Nonlinearity

PAPER

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To cite this article: Jian-Guo Liu *et al* 2019 *Nonlinearity* **32** 4346

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **32** (2019) 4346–4376 https://doi.org/10.1088/1361-6544/ab2cf1

Well-posedness and derivative blow-up for a dispersionless regularized shallow water system

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Received 25 October 2018, revised 7 May 2019
Accepted for publication 26 June 2019
Published 4 October 2019

Recommended by Professor Guido Schneider

Abstract

We study local-time well-posedness and breakdown for solutions of regularized Saint-Venant equations (regularized classical shallow water equations) recently introduced by Clamond and Dutykh. The system is linearly non-dispersive, and smooth solutions conserve an H^1-equivalent energy. No shock discontinuities can occur, but the system is known to admit weakly singular shock-profile solutions that dissipate energy. We identify a class of small-energy smooth solutions that develop singularities in the first derivatives in finite time.

Keywords: Saint-Venant equations, Green–Naghdi equations, weak solutions, nonlocal hyperbolic system, long waves, breakdown
Mathematics Subject Classification numbers: 35B44, 35B60, 35Q35, 76B15, 35L67

(Some figures may appear in colour only in the online journal)

1. Introduction

The main aim of this paper is to demonstrate singularity formation for classical solutions of a system of regularized Saint-Venant (shallow-water) equations that was introduced by Clamond and Dutykh in [5]. In conservation form in one space dimension, these regularized Saint-Venant (rSV) equations may be written

$$h_t + (hu)_x = 0, \tag{1.1}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2 + \varepsilon\mathcal{S})_x = 0, \tag{1.2}$$

$$\mathcal{S} \stackrel{\text{def}}{=} h^3(-u_{tx} - uu_{xx} + u_x^2) - gh^2\left(hh_{xx} + \frac{1}{2}h_x^2\right). \tag{1.3}$$

Here $h(x,t)$ represents the depth of the fluid, $u(x,t)$ represents the average horizontal velocity of the fluid column, g is the gravitational constant, and ε is a dimensionless regularization parameter. This system admits *weakly singular shock layer* solutions that were described in [25].

The rSV equations above were derived in [5] as a non-dispersive variant of the Green–Naghdi equations [9, 8] with zero surface tension (also called Serre equations [26]). Equations (1.1)–(1.3) follow from a least action principle for the Lagrangian with density given by

$$\frac{\mathcal{L}}{\rho} = \frac{1}{2}hu^2 - \frac{1}{2}gh^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 - \frac{1}{2}gh^2h_x^2\right) + (h_t + (hu)_x)\phi, \tag{1.4}$$

with a Lagrange multiplier field ϕ to enforce (1.1). The Green–Naghdi equations with surface tension take the same dimensional form as in (1.1)-(1.2), but with $\varepsilon\mathcal{S}$ above replaced by the quantity

$$\mathcal{S}_{\mathrm{GN}} = \frac{1}{3}h^3(-u_{tx} - uu_{xx} + u_x^2) - \gamma\left(hh_{xx} - \frac{1}{2}h_x^2\right), \tag{1.5}$$

where γ is the ratio of surface tension to density. The Green–Naghdi equations derive analogously from the Lagrangian with density

$$\frac{\mathcal{L}_{\mathrm{GN}}}{\rho} = \frac{1}{2}hu^2 - \frac{1}{2}gh^2 + \frac{1}{6}h^3u_x^2 - \frac{1}{2}\gamma h_x^2 + (h_t + (hu)_x)\phi. \tag{1.6}$$

The Green–Naghdi equations hold an important place among dispersive approximations to the full water wave equations, insofar as the small-slope assumptions they are based on are minimal and they are capable of correctly approximating large-amplitude waves. Many other dispersive water-wave models, such as the Korteweg–de Vries, Camassa–Holm, and various Boussinesq systems, can be derived from the Green–Naghdi equations by imposing further restrictions on amplitude or structure; see the treatment by Lannes [20]. Local-time well-posedness for the Green–Naghdi equations was studied in [1, 16, 21], and in [21] Li proved that they constitute an approximation to the water wave equations that is better than the classical shallow water equations (which correspond to $\epsilon = 0$ above). And recently, the Green–Naghdi system has been found to have weakly singular peakon-like traveling-wave solutions, when the Bond number $\mathrm{Bo} = gh_\infty^2/\gamma$ takes the critical value 3 [7, 23]. Yet, as far as we know, the analytical question of whether smooth solutions for the Green–Naghdi equations always exist globally in time, or whether instead singularities may develop, remains open.

