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A coupled chemotaxis-fluid model: Global existence

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Abstract

We consider a model arising from biology, consisting of chemotaxis equations coupled to viscous incompressible fluid equations through transport and external forcing. Global existence of solutions to the Cauchy problem is investigated under certain conditions. Precisely, for the chemotaxis–Navier–Stokes system in two space dimensions, we obtain global existence for large data. In three space dimensions, we prove global existence of weak solutions for the chemotaxis–Stokes system with nonlinear diffusion for the cell density.

Résumé

Nous considérons un modèle provenant de la biologie, composé d'équations de chimiotactisme couplées aux équations de fluide visqueux incompressible par le transport et le forçage externe. L'existence globale des solutions du problème de Cauchy est étudiée sous certaines conditions. Précisément, pour le système chimiotactisme–Navier–Stokes en deux dimensions d'espace, nous obtenons l'existence globale pour des données grandes. En trois dimensions d'espace, nous démontrons l'existence globale des solutions faibles pour le système chimiotactisme–Stokes avec une diffusion non-linéaire de la densité des cellules.

1. Introduction

Chemotaxis is a biological process, in which cells (e.g. bacteria) move towards a chemically more favourable environment. For example, bacteria often swim towards higher concentration of oxygen to survive. The chemical can be produced or consumed by the cells, although we are only interested in the latter in this paper.

In particular, here we consider the following: In [14], the authors observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid–air interface and they proposed this model:

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$$\begin{cases} \partial_t n + \boldsymbol{u} \cdot \nabla n = \delta \Delta n - \nabla \cdot \big(\chi(c) n \nabla c\big), \\ \partial_t c + \boldsymbol{u} \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \nu \Delta \boldsymbol{u} - n \nabla \phi, \\ \nabla \cdot \boldsymbol{u} = 0, \quad t > 0,\ \boldsymbol{x} \in \Omega. \end{cases} \tag{1.1}$$

Here, the unknowns are $n = n(t, \boldsymbol{x}) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $c = c(t, \boldsymbol{x}) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $\boldsymbol{u}(t, \boldsymbol{x}) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$ or $\mathbb{R}^2$ and $p = p(t, \boldsymbol{x}) : \mathbb{R}^+ \times \Omega \to \mathbb{R}$, denoting the cell density, chemical concentration, velocity field and pressure of the fluid, respectively. $\Omega \subset \mathbb{R}^3$ or $\mathbb{R}^2$ is a spatial domain where the cells and the fluid move and interact. Positive constants δ, μ and ν are the corresponding diffusion coefficients for the cells, chemical and fluid. $\chi(c)$ is the chemotactic sensitivity and $k(c)$ is the consumption rate of the chemical by the cells. $\phi = \phi(x)$ is a given potential function accounting the effects of external forces such as gravity. The system (1.1) is supplied with initial conditions

$$(n, c, \boldsymbol{u})|_{t=0} = \big(n_0(\boldsymbol{x}), c_0(\boldsymbol{x}), \boldsymbol{u}_0(\boldsymbol{x})\big), \quad \boldsymbol{x} \in \Omega, \tag{1.2}$$

and some proper boundary conditions. The experimental set-up corresponds to mixed-type boundary conditions [14]. However, for simplicity here we work in full space, either $\mathbb{R}^2$ or $\mathbb{R}^3$ and assume sufficient decay at infinity.

It can be seen from (1.1) that chemotaxis and fluid are coupled through both the transport of the cells and the chemical $\boldsymbol{u} \cdot \nabla n$, $\boldsymbol{u} \cdot \nabla c$ by the fluid and the external force $-n\nabla\phi$ exerted on the fluid by the cells.

In three space dimensions, we consider the chemotaxis–Stokes system with nonlinear diffusion having the following form

$$\begin{cases} \partial_t n + \boldsymbol{u} \cdot \nabla n = \delta \Delta n^m - \nabla \cdot \big(\chi(c) n \nabla c\big), \\ \partial_t c + \boldsymbol{u} \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t \boldsymbol{u} + \nabla P = \nu \Delta \boldsymbol{u} - n \nabla \phi, \\ \nabla \cdot \boldsymbol{u} = 0, \quad t > 0,\ x \in \Omega, \end{cases} \tag{1.3}$$

where compared with (1.1), the linear diffusion for n is replaced by a nonlinear diffusion. For these two systems, initial data is given by (1.2) and $\Omega = \mathbb{R}^3$ or $\mathbb{R}^2$ is supposed.

For the system (1.1) and related systems there is a local existence result [13]. Moreover, in [8] the authors proved global existence for (1.1) with the simpler Stokes equations for weak potential or small initial c. For the system (1.3) in 2D and 3D, existence issues and asymptotic behaviour are investigated in [9]. To our knowledge, these are the only results on (1.1) and (1.3).

However, attention has recently been focused on coupled kinetic-fluid systems firstly introduced in [3] which have a similar mathematical flavor; also refer to [10,6] about the studies of the Vlasov–Fokker–Planck equation coupled with the compressible or incompressible Navier–Stokes or Stokes equations, where the main tool used to prove the global existence of weak solutions or hydrodynamic limits is an existing entropy inequality.

