^{-2\alpha t}\left[\int_0^t \|f(s)\| e^{\alpha s/2} e^{\alpha s/2} ds\right]^2 dt \\
&\le \int_0^T e^{-2\alpha t}\left[\int_0^t \|f(s)\|^2 e^{\alpha s} ds\right]\left[\int_0^t e^{\alpha s} ds\right] dt \\
&\le \frac{1}{\alpha}\int_0^T e^{-\alpha t}\int_0^t \|f(s)\|^2 e^{\alpha s} ds dt \\
&= \frac{1}{\alpha}\int_0^T \|f(s)\|^2 e^{\alpha s}\int_s^T e^{-\alpha t} dt ds \\
&\le \frac{1}{\alpha^2}\int_0^T \|f(s)\|^2 ds
\end{aligned}$$

and

$$\begin{aligned}
\sup_{0<t<T} \left[\int_0^t \|f(s)\| e^{\alpha(s-t)} ds\right]^2 &\le \sup_{0<t<T} e^{-2\alpha t}\left[\int_0^t \|f(s)\|^2 e^{\alpha s} ds\right]\left[\int_0^t e^{\alpha s} ds\right] \\
&\le \sup_{0<t<T} \frac{1}{\alpha} e^{-\alpha t}\int_0^t \|f(s)\|^2 e^{\alpha s} ds \\
&\le \sup_{0<t<T} \frac{1}{\alpha} e^{-\alpha t}\int_0^t \left(\sup_{0<s<t} \|f(s)\|^2\right) e^{\alpha s} ds \\
&\le \sup_{0<t<T} \frac{1}{\alpha^2}\sup_{0<s<t} \|f(s)\|^2 \\
&= \frac{1}{\alpha^2}\sup_{0<t<T} \|f(t)\|^2. \qquad \square
\end{aligned}$$

LEMMA 3.2. *Define an operator $A\mathbf{u} = -\frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_t + \alpha\mathbf{u}) - \nu\Delta\mathbf{u}$, where the initial data $\mathbf{u}_0$ is divergence-free. Then there exist a ϵ_0 and a constant C, only depending on the domain Ω, α, and ν, such that $\forall \epsilon \le \epsilon_0$, we have*

$$\int_0^t \left(\frac{1}{\epsilon^2}\|\nabla \mathrm{div}\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla \mathrm{div}\mathbf{u}\|^2 + \|\Delta\mathbf{u}\|^2\right) dt \leqq C\int_0^t \|A\mathbf{u}\|^2 dt, \tag{3.3}$$

$$\sup_{0<t<T} \left(\frac{1}{\epsilon^2}\|\nabla \mathrm{div}\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla \mathrm{div}\mathbf{u}\|^2 + \|\Delta\mathbf{u}\|^2\right) \le C\sup_{0<t<T} \|A\mathbf{u}\|^2. \tag{3.4}$$

Proof. Define $\mathbf{w} = A\mathbf{u}$, $p = -\frac{1}{\epsilon}\mathrm{div}(\mathbf{u}_t + \alpha\mathbf{u})$, and $g = \mathrm{div}\mathbf{u}$. First we solve g from the ODE

$$g_t + \alpha g = -\epsilon p$$

with initial condition $g(0) = \text{div}\mathbf{u}_0 = 0$. The solution is

$$g(t) = -\epsilon \int_0^t p(s) e^{\alpha(s-t)} ds.$$

Let $(\mathbf{u}, p)$ satisfy the nonhomogeneous Stokes equation

$$\begin{aligned} -\nu \triangle \mathbf{u} + \nabla p &= \mathbf{w}, \\ \text{div}\mathbf{u} &= g. \end{aligned}$$

Then using the estimation for the nonhomogeneous Stokes equation (cf. [16]), we have

$$\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 \le C_0(\|\mathbf{w}\|^2 + \|\nabla g\|^2), \tag{3.5}$$

where the constant C_0 only depends on the domain Ω and ν. We now work out a few inequalities between g and p by using Lemma 3.1:

$$\int_0^T \|\nabla g\|^2 \le \epsilon^2 \int_0^T \left[\int_0^t \|\nabla p(s) e^{\alpha(s-t)}\| ds \right]^2 dt \le \frac{\epsilon^2}{\alpha^2} \int_0^T \|\nabla p\|^2 dt, \tag{3.6}$$

$$\sup_{0<t<T} \|\nabla g(t)\|^2 \le \epsilon^2 \sup_{0<t<T} \left[\int_0^t \|\nabla p\| e^{\alpha(s-t)} ds \right]^2 \le \frac{\epsilon^2}{\alpha^2} \sup_{0<t<T} \|\nabla p(t)\|^2. \tag{3.7}$$

Now we are ready to prove the results of this lemma. Take the integration from 0 to T for the inequality (3.5), and we have

$$\begin{aligned} \int_0^T \left(\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 \right) dt &\le C_0 \int_0^T \left(\|\mathbf{w}\|^2 + \|\nabla g\|^2 \right) dt \\ &\le C_0 \int_0^T \|\mathbf{w}\|^2 dt + C_0 \frac{\epsilon^2}{\alpha^2} \int_0^T \|\nabla p\|^2 dt. \end{aligned}$$

Take the sup for both sides of the inequality (3.5), and we have

$$\begin{aligned} \sup_{0<t<T} \left(\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 \right) &\le C_0 \sup_{0<t<T} \left(\|\mathbf{w}\|^2 + \|\nabla g\|^2 \right) dt \\ &\le C_0 \sup_{0<t<T} \|\mathbf{w}\|^2 + C_0 \frac{\epsilon^2}{\alpha^2} \sup_{0<t<T} \|\nabla p\|^2. \end{aligned}$$

Define $\epsilon_0 = \frac{\alpha}{\sqrt{2C_0}}$, and then for all $\epsilon \le \epsilon_0$, we have

$$\begin{aligned} \int_0^T \left(\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 \right) dt &\le 2C_0 \int_0^T \|\mathbf{w}\|^2 dt, \\ \sup_{0<t<T} \left(\|\mathbf{u}\|_2^2 + \|\nabla p\|^2 \right) &\le 2C_0 \sup_{0<t<T} \|\mathbf{w}\|^2. \end{aligned}$$

Then we notice that

$$\begin{aligned} &\int_0^T \frac{1}{\epsilon^2} \|\nabla \text{div}\mathbf{u}\|^2 ds = \int_0^T \frac{1}{\epsilon^2} \|\nabla g\|^2 ds \le \frac{1}{\alpha^2} \int_0^T \|\nabla p\|^2 ds, \\ &\int_0^T \frac{1}{\epsilon^2} \|\nabla \text{div}\mathbf{u}_t\|^2 ds = \int_0^T \frac{1}{\epsilon^2} \|\nabla g_t\|^2 ds = \int_0^T \frac{1}{\epsilon^2} \|\epsilon \nabla p + \alpha \nabla g\|^2 ds \le 4 \int_0^T \|\nabla p\|^2 ds, \\ &\sup_{0<t<T} \frac{1}{\epsilon^2} \|\nabla \text{div}\mathbf{u}\|^2 = \sup_{0<t<T} \frac{1}{\epsilon^2} \|\nabla g\|^2 \le \frac{1}{\alpha^2} \sup_{0<t<T} \|\nabla p\|^2, \\ &\sup_{0<t<T} \frac{1}{\epsilon^2} \|\nabla \text{div}\mathbf{u}_t\|^2 = \sup_{0<t<T} \frac{1}{\epsilon^2} \|\nabla g_t\|^2 \le \sup_{0<t<T} \frac{1}{\epsilon^2} \|\epsilon \nabla p + \alpha \nabla g\|^2 \le 4 \sup_{0<t<T} \|\nabla p\|^2. \end{aligned}$$

Combining the above inequalities together, we can easily conclude the lemma. ∎

Remark 3.1. Comparing with a similar lemma in [12], the improvement here is that the choice of ϵ and the constant of the bound do not depend on the time interval T, which implies that we may fix ϵ in advance and then compute the solution to any arbitrarily large time.

