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EFFECTS OF SMALL VISCOSITY AND FAR FIELD BOUNDARY CONDITIONS FOR HYPERBOLIC SYSTEMS

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Abstract. In this paper we study the effects of small viscosity term and the far-field boundary conditions for systems of convection-diffusion equations in the zero viscosity limit. The far-field boundary conditions are classified and the corresponding solution structures are analyzed. It is confirmed that the Neumann type of far-field boundary condition is preferred. On the other hand, we also identify a class of improperly coupled boundary conditions which lead to catastrophic reflection waves dominating the inlet in the zero viscosity limit. The analysis is performed on the linearized convection-diffusion model which well describes the behavior at the far field for many physical and engineering systems such as fluid dynamical equations and electro-magnetic equations. The results obtained here should provide some theoretical guidance for designing effective far field boundary conditions.

1. **Introduction.**

The purpose of this paper is to study the effects of small viscosity term and the far field boundary conditions for systems of convection-diffusion equations and provide some theoretical guidance for designing effective far field boundary conditions.

At far field, most physical quantities tend to constants. A common approach in handling the far-field in computation is to cut off the far-field domain from the computational domain and impose some far-field boundary conditions. The domain is usually large enough so that the active near domain boundary becomes insignificant and the background can be taken to be uniform and homogeneous. The underlying physical systems can then be approximated by systems of linear convection diffusion equations with constant coefficients. For simplicity, we take the artificial far field boundary to be $x = 0$ and perform the characteristic decomposition for the convection part and make the following simplifications: (1) We consider the one-dimensional case only and ignore the transversal effects. (2) We take the viscosity matrix to be identity; (3) We only consider two characteristic speeds, one

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positive and the other negative. Thus we have the following one-dimensional system of convection-diffusion equations

$$\begin{cases} \partial_t u^\varepsilon + \partial_x u^\varepsilon = \varepsilon \partial_x^2 u^\varepsilon \\ \partial_t v^\varepsilon - \partial_x v^\varepsilon = \varepsilon \partial_x^2 v^\varepsilon \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ represents the total dissipation which may arise from numerical viscosity or physical mechanism and is usually very small.

Naturally we impose the boundary conditions at the far field $x = 0$ in terms of the characteristic variables in the following general form:

$$D \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} (0,t) + E \begin{pmatrix} \partial_x u^\varepsilon \\ \partial_x v^\varepsilon \end{pmatrix} (0,t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad t \geq 0, \tag{1.2}$$

where D and E are suitable 2×2 constant matrices with $\text{Rank}(D, E) = 2$. Additionally we prescribe for (1.1) the following initial condition

$$u^\varepsilon(x,0) = u_0(x), \quad v^\varepsilon(x,0) = v_0(x), \quad x \geq 0. \tag{1.3}$$

For fixed $\varepsilon > 0$ and smooth (and compatible) initial and boundary data, the existence and uniqueness of solution to the IBVP (1.1)-(1.3) is well-known. Our interest in this paper is to study the asymptotic solution structure and analyze the effect of different boundary conditions for the IBVP (1.1)-(1.3) in the limit of small viscosity, that is, $\varepsilon \to 0$.

For small viscosity, (1.1) can be formally approximated by the following system of inviscid equations

$$\begin{cases} \partial_t u + \partial_x u = 0 \\ \partial_t v - \partial_x v = 0 \end{cases} \tag{1.4}$$

Therefore one expects that as $\varepsilon \to 0$, the solution $(u^\varepsilon, v^\varepsilon)$ of (1.1)-(1.3) should converge in some sense to an appropriate solution (u, v) of (1.4) (subject to certain initial and boundary conditions). This should be so, for example, when (1.1) is used as a numerical approximation to (1.4) and ε is the corresponding numerical viscosity.

Note that for the inviscid hyperbolic equations, the solution for the outgoing characteristic flow $v(x,t)$ is completely determined by the initial data $v(x,0)$ and therefore we can only prescribe one boundary condition for the incoming flow $u(x,t)$. As a result, in the zero viscosity limit, the solution $(u^\varepsilon, v^\varepsilon)$ of the parabolic IBVP (1.1)-(1.3) necessarily develops a boundary layer $v^{b.l.}$ in the outlet v^ε near $x = 0$. Although the boundary layer only exists in the v^ε component and its effect is confined to a narrow range near $x = 0$, the coupling of boundary conditions can make things much worse in that a further reflection wave can be induced in the inlet u^ε. The strength of the reflection wave depends on the magnitude of the boundary layer $v^{b.l.}$ and the degree of the coupling in the boundary conditions. In the worst scenario, the reflection wave can dominate the inlet u^ε and grow at order ε^{-1}, see Theorem 1.3 below. As a matter of fact, the whole purpose of designing good far field boundary conditions is to reduce the reflection wave.

The study of far field boundary conditions began with the pioneering work of Engquist and Majda [1, 2] for multidimensional inviscid systems. Their elegant recipe of design was through the expansion of the symbols in the pseudo-differential operators. There have been many studies on the consistency and stability of numerical boundary conditions either for hyperbolic systems (see for example, [4, 6, 16]) or for parabolic systems with fixed viscosity (see, for example, [11, 13, 17, 18]). In this paper, we focus on the small viscosity effect and study the solution structure and the convergence of (1.1) to (1.4) in the limit of small viscosity under various boundary conditions. We refer to [15] for the study of the convergence of (1.1) to (1.4) and the underlying boundary layer behavior in the simpler scalar case in which no reflection wave can occur. We note that several authors have also studied related nonlinear problems with Dirichlet boundary conditions, see for example, [3, 5, 8, 9, 10, 19]. Additionally we refer to [14, 20, 21, 22] for the study of the boundary layer problem in kinetic equations and hyperbolic relaxation systems.

The boundary condition (1.2) is clearly unchanged upon left multiplying an invertible 2×2 matrix. Depending on the value of $\mathrm{Rank}(E)$, it is customary to classify the above boundary condition (1.2) into the following three cases:

Case I: $\mathrm{Rank}(E) = 2$ (Neumann boundary conditions).
Case II: $\mathrm{Rank}(E) = 1$ (Mixed boundary conditions).
Case III: $\mathrm{Rank}(E) = 0$ (Dirichlet boundary conditions).

For mixed boundary conditions (Case II), we assume, without loss of generality, $e_{21} = e_{22} = 0$. It is noted that in this case, for the boundary conditions (1.2) to be meaningful, besides the condition $\mathrm{Rank}\,(D, E) = 2$, it is also necessary that $(d_{11}, d_{21}) \neq 0$ if $e_{11} = 0$, and $(d_{12}, d_{22}) \neq 0$ if $e_{12} = 0$. We distinguish the following three sub-cases.

Case II(a): $e_{12} = 0$, $e_{11} \neq 0$.
Case II(b): $e_{12} \neq 0$, $d_{21} \neq 0$.
Case II(c): $e_{12} \neq 0$, $d_{21} = 0$.

When the boundary conditions in (1.2) are decoupled or can be made so by left multiplying a 2×2 invertible matrix, the IBVP (1.1)-(1.3) is then equivalent to two scalar IBVPs for u^ε and v^ε separately and the convergence results proved in [15] can be applied. This is the case for Dirichlet (Case III) boundary conditions.

THEOREM 1.1 (Dirichlet boundary conditions). *For Dirichlet boundary conditions, there exists a unique solution (u, v) of (1.4) such that for any $T > 0$, we have*

$$\int_0^T \int_0^\infty |u^\varepsilon(x,t) - u(x,t)|^2 \, dxdt \leq C\varepsilon^2 \left(\|F\|_{H^2}^2 + \|u_0\|_{H^2}^2 \right),$$
$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t)|^2 \, dxdt \leq C\varepsilon \left(\|F\|_{L^2}^2 + \|v_0\|_{H^2}^2 \right),$$

for some constant $C = C(T) > 0$ independent of u_0, v_0, F and ε. Furthermore there exists a boundary layer $v^{b.l.}(x/\varepsilon, t)$ such that

$$\int_0^T \int_0^\infty \left| v^\varepsilon(x,t) - v(x,t) - v^{b.l.}(x/\varepsilon,t) \right|^2 \, dxdt \leq C\varepsilon^2 \left(\varepsilon\|F\|_{H^1}^2 + \|v_0\|_{H^3}^2 \right).$$

Note that in Theorem 1.1, the initial and boundary data are implicitly assumed to satisfy certain regularity and compatibility conditions, namely, $u_0 \in H^2$, $v_0 \in H^2$, $F \in H^2$ with $u_0(0) = u_0'(0) = 0$, $v_0(0) = v_0'(0) = 0$, $F(0) = F'(0) = 0$.

For typical Neumann (Case I) and mixed (Case II) boundary conditions, the boundary conditions will be genuinely coupled. In such cases, due to the presence of the boundary layer $v^{b.l.}(x/\varepsilon, t)$ in the outlet v^ε, the existence of the $\partial_x v^\varepsilon(0, t)$ term in the boundary condition (1.2) may yield a reduced boundary condition for u at the order of ε^{-1}. Consequently convergence results such as those in Theorem

1.1 may no longer be true and the question can become very complicated. This is indeed the case for Case II(c) boundary conditions. However, for all other types of boundary conditions, similar results as Theorem 1.1 still hold since either the above coupling mechanism is absent (case II(a)) or the boundary layer $v^{b.l.}$ turns out to be weak (Case I and Case II(b)).

Our main results of this paper can be stated as follows:

THEOREM 1.2 (Convergence with Optimal Error Estimates). *Let $u_0(x)$, $v_0(x)$ and $F(t)$ be sufficiently smooth and compatible. Then there exists a unique solution (u, v) of (1.4) such that for all $T > 0$, we have*
Case I: Rank$(E) = 2$ *(Neumann boundary condition)*

$$\int_0^T \int_0^\infty |u^\varepsilon(x,t) - u(x,t)|^2 \, dxdt \le C\varepsilon^2(\|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \|F\|^2_{H^1})$$
$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t)|^2 \, dxdt \le C\varepsilon^2(\|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \varepsilon\|F\|^2_{H^1})$$

Case II(a): $e_{12} = 0$, $e_{21} = e_{22} = 0$

$$\int_0^T \int_0^\infty |u^\varepsilon(x,t) - u(x,t)|^2 \, dxdt \le C\varepsilon^2(\|u_0\|^2_{H^2} + \|F\|^2_{H^2})$$
$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t)|^2 \, dxdt \le C\varepsilon(\varepsilon^2\|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \|F\|^2_{H^1})$$

Furthermore, there exists a boundary layer $v^{b.l.}(x/\varepsilon, t)$ in this case such that

$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t) - v^{b.l.}(x/\varepsilon,t)|^2 \, dxdt$$
$$\le C\varepsilon^2(\varepsilon\|u_0\|^2_{H^2} + \|v_0\|^2_{H^3} + \varepsilon\|F\|^2_{H^2}),$$

Case II(b): $e_{12} \neq 0$, $d_{21} \neq 0$, $e_{21} = e_{22} = 0$

$$\int_0^T \int_0^\infty |u^\varepsilon(x,t) - u(x,t)|^2 \, dxdt \le C\varepsilon^2(\|u_0\|^2_{H^2} + \|v_0\|^2_{H^3} + \|F\|^2_{H^2})$$
$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t)|^2 \, dxdt \le C\varepsilon^2(\|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \varepsilon\|F\|^2_{H^1})$$

THEOREM 1.3 (Case II(c): Improper boundary conditions). *Let $e_{21} = e_{22} = 0$, $e_{12} \neq 0$, $d_{21} = 0$ (and $d_{22} \neq 0$). Then there exists a unique solution (u_{-1}, v) of (1.4) such that*

$$\int_0^T \int_0^\infty |u^\varepsilon(x,t) - \varepsilon^{-1}u_{-1}(x,t)|^2 \, dxdt \le C(\|u_0\|^2_{H^2} + \|v_0\|^2_{H^3} + \|F\|^2_{H^2}),$$
$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t)|^2 \, dxdt \le C\varepsilon(\|v_0\|^2_{H^2} + \varepsilon^{1/2}\|F\|^2_{L^2}),$$

Furthermore, there exists a boundary layer $v^{b.l.}(x/\varepsilon, t)$ such that

$$\int_0^T \int_0^\infty |v^\varepsilon(x,t) - v(x,t) - v^{b.l.}(x/\varepsilon,t)|^2 \, dxdt \le C\varepsilon^2(\|v_0\|^2_{H^3} + \varepsilon\|F\|^2_{H^1})$$

Remarks:

1. In Theorem 1.2, the inviscid limit (u, v) can be obtained by solving (1.4) with initial condition $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ and the following reduced boundary condition for $u(0, t)$:

Case I: $(\det(d_1, e_2))u(0,t)+(\det E)\partial_x u(0,t) = (\det(F(t), e_2))-(\det(d_2, e_2))v_0(t)$

Case II(a): $(\det D)u(0,t) + (\det(e_1, d_2))\partial_x u(0,t) = \det(F(t), d_2)$

Case II(b): $u(0,t) = f_2(t)/d_{21} - (d_{22}/d_{21})v_0(t)$

where d_1, d_2 and e_1, e_2 are the column vectors of D and E respectively.