Concerning chemotaxis, the best-studied model is the Keller–Segel system where the chemical is produced and not consumed as in our case: For the elliptic–parabolic Keller–Segel model in $\mathbb{R}^2$, [2] summarises the results, i.e. there is a critical mass M, below M we have global existence and above M we have finite-time blow-up. For the parabolic–parabolic Keller–Segel model recent progress has been achieved in [5]. For more references on the general Keller–Segel system, the interested reader can refer to recent work [1,2,5]. Kinetic models for chemotaxis can be found in [7].

Several authors of chemotaxis literature have recently addressed the prevention of finite-time blow-up (*overcrowding*, from the modelling viewpoint) by assuming e.g. that, due to the finite size of the bacteria, the random mobility increases for large densities. This leads to a nonlinear porous-medium-like diffusion instead of a linear one, see e.g. [4,11].

Compared to the results in [8] we use weaker assumptions on the potential, i.e. no decay at infinity is required and do not need any smallness assumptions neither on the potential nor on the initial data. Moreover, compared to [9] the range for the exponent m is improved for the 3D case although in this article we need stronger assumptions on the functions f and χ.

The paper is structured as follows. In Section 2 we state the assumptions and our results on global existence. In Section 3 we prove global existence of solutions for large data by deriving an entropy, proposing a regularisation of the system and a compactness argument to pass to the limit. Finally, in Section 4 we follow the same procedure to show global existence in three dimensions for nonlinear diffusion (1.3).

Let us emphasise that several calculations are similar to [8].

2. Preliminaries

Before we establish global existence, we first need a proper notion of a weak solution:

Definition 2.1 *(Weak solution).* A triple $(c, n, \boldsymbol{u})$ is said to be a weak solution of (1.1) if

(i) $n(t, \boldsymbol{x}) \geqslant 0$, $c(t, \boldsymbol{x}) \geqslant 0$, $t \geqslant 0$, $\boldsymbol{x} \in \mathbb{R}^2$, and for any $T > 0$,

$$\begin{cases} n\big(1 + |\boldsymbol{x}| + |\ln n|\big) \in L^\infty\big(0, T; L^1\big(\mathbb{R}^2\big)\big), \quad \nabla\sqrt{n} \in L^2\big(0, T; L^2\big(\mathbb{R}^2\big)\big), \\ c \in L^\infty\big(0, T; L^1\big(\mathbb{R}^2\big) \cap L^\infty\big(\mathbb{R}^2\big) \cap H^1\big(\mathbb{R}^2\big)\big), \\ \sqrt{n}|\nabla c| \in L^2\big(0, T; L^2\big(\mathbb{R}^2\big)\big), \\ \boldsymbol{u} \in L^\infty\big(0, T; L^2\big(\mathbb{R}^2, \mathbb{R}^2\big)\big) \cap L^2\big(0, T; H^1\big(\mathbb{R}^2, \mathbb{R}^2\big)\big); \end{cases} \tag{2.1}$$

(ii)

$$\int_{\mathbb{R}^2} \psi_1(t=0) n_0 \, dx = \int_0^T \int_{\mathbb{R}^2} n\big[\partial_t \psi_1 + \nabla\psi_1 \cdot \boldsymbol{u} + \delta\Delta\psi_1 + \nabla\psi_1 \cdot \big(\chi(c)\nabla c\big)\big] \, dx \, dt,$$

$$\int_{\mathbb{R}^2} \psi_2(t=0) c_0 \, dx = \int_0^T \int_{\mathbb{R}^2} c[\partial_t \psi_2 + \nabla\psi_2 \cdot \boldsymbol{u} + \mu\Delta\psi_2] - nf(c)\psi_2 \, dx \, dt,$$

$$\int_{\mathbb{R}^2} \boldsymbol{\psi}(t=0) \cdot \boldsymbol{u}_0 \, dx = \int_0^T \int_{\mathbb{R}^2} \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi} + \nu \boldsymbol{u} \cdot \Delta\boldsymbol{\psi} + \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big) \cdot \boldsymbol{\psi} - n\nabla\phi \cdot \boldsymbol{\psi} \, dx \, dt,$$

$$\int_0^T \int_{\mathbb{R}^2} \nabla\psi_3 \cdot \boldsymbol{u} \, dx \, dt = 0$$

hold for all $\psi_1, \psi_2, \psi_3 \in C^\infty([0, T] \times \mathbb{R}^2)$ and all $\boldsymbol{\psi} \in C^\infty([0, T] \times \mathbb{R}^2, \mathbb{R}^2)$, $\nabla \cdot \boldsymbol{\psi} = 0$ where $\psi_i, \boldsymbol{\psi}$ have compact support in x and $\psi_i(T, \cdot) = \boldsymbol{\psi}(T, \cdot) = 0$.