Next we will consider the modified SRMs (2.11)–(2.12).

THEOREM 3.3. *For* (2.11)–(2.12), *we assume* (2.3). *Then there exist two constants M_1 and M_2, which only depend on the domain and N_1, such that*

$$\sup_{0<t<\infty}\left(\|\mathbf{u}\|^2+\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2\right)+\int_0^\infty\left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2+\|\mathbf{u}\|_1^2\right)ds\le M_1,$$
$$\sup_{0<t<\infty}\|\mathbf{u}\|_1^2+\int_0^\infty\left(\|\mathbf{u}_t\|^2+\frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}_t\|^2+\frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}\|^2+\|\mathbf{u}\|_2^2\right)ds\le M_2.$$

Proof. Multiplying $\mathbf{u}$ for both sides of (2.11), we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\frac{1}{2\epsilon}\frac{d}{dt}\|\mathrm{div}\mathbf{u}\|^2+\frac{\alpha}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2+\nu\|\nabla\mathbf{u}\|^2=(\mathbf{f},\mathbf{u}).$$

Using the Young inequality and integrating for both sides, we can obtain

$$\|\mathbf{u}\|^2+\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2+\int_0^t\left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2+\|\mathbf{u}\|_1^2\right)ds\le C_1\left(\|\mathbf{u}_0\|^2+\int_0^t\|\mathbf{f}\|_{-1}^2ds\right),$$

where the constant C_1 only depends on the domain Ω, α, and ν. Since the above inequality holds true for any time t and C_1 does not depend on t, it is indeed a uniform-in-time estimation. We denote $M_1=C_1(\|\mathbf{u}_0\|^2+\int_0^\infty\|\mathbf{f}\|_{-1}^2ds)$ and simplify the inequality

$$\sup_{0<t<\infty}\left(\|\mathbf{u}\|^2+\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2\right)+\int_0^\infty\left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}\|^2+\|\mathbf{u}\|_1^2\right)ds\le M_1. \tag{3.8}$$

Then let us fix a time interval $(0,T)$, where T is arbitrarily chosen. Recall the definition of operator A in Lemma 3.2. We multiply $A\mathbf{u}$ on both sides of (2.11),

$$\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2+\frac{\alpha}{2\epsilon}\frac{d}{dt}\|\mathrm{div}\mathbf{u}\|^2+\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2+\|A\mathbf{u}\|^2+\bar{b}(\mathbf{u},\mathbf{u},A\mathbf{u})=(\mathbf{f},A\mathbf{u}).$$

Since we consider the 2D case, we have

$$\bar{b}(\mathbf{u},\mathbf{u},A\mathbf{u})\le C\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_1\|\mathbf{u}\|_2^{1/2}\|A\mathbf{u}\|\le C_\delta\|\mathbf{u}\|^2\|\mathbf{u}\|_1^4+0.5\left(\|A\mathbf{u}\|^2+\delta\|\mathbf{u}\|_2^2\right).$$

Define $y(t)=\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}(t)\|^2+\|\nabla\mathbf{u}(t)\|^2$ and $k(t)=C_\delta\|\mathbf{u}\|^2\|\mathbf{u}\|_1^2$, and obtain

$$\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2+\frac{dy}{dt}+\|A\mathbf{u}\|^2\le 4\|\mathbf{f}\|^2+\delta\|\mathbf{u}\|_2^2+k(t)y(t).$$

Then we can obtain by the technique of the Gronwall lemma (letting $K(t)=\int_0^t k(s)ds$)

$$e^{-K(t)}\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2+\frac{de^{-K(t)}y}{dt}+e^{-K(t)}\left(\|A\mathbf{u}\|^2-\delta\|\mathbf{u}\|_2^2\right)\le 4e^{-K(t)}\|\mathbf{f}\|^2,$$
$$y(T)+\int_0^T e^{K(T)-K(t)}\left(\|A\mathbf{u}\|^2-\delta\|\mathbf{u}\|_2^2\right)dt\le e^{K(T)}\left(4\int_0^T\|\mathbf{f}\|^2dt+y(0)\right).$$

Then noticing that $e^{K(T)-K(t)}$ is monotonically decreasing, choosing δ to be a constant such that $\int_0^t \|A\mathbf{u}\|^2 ds \geq \int_0^t 2\delta\|\mathbf{u}\|_2^2 ds$ is true for any t (we can choose $\delta = \frac{1}{2C}$, where C is the constant in Lemma 3.2), defining $G(t) = \int_0^t (\|A\mathbf{u}\|^2 - \delta\|\mathbf{u}\|_2^2)ds$, and doing integration by parts, we have

$$\begin{aligned}\int_0^T e^{K(T)-K(t)} \left(\|A\mathbf{u}\|^2 - \delta\|\mathbf{u}\|_2^2\right) dt &= e^{K(T)-K(t)} G(t)|_0^T + \int_0^T G(t) e^{K(T)-K(t)} K'(t) dt \\ &\geq \int_0^T \left(\|A\mathbf{u}\|^2 - \delta\|\mathbf{u}\|_2^2\right) dt \\ &\geq 0.5 \int_0^T \|A\mathbf{u}\|^2 dt.\end{aligned}$$

Let $M_2 = e^{C_2 M_1^2}(\int_0^\infty \|f(t)\|^2 dt + \|\mathbf{u}(0)\|_1^2)$, where C_2 only depends on the choice of δ. Hence

$$\frac{1}{\epsilon}\|\text{div}\mathbf{u}\|^2 + \|\mathbf{u}\|_1^2 + \int_0^T \|A\mathbf{u}\|^2 dt \leq M_2. \tag{3.9}$$

Since the choice of δ is uniform for $t \in (0, \infty)$ and due to (3.8), the above inequality is uniform-in-time as well. Then we multiply $\mathbf{u}_t$ on both sides of (2.11) and obtain

$$\|\mathbf{u}_t\|^2 + (A\mathbf{u}, \mathbf{u}_t) + \bar{b}(\mathbf{u}, \mathbf{u}, \mathbf{u}_t) = (\mathbf{f}, \mathbf{u}_t).$$

By applying the Young inequality, Lemma 3.2, and inequality (3.9), we can prove this theorem. ∎

The error between the solutions of the sequential regularization formulation and the Navier–Stokes equations can be estimated uniformly in time by using Theorem 3.3 and following the technique in [7, 12]. We simply state the result here:

$$\begin{aligned}\sup_{0<t<\infty} \|\mathbf{u} - \mathbf{u}_s\|_1^2 + \int_0^\infty (\|\mathbf{u}_t - (\mathbf{u}_s)_t\|^2 + \|\mathbf{u} - \mathbf{u}_s\|_2^2 + \|\nabla(p - p_s)\|^2) dt \\ \leq (M\epsilon)^{2s} \int_0^\infty \|\nabla(p - p_0)\|^2 dt,\end{aligned} \tag{3.10}$$

where M is a constant which depends on Ω, ν, α, $\mathbf{u}_0$, $\mathbf{f}$, and p_0. The proof is exactly the same as Theorem 2.5 in [12], by noticing that all of the constants in the proof are uniform in time.