2. In Theorem 1.3, (u_{-1}, v) satisfies (1.4) with initial condition $u_{-1}(x, 0) = 0$, $v(x, 0) = v_0(x)$ and the following reduced boundary condition

Case II(c): $d_{11}u_{-1}(0,t) + e_{11}\partial_x u_{-1}(0,t) = (e_{12}/d_{22})f_2(t)) - e_{12}v_0(t)$

3. The occurrence of an $\varepsilon^{-1}u_{-1}$ term in the inlet u^ε in Theorem 1.3 originates from the boundary layer effect in the outlet v^ε and the amplification by the $\partial_x v^\varepsilon$ term in the boundary condition. It is not hard to see that for larger systems with n negative characteristic speeds and m zero characteristic speeds, through certain successive couplings in boundary conditions, an incoming wave can exhibit a singular growth term at order $\varepsilon^{-(n+m/2)}$ as $\varepsilon \to 0$.

The proof of the above theorems will be carried out in the following sections. The plan is as follows. To motivate, we will first apply the method of matched asymptotic expansions to the IBVP (1.1)-(1.3) and formally derive the leading asymptotic behavior of the solution $(u^\varepsilon, v^\varepsilon)$. Next in Section 3, we solve the viscous IBVP (1.1)-(1.3) explicitly by Laplace transform. The solution is then compared with its leading asymptotic behavior formally derived through matched asymptotic expansions and, by using Parseval's identity and careful asymptotic analysis, the desired convergence estimates are obtained in Section 4 and Section 5 for zero and nonzero initial data cases respectively.

2. Matched Asymptotic Expansions.

In order to identify the limiting behavior and the boundary layer structures of the solution for the viscous IBVP (1.1)-(1.3) as $\varepsilon \to 0$, we assume the following uniformly valid expansions in terms of ε:

$$\begin{cases} u^\varepsilon(x,t) = \varepsilon^{-1}u_{-1}(x,t) + u(x,t) + \varepsilon u_1(x,t) + \cdots \\ v^\varepsilon(x,t) = (v(x,t) + v^{b.l.}(x/\varepsilon, t)) + \varepsilon(v_1(x,t) + v_1^{b.l.}(x/\varepsilon, t)) + \cdots \end{cases} \tag{2.1}$$

with the localized boundary layers $v^{b.l.}(y, t)$ and $v_1^{b.l.}(y, t)$ exponentially decaying as $y = x/\varepsilon \to +\infty$.

Plugging the above expansions into (1.1) and matching the orders of ε, we obtain the following

$$\begin{aligned} &\partial_t u_{-1} + \partial_x u_{-1} = 0, \quad \partial_t u + \partial_x u = \partial_x^2 u_{-1}, \\ &\partial_t v - \partial_x v = 0, \quad \partial_t v_1 - \partial_x v_1 = \partial_x^2 v, \\ &\partial_y^2 v^{b.l.} = -\partial_y v^{b.l.}, \quad \partial_y^2 v_1^{b.l.} = -\partial_y v_1^{b.l.} + \partial_t v^{b.l.}. \end{aligned} \tag{2.2}$$

On the other hand, by matching (2.1) with the initial condition (1.3), we have

$$\begin{aligned} &u_{-1}(x,0) = 0, \quad u(x,0) = u_0(x), \\ &v(x,0) = v_0(x), \quad v_1(x,0) = 0, \\ &v^{b.l.}(y,0) = 0, \quad v_1^{b.l.}(y,0) = 0. \end{aligned} \tag{2.3}$$

The above equations can be solved recursively. For convenience, we represent the solutions in terms of Laplace transform [7, 12]. Recall that the Laplace transform

$\tilde{f}(x,\xi)$ of a function $f(x,t)$ is defined as

$$\tilde{f}(x,\xi) = \int_0^\infty e^{-\xi t} f(x,t)dt \tag{2.4}$$

and satisfies

$$\widetilde{\partial_t f}(x,\xi) = \xi \tilde{f}(x,\xi) - f(x,0) = \xi \tilde{f}(x,\xi) - f_0(x) \tag{2.5}$$

Throughout this paper we choose $\xi = \alpha + i\beta$ with $\alpha = \mathrm{Re}\,\xi > 0$ sufficiently large and fixed.

Applying Laplace transform on (2.2), we obtain, at the leading order,

$$\partial_x \tilde{u}_{-1} + \xi \tilde{u}_{-1} = 0, \quad \partial_x \tilde{v} - \xi \tilde{v} = -v_0(x), \quad \partial_y^2 \tilde{v}^{b.l.} = -\partial_y \tilde{v}^{b.l.} \tag{2.6}$$

and therefore

$$\begin{aligned} \tilde{u}_{-1}(x,\xi) &= \tilde{u}_{-1}(0,\xi)e^{-\xi x}, \\ \tilde{v}(x,\xi) &= \int_x^\infty e^{-\xi(\eta - x)} v_0(\eta)\, d\eta, \\ \tilde{v}^{b.l.}(y,\xi) &= \tilde{v}^{b.l.}(0,\xi)e^{-y} \end{aligned} \tag{2.7}$$

where $\tilde{u}_{-1}(0,\xi)$ and $\tilde{v}^{b.l.}(0,\xi)$ are the Laplace transform of appropriate boundary data $u_{-1}(0,t)$ and $v^{b.l.}(0,t)$ and remain to be determined.

Similarly at the next order, we have

$$\partial_x \tilde{u} + \xi \tilde{u} = u_0(x) + \partial_x^2 \tilde{u}_{-1} = u_0(x) + \xi^2 \tilde{u}_{-1}(0,\xi)e^{-\xi x} \tag{2.8}$$

$$\partial_y^2 \tilde{v}_1^{b.l.} = -\partial_y \tilde{v}_1^{b.l.} + \xi \tilde{v}^{b.l.} = -\partial_y \tilde{v}_1^{b.l.} + \xi \tilde{v}^{b.l.}(0,\xi)e^{-y} \tag{2.9}$$

With appropriate boundary data $\tilde{u}(0,\xi)$ and $\tilde{v}_1^{b.l.}(0,\xi)$, the solutions to (2.8)-(2.9) can be found to be

$$\tilde{u}(x,\xi) = \tilde{u}(0,\xi)e^{-\xi x} + \int_0^x e^{-\xi(x-\eta)} u_0(\eta)\, d\eta + x\xi^2 \tilde{u}_{-1}(0,\xi)e^{-\xi x} \tag{2.10}$$

$$\tilde{v}_1^{b.l.}(y,\xi) = \tilde{v}_1^{b.l.}(0,\xi)e^{-y} - y\xi \tilde{v}^{b.l.}(0,\xi)e^{-y} \tag{2.11}$$

We now match the boundary conditions and derive the appropriate boundary data $u_{-1}(0,t)$, $u(0,t)$, $v^{b.l.}(0,t)$ and $v_1^{b.l.}(0,t)$. This is achieved by substituting (2.1) into the boundary condition (1.2) and separating the powers of ε:

$$d_1 u_{-1} + e_1 \partial_x u_{-1} + e_2 \partial_y v^{b.l.}\big|_{x=0} = 0 \tag{2.12}$$

$$d_1 u + e_1 \partial_x u + d_2(v + v^{b.l.}) + e_2(\partial_x v + \partial_y v_1^{b.l.})\big|_{x=0} = F(t) \tag{2.13}$$

Before we solve for the desired boundary data $u_{-1}(0,t)$, $u(0,t)$, $v^{b.l.}(0,t)$, etc., we observe that

$$\partial_x \tilde{u}_{-1}(0,\xi) = -\xi \tilde{u}_{-1}(0,\xi), \quad \partial_x \tilde{u}(0,\xi) = -\xi \tilde{u}(0,\xi) + \xi^2 \tilde{u}_{-1}(0,\xi) \tag{2.14}$$

$$\partial_y \tilde{v}^{b.l.}(0,\xi) = -\tilde{v}^{b.l.}(0,\xi), \quad \partial_y \tilde{v}_1^{b.l.}(0,\xi) = -\tilde{v}_1^{b.l.}(0,\xi) - \xi \tilde{v}^{b.l.}(0,\xi) \tag{2.15}$$

Furthermore, we have

$$\tilde{v}(0,\xi) = \tilde{v}_0(\xi) = \int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta, \quad \partial_x \tilde{v}(0,\xi) = \xi \tilde{v}_0(\xi). \tag{2.16}$$

Taking the Laplace transform of (2.12)-(2.13) and using the above relations, we now obtain

$$(d_1 - \xi e_1)\tilde{u}_{-1} - e_2 \tilde{v}^{b.l.}\big|_{x=0} = 0 \tag{2.17}$$

$$\begin{aligned}(d_1-\xi e_1)\tilde{u}-e_2\tilde{v}_1^{b.l.}+e_1\xi^2\tilde{u}_{-1}+(d_2-\xi e_2)\tilde{v}^{b.l.}\big|_{x=0}\\ =\tilde{F}(\xi)-(d_2+\xi e_2)\tilde{v}_0(\xi)\end{aligned} \tag{2.18}$$

Case I: $\det E\neq 0$.

In this case, by taking $\mathrm{Re}\,\xi=\alpha>0$ sufficiently large, we have

$$\det(d_1-\xi e_1,-e_2)=\xi\det E-\det(d_1,e_2)\neq 0 \tag{2.19}$$

and (2.17)-(2.18) can be solved easily

$$\tilde{u}_{-1}(0,\xi)=0,\quad \tilde{v}^{b.l.}(0,\xi)=0, \tag{2.20}$$

$$\tilde{u}(0,\xi)=\tfrac{\det(\tilde{F}(\xi),e_2)}{\det(d_1-\xi e_1,e_2)}-\tfrac{\det(d_2,e_2)}{\det(d_1-\xi e_1,e_2)}\tilde{v}_0(\xi), \tag{2.21}$$

and

$$\tilde{v}_1^{b.l.}(0,\xi)=-\frac{\det(d_1-\xi e_1,\tilde{F}(\xi))}{\det(d_1-\xi e_1,e_2)}+\frac{\det(d_1-\xi e_1,d_2+\xi e_2)}{\det(d_1-\xi e_1,e_2)}\tilde{v}_0(\xi). \tag{2.22}$$

Case II(a): $e_{11}\neq 0$, $e_{12}=e_{21}=e_{22}=0$.

In this case, we have $\det(d_1-\xi e_1,-e_2)=0$ and the linear system (2.17)-(2.18) becomes degenerate. The first equation (2.17) reduces to

$$(d_1-\xi e_1)\tilde{u}_{-1}(0,\xi)=0 \tag{2.23}$$

which, by taking $\alpha=\mathrm{Re}\,\xi$ sufficiently large, implies

$$\tilde{u}_{-1}(0,\xi)=0 \tag{2.24}$$

With (2.24), the second equation (2.18) now becomes

$$(d_1-\xi e_1)\tilde{u}(0,\xi)+d_2\tilde{v}^{b.l.}(0,\xi)=\tilde{F}(\xi)-d_2\tilde{v}_0(\xi) \tag{2.25}$$

Next we show that by choosing $\alpha=\mathrm{Re}\,\xi>0$ sufficiently large, we have

$$\det(d_1-\xi e_1,d_2)=\det D-\xi e_{11}d_{22}\neq 0 \tag{2.26}$$

This is obvious when $d_{22}\neq 0$. On the other hand, if $d_{22}=0$, we must have $d_{21}\neq 0$. In this case, $(1.2)_2$ reduces to the following Dirichlet boundary condition for u^ε:

$$u^\varepsilon(0,t)=f_2(t)/d_{21} \tag{2.27}$$

while $(1.2)_1$ becomes

$$d_{11}u^\varepsilon(0,t)+e_{11}\partial_x u^\varepsilon(0,t)+d_{12}v^\varepsilon(0,t)=f_1(t) \tag{2.28}$$

Clearly we must also have $d_{12}\neq 0$ and hence $\det(d_1-\xi e_1,d_2)=-d_{12}d_{21}\neq 0$.