Definition of weak solution for (1.3):

Definition 2.2 *(Weak solution).* A triple $(c, n, \boldsymbol{u})$ is said to be a weak solution of (1.3) if

(i) $n(t, \boldsymbol{x}) \geqslant 0$, $c(t, \boldsymbol{x}) \geqslant 0$, $t \geqslant 0$, $\boldsymbol{x} \in \mathbb{R}^3$, and for any $T > 0$,

$$\begin{cases} n\big(1 + |\boldsymbol{x}| + |\ln n|\big) \in L^\infty\big(0, T; L^1\big(\mathbb{R}^3\big)\big), \quad \nabla n^{m/2} \in L^2\big(0, T; L^2\big(\mathbb{R}^3\big)\big), \\ c \in L^\infty\big(0, T; L^1\big(\mathbb{R}^3\big) \cap L^\infty\big(\mathbb{R}^3\big) \cap H^1\big(\mathbb{R}^3\big)\big), \\ \sqrt{n}|\nabla c| \in L^2\big(0, T; L^2\big(\mathbb{R}^3\big)\big), \\ \boldsymbol{u} \in L^\infty\big(0, T; L^2\big(\mathbb{R}^3, \mathbb{R}^3\big)\big) \cap L^2\big(0, T; H^1\big(\mathbb{R}^3, \mathbb{R}^3\big)\big); \end{cases} \tag{2.2}$$

(ii)

$$\int_{\mathbb{R}^3} \psi_1(t=0) n_0 \, dx = \int_0^T \int_{\mathbb{R}^3} n\big[\partial_t \psi_1 + \nabla\psi_1 \cdot \boldsymbol{u} + \delta n^{m-1}\Delta\psi_1 + \nabla\psi_1 \cdot \big(\chi(c)\nabla c\big)\big] \, dx \, dt,$$

$$\int_{\mathbb{R}^3} \psi_2(t=0) c_0 \, dx = \int_0^T \int_{\mathbb{R}^3} c[\partial_t \psi_2 + \nabla\psi_2 \cdot \boldsymbol{u} + \mu\Delta\psi_2] - nf(c)\psi_2 \, dx \, dt,$$

$$\int_{\mathbb{R}^3} \boldsymbol{\psi}(t=0)\cdot \boldsymbol{u}_0\,dx = \int_0^T\int_{\mathbb{R}^3} \boldsymbol{u}\cdot \partial_t\boldsymbol{\psi} + \nu\boldsymbol{u}\cdot\Delta\boldsymbol{\psi} - n\nabla\phi\cdot\boldsymbol{\psi}\,dx\,dt,$$

$$\int_0^T\int_{\mathbb{R}^3} \nabla\psi_3\cdot\boldsymbol{u}\,dx\,dt = 0$$

hold for all $\psi_1, \psi_2, \psi_3 \in C^\infty([0,T]\times\mathbb{R}^3)$ and all $\boldsymbol{\psi}\in C^\infty([0,T]\times\mathbb{R}^3,\mathbb{R}^3)$, $\nabla\cdot\boldsymbol{\psi}=0$, where $\psi_i, \boldsymbol{\psi}$ have compact support in x and $\psi_i(T,\cdot)=\boldsymbol{\psi}(T,\cdot)=0$.

Set of assumptions:

$$\text{(A)}\qquad \begin{cases} \text{(i) } \delta>0,\ \mu>0,\ \nu>0; \\ \text{(ii) } n_0(\boldsymbol{x})\geqslant 0,\ c_0(\boldsymbol{x})\geqslant 0,\ \nabla\cdot\boldsymbol{u}_0(\boldsymbol{x})=0 \text{ for all } \boldsymbol{x}\in\Omega; \\ \text{(iii) } \chi(\cdot), k(\cdot) \text{ are smooth with } k(0)=0 \text{ and } \chi(c), k(c), k'(c)\geqslant 0 \text{ for all } c\in\mathbb{R}. \end{cases}$$

(B1) $\chi'(c)\geqslant 0$, $\frac{\chi'(c)k(c)+\chi(c)k'(c)}{\chi(c)}>0$, $\frac{d^2}{dc^2}(\frac{k(c)}{\chi(c)})<0$;
(B2) $\nabla\phi\in L^\infty(\Omega)$;
(B3) $(n_0, c_0, \boldsymbol{u}_0)$ satisfies

$$n_0\big(1+|\boldsymbol{x}|+|\ln n_0|\big)\in L^1(\Omega),$$
$$c_0\leqslant c_M<\infty,\qquad c_0\in L^1(\Omega),\qquad \nabla c_0\in L^2(\Omega),\qquad \nabla\Psi(c_0)\in L^2(\Omega),\qquad u_0\in L^2(\Omega),$$

where

$$\Psi(c)=\int_0^c\sqrt{\frac{\chi(s)}{k(s)}}\,ds. \tag{2.3}$$

Remark 2.1. In [14], the experimentalists used multiples of the Heaviside step function to model χ and k. This clearly is not compatible with assumption (B1), whereas assumption (B2) includes the case used in [14].