THEOREM 3.4. *For* (2.11)–(2.12), *we assume* (2.3)–(2.4). *Then there exists a constant M_3, which depends on N_1 and N_2, such that*

$$\sup_{0<t<\infty} \left(\frac{1}{\epsilon^2}\|\nabla\text{div}\mathbf{u}_t\|^2 + \frac{1}{\epsilon^2}\|\nabla\text{div}\mathbf{u}\|^2 + \|\mathbf{u}\|_2^2 + \|\mathbf{u}_t\|^2\right) \leq M_3.$$

Proof. Multiplying $\mathbf{u}_t$ on both sides of (2.11), we have

$$\|\mathbf{u}_t\|^2 + \frac{1}{\epsilon}\|\text{div}\mathbf{u}_t\|^2 - \left(\frac{\alpha}{\epsilon}\nabla\text{div}\mathbf{u}, \mathbf{u}_t\right) - (\nu\Delta\mathbf{u}, \mathbf{u}_t) + \bar{b}(\mathbf{u}, \mathbf{u}, \mathbf{u}_t) = (\mathbf{f}, \mathbf{u}_t). \tag{3.11}$$

Choosing the time $t = 0$, applying the Young inequality, and noting that $\text{div}\mathbf{u}(0) = 0$, $\mathbf{u}(0) \in \mathbf{H}^2$, and $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, we have

$$\|\mathbf{u}_t(0)\|^2 + \frac{1}{\epsilon}\|\text{div}\mathbf{u}_t(0)\|^2 \leq C_1. \tag{3.12}$$

Then differentiating (2.11) with respect to time t, we have

$$\mathbf{u}_{tt} - \frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_{tt} + \alpha \mathbf{u}_t) - \nu \Delta \mathbf{u}_t + \bar{B}(\mathbf{u}, \mathbf{u}_t) + \bar{B}(\mathbf{u}_t, \mathbf{u}) = \mathbf{f}_t. \tag{3.13}$$

Multiplying $\mathbf{u}_t$ on both sides of (3.13) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \frac{1}{2\epsilon}\frac{d}{dt}\|\mathrm{div}\mathbf{u}_t\|^2 + \frac{\alpha}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2 + \nu\|\nabla \mathbf{u}_t\|^2 + \bar{b}(\mathbf{u}_t, \mathbf{u}, \mathbf{u}_t) + \bar{b}(\mathbf{u}, \mathbf{u}_t, \mathbf{u}_t) = (\mathbf{f}_t, \mathbf{u}_t).$$

Using

$$\begin{aligned} \left|\bar{b}(\mathbf{u}_t, \mathbf{u}, \mathbf{u}_t)\right| &\le C_2\|\mathbf{u}\|_2^2\|\mathbf{u}_t\|^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_t\|^2, \\ \bar{b}(\mathbf{u}, \mathbf{u}_t, \mathbf{u}_t) &= 0, \\ (\mathbf{f}_t, \mathbf{u}_t) &\le \frac{1}{\nu}\|\mathbf{f}_t\|_{-1}^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_t\|^2, \end{aligned}$$

and defining $y(t) = \|\mathbf{u}_t\|^2 + \frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2$, we have

$$\frac{dy}{dt} + \frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2 + \|\nabla \mathbf{u}_t\|^2 \le C_3\left(\|\mathbf{u}\|_2^2 y(t) + \|\mathbf{f}_t\|_{-1}^2\right).$$

Applying the Gronwall inequality, we obtain

$$y(t) + \int_0^t \left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2 + \|\mathbf{u}_t\|_1^2\right) dt \le C_4\left(\int_0^t \|\mathbf{f}_t\|_{-1}^2 dt + y(0)\right).$$

Using inequality (3.12) to control the right-hand side of the above inequality, we have the estimation for $\|\mathbf{u}_t\|^2$, i.e.,

$$\|\mathbf{u}_t\|^2 + \frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2 + \int_0^t \left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}_t\|^2 + \|\mathbf{u}_t\|_1^2\right) dt \le C_4 \quad \forall\, t \in (0, \infty). \tag{3.14}$$

Then multiplying $A\mathbf{u}$ on both sides of (2.11), we have

$$(\mathbf{u}_t, A\mathbf{u}) + \|A\mathbf{u}\|^2 + \bar{b}(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = (\mathbf{f}, A\mathbf{u}). \tag{3.15}$$

Since

$$\begin{aligned} |(\mathbf{u}_t, A\mathbf{u})| &\le \|\,\mathbf{u}_t\|^2 + \frac{1}{4}\|A\mathbf{u}\|^2, \\ (\mathbf{f}, A\mathbf{u}) &\le \|\mathbf{f}\|^2 + \frac{1}{4}\|A\mathbf{u}\|^2, \\ |b(\mathbf{u}, \mathbf{u}, A\mathbf{u})| &\le C_\delta\|\mathbf{u}\|_1^6 + \delta\|\mathbf{u}\|_2^2 + \frac{1}{4}\|A\mathbf{u}\|^2, \end{aligned}$$

we have

$$\frac{1}{4}\|A\mathbf{u}\|^2 \le \|\mathbf{u}_t\|^2 + C_\delta\|\mathbf{u}\|_1^6 + \delta\|\mathbf{u}\|_2^2 + \|\mathbf{f}\|^2.$$

We can choose $\delta = \frac{1}{2C}$, where C is the constant in Lemma 3.2, and then take the sup for both sides of the above inequality. Using Lemma 3.2 and noticing that $\|\mathbf{u}\|_1$, $\|\mathbf{f}\|$, and $\|\mathbf{u}_t\|$ are bounded, we have

$$\sup_{0<t<\infty}\|A\mathbf{u}\| \le M_3, \tag{3.16}$$

where the constant M_3 is uniform-in-time. Finally applying Lemma 3.2, we obtain the estimation of each term in the lemma. ∎

4. Time discretization. Let us consider the time discretization of (2.11) by the semi-implicit scheme:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} - \frac{1}{\epsilon}\nabla \operatorname{div}\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} + \alpha \mathbf{u}^{n+1}\right)$$

$$-\nu \triangle \mathbf{u}^{n+1} + \bar{B}\left(\mathbf{u}^n, \mathbf{u}^{n+1}\right) = \mathbf{f}^{n+1}, \tag{4.1}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{u}^0 = \mathbf{u}_0, \tag{4.2}$$

where $t_n = n\triangle t$ and $\mathbf{f}^n = \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}} \mathbf{f}(\tau)d\tau$.

LEMMA 4.1. *Let a_n and b_n be two sequences in a Banach space which satisfy*

$$(1+\alpha h)a_n - a_{n-1} = hb_n, \quad a_0 = 0,$$

where α and h are two positive constants. We have the inequalities

$$\sum_{n=1}^{N} \|a_n\|^2 \le \frac{1}{\alpha^2}\sum_{n=1}^{N} \|b_n\|^2,$$

$$\sup_{1\le n\le N} \|a_n\|^2 \le \frac{1}{\alpha^2} \sup_{1\le n\le N} \|b_n\|^2$$

for all integers $N > 0$.

Proof. First we express a_n in terms of b_i by induction:

$$a_n = \frac{h}{1+\alpha h} b_n + \frac{1}{1+\alpha h} a_{n-1} = \sum_{i=1}^{n} \frac{hb_i}{(1+\alpha h)^{n+1-i}}.$$

By the Holder inequality

$$\begin{aligned}\|a_n\|^2 &= \frac{h^2}{(1+\alpha h)^{2n+2}} \left\|\sum_{i=1}^{n} b_i (1+\alpha h)^i\right\|^2 \\ &\le \frac{h^2}{(1+\alpha h)^{2n+2}} \sum_{i=1}^{n} \|b_i\|^2 (1+\alpha h)^i \sum_{i=1}^{n} (1+\alpha h)^i \\ &\le \frac{h}{\alpha(1+\alpha h)^{n+1}} \sum_{i=1}^{n} \|b_i\|^2 (1+\alpha h)^i.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=1}^{N} \|a_n\|^2 &\le \sum_{n=1}^{N}\sum_{i=1}^{n} \frac{h}{\alpha(1+\alpha h)^{n+1}} \|b_i\|^2 (1+\alpha h)^i \\ &= \sum_{i=1}^{N}\sum_{n=i}^{N} \frac{h}{\alpha(1+\alpha h)^{n+1}} \|b_i\|^2 (1+\alpha h)^i \\ &\le \sum_{i=1}^{N} \frac{1}{\alpha}\|b_i\|^2 \sum_{n=1}^{\infty} \left(\frac{1}{1+\alpha h}\right)^n \\ &= \frac{1}{\alpha^2}\sum_{i=1}^{N} \|b_i\|^2,\end{aligned}$$

and

$$\begin{aligned}\|a_n\|^2 &\le \frac{h}{\alpha(1+\alpha h)^{n+1}} \sum_{i=1}^{n} (1+\alpha h)^i \sup_{1\le i\le n} \|b_i\|^2 \\ &\le \frac{1}{\alpha^2} \sup_{1\le i\le N} \|b_i\|^2.\end{aligned}$$