With (2.26), the desired boundary data $\tilde{u}(0,\xi)$ and $\tilde{v}^{b.l.}(0,\xi)$ can now be uniquely determined from (2.25):

$$\tilde{u}(0,\xi)=\frac{\det(\tilde{F}(\xi),d_2)}{\det(d_1-\xi e_1,d_2)},\quad \tilde{v}^{b.l.}(0,\xi)=-\frac{\det(\tilde{F}(\xi),d_1-\xi e_1)}{\det(d_1-\xi e_1,d_2)}-\tilde{v}_0(\xi) \tag{2.29}$$

Case II(b): $e_{12}\neq 0$, $d_{21}\neq 0$, $e_{21}=e_{22}=0$.

In this case, we have $\det(d_1-\xi e_1,-e_2)=d_{21}e_{12}\neq 0$. Therefore the linear systems (2.17) and (2.18) can be solved in exactly the same way as in Case I.

$$\tilde{u}_{-1}(0,\xi)=0,\quad \tilde{v}^{b.l.}(0,\xi)=0, \tag{2.30}$$

$$\tilde{u}(0,\xi)=\tilde{f}_2(\xi)/d_{21}-(d_{22}/d_{21})\tilde{v}_0(\xi), \tag{2.31}$$

and

$$\tilde{v}_1^{b.l.}(0,\xi)=\frac{\det(d_1-\xi e_1,\tilde{F}(\xi))}{d_{21}e_{12}}-\frac{\det(d_1-\xi e_1,d_2+\xi e_2)}{d_{21}e_{12}}\tilde{v}_0(\xi). \tag{2.32}$$

Case II(c): $e_{12} \neq 0$, $d_{21} = e_{21} = e_{22} = 0$.

In this case, again we have $\det(d_1 - \xi e_1, -e_2) = 0$ and the linear system (2.17)-(2.18) is degenerate with (2.17) now reducing to the following single equation

$$(d_{11} - \xi e_{11})\tilde{u}_{-1}(0,\xi) - e_{12}\tilde{v}^{b.l.}(0,\xi) = 0 \tag{2.33}$$

On the other hand, we have from $(2.18)_2$

$$d_{22}\tilde{v}^{b.l.}(0,\xi) = \tilde{f}_2(\xi) - d_{22}\tilde{v}_0(\xi) \tag{2.34}$$

Therefore we obtain

$$\tilde{u}_{-1}(0,\xi) = \frac{e_{12}\tilde{f}_2(\xi)}{d_{22}(d_{11} - \xi e_{11})} - \frac{e_{12}}{d_{11} - \xi e_{11}}\tilde{v}_0(\xi) \tag{2.35}$$

and

$$\tilde{v}^{b.l.}(0,\xi) = \tilde{f}_2(\xi)/d_{22} - \tilde{v}_0(\xi) \tag{2.36}$$

Note that in this case we must have $d_{22} \neq 0$ and the second boundary condition in (1.2) determines the boundary data $v^\varepsilon(0,t)$ completely

$$v^\varepsilon(0,t) = f_2(t)/d_{22} \tag{2.37}$$

On the other hand since we cannot have both boundary conditions for v^ε only, we must also have $d_{11} \neq 0$ or $e_{11} \neq 0$. Thus, by choosing α sufficiently large, we have $d_{11} - \xi e_{11} \neq 0$.

3. **Solution by Laplace Transform.** We now solve the IBVP (1.1)-(1.3) explicitly by the method of Laplace transform. Taking Laplace transform on (1.1) and using (1.3), we obtain

$$\begin{cases} \varepsilon\partial_x^2\tilde{u}^\varepsilon - \partial_x\tilde{u}^\varepsilon - \xi\tilde{u}^\varepsilon &= -u_0(x) \\ \varepsilon\partial_x^2\tilde{v}^\varepsilon + \partial_x\tilde{v}^\varepsilon - \xi\tilde{v}^\varepsilon &= -v_0(x) \end{cases} \tag{3.1}$$

On the other hand, the boundary condition (1.2) becomes

$$D\begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{v}^\varepsilon \end{pmatrix}(0,\xi) + E\begin{pmatrix} \partial_x\tilde{u}^\varepsilon \\ \partial_x\tilde{v}^\varepsilon \end{pmatrix}(0,\xi) = \begin{pmatrix} \tilde{f}_1(\xi) \\ \tilde{f}_2(\xi) \end{pmatrix} \tag{3.2}$$

With appropriate boundary data $(\tilde{u}^\varepsilon(0,\xi), \partial_x\tilde{u}^\varepsilon(0,\xi))$ and $(\tilde{v}^\varepsilon(0,\xi), \partial_x\tilde{v}^\varepsilon(0,\xi))$, the solution to (3.1) can be expressed in the following form [15]

$$\begin{cases} \tilde{u}^\varepsilon(x,\xi) &= A_+(x,\xi,\varepsilon)e^{\omega_{1+}x} + A_-(x,\xi,\varepsilon)e^{\omega_{1-}x} \\ \tilde{v}^\varepsilon(x,\xi) &= B_+(x,\xi,\varepsilon)e^{\omega_{2+}x} + B_-(x,\xi,\varepsilon)e^{\omega_{2-}x} \end{cases} \tag{3.3}$$

where

$$\omega_{1\pm}(\xi,\varepsilon) = \frac{1 \pm \sqrt{1+4\varepsilon\xi}}{2\varepsilon}, \quad \omega_{2\pm}(\xi,\varepsilon) = \frac{-1 \pm \sqrt{1+4\varepsilon\xi}}{2\varepsilon} \tag{3.4}$$

and

$$\begin{aligned}
A_+ &= \frac{1}{\omega_{1+}-\omega_{1-}}\left(-\omega_{1-}\tilde{u}^\varepsilon(0,\xi)+\partial_x\tilde{u}^\varepsilon(0,\xi)-\varepsilon^{-1}\int_0^x e^{-\omega_{1+}\eta}u_0(\eta)\,d\eta\right)\\
A_- &= \frac{1}{\omega_{1+}-\omega_{1-}}\left(\omega_{1+}\tilde{u}^\varepsilon(0,\xi)-\partial_x\tilde{u}^\varepsilon(0,\xi)+\varepsilon^{-1}\int_0^x e^{-\omega_{1-}\eta}u_0(\eta)\,d\eta\right)\\
B_+ &= \frac{1}{\omega_{2+}-\omega_{2-}}\left(-\omega_{2-}\tilde{v}^\varepsilon(0,\xi)+\partial_x\tilde{v}^\varepsilon(0,\xi)-\varepsilon^{-1}\int_0^x e^{-\omega_{2+}\eta}v_0(\eta)\,d\eta\right)\\
B_- &= \frac{1}{\omega_{2+}-\omega_{2-}}\left(\omega_{2+}\tilde{v}^\varepsilon(0,\xi)-\partial_x\tilde{v}^\varepsilon(0,\xi)+\varepsilon^{-1}\int_0^x e^{-\omega_{2-}\eta}v_0(\eta)\,d\eta\right)
\end{aligned}$$

The following estimates can be proved directly.

LEMMA 3.1. *For all $\varepsilon > 0$ and $\alpha = \operatorname{Re}\xi > 0$, we have*

$$\operatorname{Re}\omega_{1+} > 0, \quad \operatorname{Re}\omega_{1-} < 0, \quad \operatorname{Re}\omega_{2+} > 0, \quad \operatorname{Re}\omega_{2-} < 0. \tag{3.5}$$

Clearly the boundary data $(\tilde{u}^\varepsilon(0,\xi), \partial_x\tilde{u}^\varepsilon(0,\xi))$ and $(\tilde{v}^\varepsilon(0,\xi), \partial_x\tilde{v}^\varepsilon(0,\xi))$ have to satisfy the boundary condition (3.2). On the other hand, in order to determine a unique solution $(\tilde{u}^\varepsilon(\cdot,\xi), \tilde{v}^\varepsilon(\cdot,\xi)) \in L^2(\mathbb{R}^+)$, we also need to impose the following boundary condition at $x = +\infty$:

$$\tilde{u}^\varepsilon(\infty,\xi) = 0, \quad \tilde{v}^\varepsilon(\infty,\xi) = 0 \tag{3.6}$$

which, by (3.3) and Lemma 3.1, yields,

$$A_+(\infty,\xi,\varepsilon) = 0, \quad B_+(\infty,\xi,\varepsilon) = 0 \tag{3.7}$$

or equivalently,

$$\begin{cases} \partial_x\tilde{u}^\varepsilon(0,\xi) = \omega_{1-}\tilde{u}^\varepsilon(0,\xi) + \varepsilon^{-1}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)d\eta \\ \partial_x\tilde{v}^\varepsilon(0,\xi) = \omega_{2-}\tilde{v}^\varepsilon(0,\xi) + \varepsilon^{-1}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)d\eta \end{cases} \tag{3.8}$$

Substituting (3.8) into (3.2), we now obtain

$$\begin{aligned}
&(d_1+\omega_{1-}e_1)\tilde{u}^\varepsilon(0,\xi) + (d_2+\omega_{2-}e_2)\tilde{v}^\varepsilon(0,\xi) \\
&= \tilde{F}(\xi) - e_1\varepsilon^{-1}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\,d\eta - e_2\varepsilon^{-1}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)\,d\eta
\end{aligned} \tag{3.9}$$

LEMMA 3.2. *For $\alpha = \operatorname{Re}\xi > 0$ (sufficiently large and fixed), there exists an $\varepsilon_0 > 0$ (sufficiently small) such that the following estimate*

$$\Delta = \det(d_1 + \omega_{1-}(\xi,\varepsilon)e_1, d_2 + \omega_{2-}(\xi,\varepsilon)e_2) \neq 0 \tag{3.10}$$

holds in all cases independent of $0 < \varepsilon \leq \varepsilon_0$ and $\beta \in \mathbb{R}$.

Proof. First we note that by using

$$\operatorname{Re}\sqrt{1+4\varepsilon\xi} \geq \sqrt{1+4\varepsilon\alpha} \tag{3.11}$$

we obtain $(i = 1, 2)$

$$|\operatorname{Re}\omega_{i\pm}(\xi,\varepsilon)| \geq \frac{\alpha}{\sqrt{1+4\varepsilon\alpha}} \to \infty \quad \text{as} \quad \alpha\to\infty \tag{3.12}$$

and hence

$$|\omega_{i\pm}(\xi,\varepsilon)| \to \infty \quad \text{as} \quad \alpha\to\infty \tag{3.13}$$

independent of $\beta \in \mathbb{R}$ and $\varepsilon > 0$ (without loss of generality, we assume $\varepsilon \leq 1$).

We now prove the estimate (3.10) separately for the cases I, II(a)-(c).

Case I: $\det E \neq 0$.

In this case, we have

$$\begin{aligned}\Delta &= (\det E)\omega_{1-}\omega_{2-} + \det(d_1,e_2)\omega_{2-} + \det(e_1,d_2)\omega_{1-} + \det D \\ &\approx (\det E)\omega_{1-}\omega_{2-} = (\det E)\varepsilon^{-1}\xi \to \infty \quad \text{as} \quad \alpha \to \infty. \end{aligned} \tag{3.14}$$

Case II(a): $e_{11} \neq 0$, $e_{12} = e_{21} = e_{22} = 0$.

In this case, we have

$$\Delta = d_{22}e_{11}\omega_{1-} + \det D. \tag{3.15}$$

For $d_{22} \neq 0$, clearly we have $|\Delta| \to \infty$ as $\alpha \to \infty$. For $d_{22} = 0$, we must have $d_{21} \neq 0$ and $d_{12} \neq 0$ (see the proof of (2.26)) and hence $\Delta = \det D = -d_{12}d_{21} \neq 0$.

Case II(b): $e_{12} \neq 0$, $d_{21} \neq 0$, $e_{21} = e_{22} = 0$.