These assumptions enable us to obtain an entropy functional which is at most exponentially growing in time:

Theorem 2.1 *(2D). For $\Omega=\mathbb{R}^2$, under the assumptions* (A) *and* (B1)–(B3)*, the system* (1.1) *has a global-in-time weak solutions. Moreover, we have this entropy estimate*

$$\frac{d}{dt}\mathcal{E}+\mathcal{D}\leqslant C+C\mathcal{E} \tag{2.4}$$

with

$$\mathcal{E}:=\int_{\mathbb{R}^2} n\ln(n)+2n\big(1+|x|^2\big)^{1/2}+\frac{1}{2}\big|\nabla\psi(c)\big|^2+\frac{1}{2}|\boldsymbol{u}|^2\,dx \tag{2.5}$$

and

$$\mathcal{D}:=\delta\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\,dx+\mu\sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi-\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2dx+\nu\int_{\mathbb{R}^2}|\nabla\boldsymbol{u}|^2\,dx$$
$$+\lambda_1\mu\int_{\mathbb{R}^2}|\nabla\Psi|^4\,dx+\lambda_2\int n\nabla\Psi\cdot\nabla\Psi\,dx. \tag{2.6}$$

Theorem 2.2 *(3D). For $\Omega = \mathbb{R}^3$, under the assumptions* (A) *and* (B1)–(B3)*, we have global-in-time weak solutions for* (1.3) *with $m = 4/3$.*

Remark 2.2. Compared to [8], the free energy functional is simpler, but instead of linear growth as before, it grows exponentially in time. Moreover, the free energy functional combines all equations in (1.1), whereas in [9] the functional decouples from the fluid.

Remark 2.3. For a square domain Ω with periodic boundary conditions Theorem 2.2 holds for any m between $4/3$ and 2.

3. Results in 2D

3.1. Positivity of n and c

Notice that the assumptions (A) and (B1)–(B3) imply that c and n preserve the nonnegativity of the initial data by the maximum principle. Moreover, we have $c \leqslant c_M$.

3.2. Entropy

Multiplying Eq. $(1.1)_1$ with $1 + \ln(n)$ and integrating gives

$$\frac{d}{dt}\int_{\mathbb{R}^2} n\ln(n)\,dx + \delta\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\,dx = \int_{\mathbb{R}^2}\nabla n\cdot\big(\chi(c)\nabla c\big)\,dx. \tag{3.1}$$

Multiplying Eq. $(1.1)_2$ with Ψ', then with $\Delta\Psi$ and integrating, gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\Psi\|^2 + \mu\|\Delta\Psi\|^2 &= \mu\int_{\mathbb{R}^2}\Psi''(c)\Delta\Psi|\nabla c|^2 + \Delta\Psi\boldsymbol{u}\cdot\nabla\Psi + \sqrt{k(c)\chi(c)}n\Delta\Psi\,dx\\
&= -\mu\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k}{\chi}}\Delta\Psi|\nabla\Psi|^2 + \Delta\Psi\boldsymbol{u}\cdot\nabla\Psi - n\sqrt{k\chi}'\nabla c\cdot\nabla\Psi - \sqrt{k\chi}\nabla n\cdot\nabla\Psi\,dx\\
&= -\mu\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k}{\chi}}\Delta\Psi|\nabla\Psi|^2 + \Delta\Psi\boldsymbol{u}\cdot\nabla\Psi - n\frac{\sqrt{k\chi}'}{\Psi'}\nabla\Psi\cdot\nabla\Psi - \chi\nabla n\cdot\nabla c\,dx \end{aligned}\tag{3.2}$$

where from the assumptions (A) and (B1)–(B3), it holds that

$$\frac{d}{dc}\sqrt{k(c)\chi(c)}\sqrt{\frac{k(c)}{\chi(c)}} = \frac{\chi'(c)k(c) + \chi(c)k'(c)}{2\chi(c)} > 0.$$

We remark that Ψ is chosen such that the contribution of the $n\nabla c$-term in (3.1) cancels part of the contribution of the $nk(c)$-term in (3.2). Noticing the identity

$$\nabla\cdot\big(|\nabla\Psi|^2\nabla\Psi\big) = |\nabla\Psi|^2\Delta\Psi + \nabla\big(|\nabla\Psi|^2\big)\cdot\nabla\Psi,$$

it follows from integration by parts that

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}|\nabla\Psi|^2\Delta\Psi\,dx &= -\int_{\mathbb{R}^2}\frac{d^2}{dc^2}\sqrt{\frac{k(c)}{\chi(c)}}\sqrt{\frac{k(c)}{\chi(c)}}|\nabla\Psi|^4\,dx\\
&\quad - 2\sum_{ij}\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\partial_i\partial_j\Psi\,dx.\end{aligned}$$

Therefore, it holds that

$$\int_{\mathbb{R}^2} \left|\nabla^2\Psi\right|^2 dx + \int_{\mathbb{R}^2} \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}|\nabla\Psi|^2\Delta\Psi\, dx$$
$$= \sum_{ij}\int_{\mathbb{R}^2}\left(|\partial_i\partial_j\Psi|^2 - 2\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\partial_i\partial_j\Psi - \frac{d^2}{dc^2}\sqrt{\frac{k(c)}{\chi(c)}}\sqrt{\frac{k(c)}{\chi(c)}}|\partial_i\Psi|^2|\partial_j\Psi|^2\right)dx$$
$$= \sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2}\frac{d^2}{dc^2}\frac{k(c)}{\chi(c)}|\nabla\Psi|^4\, dx, \tag{3.3}$$

where we used the identity

$$\left(\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\right)^2 + \frac{d^2}{dc^2}\sqrt{\frac{k(c)}{\chi(c)}}\sqrt{\frac{k(c)}{\chi(c)}} = \frac{1}{2}\frac{d^2}{dc^2}\left(\frac{k(c)}{\chi(c)}\right).$$