Smooth solutions of the rSV equations also satisfy a conservation law for energy, in the form

$$\mathcal{E}_t^\epsilon + \mathcal{Q}_x^\epsilon = 0, \tag{1.7}$$

where

$$\mathcal{E}^\epsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2\right), \tag{1.8}$$

$$\mathcal{Q}^\epsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^3 + gh^2u + \varepsilon\left(\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2 + \mathcal{S}\right)u + gh^3h_xu_x\right). \tag{1.9}$$

For $\epsilon > 0$ and provided the fluid depth h remains larger than a positive constant, this energy controls the L^2 norms of the derivatives of both h and u, precluding shock formation. By comparison, the Green–Naghdi energy, given by

$$\mathcal{E}_{\text{GN}} \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{1}{6}h^3u_x^2 + \frac{1}{2}\gamma h_x^2, \tag{1.10}$$

fails to control h_x for $\gamma = 0$, so one might guess the Green–Naghdi equations without surface tension are 'less regularizing' than the rSV equations.

When linearized about any constant state $(h_\star, u_\star)$, the opposite seems to be the case, however. For the linearized rSV equations, the phase velocity of linear waves is independent of frequency. Thus the rSV equations are linearly *dispersionless*—they appear to lack a linear dispersive regularization mechanism.

This dispersionless nature of the rSV equations and the tendency of numerically computed solutions not to generate oscillations and discontinuities were primary reasons given by Clamond and Dutykh [5] for interest in studying these equations. These authors pointed out, in fact, that the rSV equations are less accurate than the Green–Naghdi equations for approximating the exact water-wave dispersion relation (with zero surface tension) in the long-wave regime, and only as accurate as the classical shallow-water system. We note, however, that actually there is a physical regime where the rSV equations do approximate the linear dispersion of water waves as accurately as Green–Naghdi equations. From (1.1)–(1.5) above, clearly both systems yield the same linearization at depth $h_\star$ when both ε and the inverse Bond number $\text{Bo}^{-1} = \gamma/gh_\star^2$ take the value $\frac{1}{3}$. (It is well-known that linear dispersion vanishes in the Korteweg–de Vries approximation at this critical value of Bond number, and the same is clear from the dispersion relation for Green–Naghdi equations with surface tension in [8, equation (6.10)].) Even in this case, though, the nonlinear factor gh^2 in (1.3) makes the rSV system formally less accurate than Green–Naghdi as a weakly nonlinear water-wave approximation, unless the amplitude variation is so small that the differences with (1.5) are of the same order as the terms neglected there.

What is more interesting for present purposes is the fact that the rSV equations admit a new kind of traveling wave, which is a weakly singular analog of classical shallow-water shock waves: it is shown in [25] that for every such classical shock, the rSV equations admit a corresponding non-oscillatory traveling wave solution which is continuous but only piecewise smooth, having a weak singularity at a single point where the energy is dissipated as it is for the classical shock. Numerical evidence provided in [25] suggests that a weak singularity can develop from a smooth solution and start to dissipate energy after some positive time.

It is the purpose of this paper to partly address the question of well-posedness and whether singularity formation occurs for smooth solutions of the rSV equations. Our goals are: (i) to provide a basic theory of local-time well-posedness and lifespan for classical solutions with sufficient Sobolev regularity; (ii) to prove that depth h remains strictly positive for small-energy perturbations of a constant state; and (iii) to identify initial data for which no classical solution can exist globally in time. From our continuation criteria for solutions we infer that the sup norms of both h_x and u_x blow up as t approaches the maximal time of existence.