In this case, we have

$$\begin{aligned}\Delta &= \det(d_1,e_2)\omega_{2-} + \det(e_1,d_2)\omega_{1-} + \det D \\ &= \frac{1}{2\varepsilon}\left\{2d_{21}e_{12} + (d_{21}e_{12} - e_{11}d_{22})\left(\sqrt{1+4\varepsilon\xi} - 1\right) + 2\varepsilon \det D\right\}\end{aligned}$$

In the case $d_{21}e_{12} - e_{11}d_{22} = 0$, it is clear that since $e_{12} \neq 0$, $d_{21} \neq 0$, by taking ε sufficiently small, we have $|\Delta| \geq O(1)\varepsilon^{-1}$ independent of ξ.

Next we assume $d_{21}e_{12} - e_{11}d_{22} \neq 0$. Since $\sqrt{1+4\varepsilon\xi}$ is continuous in the half plane $\operatorname{Re}\xi \geq 0$, we have

$$\left|\sqrt{1+4\varepsilon\xi} - 1\right| \leq o(1)$$

and hence

$$|\Delta| \geq \frac{1}{2\varepsilon}\left\{2|d_{21}e_{12}| - |d_{21}e_{12} - e_{11}d_{22}|\left|\sqrt{1+4\varepsilon\xi} - 1\right| - 2\varepsilon|\det D|\right\} \geq O(1)\varepsilon^{-1}$$

for $\varepsilon|\xi| \leq \delta_0$ with δ_0 sufficiently small.

For $\varepsilon|\xi| > \delta_0$, since $\alpha > 0$ is fixed, we may choose ε_0 sufficiently small such that $\varepsilon\alpha < \delta_0/2$ and hence $\varepsilon|\beta| > \delta_0/2$. By using the following estimate

$$\left|\operatorname{Im}\sqrt{1+4\varepsilon\xi}\right| = \frac{2\varepsilon|\beta|}{\operatorname{Re}\sqrt{1+4\varepsilon\xi}} \geq \frac{2\varepsilon|\beta|}{|\sqrt{1+4\varepsilon\xi}|} = \frac{2\varepsilon|\beta|}{\sqrt{(1+4\varepsilon\alpha)^2 + (4\varepsilon\beta)^2}}$$

and the monotonicity (in $\varepsilon|\beta|$) of the right hand side in the above inequality, we obtain again

$$|\Delta| \geq |\operatorname{Im}\Delta| \geq O(1)\varepsilon^{-1}\left|\operatorname{Im}\sqrt{1+4\varepsilon\xi}\right| \geq O(1)\varepsilon^{-1}$$

Case II(c): $e_{12} \neq 0$, $d_{21} = e_{21} = e_{22} = 0$.

In this case,

$$\Delta = d_{22}(d_{11} + e_{11}\omega_{1-}) \tag{3.16}$$

Since we must have $d_{22} \neq 0$ and $(d_{11}, e_{11}) \neq (0,0)$, by choosing α sufficiently large, we can always guarantee $\Delta \neq 0$.

The proof of Lemma 3.2 is now complete.

With Lemma 3.2, we can now solve (3.9) to obtain the desired boundary data $(\tilde{u}^\varepsilon(0,\xi), \tilde{v}^\varepsilon(0,\xi))$:

$$\begin{aligned}\tilde{u}^\varepsilon(0,\xi) &= \Delta^{-1}\det(\tilde{F}(\xi), d_2+\omega_{2-}e_2)\\ &\quad -(\varepsilon\Delta)^{-1}\det(e_1, d_2+\omega_{2-}e_2)\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\,d\eta\\ &\quad -(\varepsilon\Delta)^{-1}\det(e_2,d_2)\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)\,d\eta \end{aligned}\tag{3.17}$$

$$\begin{aligned}\tilde{v}^\varepsilon(0,\xi) &= -\Delta^{-1}\det(\tilde{F}(\xi), d_1+\omega_{1-}e_1)\\ &\quad +(\varepsilon\Delta)^{-1}\det(e_1, d_1)\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\,d\eta\\ &\quad +(\varepsilon\Delta)^{-1}\det(e_2,d_1+\omega_{1-}e_1)\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)\,d\eta \end{aligned}\tag{3.18}$$

and hence the following solution representations for $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon$

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &= \tilde{u}^\varepsilon_{\mathrm{I}}(x,\xi)+\tilde{u}^\varepsilon_{\mathrm{II}}(x,\xi)+\tilde{u}^\varepsilon_{\mathrm{III}}(x,\xi)+\tilde{u}^\varepsilon_{\mathrm{IV}}(x,\xi)+\tilde{u}^\varepsilon_{\mathrm{V}}(x,\xi)\\ \tilde{v}^\varepsilon(x,\xi) &= \tilde{v}^\varepsilon_{\mathrm{I}}(x,\xi)+\tilde{v}^\varepsilon_{\mathrm{II}}(x,\xi)+\tilde{v}^\varepsilon_{\mathrm{III}}(x,\xi)+\tilde{v}^\varepsilon_{\mathrm{IV}}(x,\xi)+\tilde{v}^\varepsilon_{\mathrm{V}}(x,\xi)\end{aligned}\tag{3.19}$$

where

$$\begin{aligned}\tilde{u}^\varepsilon_{\mathrm{I}}(x,\xi) &= \Delta^{-1}\det(\tilde{F}, d_2+\omega_{2-}e_2)e^{\omega_{1-}x}\\ \tilde{v}^\varepsilon_{\mathrm{I}}(x,\xi) &= -\Delta^{-1}\det(\tilde{F}(\xi), d_1+\omega_{1-}e_1)e^{\omega_{2-}x}\end{aligned}\tag{3.20}$$

$$\begin{aligned}\tilde{u}^\varepsilon_{\mathrm{II}}(x,\xi) &= \frac{1}{\varepsilon(\omega_{1+}-\omega_{1-})}\int_0^x e^{\omega_{1-}(x-\eta)}u_0(\eta)d\eta\\ &\quad +\frac{1}{\varepsilon(\omega_{1+}-\omega_{1-})}\int_x^\infty e^{\omega_{1+}(x-\eta)}u_0(\eta)d\eta\end{aligned}\tag{3.21}$$

$$\begin{aligned}\tilde{v}^\varepsilon_{\mathrm{II}}(x,\xi) &= \frac{1}{\varepsilon(\omega_{2+}-\omega_{2-})}\int_0^x e^{\omega_{2-}(x-\eta)}v_0(\eta)d\eta\\ &\quad +\frac{1}{\varepsilon(\omega_{2+}-\omega_{2-})}\int_x^\infty e^{\omega_{2+}(x-\eta)}v_0(\eta)d\eta\end{aligned}\tag{3.22}$$

$$\begin{aligned}\tilde{u}^\varepsilon_{\mathrm{III}}(x,\xi) &= -\frac{1}{\varepsilon(\omega_{1+}-\omega_{1-})}e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)d\eta\\ \tilde{v}^\varepsilon_{\mathrm{III}}(x,\xi) &= -\frac{1}{\varepsilon(\omega_{2+}-\omega_{2-})}e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)d\eta\end{aligned}$$

$$\begin{aligned}\tilde{u}^\varepsilon_{\mathrm{IV}}(x,\xi) &= -(\varepsilon\Delta)^{-1}\det(e_1, d_2+\omega_{2-}e_2)e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)d\eta\\ \tilde{v}^\varepsilon_{\mathrm{IV}}(x,\xi) &= (\varepsilon\Delta)^{-1}\det(e_1,d_1)e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)d\eta\end{aligned}\tag{3.23}$$

$$\begin{aligned}\tilde{u}^\varepsilon_{\mathrm{V}}(x,\xi) &= -(\varepsilon\Delta)^{-1}\det(e_2,d_2)e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)d\eta\\ \tilde{v}^\varepsilon_{\mathrm{V}}(x,\xi) &= (\varepsilon\Delta)^{-1}\det(e_2, d_1+\omega_{1-}e_1)e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)d\eta\end{aligned}\tag{3.24}$$

In view of the results in the scalar case [15], it is not difficult to see that the first part $(\tilde{u}^\varepsilon_{\mathrm{I}}, \tilde{v}^\varepsilon_{\mathrm{I}})$ in the above solution decomposition corresponds to the Laplace transform of the solution of the IBVP (1.1)-(1.3) with zero initial data, that is, $u_0(x) = v_0(x) = 0$. All other terms depend on $(u_0(x), v_0(x))$ and vanish when $u_0(x) = v_0(x) = 0$.

Similarly it can be easily checked that the second part $(\tilde{u}^\varepsilon_{\mathrm{II}}, \tilde{v}^\varepsilon_{\mathrm{II}})$, corresponds to the Laplace transform of the solution of the Cauchy problem of (1.1) with initial data

$$(u^\varepsilon_{\mathrm{II}}(x,0), v^\varepsilon_{\mathrm{II}}(x,0)) = \begin{cases} (u_0(x), v_0(x)) & x \ge 0 \\ (0,0) & x < 0 \end{cases} \tag{3.25}$$

and the third part $(\tilde{u}^\varepsilon_{\mathrm{III}}, \tilde{v}^\varepsilon_{\mathrm{III}})$ corresponds to the Laplace transform of the solution of (1.1) with zero initial condition

$$u^\varepsilon_{\mathrm{III}}(x,0) = 0, \quad v^\varepsilon_{\mathrm{III}}(x,0) = 0 \tag{3.26}$$

and the following (decoupled) Dirichlet boundary condition

$$u^\varepsilon_{\mathrm{III}}(0,t) = -u^\varepsilon_{\mathrm{II}}(0,t), \quad v^\varepsilon_{\mathrm{III}}(0,t) = -v^\varepsilon_{\mathrm{II}}(0,t) \tag{3.27}$$

Note that these two parts have exactly the same form as in the scalar case, see [15].

Finally, by linearity, it is clear that the last two parts $(\tilde{u}^\varepsilon_{\mathrm{IV}} + \tilde{u}^\varepsilon_{\mathrm{V}}, \tilde{v}^\varepsilon_{\mathrm{IV}} + \tilde{v}^\varepsilon_{\mathrm{V}})$ corresponds to the Laplace transform of the solution of (1.1) with zero initial condition

$$u^\varepsilon_{\mathrm{IV}}(x,0) + u^\varepsilon_{\mathrm{V}}(x,0) = 0, \quad v^\varepsilon_{\mathrm{IV}}(x,0) + v^\varepsilon_{\mathrm{V}}(x,0) = 0$$

and the following boundary condition

$$D \begin{pmatrix} u^\varepsilon_{\mathrm{IV}} + u^\varepsilon_{\mathrm{V}} \\ v^\varepsilon_{\mathrm{IV}} + v^\varepsilon_{\mathrm{V}} \end{pmatrix}(0,t) + E\partial_x \begin{pmatrix} u^\varepsilon_{\mathrm{IV}} + u^\varepsilon_{\mathrm{V}} \\ v^\varepsilon_{\mathrm{IV}} + v^\varepsilon_{\mathrm{V}} \end{pmatrix}(0,t) = -E\partial_x \begin{pmatrix} u^\varepsilon_{\mathrm{II}} + u^\varepsilon_{\mathrm{III}} \\ v^\varepsilon_{\mathrm{II}} + v^\varepsilon_{\mathrm{III}} \end{pmatrix}(0,t).$$

We will consider these five parts separately in the next two sections. The convergence analysis for the first part $(u^\varepsilon_{\mathrm{I}}, v^\varepsilon_{\mathrm{I}})$ is the most straightforward and will be carried out in Section 4. The main difficulty is with the nonzero initial data effect, particularly the last part $(u^\varepsilon_{\mathrm{V}}, v^\varepsilon_{\mathrm{V}})$. This will be done in Section 5.