Thus, adding (3.1) and (3.2), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} n\ln(n)\, dx + \frac{1}{2}\frac{d}{dt}\left\|\nabla\Psi(c)\right\|^2 + \delta\int_{\mathbb{R}^2}\frac{|\nabla n|^2}{n}\, dx$$
$$+ \mu\sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^2}\frac{d^2}{dc^2}\frac{k(c)}{\chi(c)}|\nabla\Psi|^4\, dx$$
$$+ \int_{\mathbb{R}^2} n\frac{\sqrt{k\chi}'}{\Psi'}\nabla\Psi\cdot\nabla\Psi\, dx = -\int_{\mathbb{R}^2}\sum_{ij}\partial_i\Psi\partial_j\Psi\partial_i u_j\, dx. \tag{3.4}$$

Now working on the right-hand side, we have with Young's inequality

$$-\int_{\mathbb{R}^2}\sum_{ij}\partial_i\Psi\partial_j\Psi\partial_i u_j\, dx \leqslant \lambda_1\mu\int_{\mathbb{R}^2}|\nabla\Psi|^4\, dx + C\|\nabla\boldsymbol{u}\|^2$$

where we define

$$2\lambda_0 := \min_{0\leqslant c\leqslant c_M}\frac{\chi'(c)k(c) + \chi(c)k'(c)}{2\chi(c)},$$
$$2\lambda_1 := \min_{0\leqslant c\leqslant c_M} -\frac{1}{2}\frac{d^2}{dc^2}\left(\frac{k(c)}{\chi(c)}\right),$$

by the assumptions (A) and (B1)–(B3), $\lambda_0 > 0$ and $\lambda_1 > 0$. Therefore we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} n\ln(n)\, dx + \frac{1}{2}\frac{d}{dt}\left\|\nabla\Psi(c)\right\|^2 + 4\delta\|\nabla\sqrt{n}\|^2$$
$$+ \mu\sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2 dx + \lambda_1\mu\int_{\mathbb{R}^2}|\nabla\Psi|^4\, dx$$
$$+ 2\lambda_0\int_{\mathbb{R}^2} n|\nabla\Psi|^2\, dx \leqslant C\|\nabla\boldsymbol{u}\|^2. \tag{3.5}$$

The left-hand side already looks like an entropy and its entropy dissipation. To deal with the right-hand side, we will use $(1.1)_3$, but first we need to control the behaviour of n as $|x| \to \infty$.

Moment control The goal is to obtain a bound of the first-order spatial moment of $n(t,\boldsymbol{x})$, in order to bound $\int n\ln(n)$ from below. For this purpose, multiplying $(1.1)_1$ by the smooth function $\xi=(1+|\boldsymbol{x}|^2)^{1/2}$ and integrating, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2}\xi n\,dx=\int_{\mathbb{R}^2}n\boldsymbol{u}\cdot\nabla\xi\,dx+\delta\int_{\mathbb{R}^2}n\Delta\xi\,dx+\int_{\mathbb{R}^2}\sqrt{k(c)\chi(c)}n\nabla\Psi\cdot\nabla\xi\,dx.$$

Next, we estimate each term on the r.h.s. of the above identity. Notice that $\|\nabla\xi\|_{L^\infty}$ and $\|\Delta\xi\|_{L^\infty}$ are finite constants. For the first term, it follows from the Cauchy–Schwarz inequality, the Gagliardo–Nirenberg–Sobolev inequality and the mass conservation for n that

$$\begin{aligned}\int_{\mathbb{R}^2}n\boldsymbol{u}\cdot\nabla\xi\,dx&\leqslant\|\nabla\xi\|_{L^\infty}\|n\|\|\boldsymbol{u}\|\leqslant C\|n\|_{L^1}^{1/2}\|\nabla\sqrt{n}\|\|\boldsymbol{u}\|\\&\leqslant\delta\|\nabla\sqrt{n}\|^2+C\|n_0\|_{L^1}\|\boldsymbol{u}\|^2.\end{aligned}$$

For the third term, it follows from the Cauchy–Schwarz inequality that

$$\int_{\mathbb{R}^2}\sqrt{k(c)\chi(c)}n\nabla\Psi\cdot\nabla\xi\,dx\leqslant\lambda_0\|\sqrt{n}\nabla\Psi\|^2+\frac{C}{\lambda_0}\Big(\sup_{0\leqslant c\leqslant c_M}\chi(c)\Big)\big\|\sqrt{k(c)n}\big\|^2.$$

Collecting the above estimates, it therefore holds that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\xi n\,dx&\leqslant C\delta\|n_0\|_{L^1}+\frac{C}{\lambda_0}\Big(\sup_{0\leqslant c\leqslant c_M}\chi(c)\Big)\big\|k(c)n\big\|_{L^1}+C\|n_0\|_{L^1}\|\boldsymbol{u}\|^2\\&\quad+\delta\|\nabla\sqrt{n}\|^2+\lambda_0\|\sqrt{n}\nabla\Psi\|^2.\end{aligned}\tag{3.6}$$