Our local-time well-posedness theorem (theorem 3.1 in section 3) deals with (possibly large) initial perturbations of a constant state in $H^s(\mathbb{R})$ for some real $s > \frac{3}{2}$, such that the depth h is initially uniformly positive. The depth remains uniformly positive for small-energy perturbations—this follows from energy conservation and proposition 2.1 in section 2, which is essentially a Sobolev-type inequality. In section 4 we establish criteria for finite-time blow-up based on the sup norm of the derivatives h_x and u_x and/or the vanishing of h. Our main blow-up argument (the proof of theorem 5.2 in section 5) identifies a class of small-energy initial data, defined by a few explicit inequalities (all listed in lemma 5.6), for which h_x and u_x must both blow up in sup norm. The nature of the blow up is that the derivative of one of the classical shallow-water Riemann invariants $R_\pm = u \pm 2\sqrt{gh}$ blows up to $-\infty$ while remaining bounded above, while the derivative of the other Riemann invariant remains bounded.

To give some insight into how our analysis will proceed, observe that in the momentum equation (1.2), there are two terms involving time derivatives. It is natural to combine them and transform the momentum equation (1.2) into a standard evolution equation for u. For a smooth function $w : \mathbb{R} \to \mathbb{R}$, define

$$\mathcal{I}_h(w) = hw - \varepsilon(h^3 w_x)_x$$

or, in terms of composition of operators,

$$\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x. \tag{1.11}$$

Formally acting $\mathcal{I}_h^{-1}$ on both sides of the momentum equation (1.2), one obtains

$$u_t + gh_x + uu_x + \varepsilon\mathcal{I}_h^{-1}\partial_x\left(2h^3u_x^2 - \frac{1}{2}gh^2h_x^2\right) = 0. \tag{1.12}$$

This is the standard evolution equation for horizontal velocity in the classical shallow-water system plus a nonlocal term. Because we expect the operator $\mathcal{I}_h^{-1}$ gains two derivatives, the nonlocal term is formally of *order zero* and represents a lower-order perturbation to the classical shallow-water system.

This is an important difference with the Green–Naghdi system as treated by Israwi in [16] without surface tension. The system with constant surface tension $\gamma > 0$ appears no better, even though the Green–Naghdi energy in (1.10) controls the H^1 norm of h in that case. For, instead of (1.12), the momentum equation takes the form

$$u_t + gh_x + uu_x + \mathcal{I}_h^{-1}\partial_x\left(\frac{2}{3}h^3u_x^2 + (\frac{1}{3}gh^2 - \gamma)hh_{xx} + \frac{1}{2}\gamma h_x^2\right) = 0.$$

The trouble is that the nonlocal term is stronger here than in (1.12), remaining formally of order one in h, except when linearized at the constant depth $h_\star$ where $\gamma = \frac{1}{3}gh_\star^2$, corresponding to Bond number $\mathrm{Bo} = 3$.

For the rSV system, then, equations (1.1) and (1.12) constitute a nonlocal hyperbolic system for which we are able to use a standard shallow-water symmetrizer to study well-posedness, and study blowup using (coupled) Ricatti-type equations for the derivatives of classical Riemann invariants. It turns out that, in addition to coupling the pair of Ricatti-type equations, the nonlocal terms contain a local part that alters the main quadratic terms. This important contribution to the Ricatti-type equations appears to change the nature of blowup profiles as compared with the classical shallow-water case. We discuss this difference heuristically in section 6 below.

The rSV equations that we study in this paper also loosely resemble a number of 2-component systems that generalize the Camassa–Holm equation; see [4, 13, 15, 17, 19] for studies of such systems. One of the more extensively studied systems of this kind, appearing in [4, 17, 19], is an integrable 2-component Camassa–Holm system that can be written in the form

$$h_t + (hu)_x = 0, \tag{1.13}$$

$$u_t + 3uu_x - u_{txx} - 2u_xu_{xx} - uu_{xxx} + ghh_x = 0. \tag{1.14}$$

In the context of shallow-water theory, this system has been derived by Constantin and Ivanov [6] (see also [14]). For this system, derivative blow-up does not occur—smooth solutions exist globally in time for all smooth initial data for which h is initially strictly positive, see [6, 10–12].

An interesting question that remains open is whether the rSV equations admit globally defined weak solutions for arbitrary initial perturbations small in $H^1(\mathbb{R})$. The rSV system does admit energy-conserving small-energy traveling waves with cusp singularities, as described in [25]. The scalar Camassa–Holm equation, which famously admits weak solutions that include peakon traveling waves, has global existence for weak solutions that may conserve the H^1 energy [2] or dissipate it [3]. An expected difference between the scalar Camassa–Holm equation and the rSV system, however, is that in general we do not expect weak rSV solutions to conserve energy globally in time, due to the presence of energy-dissipating weakly singular traveling waves.