4. **Convergence Proof: Zero Initial Data Case.** With the explicit solution representation obtained in the last section, we are now ready to prove the convergence estimates as stated in Theorem 1.2 and Theorem 1.3. For simplicity of presentation, we consider the zero initial data case first and assume $u_0(x) \equiv 0$, $v_0(x) \equiv 0$ in this section. For convenience, we drop the subscripts and still use u^ε and v^ε instead of $u^\varepsilon_{\mathrm{I}}$ and $v^\varepsilon_{\mathrm{I}}$. Therefore, we have,

$$\begin{aligned} \tilde{u}^\varepsilon(x,\xi) &= \Delta^{-1} \det(\tilde{F}, d_2 + \omega_{2-} e_2) e^{\omega_{1-} x} \\ \tilde{v}^\varepsilon(x,\xi) &= -\Delta^{-1} \det(\tilde{F}, d_1 + \omega_{1-} e_1) e^{\omega_{2-} x} \end{aligned} \tag{4.1}$$

where

$$\Delta = \det(d_1 + \omega_{1-} e_1, d_2 + \omega_{2-} e_2) \tag{4.2}$$

satisfies (see the proof of Lemma 3.2)

$$|\Delta| \ge \begin{cases} O(1)\varepsilon^{-1}|\xi| & \text{Case I} \\ O(1)\varepsilon^{-1} & \text{Case II(b)} \\ O(1) & \text{Case II(a) and Case II(c)} \end{cases} \tag{4.3}$$

for some constant $O(1)$ independent of ε and β.

The following uniform estimates can be found in [15] and will be used extensively in our proof.

Lemma 4.1 (Asymptotic estimates on $\omega_\pm(\xi,\varepsilon)$). *There exists a constant $O(1)$ independent of ε and β such that*

$$\frac{1}{\operatorname{Re}\omega_{1+}(\xi,\varepsilon)} \le O(1)\varepsilon, \quad \frac{1}{-\operatorname{Re}\omega_{1-}(\xi,\varepsilon)} \le O(1) \tag{4.4}$$

$$\frac{1}{\operatorname{Re}\omega_{2+}(\xi,\varepsilon)} \le O(1), \quad \frac{1}{-\operatorname{Re}\omega_{2-}(\xi,\varepsilon)} \le O(1)\varepsilon \tag{4.5}$$

$$|\omega_{1+}(\xi,\varepsilon) - 1/\varepsilon| \le O(1)|\xi|, \quad |\omega_{1-}(\xi,\varepsilon) + \xi| \le O(1)\varepsilon|\xi|^2 \tag{4.6}$$

$$|\omega_{2+}(\xi,\varepsilon) - \xi| \le O(1)\varepsilon|\xi|^2, \quad |\omega_{2-}(\xi,\varepsilon) + 1/\varepsilon| \le O(1)|\xi| \tag{4.7}$$

We now consider each case separately.

Case I:

First we note that by taking formal pointwise limit $\varepsilon \to 0$, we obtain from (4.1)

$$\tilde{u}^\varepsilon(x,\xi) \sim \frac{\det(\tilde{F}(\xi), e_2)}{\det(d_1 - \xi e_1, e_2)} e^{-\xi x}, \quad \tilde{v}^\varepsilon(x,\xi) \sim \varepsilon \frac{\det(\tilde{F}(\xi), d_1 - \xi e_1)}{\det(d_1 - \xi e_1, e_2)} e^{-x/\varepsilon} \tag{4.8}$$

This is clearly consistent with the formal expansion results obtained in Section 2:

$$\tilde{u}_{-1}(x,\xi) = 0, \quad \tilde{u}(x,\xi) = \frac{\det(\tilde{F}(\xi), e_2)}{\det(d_1 - \xi e_1, e_2)} e^{-\xi x} \tag{4.9}$$

$$\tilde{v}(x,\xi) = 0, \quad \tilde{v}^{b.l.}(y,\xi) = 0, \quad \tilde{v}_1^{b.l.}(y,\xi) = \frac{\det(\tilde{F}(\xi), d_1 - \xi e_1)}{\det(d_1 - \xi e_1, e_2)} e^{-y} \tag{4.10}$$

The above convergence can be justified easily by using Parseval's relation [12]. First we observe that by using (4.3) and Lemma 4.1, we have

$$|\tilde{v}^\varepsilon(x,\xi)| \le O(1) \left|\tilde{F}(\xi)/\omega_{2-}\right| e^{\operatorname{Re}\omega_{2-}x} \le O(1)\varepsilon \left|\tilde{F}(\xi)\right| e^{\operatorname{Re}\omega_{2-}x} \tag{4.11}$$

and therefore,

$$\begin{aligned} \int_0^\infty \int_0^\infty |v^\varepsilon(x,t)|^2 e^{-2\alpha t} dx dt &= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty |\tilde{v}^\varepsilon(x,\alpha + i\beta)|^2 dx d\beta \\ &\le O(1)\varepsilon^2 \sup_\beta \frac{1}{-\operatorname{Re}\omega_{2-}(\xi,\varepsilon)} \int_{-\infty}^\infty \left|\tilde{F}(\alpha + i\beta)\right|^2 d\beta \\ &\le O(1)\varepsilon^3 \int_0^\infty |F(t)|^2 e^{-2\alpha t}\, dt \le O(1)\varepsilon^3 ||F||_{L^2}^2. \end{aligned} \tag{4.12}$$

This proves rigorously the asymptotic convergence $v^\varepsilon \to v = 0$ in the zero viscosity limit $\varepsilon \to 0$. The leading boundary layer $v^{b.l.}$ also vanishes in the present case.

Next we consider the u^ε component. We have seen that the convergence $\tilde{u}^\varepsilon \to \tilde{u}$ holds for all x and ξ as $\varepsilon \to 0$. On the other hand, by direct integration, it can be easily checked that

$$\int_0^\infty \int_{-\infty}^\infty \left(|\tilde{u}^\varepsilon(x,\xi)|^2 + |\tilde{u}(x,\xi)|^2\right) dx d\beta \le O(1) \int_{-\infty}^\infty \left|\tilde{F}(\xi)\right|^2 d\beta \tag{4.13}$$

Therefore, by Lebesgue's dominated convergence theorem (and assuming $F \in L^2$), we obtain

$$\int_{-\infty}^\infty \int_0^\infty |\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)|^2\, dx d\beta \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{4.14}$$

which, by Parseval's identity, implies

$$\int_0^\infty \int_0^\infty |u^\varepsilon(x,t) - u(x,t)|^2 e^{-2\alpha t}\, dxdt \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.15}$$

In order to obtain optimal convergence rate, we rewrite $\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)$ as

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi) &= \Delta^{-1}\det(\tilde{F}, d_2)e^{\omega_{1-}x} \\ &\quad + \left(\frac{\omega_{2-}}{\Delta} - \frac{1}{\det(d_1 - \xi e_1, e_2)}\right)\det(\tilde{F}, e_2)e^{\omega_{1-}x} \\ &\quad + \frac{1}{\det(d_1 - \xi e_1, e_2)}\det(\tilde{F}, e_2)\left(e^{\omega_{1-}x} - e^{-\xi x}\right)\end{aligned} \tag{4.16}$$

Next, we note that by choosing α sufficiently large and ε sufficiently small, we have (see Lemma 3.2)

$$|\Delta| \geq O(1)|\omega_{1-}\omega_{2-}| \geq O(1)\varepsilon^{-1}|\xi|, \quad |\det(d_1 - \xi e_1, e_2)| \geq O(1)|\xi| \tag{4.17}$$

Furthermore, by using Lemma 4.1, we have

$$\left|\frac{\omega_{2-}}{\Delta} - \frac{1}{\det(d_1 - \xi e_1, e_2)}\right| \leq O(1)\varepsilon|\xi| \tag{4.18}$$

and

$$\int_0^\infty \left|e^{\omega_{1-}x} - e^{-\xi x}\right|^2 dx \leq O(1)\varepsilon^2|\xi|^4 \tag{4.19}$$

Combining the above, we obtain easily

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty |\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)|^2\, dxd\beta &\leq O(1)\varepsilon^2 \int_{-\infty}^\infty |\xi|^2 \left|\tilde{F}(\xi)\right|^2 d\beta \\ \leq O(1)\varepsilon^2 \int_0^\infty |F'(t)|^2 e^{-2\alpha t}dt &\leq O(1)\varepsilon^2 ||F||_{H^1}^2\end{aligned} \tag{4.20}$$

Case II(a):

In this case, we have

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &= \Delta^{-1}\det(\tilde{F}, d_2)e^{\omega_{1-}x} \\ \tilde{v}^\varepsilon(x,\xi) &= -\Delta^{-1}\det(\tilde{F}, d_1 + \omega_{1-}e_1)e^{\omega_{2-}x}\end{aligned} \tag{4.21}$$

with

$$\Delta = \det(d_1 + \omega_{1-}e_1, d_2) = d_{22}e_{11}\omega_{1-} + \det D \tag{4.22}$$

Taking the formal pointwise limit $\varepsilon \to 0$, we have

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &\sim \tilde{u}(x,\xi) = \frac{1}{\det(d_1 - \xi e_1, d_2)}\det(\tilde{F}, d_2)e^{-\xi x} \\ \tilde{v}^\varepsilon(x,\xi) &\sim \tilde{v}^{b.l.}(x,\xi) = -\frac{1}{\det(d_1 - \xi e_1, d_2)}\det(\tilde{F}, d_1 - \xi e_1)e^{-x/\varepsilon}\end{aligned} \tag{4.23}$$

The proof of the convergence $\tilde{u}^\varepsilon \to \tilde{u}$ as $\varepsilon \to 0$ is similar to that in Case I. First we note that, similar to (4.16), $\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)$ can be written as

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi) &= \left(\Delta^{-1} - \det(d_1 - \xi e_1, d_2)^{-1}\right)\det(\tilde{F}, d_2)e^{-\xi x} \\ &\quad + \det(d_1 - \xi e_1, d_2)^{-1}\det(\tilde{F}, d_2)\left(e^{\omega_{1-}x} - e^{-\xi x}\right)\end{aligned} \tag{4.24}$$

Next, by using Lemma 4.1 and the following estimates

$$|\Delta| \geq O(1), \quad |\det(d_1 - \xi e_1, d_2)| \geq O(1) \tag{4.25}$$

we have

$$\left|\Delta^{-1} - \det(d_1 - \xi e_1, d_2)^{-1}\right| \leq O(1)\varepsilon|\xi| \tag{4.26}$$

Therefore, using (4.26) and (4.19) (and assuming $F(t) \in H^2$ with $F(0) = F'(0) = 0$), we obtain the following convergence in u^ε

$$\int_0^\infty \int_{-\infty}^\infty |\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)|^2 \, dx d\beta \leq O(1)\varepsilon^2 \int_{-\infty}^\infty |\xi|^4 \left|\tilde{F}(\xi)\right|^2 d\beta$$
$$\leq O(1)\varepsilon^2 \int_0^\infty \left(|F'(t)|^2 + |F''(t)|^2\right) e^{-2\alpha t} dt \leq O(1)\varepsilon^2 ||F||_{H^2}^2 \tag{4.27}$$

We now turn to the v^ε component. First it is clear that by using (4.25) and the estimate $|\omega_{1-}(\xi,\varepsilon)| \leq O(1)|\xi|$, we have from (4.21),

$$|\tilde{v}^\varepsilon(x,\xi)| \leq O(1) \left|\xi \tilde{F(\xi)} e^{\omega_{2-}(\xi,\varepsilon)x}\right| \tag{4.28}$$

Therefore, similar to (4.12), we obtain (assuming $F \in H^1$ with $F(0) = 0$),

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty |\tilde{v}^\varepsilon(x,\xi)|^2 \, dx d\beta &\leq O(1) \sup_\beta \frac{1}{-\mathrm{Re}\,\omega_{2-}(\xi,\varepsilon)} \int_{-\infty}^\infty \left|\xi \tilde{F}(\xi)\right|^2 d\beta \\ &\leq O(1)\varepsilon ||F||_{H^1}^2 \end{aligned} \tag{4.29}$$

This again proves rigorously the asymptotic convergence of $v^\varepsilon \to v = 0$ in the zero viscosity limit $\varepsilon \to 0$. Note that, due to the presence of the boundary layer $v^{b.l.}$ (see (4.23)), the convergence rate of $v^\varepsilon \to v = 0$ is now one order lower than in Case I, see (4.12).