We have

$$\int_{\mathbb{R}^2}n\ln n\,dx=\int_{\mathbb{R}^2}n|\ln n|\,dx-2\int_{\mathbb{R}^2}n\ln\frac{1}{n}\chi_{n\leqslant1}\,dx,$$

and the estimate

$$\begin{aligned}0\leqslant\int_{\mathbb{R}^2}n\ln\frac{1}{n}\chi_{n\leqslant1}\,dx&=\int_{\mathbb{R}^2}n\ln\frac{1}{n}\chi_{e^{-\xi}<n\leqslant1}\,dx+\int_{\mathbb{R}^2}n\ln\frac{1}{n}\chi_{n\leqslant e^{-\xi}}\,dx\\&\leqslant\int_{\mathbb{R}^2}\xi n\,dx+C\int_{\mathbb{R}^2}n^{1/2}\chi_{n\leqslant e^{-\xi}}\,dx\\&\leqslant\int_{\mathbb{R}^2}\xi n\,dx+C.\end{aligned}$$

Now multiplying Eq. $(1.1)_3$ with $\boldsymbol{u}$, integrating

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2+\nu\|\nabla\boldsymbol{u}\|^2&\leqslant\|\nabla\phi\|_\infty\|n\|\|\boldsymbol{u}\|\leqslant C\|\nabla\phi\|_\infty\|n_0\|_{L^1}^{1/2}\|\nabla\sqrt{n}\|\|\boldsymbol{u}\|\\&\leqslant\delta\|\nabla\sqrt{n}\|^2+C\|\nabla\phi\|_\infty^2\|n_0\|_{L^1}\|\boldsymbol{u}\|^2,\end{aligned}\tag{3.7}$$

and adding it to (3.5), we obtain

$$\begin{aligned}&\frac{d}{dt}\int n\big(\ln(n)+2\xi\big)\,dx+\frac{d}{dt}\frac{1}{2}\big\|\nabla\Psi(c)\big\|^2+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2+2\delta\|\nabla\sqrt{n}\|^2+\nu\|\nabla\boldsymbol{u}\|^2\\&\quad+\mu\sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi-\frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2dx+\lambda_1\int_{\mathbb{R}^2}|\nabla\Psi|^4\,dx\\&\quad+\lambda_0\int_{\mathbb{R}^2}n|\nabla\Psi|^2\,dx=C+C\|\boldsymbol{u}\|^2.\end{aligned}\tag{3.8}$$

So we achieve an estimate of the form:

$$\frac{d}{dt}\mathcal{E} + \mathcal{D} \leqslant C + C\mathcal{E} \tag{3.9}$$

with

$$\mathcal{E} := \int_{\mathbb{R}^2} n\bigl(\ln(n) + 2\xi\bigr) + \frac{1}{2}\bigl|\nabla\Psi(c)\bigr|^2 + \frac{1}{2}|\boldsymbol{u}|^2\,dx \tag{3.10}$$

and

$$\begin{aligned}\mathcal{D} := {} & 2\delta\|\nabla\sqrt{n}\|^2 + \nu\|\nabla\boldsymbol{u}\|^2 + \mu\sum_{ij}\int_{\mathbb{R}^2}\left|\partial_i\partial_j\Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}}\partial_i\Psi\partial_j\Psi\right|^2 dx \\ & + \lambda_1\mu\int_{\mathbb{R}^2}|\nabla\Psi|^4\,dx + \lambda_0\int_{\mathbb{R}^2} n|\nabla\Psi|^2\,dx.\end{aligned} \tag{3.11}$$

3.3. Regularisation

With this entropy at hand, the next step is to establish a regularised system which is consistent with the entropy: Let us take a mollifier σ^ϵ and define the regularised system as

$$\begin{cases} \partial_t n^\epsilon + \boldsymbol{u}^\epsilon\cdot\nabla n^\epsilon = \delta\Delta n^\epsilon - \nabla\cdot\bigl(n^\epsilon\bigl[\bigl(\chi\bigl(c^\epsilon\bigr)\nabla c^\epsilon\bigr)*\sigma^\epsilon\bigr]\bigr), \\ \partial_t c^\epsilon + \boldsymbol{u}^\epsilon\cdot\nabla c^\epsilon = \mu\Delta c^\epsilon - k\bigl(c^\epsilon\bigr)\bigl[n^\epsilon*\sigma^\epsilon\bigr], \\ \partial_t\boldsymbol{u}^\epsilon + \nabla p^\epsilon = \nu\Delta\boldsymbol{u}^\epsilon - \bigl(\boldsymbol{u}^\epsilon\cdot\nabla\bigr)\boldsymbol{u}^\epsilon - \bigl(n^\epsilon\nabla\phi\bigr)*\sigma^\epsilon, \\ \nabla\cdot\boldsymbol{u}^\epsilon = 0, \quad t>0,\ x\in\mathbb{R}^2, \end{cases} \tag{3.12}$$

with initial data

$$\bigl(n^\epsilon, c^\epsilon, \boldsymbol{u}^\epsilon\bigr)\big|_{t=0} = \bigl(n_0*\sigma^\epsilon, c_0*\sigma^\epsilon, \boldsymbol{u}_0*\sigma^\epsilon\bigr), \quad x\in\mathbb{R}^2. \tag{3.13}$$