Then this lemma follows by taking the sup for both sides of the inequality. ∎

LEMMA 4.2. *Assume that the time step is* $\triangle t$. *Let* N *be any positive integer. Define* $A\mathbf{u}^{n+1} = -\frac{1}{\epsilon}\nabla\text{div}(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} + \alpha\mathbf{u}^{n+1}) - \nu\triangle\mathbf{u}^{n+1}$, $n = 1, 2, 3, \ldots$, *and* $\mathbf{u}^0 = \mathbf{u}(0)$. *Then there exist a constant* ϵ_0, *which depends on* Ω, α *and* ν, *such that, when* $\epsilon \le \epsilon_0$, *we have*

$$\triangle t \sum_{n=1}^{N} \left(\|\mathbf{u}^n\|_2^2 + \frac{1}{\epsilon^2}\|\nabla\text{div}\mathbf{u}^n\|^2 + \frac{1}{\epsilon^2}\left\|\nabla\text{div}\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\triangle t}\right\|^2 \right) \le C\triangle t \sum_{n=1}^{N} \|A\mathbf{u}^n\|^2,$$

$$\sup_{1\le n\le N} \left(\|\mathbf{u}^n\|_2 + \frac{1}{\epsilon}\|\nabla\text{div}\mathbf{u}^n\| + \frac{1}{\epsilon}\left\|\nabla\text{div}\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\triangle t}\right\| \right) \le C \sup_{1\le n\le N} \|A\mathbf{u}^n\|,$$

where the constant C *does not depend on the choice of* ϵ *and* N.

Proof. Define $\mathbf{w}^n = A\mathbf{u}^n$, $p^n = -\frac{1}{\epsilon}\text{div}(\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\triangle t} + \alpha\mathbf{u}^n)$, and $g^n = \text{div}\mathbf{u}^n$; then

$$\begin{aligned}-\nu\triangle\mathbf{u}^n + \nabla p^n &= \mathbf{w}^n, \\ \text{div}\mathbf{u}^n &= g^n,\end{aligned}$$

and

$$(1+\alpha\triangle t)g^n - g^{n-1} = -\epsilon\triangle t p^n.$$

We have the inequality for the Stokes equations (see (3.5))

$$\|\mathbf{u}^n\|_2^2 + \|\nabla p^n\|^2 \le C_0 \left(\|\mathbf{w}^n\|^2 + \|\nabla g^n\|^2\right). \tag{4.3}$$

Since $\mathbf{u}^n$ satisfies the homogeneous Dirichlet boundary condition, we can choose the Banach space as $H^1\cap L_0^2$ with the norm $\|\nabla f\|^2$, and notice that $\mathbf{u}(0)$ is divergence-free; then we apply Lemma 3.1,

$$\begin{aligned}\sum_{n=1}^{N} \|\nabla g^n\|^2 &\le \frac{\epsilon^2}{\alpha^2} \sum_{n=1}^{N} \|\nabla p^n\|^2, \\ \sup_{1\le n\le N} \|\nabla g^n\|^2 &\le \frac{\epsilon^2}{\alpha^2} \sup_{1\le n\le N} \|b_n\|^2.\end{aligned}$$

Sum up inequality (4.3),

$$\begin{aligned}\triangle t \sum_{n=1}^{N} \left(\|\mathbf{u}^n\|_2^2 + \|\nabla p^n\|^2\right) &\le C_0\triangle t \sum_{n=1}^{N} \left(\|\mathbf{w}^n\|^2 + \|\nabla g^n\|^2\right) \\ &\le \triangle t \sum_{n=1}^{N} \left(C_0\|\mathbf{w}^n\|^2 + \frac{C_0\epsilon^2}{\alpha^2}\|\nabla p^n\|^2 \right),\end{aligned}$$

and take the sup of (4.3),

$$\sup_{1\le n\le N}\left(\|\mathbf{u}^n\|_2^2+\|\nabla p^n\|^2\right)\le C_0\sup_{1\le n\le N}\left(\|\mathbf{w}^n\|^2+\|\nabla g^n\|^2\right)$$
$$\le C_0\sup_{1\le n\le N}\left(\|\mathbf{w}^n\|^2+\frac{C_0\epsilon^2}{\alpha^2}\sup_{1\le n\le N}\|\nabla p^n\|^2\right).$$

Let $\epsilon_0=\frac{\alpha}{2\sqrt{C_0}}$. Then $\forall\epsilon\le\epsilon_0$, we have

$$\triangle t\sum_{n=1}^{N}\left(\|\mathbf{u}^n\|_2^2+\|\nabla p^n\|^2\right)\le 2C_0\triangle t\sum_{n=1}^{N}\|\mathbf{w}^n\|^2,$$
$$\sup_{1\le n\le N}\left(\|\mathbf{u}^n\|_2^2+\|\nabla p^n\|^2\right)\le 2C_0\sup_{1\le n\le N}\|\mathbf{w}^n\|^2.$$

Notice that

$$\triangle t\sum_{n=1}^{N}\frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}^n\|^2=\triangle t\sum_{n=1}^{N}\frac{1}{\epsilon^2}\|\nabla g^n\|^2\le\frac{\triangle t}{\alpha^2}\sum_{n=1}^{N}\|\nabla p^n\|^2,$$
$$\triangle t\sum_{n=1}^{N}\frac{1}{\epsilon^2}\left\|\nabla\mathrm{div}\left(\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\triangle t}\right)\right\|^2=\triangle t\sum_{n=1}^{N}\left\|\frac{\alpha}{\epsilon}\nabla\mathrm{div}\mathbf{u}^n+\nabla p^n\right\|^2\le 4\triangle t\sum_{n=1}^{N}\|\nabla p^n\|^2,$$
$$\sup_{1\le n\le N}\frac{1}{\epsilon^2}\|\nabla\mathrm{div}\mathbf{u}^n\|^2=\sup_{1\le n\le N}\frac{1}{\epsilon^2}\|\nabla g^n\|^2\le\sup_{1\le n\le N}\frac{1}{\alpha^2}\|\nabla p^n\|^2,$$
$$\sup_{1\le n\le N}\frac{1}{\epsilon^2}\left\|\nabla\mathrm{div}\left(\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\triangle t}\right)\right\|^2=\sup_{1\le n\le N}\left\|\frac{\alpha}{\epsilon}\nabla\mathrm{div}\mathbf{u}^n+\nabla p^n\right\|^2\le 4\sup_{1\le n\le N}\|\nabla p^n\|^2.$$

Combining all of the above inequalities together, we have the lemma. ◻

LEMMA 4.3. *For* (4.1), *if we assume* (2.3)–(2.4), *then there exist constants* M_1, *which depends on* N_1, *and* M_2, *which depends on* N_1 *and* N_2, *such that*

$$\sup_n\|\mathbf{u}^n\|_1^2+\triangle t\sum_1^\infty\left(\|\mathbf{u}^i\|_2^2+\|\nabla\mathrm{div}\mathbf{u}^i\|^2+\left\|\nabla\mathrm{div}\frac{\mathbf{u}^i-\mathbf{u}^{i-1}}{\triangle t}\right\|^2+\left\|\frac{\mathbf{u}^i-\mathbf{u}^{i-1}}{\triangle t}\right\|^2\right)\le M_1,$$
$$\sup_n\|\mathbf{u}^n\|_2\le M_2.$$