The validity of the leading boundary layer $v^{b.l.}$ can be proved as follows. First, by using (4.25) and Lemma 4.1, it can be easily checked that

$$\begin{aligned} \left|\tilde{v}^\varepsilon(x,\xi) - \tilde{v}^{b.l.}(x,\xi)\right| &\leq O(1)\varepsilon|\xi|^2 \left|\tilde{F}(\xi)\right| |e^{\omega_{2-}x}| \\ &\quad + O(1)|\xi| \left|\tilde{F}(\xi)\right| \left|e^{\omega_{2-}x} - e^{-x/\varepsilon}\right| \end{aligned} \tag{4.30}$$

Next, from Lemma 4.1, it follows

$$\int_0^\infty |e^{\omega_{2-}x}|^2 \, dx \leq O(1)\varepsilon \tag{4.31}$$

$$\int_0^\infty \left|e^{\omega_{2-}x} - e^{-x/\varepsilon}\right|^2 dx \leq O(1)\varepsilon^3|\xi|^2 \tag{4.32}$$

and hence

$$\int_0^\infty |\tilde{v}^\varepsilon(x,\xi) - \tilde{v}^{b.l.}(x,\xi)|^2 \, dx \leq O(1)\varepsilon^3|\xi|^4|\tilde{F}(\xi)|^2 \tag{4.33}$$

The desired boundary layer estimate

$$\int_0^\infty \int_{-\infty}^\infty |\tilde{v}^\varepsilon(x,\xi) - \tilde{v}^{b.l.}(x,\xi)|^2 dx d\beta \leq O(1)\varepsilon^3 \|F\|_{H^2}^2 \tag{4.34}$$

now follows easily from Parseval's relation (assuming $F \in H^2$ with $F(0) = F'(0) = 0$).

Case II(b):

In this case, we have

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &= \Delta^{-1}\det(\tilde{F}, d_2+\omega_{2-}e_2)e^{\omega_{1-}x}\\ \tilde{v}^\varepsilon(x,\xi) &= -\Delta^{-1}\det(\tilde{F}, d_1+\omega_{1-}e_1)e^{\omega_{2-}x}\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &\sim \tilde{u}(x,\xi) = \tilde{f}_2(\xi)/d_{21}e^{-\xi x}\\ \tilde{v}^\varepsilon(x,\xi) &\sim \tilde{v}^{b.l.}(x,\xi) = -\varepsilon\det(\tilde{F}, d_1-\xi e_1)/(d_{21}e_{12})e^{-x/\varepsilon}\end{aligned} \tag{4.36}$$

where

$$\Delta = \det(d_1,e_2)\omega_{2-} + \det(e_1,d_2)\omega_{1-} + \det D \tag{4.37}$$

First we observe that by using $|\Delta| \geq O(1)\varepsilon^{-1}$ and $|\omega_{1-}(\xi,\varepsilon)| \leq O(1)|\xi|$, it follows immediately that

$$|\tilde{v}^\varepsilon(x,\xi)| \leq O(1)\varepsilon|\xi|\left|\tilde{F}(\xi)e^{\omega_{2-}x}\right| \tag{4.38}$$

and therefore

$$\int_0^\infty\int_{-\infty}^\infty |\tilde{v}^\varepsilon(x,\xi)|^2\,d\beta \leq O(1)\varepsilon^3\|F\|_{H^1}^2 \tag{4.39}$$

As in Case I, the above estimate establishes the convergence of $v^\varepsilon \to v = 0$ in the zero viscosity limit. It also shows that the boundary layer $v^{b.l.}$ vanishes at the leading order.

Next we prove the convergence of $u^\varepsilon \to u$. Similarly to (4.16), we have

$$\begin{aligned}\tilde{u}^\varepsilon - \tilde{u} &= \Delta^{-1}\det(\tilde{F}, d_2)e^{\omega_{1-}x} + \left(\frac{\omega_{2-}}{\Delta} - \frac{1}{\det(d_1,e_2)}\right)\det(\tilde{F}, e_2)e^{\omega_{1-}x}\\ &\quad + \frac{1}{\det(d_1,e_2)}\det(\tilde{F},e_2)\left(e^{\omega_{1-}x} - e^{-\xi x}\right)\end{aligned} \tag{4.40}$$

Furthermore, similar to (4.18), it holds,

$$\left|\frac{\omega_{2-}}{\Delta} - \frac{1}{\det(d_1,e_2)}\right| = \frac{|\omega_{1-}\det(e_1,d_2]+\det D|}{|\Delta\det(d_1,e_2)|} \leq O(1)\varepsilon|\xi| \tag{4.41}$$

Therefore, similar to (4.20), by using (4.19), (4.41) and $|\Delta| \geq O(1)\varepsilon^{-1}$ (and assuming $F \in H^2$ with $F(0) = F'(0) = 0$), we obtain

$$\int_0^\infty\int_{-\infty}^\infty |\tilde{u}^\varepsilon(x,\xi) - \tilde{u}(x,\xi)|^2\,dxd\beta \leq O(1)\varepsilon^2\|F\|_{H^2}^2 \tag{4.42}$$

Case II(c):

In this case, we have

$$\Delta = \det(d_1+\omega_{1-}e_1, d_2) = d_{22}(d_{11}+\omega_{1-}e_{11}), \tag{4.43}$$

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &= \frac{-\omega_{2-}e_{12}\tilde{f}_2 + \det(\tilde{F},d_2)}{d_{22}(d_{11}+\omega_{1-}e_{11})}e^{\omega_{1-}x}\\ \tilde{v}^\varepsilon(x,\xi) &= \tilde{f}_2(\xi)/d_{22}e^{\omega_{2-}x}\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &\sim \varepsilon^{-1}\tilde{u}_{-1}(x,\xi) = \varepsilon^{-1}\frac{e_{12}\tilde{f}_2(\xi)}{d_{22}(d_{11}-\xi e_{11})}e^{-\xi x}\\ \tilde{v}^\varepsilon(x,\xi) &\sim \tilde{v}^{b.l.}(x,\xi) = \tilde{f}_2(\xi)/d_{22}e^{-x/\varepsilon}\end{aligned} \tag{4.45}$$

Note that in this case, since we have a decoupled Dirichlet boundary condition for the outflow v^ε, that is, $v^\varepsilon(0,t) = f_2(t)/d_{22}$, the validity of the above boundary layer structure for $v^{b.l.}$ (and the convergence of $v^\varepsilon \to v = 0$) follows directly from Theorem 1.1. Indeed, using (4.32) and assuming $f_2 \in H^1$ with $f_2(0) = 0$, we obtain immediately

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty |\tilde{v}^\varepsilon(x,\xi) - \tilde{v}^{b.l.}(x,\xi)|^2 dxd\beta \\ &\leq O(1)\varepsilon^3 \int_{-\infty}^\infty |\xi|^2 \left|\tilde{f}_2(\xi)\right|^2 d\beta \leq O(1)\varepsilon^3 \|f_2\|_{H^1}^2 \end{aligned} \tag{4.46}$$

The u^ε component, on the other hand, now includes an expansive growth term at the order of ε^{-1}, see (4.45). By using the estimates $|\omega_{2-}(\xi,\varepsilon) + 1/\varepsilon| \leq O(1)|\xi|$ and $|\omega_{1-}(\xi,\varepsilon) + \xi| \leq O(1)\varepsilon|\xi|^2$, it can be easily checked that

$$\begin{aligned}&\left|\tilde{u}^\varepsilon(x,\xi) - \varepsilon^{-1}\tilde{u}_{-1}(x,\xi)\right| \\ &\leq O(1)|\xi| \left|\tilde{F}(\xi)e^{\omega_{1-}x}\right| + O(1)\varepsilon^{-1}\left|\tilde{f}_2(\xi)\left(e^{\omega_{1-}x} - e^{-\xi x}\right)\right| \end{aligned} \tag{4.47}$$

and therefore we obtain

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde{u}^\varepsilon(x,\xi) - \varepsilon^{-1}\tilde{u}_{-1}(x,\xi)\right|^2 dxd\beta \leq O(1)\|F\|_{H^2}^2 \tag{4.48}$$

5. **Convergence Proof: Nonzero Initial Data Case.** We now turn to the nonzero initial data effect in the viscous IBVP (1.1)-(1.3) and consider the remaining parts in the solution representations (3.19). Without confusion, we assume $F \equiv 0$ in this section. Then it is clear that

$$\begin{aligned}\tilde{u}^\varepsilon(x,\xi) &= \tilde{u}^\varepsilon_{\mathrm{II}}(x,\xi) + \tilde{u}^\varepsilon_{\mathrm{III}}(x,\xi) + \tilde{u}^\varepsilon_{\mathrm{IV}}(x,\xi) + \tilde{u}^\varepsilon_{\mathrm{V}}(x,\xi) \\ \tilde{v}^\varepsilon(x,\xi) &= \tilde{v}^\varepsilon_{\mathrm{II}}(x,\xi) + \tilde{v}^\varepsilon_{\mathrm{III}}(x,\xi) + \tilde{v}^\varepsilon_{\mathrm{IV}}(x,\xi) + \tilde{v}^\varepsilon_{\mathrm{V}}(x,\xi)\end{aligned} \tag{5.1}$$

solves the following IBVP

$$\begin{cases} \partial_t u^\varepsilon + \partial_x u^\varepsilon_x = \varepsilon\partial_x^2 u^\varepsilon \\ \partial_t v^\varepsilon - \partial_x v^\varepsilon_x = \varepsilon\partial_x^2 v^\varepsilon \end{cases} \tag{5.2}$$

$$u^\varepsilon(x,0) = u_0(x), \quad v^\varepsilon(x,0) = v_0(x) \tag{5.3}$$

$$D\begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}(0,t) + E\begin{pmatrix} \partial_x u^\varepsilon \\ \partial_x v^\varepsilon \end{pmatrix}(0,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.4}$$

5.1. **Estimates on** $(\tilde{u}^\varepsilon_{\mathrm{II}}, \tilde{v}^\varepsilon_{\mathrm{II}})$ **and** $(\tilde{u}^\varepsilon_{\mathrm{III}}, \tilde{v}^\varepsilon_{\mathrm{III}})$**.** Recall that the first two parts $(\tilde{u}^\varepsilon_{\mathrm{II}}, \tilde{v}^\varepsilon_{\mathrm{II}})$ and $(\tilde{u}^\varepsilon_{\mathrm{III}}, \tilde{v}^\varepsilon_{\mathrm{III}})$ in (5.1) correspond to the solution of the Cauchy problem (1.1), (3.25) and the decoupled IBVP (1.1), (3.26)-(3.27) respectively. Let

$$\tilde{u}_{\mathrm{II}}(x,\xi) = \int_0^x e^{-\xi(x-\eta)} u_0(\eta)\, d\eta, \quad \tilde{v}_{\mathrm{II}}(x,\xi) = \int_x^\infty e^{\xi(x-\eta)} v_0(\eta)\, d\eta, \tag{5.5}$$

and

$$\tilde{v}^{b.l.}_{\mathrm{III}}(x/\varepsilon,\xi) = -e^{-x/\varepsilon}\tilde{v}_0(\xi) = -e^{-x/\varepsilon}\int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta \tag{5.6}$$

Then by applying the convergence results in the scalar case [15], we can obtain the following convergence estimates for $(\tilde{u}^\varepsilon_{\mathrm{II}}, \tilde{v}^\varepsilon_{\mathrm{II}})$ and $(\tilde{u}^\varepsilon_{\mathrm{III}}, \tilde{v}^\varepsilon_{\mathrm{III}})$:

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde{u}^\varepsilon_{\mathrm{II}}(x,\xi) - \tilde{u}_{\mathrm{II}}(x,\xi)\right|^2 dxd\beta \leq O(1)\varepsilon^2 \|u_0\|_{H^2}^2 \tag{5.7}$$

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde v^\varepsilon_{\mathrm{II}}(x,\xi) - \tilde v_{\mathrm{II}}(x,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon^2 \|v_0\|^2_{H^2} \tag{5.8}$$

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde u^\varepsilon_{\mathrm{III}}(x,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon^2 \|u_0\|^2_{H^2} \tag{5.9}$$

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde v^\varepsilon_{\mathrm{III}}(x,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon \|v_0\|^2_{L^2} \tag{5.10}$$

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde v^\varepsilon_{\mathrm{III}}(x,\xi) - \tilde v^{b.l.}_{\mathrm{III}}(x/\varepsilon,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon^3 \|v_0\|^2_{H^3} \tag{5.11}$$

5.2. **Estimates on** $(\tilde u^\varepsilon_{\mathrm{IV}}, \tilde v^\varepsilon_{\mathrm{IV}})$**.** Next we consider the third part $(\tilde u^\varepsilon_{\mathrm{IV}}, \tilde v^\varepsilon_{\mathrm{IV}})$ in (5.1):

$$\begin{aligned} \tilde u^\varepsilon_{\mathrm{IV}}(x,\xi) &= -(\varepsilon\Delta)^{-1}\det(e_1, d_2 + \omega_{2-}e_2)e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)d\eta \\ \tilde v^\varepsilon_{\mathrm{IV}}(x,\xi) &= (\varepsilon\Delta)^{-1}\det(e_1, d_1)e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\, d\eta \end{aligned} \tag{5.12}$$

We will show that the effect of $(\tilde u^\varepsilon_{\mathrm{IV}}, \tilde v^\varepsilon_{\mathrm{IV}})$ is negligible and the following estimates

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde u^\varepsilon_{\mathrm{IV}}(x,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon^2 \|u_0\|^2_{H^2} \tag{5.13}$$

$$\int_0^\infty \int_{-\infty}^\infty \left|\tilde v^\varepsilon_{\mathrm{IV}}(x,\xi)\right|^2 \, dxd\beta \le O(1)\varepsilon^3 \|u_0\|^2_{H^2} \tag{5.14}$$

hold in all cases provided the initial data $u_0(x)$ is twice differentiable and satisfies $u_0(0) = u_0'(0) = 0$.