Entropy for regularisation To increase readability, we write k, χ and Ψ instead of $k(c^\epsilon)$, $\chi(c^\epsilon)$ and $\Psi(c^\epsilon)$: Multiplying Eq. $(1.1)_1$ with $1+\ln(n^\epsilon)$ and integrating gives

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\epsilon\ln\bigl(n^\epsilon\bigr)\,dx + \delta\int_{\mathbb{R}^2}\frac{|\nabla n^\epsilon|^2}{n^\epsilon}\,dx = \int_{\mathbb{R}^2}\nabla n^\epsilon\cdot\bigl[\bigl(\chi\nabla c^\epsilon\bigr)*\sigma^\epsilon\bigr]\,dx. \tag{3.14}$$

Multiplying Eq. $(1.1)_2$ with Ψ', then with $\Delta\Psi$ and integrating, gives

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\nabla\Psi\|^2 + \mu\|\Delta\Psi\|^2 &= \mu\int_{\mathbb{R}^2}\Psi''\bigl(c^\epsilon\bigr)\Delta\Psi\bigl|\nabla c^\epsilon\bigr|^2 + \Delta\Psi\boldsymbol{u}^\epsilon\cdot\nabla\Psi + \sqrt{k\chi}\bigl[n^\epsilon*\sigma^\epsilon\bigr]\Delta\Psi\,dx \\ &= -\mu\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k}{\chi}}\Delta\Psi|\nabla\Psi|^2 + \Delta\Psi\boldsymbol{u}^\epsilon\cdot\nabla\Psi - n^\epsilon*\sigma^\epsilon\sqrt{k\chi}'\nabla c^\epsilon\cdot\nabla\Psi \\ &\quad - \sqrt{k\chi}\nabla\bigl(n^\epsilon*\sigma^\epsilon\bigr)\cdot\nabla\Psi\,dx \\ &= -\mu\int_{\mathbb{R}^2}\frac{d}{dc}\sqrt{\frac{k}{\chi}}\Delta\Psi|\nabla\Psi|^2 + \Delta\Psi\boldsymbol{u}^\epsilon\cdot\nabla\Psi - n^\epsilon*\sigma^\epsilon\frac{\sqrt{k\chi}'}{\Psi'}\nabla\Psi\cdot\nabla\Psi \\ &\quad - \chi\nabla\bigl(n^\epsilon*\sigma^\epsilon\bigr)\cdot\nabla c^\epsilon\,dx.\end{aligned} \tag{3.15}$$

Therefore the same entropy works also for the regularised system and we have the following bounds uniform in ϵ:

1. $0\leqslant n^\epsilon$ and $0\leqslant c^\epsilon\leqslant c_M$.
2. $n^\epsilon|\ln(n^\epsilon)|$ bounded in $L^\infty((0,T),L^1(\mathbb{R}^2))$.

3. $\nabla\sqrt{n^\epsilon}$ bounded in $L^2((0,T),L^2(\mathbb{R}^2))$.
4. n^ϵ bounded in $L^2((0,T)\times\mathbb{R}^2)$.
5. $\boldsymbol{u}^\epsilon$ bounded in $L^\infty((0,T),L^2(\mathbb{R}^2,\mathbb{R}^2))\cap L^2((0,T),H^1(\mathbb{R}^2,\mathbb{R}^2))$.
6. $\boldsymbol{u}_t^\epsilon$ bounded in $L^2((0,T),H^{-1}(\mathbb{R}^2,\mathbb{R}^2))$.
7. c^ϵ bounded in $L^\infty((0,T),H^1(\mathbb{R}^2))\cap L^2((0,T),H^2(\mathbb{R}^2))$.
8. c_t^ϵ bounded in $L^2((0,T)\times\mathbb{R}^2)$.
9. n_t^ϵ bounded in $L^2((0,T),H^{-3}(\mathbb{R}^2))$.

1)–3) can be directly seen from the entropy.
For 4) we use the Gagliardo–Nirenberg–Sobolev inequality $\|n^\epsilon\|\leqslant C\|\nabla\sqrt{n^\epsilon}\|\|n^\epsilon\|_{L^1}^{1/2}$.
6) can be shown as standard regularity results for Stokes, see e.g. [12].
7) and 8) can be both obtained using parabolic regularity for the c-equation together with the L^2-space–time bound on $nk(c)$ and the regularity of $\boldsymbol{u}$.
For 9), we consider

$$\begin{aligned}\langle\varphi,n_t^\epsilon\rangle&=\delta\langle\Delta\varphi,n^\epsilon\rangle+\langle\nabla\varphi,n^\epsilon\boldsymbol{u}^\epsilon\rangle+\langle\nabla\varphi,n^\epsilon\big[\big(\chi\big(c^\epsilon\big)\nabla c^\epsilon\big)*\sigma^\epsilon\big]\rangle\\&\leqslant\delta\|\Delta\varphi\|\|n^\epsilon\|+C\|\varphi\|_{H^3(\mathbb{R}^2)}\|n^\epsilon\|\|\boldsymbol{u}^\epsilon\|+C\|\varphi\|_{H^3(\mathbb{R}^2)}\|n^\epsilon\|\|\nabla c^\epsilon\|.\end{aligned}$$

Therefore we have

$$\int_0^T\|n_t^\epsilon\|^2_{H^{-3}(\mathbb{R}^2)}\,dt\leqslant C\int_0^T\|n^\epsilon\|^2\,dt.$$

Compactness This will be done in detail for the more involved case of $\mathbb{R}^3$ at the end of the next section.