Proof. We will follow the proof of Lemmas 3.3 and 4.5 of [12]. First we notice that conditions (2.3)–(2.4) imply

$$\triangle t\sum_{n=0}^{\infty}\|\mathbf{f}^n\|^2\le N_1,\qquad\sup_{0\le n<\infty}\|\mathbf{f}^n\|^2\le N_2,\qquad\triangle t\sum_{n=0}^{\infty}\left\|\frac{\mathbf{f}^{n+1}-\mathbf{f}^n}{\triangle t}\right\|^2\le N_2.$$

Multiplying $\mathbf{u}^{n+1}$ on both sides of (4.1) and using identity $(b-a,b)=\frac{1}{2}(\|b\|^2-\|a\|^2+\|b-a\|^2)$, we obtain

$$\|\mathbf{u}^n\|^2+\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}^n\|^2+\triangle t\sum_0^\infty\left(\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}^n\|^2+\|\mathbf{u}^n\|_1^2\right)\le C_1. \tag{4.4}$$

Then we multiply $A\mathbf{u}^{n+1}$ on both sides of (4.1), define $y^n=\frac{1}{\epsilon}\|\mathrm{div}\mathbf{u}^n\|+\|\mathbf{u}^n\|_1^2$, $k^n=\|\mathbf{u}^n\|^2\|\mathbf{u}^{n+1}\|_1^2$, and we have

$$\frac{y^{n+1}-y^n}{\triangle t}+\left\|A\mathbf{u}^{n+1}\right\|^2\le\delta\left\|\mathbf{u}^{n+1}\right\|_2^2+C_\delta\left(k^ny^n+\left\|\mathbf{f}^{n+1}\right\|^2\right).$$

We let δ be suitably small such that $\delta \sum_1^N \|\mathbf{u}\|_2^2 \leq 0.5 \sum_1^N \|A\mathbf{u}^n\|^2$ holds for all N (one choice is $\delta = \frac{1}{2C}$, where C is constant in Lemma 4.2) and denote $K^n = \Pi_0^n(1+C_\delta \triangle t k^i)$, $a^n = 0.5\triangle t\|A\mathbf{u}^n\|^2$, $b^n = \triangle t(0.5\|A\mathbf{u}^n\|^2 - \delta\|\mathbf{u}\|_2^2)$, and $d^n = C_\delta \triangle t\|\mathbf{f}^n\|^2$. Then we can rewrite the above inequality as

$$y^{n+1} + a^{n+1} + b^{n+1} \leq \frac{K^n}{K^{n-1}} y^n + d^{n+1}.$$

By eliminating $y^1, y^2, \ldots, y^n$ from above the inequalities, we obtain

$$y^{n+1} + \sum_1^{n+1} a^i + K^n \sum_1^{n+1} \frac{b^i}{K^{i-1}} \leq K^n y^0 + K^n \sum_1^{n+1} d^i.$$

Using identity

$$\sum_1^{n+1} \frac{b^i}{K^{i-1}} = \frac{1}{K^n} \sum_{i=1}^{n+1} b^i + \sum_{i=2}^{n} \left[\left(\frac{1}{K^{i-1}} - \frac{1}{K^i} \right) \sum_{j=1}^{i} b^j \right] + \frac{b^1}{K^1}$$

and noticing that K^n is monotone increasing, $\sum_{j=1}^i b^j \geq 0$, we conclude that

$$\frac{1}{\epsilon}\|\text{div}\mathbf{u}^n\|^2 + \|\mathbf{u}\|_1^2 + \triangle t \sum_1^N \|A\mathbf{u}^n\|^2 \leq M_1. \tag{4.5}$$

Multiplying $\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t}$ on both sides of (4.1), applying Lemma 4.2 and the above inequality, and noticing that the choice of N does not affect the constant on the right-hand side, we have

(4.6)

$$\sup_n \|\mathbf{u}^n\|_1^2 + \triangle t \sum_1^\infty \left(\|\mathbf{u}^n\|_2^2 + \|\nabla\text{div}\mathbf{u}^n\|^2 + \left\|\nabla\text{div}\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\triangle t}\right\|^2 + \left\|\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\triangle t}\right\|^2 \right) \leq M_1.$$

Finally, the proof of the second inequality $\|\mathbf{u}\|_2 \leq M_2$ can be similarly done (cf. Lemma 4.5 of [12]). ◻

In addition the existence and uniqueness of the solution to (2.11)–(2.12) can be shown after we obtain a priori estimates for the time discrete solution (i.e., Lemma 4.3). We ignore the proof here since the steps are similar to those in [12] for the finite time case.

Define the error function $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}(t_n)$; then $\mathbf{e}^n$ satisfies

$$\begin{aligned} &\frac{\mathbf{e}^{n+1} - \mathbf{e}^n}{\triangle t} - \frac{1}{\epsilon}\mathbf{grad}\,\text{div}\left(\frac{\mathbf{e}^{n+1} - \mathbf{e}^n}{\triangle t} + \alpha\mathbf{e}^{n+1}\right) - \nu\triangle\mathbf{e}^{n+1} \\ &+ \bar{B}(\mathbf{u}^n, \mathbf{u}^{n+1}) - \bar{B}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1})) = \mathbf{r}^{n+1}, \end{aligned} \tag{4.7}$$

where initial data $\mathbf{e}^0 = 0$ and the remainder term

$$\mathbf{r}^{n+1} = \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}} (\eta - t_n) \left[\mathbf{u}_{tt}(\eta) - \frac{1}{\epsilon}\mathbf{grad}\,\text{div}\mathbf{u}_{tt}(\eta) \right] d\eta.$$

To obtain the full first-order error estimation, we need the remainder to be bounded; i.e.,

$$\sum_1^\infty \|\mathbf{r}^n\|^2 \leq C\triangle t^2. \tag{4.8}$$

THEOREM 4.4. *Assume conditions* (2.3)–(2.4) *and* (4.8)*; we can obtain error estimates of* $\mathbf{e}^n$*,*

$$\sup_{0\le n<\infty} \|\mathbf{e}^n\|_1^2 + \triangle t \sum_0^\infty \left(\frac{1}{\epsilon^2} \|\nabla \mathrm{div} \mathbf{e}^n\|^2 + \|\mathbf{e}^n\|_2^2 \right) \le C \triangle t^2.$$

Proof. Since the proof is almost same as Theorem 4.6 of [12], we give a sketch here. Due to the bilinear property of $\bar{B}$, we have

$$\begin{aligned} \bar{B}\left(\mathbf{u}^n, \mathbf{u}^{n+1}\right) &- \bar{B}(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1})) \\ &= \bar{B}\left(\mathbf{u}^n - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}\right) + \bar{B}\left(\mathbf{u}^{n+1}, \mathbf{e}^{n+1}\right) + \bar{B}\left(\mathbf{e}^{n+1}, \mathbf{u}(t_{n+1})\right). \end{aligned}$$

Then multiplying $\mathbf{e}^{n+1}$ on both sides of (4.7), we obtain

$$\sup_{0\le n<\infty} \|\mathbf{e}^n\|^2 + \triangle t \sum_0^\infty \left(\frac{1}{\epsilon} \|\mathrm{div} \mathbf{e}^n\|^2 + \|\mathbf{e}^n\|_1^2 \right) \le C_1 \triangle t^2.$$

Then we multiply $A\mathbf{e}^{n+1}$ on both sides of (4.7) and notice $\|\mathbf{u}(t)\|_2$ and $\|\mathbf{u}^n\|_2$ to be uniformly bounded; we can conclude the result. ∎

Remark 4.1. By the same proof of Lemma 4.3 in [12], we can extend the result to a large time interval; i.e., with the same condition (we will simplify the condition $(*)\, \mathbf{g} = \mathbf{f}(0) + \nu \triangle \mathbf{u}_0 - \bar{B}(\mathbf{u}_0, \mathbf{u}_0) \in \mathbf{H}^1$ later) of Lemma 4.3 in [12], we have

$$\sup_{0<t<\infty} \|\mathbf{u}_t\|_1^2 + \int_0^\infty \frac{1}{\epsilon^2} \|\nabla \mathrm{div} \mathbf{u}_{tt}\|^2 + \|\mathbf{u}_t\|_2^2 + \|\mathbf{u}_{tt}\|^2 dt \le C. \tag{4.9}$$

From the above estimation and Remark 4.1 in [12], we know that the condition $(*)$ and inequality (4.8) are true when the global compatibility condition holds (cf. [6]). For simplicity, we just assume this global compatibility holds.