To prove (5.13) and (5.14), we first note that by using (4.3), we have in all cases

$$\left|(\varepsilon\Delta)^{-1}\det(e_1, d_1)\right| \le O(1)\varepsilon^{-1} \tag{5.15}$$

and

$$\left|(\varepsilon\Delta)^{-1}\det(e_1, d_2 + \omega_{2-}e_2)\right| \le O(1)\varepsilon^{-1} \tag{5.16}$$

Furthermore, we have for all $\beta \in \mathbb{R}$,

$$\int_0^\infty \left|e^{\omega_{1-}x}\right|^2 \, dx = \frac{1}{-2\mathrm{Re}\,\omega_{1-}} \le O(1) \tag{5.17}$$

and

$$\int_0^\infty \left|e^{\omega_{2-}x}\right|^2 \, dx = \frac{1}{-2\mathrm{Re}\,\omega_{2-}} \le O(1)\varepsilon \tag{5.18}$$

To finish the proof of (5.13) and (5.14), now it remains to show that

$$\int_{-\infty}^\infty \left|\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\, d\eta\right|^2 \, d\beta \le O(1)\varepsilon^4 ||u_0||^2_{H^2} \tag{5.19}$$

By using Cauchy-Schwarz, it is easy to see that

$$\left|\int_0^\infty e^{-\omega_{1+}\eta}u_0(\eta)\, d\eta\right|^2 \le \int_0^\infty \left|e^{-\omega_{1+}\eta}\right|^2 \, d\eta \int_0^\infty |u_0(\eta)|^2 \, d\eta \le O(1)\varepsilon ||u_0||^2_{L^2} \tag{5.20}$$

However this does not lead to (5.19). Notice that the right hand side of (5.20) is not even integrable with respect to β.

To overcome this difficulty and also to obtain the desired convergence rate, we need to assume that $u_0(x)$ is twice differentiable and satisfies the compatibility condition $u_0(0) = u_0'(0) = 0$. Then a simple integration by parts yields

$$\int_0^\infty e^{-\omega_{1+}\eta} u_0(\eta)\, d\eta = \frac{1}{\omega_{1+}} \int_0^\infty e^{-\omega_{1+}\eta} u_0'(\eta)\, d\eta = \frac{1}{\omega_{1+}^2} \int_0^\infty e^{-\omega_{1+}\eta} u_0''(\eta)\, d\eta \quad (5.21)$$

The new integral term on the right hand side of (5.21) can be similarly estimated as in (5.20) and hence

$$\left| \int_0^\infty e^{-\omega_{1+}\eta} u_0''(\eta)\, d\eta \right|^2 \le O(1)\varepsilon ||u_0||_{H^2} ^2 \quad (5.22)$$

On the other hand, the extra integrated factor $1/\omega_{1+}(\xi,\varepsilon)^2$ now gives the desired convergence rate and integrability with respect to β since

$$\int_{-\infty}^\infty \frac{1}{|\omega_{1+}(\xi,\varepsilon)|^4}\, d\beta \le O(1)\varepsilon^4 \int_{-\infty}^\infty \frac{1}{1+(4\varepsilon\beta)^2}\, d\beta \le O(1)\varepsilon^3 \quad (5.23)$$

The desired estimate (5.19) now follows by combining (5.21)-(5.23). Furthermore we remark that the integral $\int_0^\infty e^{-\omega_{1+}\eta} u_0(\eta)\, d\eta$ and hence $(\tilde{u}^\varepsilon_{\rm IV}, \tilde{v}^\varepsilon_{\rm IV})$ can be arbitrarily small provided the initial data $u_0(x)$ is sufficiently smooth and compatible at $x = 0$. This can be proved by applying additional integration by parts as in (5.21).

5.3. **Estimates on $(\tilde{u}^\varepsilon_{\rm V}, \tilde{v}^\varepsilon_{\rm V})$.** We now turn to the last part $(\tilde{u}^\varepsilon_{\rm V}, \tilde{v}^\varepsilon_{\rm V})$ in the solution representation (5.1). It is clear that from (3.24), we have

$$\begin{aligned} \tilde{u}^\varepsilon_{\rm V}(x,\xi) &= -(\varepsilon\Delta)^{-1} \det(e_2, d_2) e^{\omega_{1-}x} \int_0^\infty e^{-\omega_{2+}\eta} v_0(\eta) d\eta \\ \tilde{v}^\varepsilon_{\rm V}(x,\xi) &= (\varepsilon\Delta)^{-1} \det(e_2, d_1 + \omega_{1-}e_1) e^{\omega_{2-}x} \int_0^\infty e^{-\omega_{2+}\eta} v_0(\eta) d\eta \end{aligned} \quad (5.24)$$

Note that by taking formal pointwise limit, we can obtain from (5.24) the following asymptotic behavior for $(\tilde{u}^\varepsilon_{\rm V}, \tilde{v}^\varepsilon_{\rm V})$ as $\varepsilon \to 0$:

Case I:

$$\begin{aligned} \tilde{u}^\varepsilon_{\rm V}(x,\xi) &\sim \tilde{u}_{\rm V}(x,\xi) = -\frac{\det(e_2, d_2)}{\det(e_2, d_1 - \xi e_1)} e^{-\xi x} \int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta \\ \tilde{v}^\varepsilon_{\rm V}(x,\xi) &\sim \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi) = e^{-x/\varepsilon} \int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta \end{aligned} \quad (5.25)$$

Case II(a):

$$\tilde{u}^\varepsilon_{\rm V}(x,\xi) = \tilde{u}_{\rm V}(x,\xi) \equiv 0, \quad \tilde{v}^\varepsilon_{\rm V}(x,\xi) = \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi) \equiv 0 \quad (5.26)$$

Case II(b):

$$\begin{aligned} \tilde{u}^\varepsilon_{\rm V}(x,\xi) &\sim \tilde{u}_{\rm V}(x,\xi) = -d_{22}/d_{21} e^{-\xi x} \int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta \\ \tilde{v}^\varepsilon_{\rm V}(x,\xi) &\sim \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi) = e^{-x/\varepsilon} \int_0^\infty e^{-\xi\eta} v_0(\eta)\, d\eta \end{aligned} \quad (5.27)$$

Case II(c):

$$\begin{aligned} \tilde{u}^{\varepsilon}_{\mathrm{V}}(x,\xi) &\sim \varepsilon^{-1}\tilde{u}_{-1}(x,\xi) = -\frac{e_{12}}{\varepsilon(d_{11}-\xi e_{11})}e^{-\xi x}\int_0^\infty e^{-\xi\eta}v_0(\eta)\,d\eta \\ \tilde{v}^{\varepsilon}_{\mathrm{V}}(x,\xi) &= \tilde{v}^{b.l.}_{\mathrm{V}}(x/\varepsilon,\xi) = 0 \end{aligned} \tag{5.28}$$

It is worth mentioning that although the expression of $(\tilde{u}^{\varepsilon}_{\mathrm{V}}, \tilde{v}^{\varepsilon}_{\mathrm{V}})$ is very similar to that of $(\tilde{u}^{\varepsilon}_{\mathrm{IV}}, \tilde{v}^{\varepsilon}_{\mathrm{IV}})$, the integral $\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)\,d\eta$ is no longer arbitrarily small as $\varepsilon \to 0$. As a result, the asymptotic behavior of $(\tilde{u}^{\varepsilon}_{\mathrm{V}}, \tilde{v}^{\varepsilon}_{\mathrm{V}})$ now depends crucially on Δ or the type of boundary conditions, see (4.3) or Lemma 3.2.

In view of the convergence estimates already proved in the previous subsections for $(\tilde{u}^{\varepsilon}_{\mathrm{II}}, \tilde{v}^{\varepsilon}_{\mathrm{II}})$, $(\tilde{u}^{\varepsilon}_{\mathrm{III}}, \tilde{v}^{\varepsilon}_{\mathrm{III}})$ and $(\tilde{u}^{\varepsilon}_{\mathrm{IV}}, \tilde{v}^{\varepsilon}_{\mathrm{IV}})$, it is easy to see that the above formal asymptotic convergence results for $(\tilde{u}^{\varepsilon}_{\mathrm{V}}, \tilde{v}^{\varepsilon}_{\mathrm{V}})$ are clearly consistent with those obtained in Section 2 through matched asymptotic expansions. In particular, we note that for type I and type II(b) boundary conditions, the boundary layer effect in the last part $\tilde{v}^{\varepsilon}_{\mathrm{V}}$ exactly offsets that in the second part $\tilde{v}^{\varepsilon}_{\mathrm{III}}$, that is, $\tilde{v}^{b.l.}_{\mathrm{V}} = -\tilde{v}^{b.l.}_{\mathrm{III}}$ Therefore for such boundary conditions the total boundary layer effect for the outlet v^{ε} vanishes at the leading order. In the rest of this subsection, we will show that for these two types of boundary conditions, it holds that

$$\int_0^\infty\int_{-\infty}^\infty \left|\tilde{u}^{\varepsilon}_{\mathrm{V}}(x,\xi) - \tilde{u}_{\mathrm{V}}(x,\xi)\right|^2\,dxd\beta \le \begin{cases} O(1)\varepsilon^2||v_0||^2_{H^3} & \text{Case I} \\ O(1)\varepsilon^2||v_0||^2_{H^3} & \text{Case II(b)} \end{cases} \tag{5.29}$$

and

$$\int_0^\infty\int_{-\infty}^\infty \left|\tilde{v}^{\varepsilon}_{\mathrm{V}}(x,\xi) - \tilde{v}^{b.l.}_{\mathrm{V}}(x/\varepsilon,\xi)\right|^2\,dxd\beta \le O(1)\varepsilon^3||v_0||^2_{H^3} \tag{5.30}$$

provided $v_0 \in H^3$ and satisfies $v_0(0) = v_0'(0) = v_0''(0) = 0$.

For type II(c) boundary conditions, we will show that

$$\int_0^\infty\int_{-\infty}^\infty \left|\tilde{u}^{\varepsilon}_{\mathrm{V}}(x,\xi) - \varepsilon^{-1}\tilde{u}_{-1}(x,\xi)\right|^2\,dxd\beta \le O(1)||v_0||^2_{H^3} \tag{5.31}$$

therefore confirming the secular growth term (at the order of ε^{-1}) in the last part $\tilde{u}^{\varepsilon}_{\mathrm{V}}$.

We now set out to prove (5.29)-(5.30) and (5.31). For simplicity, we assume as usual $v_0 \in H^3$ with $v_0(0) = v_0'(0) = v_0''(0) = 0$ and shall not concern ourselves with possibly weaker or minimum assumptions on v_0.