4. 3D

4.1. Entropy

From Eq. $(1.3)_1$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}n\ln(n)\,dx+\int_{\mathbb{R}^3}\frac{4}{m}\big|\nabla n^{m/2}\big|^2\,dx=\int_{\mathbb{R}^3}\nabla n\cdot\big(\chi(c)\nabla c\big)\,dx.\tag{4.1}$$

Then for $m=4/3$ all the calculations are the same as for the 2D case, except for (3.7). Instead we use $\|n\|\leqslant C\|\nabla n^{2/3}\|\|n\|_{L^1}^{1/3}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2+\nu\|\nabla\boldsymbol{u}\|^2\leqslant\|\nabla\phi\|_\infty C\big\|\nabla n^{2/3}\big\|\|n\|_{L^1}^{1/3}\|\boldsymbol{u}\|.$$

As far as the a priori estimates are concerned, the Gagliardo–Nirenberg–Sobolev inequality to show 4) is now $\|n\|\leqslant C\|\nabla n^{2/3}\|\|n\|_{L^1}^{1/3}$.

4.2. Compactness

We apply Aubin–Lions with the following spaces

$$H^2\big(\Omega'\big)\hookrightarrow H^1\big(\Omega'\big)\hookrightarrow L^2\big(\Omega'\big)\tag{4.2}$$

for all smooth, bounded domains $\Omega'\subset\mathbb{R}^3$.

Together with the a priori estimates, it follows that $\boldsymbol{u}^\epsilon$ converges strongly in $L^2([0,T];L^2_{loc}(\mathbb{R}^3))$ and that c^ϵ converges strongly in $L^2([0,T];H^1_{loc}(\mathbb{R}^3))$. For n^ϵ, we obtain weak convergence in $L^2([0,T];L^2_{loc}(\mathbb{R}^3))$ and weak convergence of $(n^\epsilon)^{4/3}$ in $L^1([0,T];L^1_{loc}(\mathbb{R}^3))$. Moreover, $k(c^\epsilon)$ and $\chi(c^\epsilon)$ converge almost everywhere.

So we can pass to the weak limit in $L^1([0,T]; L^1_{loc}(\mathbb{R}^3))$ in the terms n^m, $n \cdot \boldsymbol{u}$ and $n \cdot (\chi(c)\nabla c)$ from the definition of weak solution for the n-equation. Similarly, we can pass to the weak limit in all terms of the c and $\boldsymbol{u}$-equation.

So the limit is indeed a weak solution of (1.3).

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References

[1] A. Blanchet, J.A. Carrillo, N. Masmoudi, Infinite time aggregation for the critical Patlak–Keller–Segel model in $\mathbb{R}^2$, Comm. Pure Appl. Math. 61 (2008) 1449–1481.
[2] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller–Segel model: Optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations 2006 (2006) 1–33.
[3] R. Caflisch, G. Papanicolaou, Dynamic theory of suspensions with Brownian effects, SIAM J. Appl. Math. 43 (1983) 885–906.
[4] V. Calvez, J. Carrillo, Volume effects in the Keller–Segel model: energy estimates preventing blow-up, J. Math. Pures Appl. 86 (2006) 155–175.
[5] V. Calvez, L. Corrias, The parabolic–parabolic Keller–Segel model in $\mathbb{R}^2$, Commun. Math. Sci. 6 (2008) 417–447.
[6] J. Carrillo, T. Goudon, Stability and asymptotic analysis of a fluid–particle interaction model, Comm. Partial Differential Equations 31 (2006) 1349–1379.
[7] F. Chalub, Y. Dolak-Struss, P. Markowich, D. Oelz, C. Schmeiser, A. Soreff, Model hierarchies for cell aggregation by chemotaxis, Math. Models Methods Appl. Sci. 16 (2006) 1173–1197.
[8] R.-J. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis–fluid equations, Comm. Partial Differential Equations 35 (9) (2010) 1635–1673.
[9] M.D. Francesco, A. Lorz, P. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dyn. Syst. Ser. A 28 (4) (2010) 1437–1453.
[10] K. Hamdache, Global existence and large time behaviour of solutions for the Vlasov–Stokes equations, Japan J. Indust. Appl. Math. 15 (1998) 51–74.
[11] R. Kowalczyk, Preventing blow-up in a chemotaxis model, J. Math. Anal. Appl. 305 (2005) 566–588.
[12] P.-L. Lions, Mathematical Topics in Fluid Mechanics. Volume 1: Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, 1996.
[13] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci. 20 (6) (2010) 987–1004.
[14] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.