2. An energy criterion for uniform positivity of depth

We begin the analysis of solutions of the rSV equations (1.1)–(1.3) by establishing an explicit energy criterion that ensures the uniform positivity of the depth h for small $H^1(\mathbb{R})$ perturbations of any given constant state $(h_\star, u_\star)$ with $h_\star > 0$, $u_\star \in \mathbb{R}$. The proof resembles the proof of the Sobolev inequality for the H^1 norm, and exploits the simple idea that for the surface to reach the bottom, relative energy has to be sufficiently large. Our criterion has no apparent analog for the Green–Naghdi system with $\gamma = 0$ or the two-component Camassa–Holm system mentioned above, because the energies for those systems do not control the integral of h_x^2.

Formally, a smooth solution (h, u) of the rSV equations defined for all $x \in \mathbb{R}$, with the property that $(h - h_\star, u - u_\star) \in H^1(\mathbb{R})$ for all t, conserves the relative energy

$$E_\star = \int_{\mathbb{R}} \frac{1}{2}h(u-u_\star)^2 + \frac{1}{2}g(h-h_\star)^2 + \frac{1}{2}\varepsilon\left(h^3u_x^2 + gh^2h_x^2\right)\,\mathrm{d}x. \tag{2.1}$$

By fixing t and discarding the terms involving u, we infer the following.

Proposition 2.1. *Let $h_\star > 0$, $u_\star \in \mathbb{R}$ and suppose $(h - h_\star, u - u_\star) \in H^1(\mathbb{R})$. Then*

(a) For all $x \in \mathbb{R}$ we have

$$E_\star \geqslant \frac{g\sqrt{\epsilon}}{3}(h(x) - h_\star)^2(2h(x) + h_\star). \tag{2.2}$$

(b) If $E_\star < \frac{1}{3}g\sqrt{\epsilon}h_\star^3$, then we have $h(x) \geqslant h_E > 0$ for all $x \in \mathbb{R}$, where h_E is the unique solution in $(0, h_\star)$ of

$$E_\star = \frac{g\sqrt{\epsilon}}{3}(h_E - h_\star)^2(2h_E + h_\star). \tag{2.3}$$

Proof. Because $\frac{1}{2}(a^2 + b^2) \geqslant \pm ab$, for any $x \in \mathbb{R}$ we have

$$\begin{aligned} E_\star &\geqslant g\sqrt{\epsilon}\left(\int_{-\infty}^{x}(h - h_\star)hh_x\,\mathrm{d}x - \int_x^{\infty}(h - h_\star)hh_x\,\mathrm{d}x\right) \\ &= \frac{g\sqrt{\epsilon}}{3}(h(x) - h_\star)^2(2h(x) + h_\star). \end{aligned}$$

This proves (a). To deduce (b), note that the map $w \mapsto (w - h_\star)^2(2w + h_\star)$ is strictly decreasing for $w \in (0, h_\star)$. □

Remark 2.2.

(i) The lower bound $h(x) \geqslant h_E$ in part (b) is sharp, as one can see by choosing $h(x)$ to be an even function, determined on $[0, \infty)$ as the solution of

$$\sqrt{\varepsilon}hh_x = h_\star - h, \quad h(0) = h_E.$$

(ii) For periodic functions on $\mathbb{R}$ having finite period L, the same estimates hold with $E_\star$ obtained by integrating over a single period and with $h_\star$ replaced by the average value of h over one period. One alters the proof by replacing the endpoints $-\infty$ and ∞ by points a and $a + L$ where $h(a) = h_\star$.

(iii) Using the upper bound in case (a), the lower bound in case (b) implies that $(h(x) - h_\star)^2 \leqslant (h_E - h_\star)^2$, whence $h(x) \leqslant 2h_* - h_E$ for all x.

(iv) The part of the relative energy that we are using to bound the depth from below corresponds in principle to potential energy of the fluid. In an exact physical fluid model with zero surface tension, however, it is possible to perturb a flat fluid surface to reach the bottom with a small change in potential energy, by creating a downward cusp on a tiny horizontal length scale.