Case I:

First, by using a similar integration by parts as in (5.21), we can get

$$\tilde{u}^{\varepsilon}_{\mathrm{V}}(x,\xi) = -\frac{\det(e_2,d_2)}{\varepsilon\Delta\omega_{2+}^2}e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0''(\eta)d\eta \tag{5.32}$$

$$\tilde{u}_{\mathrm{V}}(x,\xi) = -\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}e^{-\xi x}\int_0^\infty e^{-\xi\eta}v_0''(\eta)\,d\eta \tag{5.33}$$

Therefore, we have

$$
\begin{aligned}
&\tilde{u}^{\varepsilon}_{\rm V}(x,\xi) - \tilde{u}_{\rm V}(x,\xi)\\
=& -\det(e_2,d_2)\left(\frac{1}{\varepsilon\Delta\omega_{2+}^2} - \frac{1}{\det(e_2,d_1-\xi e_1)\xi^2}\right)e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0''(\eta)d\eta\\
&-\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}\left(e^{\omega_{1-}x}-e^{-\xi x}\right)\int_0^\infty e^{-\omega_{2+}\eta}v_0''(\eta)\,d\eta\\
&-\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}e^{-\xi x}\int_0^\infty \left(e^{-\omega_{2+}\eta}-e^{-\xi\eta}\right)v_0''(\eta)\,d\eta
\end{aligned}
\tag{5.34}
$$

Note that the last two terms in (5.34) can be easily estimated by routine calculations. For example, let us consider the last term in (5.34). First, it is clear that for type I boundary conditions, it holds that

$$
\left|\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}\right| \le O(1)|\xi|^{-3}
\tag{5.35}
$$

Next by using Cauchy-Schwarz, we have

$$
\begin{aligned}
&\left|\int_0^\infty \left(e^{-\omega_{2+}\eta}-e^{-\xi\eta}\right)v_0''(\eta)\,d\eta\right|^2 \qquad (5.36)\\
\le& \int_0^\infty \left|e^{-\omega_{2+}\eta}-e^{-\xi\eta}\right|^2\,d\eta\int_0^\infty \left|v_0''(\eta)\right|^2\,d\eta \le O(1)\varepsilon^2|\xi|^4||v_0||_{H^2}^2
\end{aligned}
$$

Therefore it follows

$$
\begin{aligned}
&\int_0^\infty\int_{-\infty}^\infty\left|\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}e^{-\xi x}\int_0^\infty\left(e^{-\omega_{2+}\eta}-e^{-\xi\eta}\right)v_0''(\eta)\,d\eta\right|^2\,dxd\beta\\
\le& O(1)\varepsilon^2||v_0||_{H^2}^2\int_0^\infty\int_{-\infty}^\infty|\xi|^{-2}\left|e^{-\xi x}\right|^2\,dxd\beta \le O(1)\varepsilon^2||v_0||_{H^2}^2
\end{aligned}
\tag{5.37}
$$

Similarly, it can be proved that

$$
\begin{aligned}
&\int_0^\infty\int_{-\infty}^\infty\left|\frac{\det(e_2,d_2)}{\det(e_2,d_1-\xi e_1)\xi^2}\left(e^{\omega_{1-}x}-e^{-\xi x}\right)\int_0^\infty e^{-\omega_{2+}\eta}v_0''(\eta)\,d\eta\right|^2\,dxd\beta\\
\le& O(1)\varepsilon^2||v_0||_{H^2}^2
\end{aligned}
\tag{5.38}
$$

To finish the proof of (5.34), we now only have to show that

$$
\left|\frac{1}{\varepsilon\Delta\omega_{2+}^2} - \frac{1}{\det(e_2,d_1-\xi e_1)\xi^2}\right| \le O(1)\varepsilon|\xi|^{-1}
\tag{5.39}
$$

For this purpose, we will need the following slightly stronger estimates than given in Lemma 4.1:

$$
|\omega_{2+}| \approx O(1)|\xi|/\sqrt{|1+4\varepsilon\xi|},\quad |\omega_{2+}-\xi| \le O(1)\varepsilon|\xi|^2/|1+4\varepsilon\xi|
\tag{5.40}
$$

Then it can be easily checked that

$$
\left|\det(e_2,d_1-\xi e_1)\xi^2 - \varepsilon\Delta\omega_{2+}^2\right| \le O(1)\varepsilon|\xi|^4/|1+4\varepsilon\xi|
\tag{5.41}
$$

The desired estimate (5.39) now follows easily. This finishes the proof of (5.29).

Next we prove the boundary layer estimate (5.30). It is clear that by using integration by parts (three times), one can rewrite $\tilde{v}^{\varepsilon}_{\rm V}(x,\xi) - \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi)$ as

$$\begin{aligned}\tilde{v}^{\varepsilon}_{\rm V}(x,\xi) - \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi) &= \left(\frac{\det(e_2, d_1+\omega_{1-}e_1)}{\varepsilon\Delta\omega_{2+}^3} - \frac{1}{\xi^3}\right) e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta) d\eta \\ &\quad + \frac{1}{\xi^3}\left(e^{\omega_{2-}x} - e^{-x/\varepsilon}\right)\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta)\, d\eta \\ &\quad + \frac{1}{\xi^3} e^{-x/\varepsilon}\int_0^\infty \left(e^{-\omega_{2+}\eta} - e^{-\xi\eta}\right) v_0'''(\eta)\, d\eta \qquad (5.42)\end{aligned}$$

By using

$$\left|\frac{\det(e_2, d_1+\omega_{1-}e_1)}{\varepsilon\Delta\omega_{2+}^3} - \frac{1}{\xi^3}\right| \le O(1)\varepsilon|\xi|^{-1} \qquad (5.43)$$

and a similar analysis as in the above, the desired boundary layer estimate (5.30) can be proved easily.

Finally we remark that by exploiting the formal asymptotic expansion results obtained in Section 2, it is actually more convenient to prove the following convergence estimate

$$\int_0^\infty \int_{-\infty}^\infty |\tilde{v}^{\varepsilon}_{\rm III}(x,\xi) + \tilde{v}^{\varepsilon}_{\rm V}(x,\xi)|\, dx d\beta \le O(1)\varepsilon^3 ||v_0||^2_{H^2} \qquad (5.44)$$

directly for both type I and type II(b) boundary conditions. Note that (5.44) implies the boundary layer estimate (5.30) and requires only $v_0 \in H^2$ and $v_0(0) = v_0'(0) = 0$.

To prove (5.44), we only have to note that for both type I and type II(b) boundary conditions, it holds

$$\left|\frac{1}{\varepsilon(\omega_{2+}-\omega_{2-})} - \frac{\det(e_2, d_1+\omega_{1-}e_1)}{\varepsilon\Delta}\right| \le O(1)\varepsilon|\xi|^{-1}\,|\omega_{2+}|^2 \qquad (5.45)$$

Then by using an integration by parts, namely,

$$\int_0^\infty e^{-\omega_{2+}\eta} v_0(\eta)\, d\eta = \frac{1}{\omega_{2+}^2}\int_0^\infty e^{-\omega_{2+}\eta} v_0''(\eta)\, d\eta, \qquad (5.46)$$

(5.44) follows immediately.

Case II(b):

In this case, we have

$$\begin{aligned}\tilde{u}^{\varepsilon}_{\rm V}(x,\xi) - \tilde{u}_{\rm V}(x,\xi) &= -\frac{d_{22}}{d_{21}}\left(\frac{e_{12}d_{21}}{\varepsilon\Delta\omega_{2+}^3} - \frac{1}{\xi^3}\right) e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta) d\eta \\ &\quad - \frac{d_{22}}{d_{21}\xi^3}\left(e^{\omega_{1-}x} - e^{-\xi x}\right)\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta)\, d\eta \\ &\quad - \frac{d_{22}}{d_{21}\xi^3} e^{-\xi x}\int_0^\infty \left(e^{-\omega_{2+}\eta} - e^{-\xi\eta}\right) v_0'''(\eta)\, d\eta \qquad (5.47)\end{aligned}$$

and

$$\begin{aligned}\tilde{v}^{\varepsilon}_{\rm V}(x,\xi) - \tilde{v}^{b.l.}_{\rm V}(x/\varepsilon,\xi) &= \left(\frac{e_{12}d_{21}}{\varepsilon\Delta\omega_{2+}^3} - \frac{1}{\xi^3}\right) e^{\omega_{2-}x}\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta) d\eta \\ &\quad + \frac{1}{\xi^3}\left(e^{\omega_{2-}x} - e^{-x/\varepsilon}\right)\int_0^\infty e^{-\omega_{2+}\eta} v_0'''(\eta)\, d\eta \\ &\quad + \frac{1}{\xi^3} e^{-x/\varepsilon}\int_0^\infty \left(e^{-\omega_{2+}\eta} - e^{-\xi\eta}\right) v_0'''(\eta)\, d\eta \qquad (5.48)\end{aligned}$$

Note that

$$\begin{aligned}\left|e_{12}d_{21}\xi^3 - \varepsilon\Delta\omega_{2+}^3\right| &= \left|e_{12}d_{21}\xi(\xi^2-\omega_{2+}^2) - \varepsilon\omega_{2+}^3 \det(d_1+\omega_{1-}e_1, d_2)\right| \\ &\leq O(1)\varepsilon|\xi|^4/|1+4\varepsilon\xi| \qquad (5.49)\end{aligned}$$

and therefore

$$\left|\frac{e_{12}d_{21}}{\varepsilon\Delta\omega_{2+}^3} - \frac{1}{\xi^3}\right| \leq O(1)\varepsilon|\xi|^{-1} \qquad (5.50)$$

The rest of the proof is the same as in Case I.

Case II(c):

In this case, we have

$$\tilde{u}_{\mathrm{V}}^{\varepsilon}(x,\xi) = -\frac{e_{12}}{\varepsilon(d_{11}+e_{11}\omega_{1-})}e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0(\eta)d\eta \qquad (5.51)$$

$$\tilde{u}_{\mathrm{V}}(x,\xi) = \varepsilon^{-1}\tilde{u}_{-1}(x,\xi) = -\frac{e_{12}}{\varepsilon(d_{11}-\xi e_{11})}e^{-\xi x}\int_0^\infty e^{-\xi\eta}v_0(\eta)\,d\eta \qquad (5.52)$$

Using a similar integration by parts, we then have

$$\begin{aligned}&\varepsilon\tilde{u}_{\mathrm{V}}^{\varepsilon}(x,\xi) - \tilde{u}_{-1}(x,\xi) \\ &= -\left(\frac{e_{12}}{(d_{11}+\omega_{1-}e_{11})\omega_{2+}^3} - \frac{e_{12}}{(d_{11}-\xi e_{11})\xi^3}\right)e^{\omega_{1-}x}\int_0^\infty e^{-\omega_{2+}\eta}v_0'''(\eta)d\eta \\ &\quad -\frac{e_{12}}{(d_{11}-\xi e_{11})\xi^3}\left(e^{\omega_{1-}x}-e^{-\xi x}\right)\int_0^\infty e^{-\omega_{2+}\eta}v_0'''(\eta)\,d\eta \\ &\quad -\frac{e_{12}}{(d_{11}-\xi e_{11})\xi^3}e^{-\xi x}\int_0^\infty \left(e^{-\omega_{2+}\eta}-e^{-\xi\eta}\right)v_0'''(\eta)\,d\eta \qquad (5.53)\end{aligned}$$

It is easy to see that

$$\begin{aligned}\left|(d_{11}+\omega_{1-}e_{11})\omega_{2+}^3 - (d_{11}-\xi e_{11})\xi^3\right| &= \left|d_{11}\left(\omega_{2+}^3-\xi^3\right) + e_{11}\left(\xi^4-\omega_{2+}^4\right)\right| \\ &\leq O(1)\varepsilon\left(|d_{11}|+|e_{11}|\cdot|\xi|\right)|\xi|^4/|1+4\varepsilon\xi| \qquad (5.54)\end{aligned}$$

and hence

$$\left|\frac{1}{(d_{11}+\omega_{1-}e_{11})\omega_{2+}^3} - \frac{1}{(d_{11}-\xi e_{11})\xi^3}\right| \leq O(1)\varepsilon|\xi|^{-1} \qquad (5.55)$$

Therefore, by the same analysis as before, we can get

$$\int_0^\infty\int_{-\infty}^\infty \left|\varepsilon\tilde{u}_{\mathrm{V}}^{\varepsilon}(x,\xi) - \tilde{u}_{-1}(x,\xi)\right|^2\,dxd\beta \leq O(1)\varepsilon^2||v_0||_{H^3}^2 \qquad (5.56)$$

or equivalently

$$\int_0^\infty\int_{-\infty}^\infty \left|\tilde{u}_{\mathrm{V}}^{\varepsilon}(x,\xi) - \varepsilon^{-1}\tilde{u}_{-1}(x,\xi)\right|^2\,dxd\beta \leq O(1)||v_0||_{H^3}^2 \qquad (5.57)$$

This finishes the proof of (5.31) for type II(c) boundary conditions.

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