Now we can compare the time discrete solution for the sequential regularization formulation with the solution for the Navier–Stokes equations. We denote s as the SRM index and n as the time index (see the detail of this implementation in section 5.1). We combine inequality (3.10) with Theorem 4.4 and obtain the following estimation.

THEOREM 4.5. *Assume conditions* (2.3)–(2.4) *and the global compatibility condition at* $t = 0$*. Let* $\mathbf{u}_s^i$ *and* $\mathbf{u}$ *be the solutions to the time discrete sequential regularization reformulation and the original Navier–Stokes equation, respectively. Then we have*

$$\begin{aligned} \|\mathbf{u}_s^n - \mathbf{u}(t_n)\|_1^2 + \triangle t \sum_0^n \left(\frac{1}{\epsilon^2} \left\|\nabla \mathrm{div}\left(\mathbf{u}_s^i - \mathbf{u}(t_i)\right)\right\|^2 + \left\|\mathbf{u}_s^i - \mathbf{u}(t_i)\right\|_2^2 \right) \\ \le C\left(\triangle t^2 + \left(M\epsilon^2\right)^s\right). \end{aligned}$$

Proof. We only need to check if all of the conditions for inequality (3.10) and Theorem 4.4 are satisfied. Since the sequential regularization formulation replaces the external force $\mathbf{f}$ by $\mathbf{f} - \nabla p_s$ at the $(s+1)$st step of the SRM iteration, we need to check the following inequalities:

$$\int_0^\infty \|\nabla p_s\|^2 dt \le C_s, \tag{4.10}$$

$$\sup_{0<t<\infty} \|\nabla p_s\| + \int_0^\infty \|\nabla (p_s)_t\|^2 dt \le C_s, \tag{4.11}$$

where C_s is a constant which does not depend on ϵ. The case of $s = 0$ is not a problem since we can choose p_0 freely. When $s = 1, 2, 3, \dots$, we have $p_s = p_{s-1} - \frac{1}{\epsilon}(\mathrm{div}(\mathbf{u}_s)_t + \alpha \mathrm{div}\mathbf{u}_s)$ from (1.5). Hence we just need to show that

$$\int_0^\infty \left\|\frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_s)_t\right\|^2 + \left\|\frac{1}{\epsilon}\nabla \mathrm{div}\mathbf{u}_s\right\|^2 dt \le C_s, \tag{4.12}$$

$$\sup_{0<t<\infty}\left(\left\|\frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_s)_t\right\| + \left\|\frac{1}{\epsilon}\nabla \mathrm{div}\mathbf{u}_s\right\|\right) + \int_0^\infty \left\|\frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_s)_{tt}\right\|^2 + \left\|\frac{1}{\epsilon}\nabla \mathrm{div}(\mathbf{u}_s)_t\right\|^2 dt \le C_s. \tag{4.13}$$

By mathematical induction, inequality (4.12) is simply a corollary of Theorem 3.3, and inequality (4.13) is a corollary of Theorem 3.4 and estimation (4.9). □

Remark 4.2. The analysis in this chapter is based on a semi-implicit scheme. In practice we are also interested in the fully explicit treatment of the nonlinear convection term. If $\|\mathbf{u}^n\|$ is uniformly bounded for n, the theorem in this chapter is also true for the fully explicit treatment of the nonlinear convection term. Without any specific assumption, $\|\mathbf{u}^n\|$ only remains bounded in a short time interval (see Lemma 3.5 in [12]). But in practice, the velocity of many flow problems is stably away from blow up. And since the fully explicit treatment is easier to implement, we will use it in the next chapter. For other schemes, for example, the Crank–Nicolson scheme, there is no essential difference for analysis, and the accepted order can be achieved by assuming enough regularity of the solution (see the discussion in [12, 11]).

5. Numerical examples. In this section, we will present results of numerical experiments of long time simulation of two-dimensional viscous flows.

5.1. Implementation. Let $\mathbf{V}_h$ be the finite element space; the full discrete scheme of SRM can be represented as follows: given an initial guess p_0^n, where $n = 0, 1, 2, \dots$, for $s = 1, 2, \dots$, we solve $(\mathbf{u}_s^{n+1}, p_s^{n+1})$ from the system

$$\begin{aligned}
&\left(\frac{\mathbf{u}_s^{n+1} - \mathbf{u}_s^n}{\triangle t}, \mathbf{v}\right) + \frac{1}{\epsilon}\left(\mathrm{div}\frac{\mathbf{u}_s^{n+1} - \mathbf{u}_s^n}{\triangle t}, \mathrm{div}\mathbf{v}\right) + \frac{\alpha}{\epsilon}\left(\mathrm{div}\mathbf{u}_s^{n+1}, \mathrm{div}\mathbf{v}\right)\\
&+\nu\left(\nabla \mathbf{u}_s^{n+1}, \nabla \mathbf{v}\right) + \bar{b}\left(\mathbf{u}_s^n, \mathbf{u}_s^n, \mathbf{v}\right) = \left(\mathbf{f}_s^{n+1}, \mathbf{v}\right) - \left(p_{s-1}^{n+1}, \mathrm{div}\mathbf{v}\right) \quad \forall \mathbf{v} \in \mathbf{V}_h,\\
&p_s^{n+1} = \frac{1}{\epsilon}\left(\mathrm{div}\frac{\mathbf{u}_s^{n+1} - \mathbf{u}_s^n}{\triangle t} + \alpha \mathrm{div}\mathbf{u}_s^{n+1}\right).
\end{aligned}$$

The above scheme is as follows: starting from $s = 0$ for each n, find $\mathbf{u}_1^n$ and p_1^n at time t_n, and then repeat this procedure to find $\mathbf{u}_2^n$, p_2^n, and so on. Since we need to store the values of $\mathbf{u}_1^n$ and p_1^n for all n time steps before the next SRM iteration, the storage may be very expensive since we compute the solution to a very long time. It is *not* what we do in practice. This difficulty can be overcome by an equivalent implementation.

We make an observation: the solution $(\mathbf{u}_s^{n+1}, p_s^{n+1})$ only depends on $\mathbf{u}_s^n$ and p_{s-1}^{n+1}. Now fix the number of iterations in advance according to the accuracy requirement (numerical experiments indicate that $s = 3$ or 4 may be enough in most cases). Since the initial guess p_0^n and the initial data $\mathbf{u}_s^0$ are known, we can rearrange the order of the computation as follows. Starting from $n = 1$, find $(\mathbf{u}_1^1, p_1^1)$ by using information p_0^1 and $\mathbf{u}_1^0$, and find $(\mathbf{u}_2^1, p_2^1)$ by using information p_1^1 and $\mathbf{u}_2^0, \dots$, up to $(\mathbf{u}_s^1, p_s^1)$. Save all the data $(\mathbf{u}_s^1, p_s^1)$, and move to $n = 2$. Compute $(\mathbf{u}_1^2, p_1^2)$ by using p_0^2 and $\mathbf{u}_1^1$ up