3. Local well-posedness, and scaling of lifespan

In this section, we will establish finite-time existence and uniqueness for solutions of the initial-value problem for the rSV system that have finite energy relative to a constant state $(h_\star, 0)$ with $h_\star > 0$. (We take $u_\star = 0$ without loss due to Galilean invariance of the system.) We will pay particular attention to how the existence time (lifespan of the solution) varies according to the value of the nonlinearity parameter $\alpha = a/h_\star$, where the parameter a indicates the amplitude of the perturbation. For example, in the inviscid Burgers equation $u_t + uu_x = 0$, a Ricatti-type calculation for u_x shows that the existence time for smooth solutions is proportional to $1/\alpha$.

For this reason, we make the following change of variables, writing

$$h = h_\star + \alpha\eta, \quad \text{and replacing } u \text{ by } \alpha u. \tag{3.1}$$

Here and below we retain the notation $h = h_\star + \alpha\eta$ for brevity, however. The scaled pair (η, u) now satisfies the following system:

$$\eta_t + (hu)_x = 0, \tag{3.2}$$

$$h(u_t + g\eta_x + \alpha u u_x) + \varepsilon\alpha\tilde{\mathcal{S}}_x = 0, \tag{3.3}$$

$$\alpha\tilde{\mathcal{S}} = h^3(-u_{tx} - \alpha u u_{xx} + \alpha u_x^2) - gh^2\left(h\eta_{xx} + \frac{1}{2}\alpha\eta_x^2\right). \tag{3.4}$$

In terms of $\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x$, we observe that we can reformulate the momentum equation (3.3) as

$$u_t + g\eta_x + \alpha u u_x + \varepsilon\alpha\mathcal{I}_h^{-1}\partial_x\left(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\right) = 0. \tag{3.5}$$

Equations (3.2), (3.5) form a (nonlocal) hyperbolic system that takes the form

$$W_t + B(W)W_x + F(W) = 0, \tag{3.6}$$

with $W = (\eta, u)^T$ and where

$$B(W) = \begin{pmatrix} \alpha u & h \\ g & \alpha u \end{pmatrix}, \quad F(W) = \begin{pmatrix} 0 \\ f(W) \end{pmatrix}, \tag{3.7}$$

with

$$f(W) = \varepsilon\alpha\mathcal{I}_h^{-1}\partial_x\left(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\right). \tag{3.8}$$

For this system, we shall use a standard iteration scheme for symmetrizable hyperbolic systems to prove the main theorem of this section. We remark that both of the parameters α and ε are dimensionless, and there is some interest in understanding how solutions behave in the regime when one or both parameters become small.

Theorem 3.1. *Fix $h_\star > 0$. Let $s > \frac{3}{2}$ be real, and let $\varepsilon, \alpha \in (0, 1]$. Assume the initial data $W^0 = (\eta^0, u^0)^T \in H^s(\mathbb{R})$ and satisfies*

$$h^0_{\min} \stackrel{\text{def}}{=} \inf_{x\in\mathbb{R}}(h_\star + \alpha\eta^0(x)) > 0. \tag{3.9}$$

Then there exists $T_0 = T_0(s, \|W^0\|_{H^s}, h^0_{\min}) > 0$ independent of ε and α, such that the regularized shallow-water system (3.6) admits a unique solution

$$W = (\eta, u)^T \in C([0, T_0/\alpha]; H^s(\mathbb{R})) \cap C^1([0, T_0/\alpha]; H^{s-1}(\mathbb{R})),$$

having the initial condition W^0 and preserving the positive depth condition

$$\inf_{x\in\mathbb{R}} h(x,t) > 0.$$

Moreover, the following conservation of energy property holds: $\tilde{E}_\star$ =const, where

$$\tilde{E}_\star = \frac{1}{2}\int_{\mathbb{R}} hu^2 + g\eta^2 + \varepsilon\left(h^3u_x^2 + gh^2\eta_x^2\right) \mathrm{d}x. \tag{3.10}$$

Remark 3.2.

(i) In theorem 3.1, the dependence on $h^0_{\min}$ can be dropped if the initial relative energy $E_\star = \alpha^2 \tilde{E}_\star$ is so small that proposition 2.1(b) applies.

(ii) Although continuous dependence on the initial data is not mentioned in the theorem, we do have it in the following sense: for all initial data $\tilde{W}^0$ satisfying $\|\tilde{W}^0\|_{H^s} \leqslant 2\|W^0\|_{H^s}$ with uniformly positive depth $\tilde{h} \geqslant h_{\star\star} > 0$, the corresponding solution $\tilde{W}$ satisfies

$$\|\tilde{W} - W\|_{L^\infty([0,T];H^{s-1})} \leqslant C(s, h_{\star\star}, \|W\|_{L^\infty([0,T];H^s)})\|\tilde{W}^0 - W^0\|_{H^s}, \tag{3.11}$$

on any common time interval of existence where $\tilde{h}, h \geqslant h_{\star\star} > 0$. The proof follows in a standard way analogous to the convergence proof of the iteration scheme for existence; see [22, 24] for details.

(iii) When $s \geqslant 2$, the relative energy satisfies the following conservation law in a strong L^2 sense:

$$\tilde{\mathcal{E}}^\varepsilon_t + \tilde{\mathcal{Q}}^\varepsilon_x = 0, \tag{3.12}$$

with

$$\tilde{\mathcal{E}}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}g\eta^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2\eta_x^2\right), \tag{3.13}$$

$$\tilde{\mathcal{Q}}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}\alpha hu^3 + g\eta hu + \varepsilon\left(\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2\eta_x^2 + \tilde{\mathcal{S}}\right)\alpha u + gh^3\eta_x u_x\right), \tag{3.14}$$

where we find using (3.5) that $\tilde{\mathcal{S}}$ from (3.4) satisfies

$$\tilde{\mathcal{S}} = \left(I + \varepsilon h^3\partial_x\mathcal{I}_h^{-1}\partial_x\right)\left(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\right). \tag{3.15}$$

(For $s \geqslant 2$ this expression will belong to $H^1(\mathbb{R})$.)

The proof of theorem 3.1 is structured as follows: section 3.1 contains preliminary estimates, including technical analysis of the operator $\mathcal{I}_h$. Section 3.2 analyzes the iteration step in the iteration scheme and establishes the needed *a priori* energy estimates. The main proof of theorem 3.1 is presented in section 3.3.

3.1. Preliminary results

The elliptic operator $\mathcal{I}_h$ plays an important role in the energy estimate and well-posedness of the regularized shallow-water system. In this subsection, we shall introduce the main technical tools to handle $\mathcal{I}_h$ and the nonlocal term in (3.5).

Let $D = \partial_x$ and let $\Lambda = (I - \partial_x^2)^{1/2}$ be the operator associated with Fourier symbol $(1+\xi^2)^{1/2}$, so that $\widehat{\Lambda u} = (1+\xi^2)^{1/2}\hat{u}$ for all tempered distributions u. We will make use of two well-known harmonic analysis results which we cite here without proofs. The first one is a Kato–Ponce commutator estimate [18],

$$\|[\Lambda^s, \phi]\psi\|_{L^2} \leqslant C(s)\left(\|D\phi\|_{L^\infty}\|\Lambda^{s-1}\psi\|_{L^2} + \|\Lambda^s\phi\|_{L^2}\|\psi\|_{L^\infty}\right) \tag{3.16}$$

valid for all $\phi \in H^s(\mathbb{R})$, $D\phi \in L^\infty(\mathbb{R})$ and $\psi \in H^{s-1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for all real $s \geqslant 0$. The second one is a classical 'tame' product estimate (also proved in [18]),

$$\|\Lambda^s(\phi\psi)\|_{L^2} \leqslant C(s)(\|\phi\|_{L^\infty}\|\Lambda^s\psi\|_{L^2} + \|\Lambda^s\phi\|_{L^2}\|\psi\|_{L^\infty}) \tag{3.17}$$

valid for all $\phi, \psi \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and all real $s \geqslant 0$.

The following lemma establishes the invertibility of $\mathcal{I}_h$ and bounds $\mathcal{I}_h^{-1}\partial_x$. It improves the bounds on $\mathcal{I}_h^{-1}\partial_x$ claimed in lemma 2 of [16] in two ways, bounding one derivative more and providing a tame estimate that is needed for the blow-up analysis in section 5.