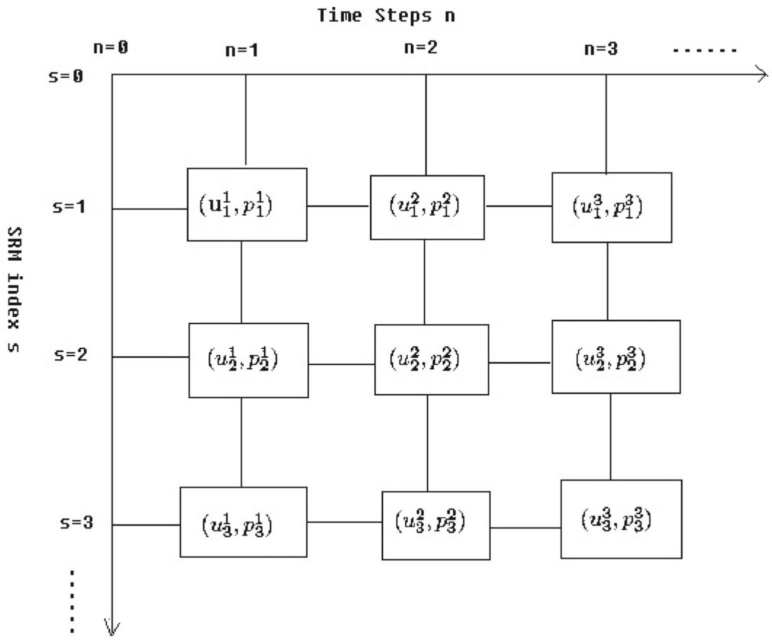

FIG. 5.1. *Implementation instruction.*

to $(\mathbf{u}_s^2, p_s^2)$. We can repeat this procedure at all time steps until the terminal time is reached (cf. [8]). The procedure is illustrated in Figure 5.1. The original SRM is to do the computation row by row, and the practical one is to do the computation column by column. It solves the storage problem since we only need to store $(\mathbf{u}_1^n, p_1^n)$, ..., $(\mathbf{u}_s^n, p_s^n)$ with $s = 3$ or 4 for computations at the next time step $n + 1$.

At every time step and SRM iteration, we need to solve a PDE of the form

$$\mathbf{u} - \frac{1 + \alpha\triangle t}{\epsilon}\nabla \text{div}\mathbf{u} - \nu\triangle t\Delta\mathbf{u} = \mathbf{g}.$$

Due to the noncommutativity of ∇div and Δ, the fast Poisson solver does not apply here. We will solve it by a direct solver. Since the equation does not depend on the time step n and SRM index s, we can discrete this PDE into matrix form and do the LU or the Choleski factorization at the initial time. This factorization at the later time would be the same as that of the initial time, and thus we simply do backward and forward substitutions in computations of all other time steps except the initial time. This implementation could save a lot of computational time. In our numerical examples, we will use the P_2 finite element for the spatial discretization. Let Ω be a convex polygon and $\mathcal{T}_h$ be a quasi-regular triangulation of Ω, where h represents the maximal diameters of the triangles in the triangulation. Define

$$\mathbf{V}_h = \left\{\mathbf{v} \in \mathbf{C}^0, \mathbf{v} = 0 \text{ on } \partial\Omega,\ \mathbf{v}|_{T_i} \text{ is a polynomial of degree } \le 2\right\}.$$

5.2. Cavity flow. The first example is lid-driven cavity flow in a unit square. To find out the steady state solution, we use time-dependent Navier–Stokes equations and compute them to a long time. The external force is $\mathbf{f} = 0$, and the boundary condition is $\mathbf{u} = (1, 0)$ at the top side of the square $\mathbf{u} = 0$ at the other three sides. We choose $\alpha = 1$, $\triangle t = 0.005$, $\epsilon = 0.01$, $s = 3$, and the P_2 element on a 13748 triangular mesh. We do the computation for Reynolds numbers Re = 1000 and Re = 5000,

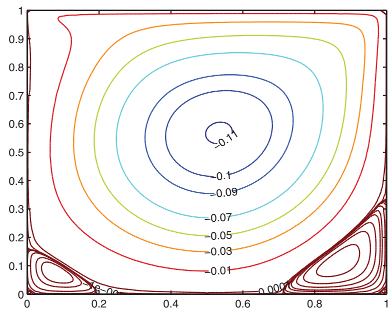

FIG. 5.2. *Streamline,* Re = 1000.

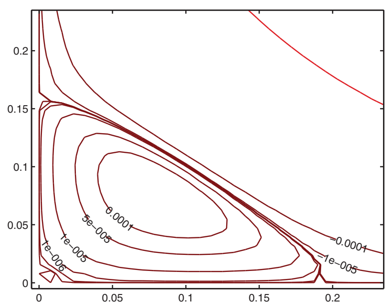

FIG. 5.3. *Left eddy,* Re = 1000.

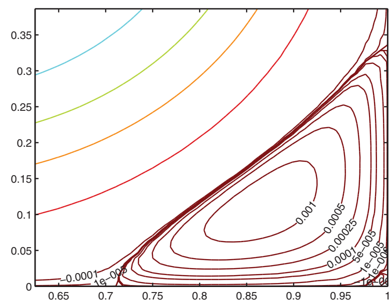

FIG. 5.4. *Right eddy,* Re = 1000.

respectively. We will compare the computational results with the benchmark solution in [4] as well.

- Re = 1000. The solution approaches the steady state around $T = 100$. Figure 5.2 is the streamline profile. There are three eddies in the streamline figure. We zoom in on the left and the right eddies in Figures 5.3 and 5.4 and depict the value of the velocity u at vertical line $x = 0.5$ and v at horizontal line $y = 0.5$ in Figures 5.5 and 5.6, respectively.
- Re = 5000. When Reynolds number is larger, we must compute the solution to a longer time T to ensure that the flow approaches steady state. Meanwhile we need to choose a smaller time step $\triangle t$ to maintain the stability. It increases the computational cost. We can use the steady state solution at a smaller Reynolds number as our initial condition to reduce the computational time. Figure 5.7 is the streamline profile. There are four eddies in the streamline figure. Figures 5.8, 5.9, and 5.10 are zoomed figures of the left bottom eddy, right bottom eddy, and left top eddy, respectively. Figures 5.11 and 5.12 depict the value of u at vertical line $x = 0.5$ and v at horizontal line $y = 0.5$, respectively.

5.3. Flow past a cylinder. The second example is flow past a cylinder. The domain is unbounded. It is not possible to do numerical computation in an infinite domain, so we must do a truncation. The computational domain is a rectangle excluding a disk (the cross section of the cylinder) inside. The x coordinate of the rectangle is from 0 to 50, the y coordinate is from -4 to 4, and the disk is centered at $(4, 0)$ with diameter 1. The boundary condition can be represented as follows: the inflow at the left side has a constant velocity which is parallel to the x-axis ($\mathbf{u} = (1, 0)$); the flow at the upper and lower boundaries is parallel to the x-axis, and its velocity

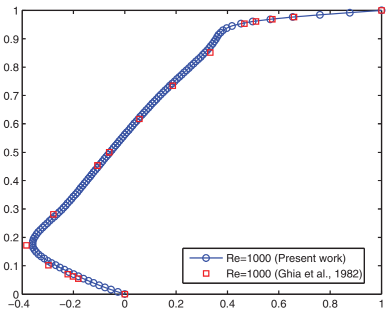

FIG. 5.5. $u(0.5, y)$, Re $= 1000$.

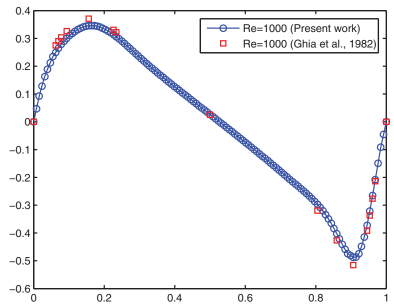

FIG. 5.6. $v(x, 0.5)$, Re $= 1000$.

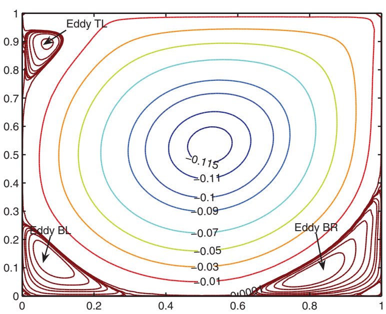

FIG. 5.7. *Streamline,* Re $= 5000$.

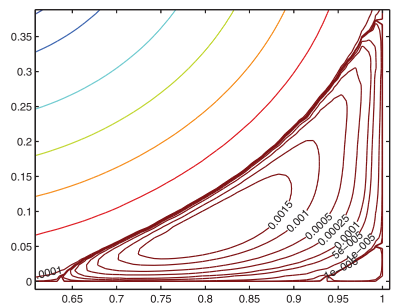

FIG. 5.8. *Right eddy,* Re $= 5000$.

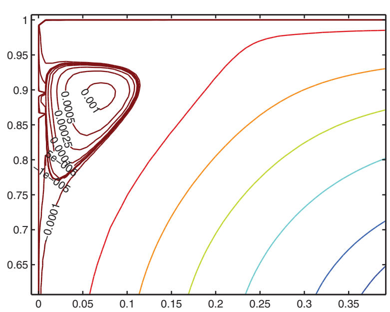

FIG. 5.9. *Top eddy,* Re $= 5000$.

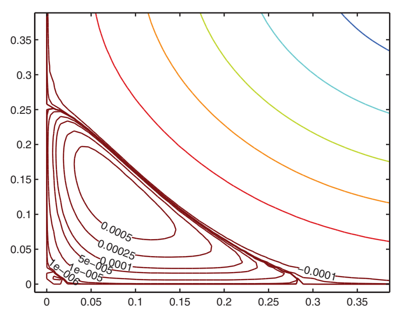

FIG. 5.10. *Left eddy,* Re $= 5000$.

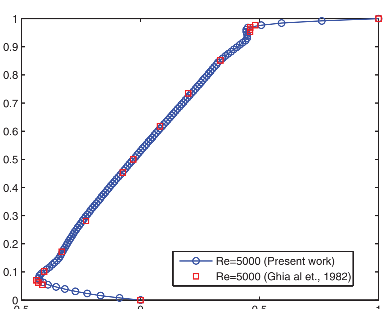

FIG. 5.11. $u(0.5, y)$, Re $= 5000$.

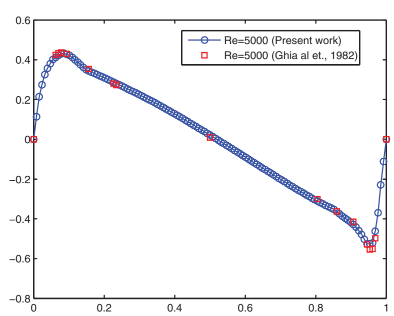

FIG. 5.12. $v(x, 0.5)$, Re $= 5000$.

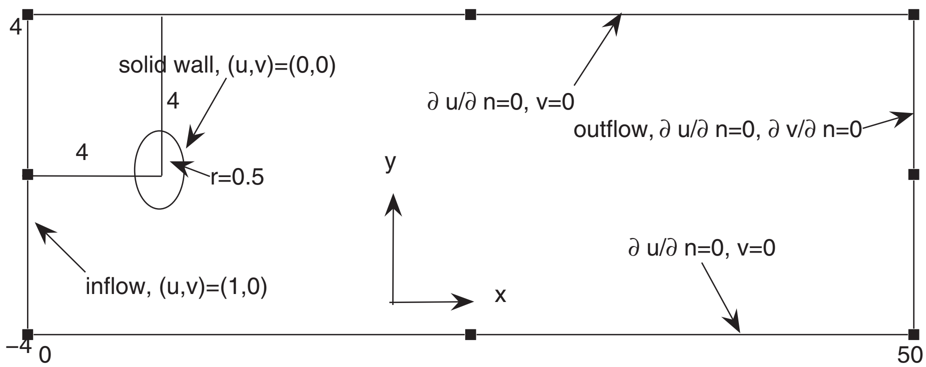

FIG. 5.13. *Flow past disk: geometry and boundary conditions.*

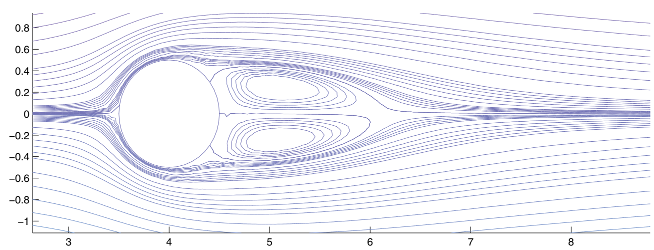

FIG. 5.14. Re = 40, $T = 10$.

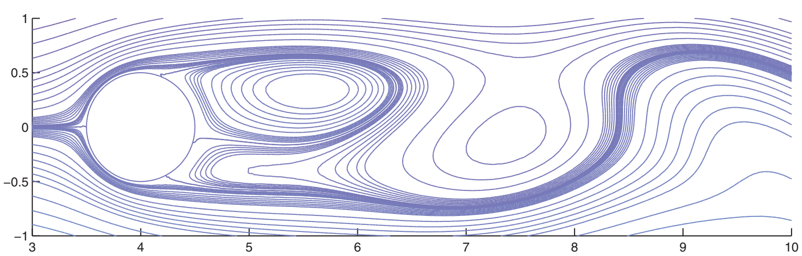

FIG. 5.15. Re = 100, $T = 35$.

does not change along the y-direction ($\frac{\partial u}{\partial \mathbf{n}} = 0, v = 0$); the boundary of the disk is a solid wall with a no-slip boundary condition ($\mathbf{u} = \mathbf{0}$); the outflow at the right side is imposed by a natural boundary condition (homogeneous Neumann boundary condition, $\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0$). Figure 5.13 is a diagram for the computational domain and the boundary conditions. It is well known that the Reynolds number plays a crucial role in this problem. We will compute the solution with two different Reynolds numbers (Re = 40 and Re = 100).

- Re = 40. In this case, the flow approaches steady state after a certain time. Figure 5.14 depicts the streamline after the flow reaches its steady state. There are two symmetric eddies behind the disk.
- Re = 100. When Re = 100, the flow does not have a steady state. Figures 5.15–5.17 depict the streamlines at different times. From these figures, we can see how the flow separates.

All of our computational results are pretty close to existing results.

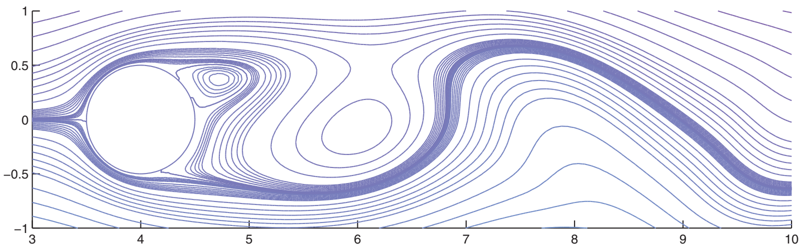

Fig. 5.16. Re = 100, $T = 45$.

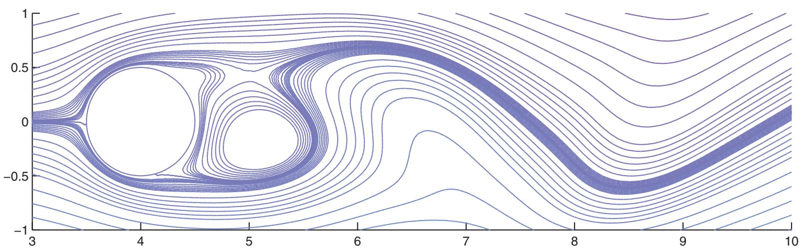

Fig. 5.17. Re = 100, $T = 60$.

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