Lemma 3.3. *Let $h_{\star\star} > 0$ and $\varepsilon \in (0, 1]$ and suppose $h \in W^{1,\infty}(\mathbb{R})$ satisfies*

$$h(x) \geqslant h_{\star\star} \quad \text{for all } x \in \mathbb{R}. \tag{3.18}$$

Then:

(1) The operator $\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x$ from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ is an isomorphism.
(2) Let $s \geqslant 0$ and $h - h_\star \in H^s(\mathbb{R})$. Then for any $\psi \in H^s(\mathbb{R})$, the function

$$w = \epsilon\mathcal{I}_h^{-1}\partial_x\psi$$

belongs to $H^{1+s}(\mathbb{R})$ and satisfies the estimate

$$\|w\|_{H^{1+s}} \leqslant \hat{C}_1\Big(\|\psi\|_{H^s} + \|h - h_\star\|_{H^s}(\epsilon^{-1/2}\|w\|_{L^\infty} + \|w_x\|_{L^\infty})\Big), \tag{3.19}$$

where $\hat{C}_1 = C(s, h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})$ independent of ψ, ϵ and α.
(3) If furthermore $s > \frac{1}{2}$, then

$$\|w\|_{H^{1+s}} \leqslant \hat{C}_2\,\|\psi\|_{H^s}(1 + \|h - h_\star\|_{H^s}) \tag{3.20}$$

where $\hat{C}_2 = C(s, h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})$ independent of ψ, ϵ and α.

Remark 3.4. The estimate (3.19) will be improved below in lemma 4.5, to provide bounds on $\|w\|_{L^\infty}$ and $\|w_x\|_{L^\infty}$ in terms of ψ that will be used to prove a blow-up criterion.

Proof.

1. The idea is that $\mathcal{I}_h$ is in essence a very well-behaved elliptic operator such that the basic Lax–Milgram approach works on it.

   We define the bilinear mapping $a : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ such that

$$a(u, v) = (hu, v)_{L^2} + \varepsilon(h^3u_x, v_x)_{L^2} \quad \forall\, u, v \in H^1(\mathbb{R}). \tag{3.21}$$

   Next, we will show that a is not only bounded but also coercive. We have

$$\begin{aligned}|a(u, v)| &\leqslant \|h\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} + \varepsilon\|h\|^3_{L^\infty}\|u\|_{H^1}\|v\|_{H^1}\\ &\leqslant C(h)\|u\|_{H^1}\|v\|_{H^1}.\end{aligned}$$

   and by (3.18)

$$|a(u, u)| \geqslant h_{\star\star}\|u\|^2_{L^2} + \varepsilon(h_{\star\star})^3\|u_x\|^2_{L^2} \geqslant \varepsilon C(h_{\star\star})\|u\|^2_{H^1}. \tag{3.22}$$

   So by Lax–Milgram, there is a bounded bijective linear operator $\tilde{I} : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ such that

$$a(u, v) = \langle \tilde{I}u, v\rangle_{H^{-1}\times H^1} \quad \forall\, u, v \in H^1(\mathbb{R}). \tag{3.23}$$

Therefore, given any $f \in L^2(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$, $u := \tilde{I}^{-1}f$ satisfies

$$(f, v)_{L^2} = \langle f, v \rangle_{H^{-1} \times H^1} = a(u, v) \quad \forall\, v \in H^1(\mathbb{R}). \tag{3.24}$$

It follows that the distributional derivative $(\epsilon h^3 u_x)_x = f - hu \in L^2(\mathbb{R})$, whence $h^3 u_x \in H^1(\mathbb{R})$. Hence $u \in H^2(\mathbb{R})$ and $\mathcal{I}_h u = f$. It follows $\mathcal{I}_h$ is an isomorphism.

2. Let $\|\cdot\|_{H^1_\epsilon}$ be the norm on $H^1(\mathbb{R})$ (equivalent to $\|\cdot\|_{H^1}$ but not uniformly in ε) determined by

$$\|u\|^2_{H^1_\epsilon} := \|u\|^2_{L^2} + \varepsilon \|u_x\|^2_{L^2}. \tag{3.25}$$

Consider $\phi, \psi \in C_c^\infty(\mathbb{R})$ and $u \in H^2(\mathbb{R})$ such that

$$\mathcal{I}_h u = \phi + \sqrt{\varepsilon}\psi_x. \tag{3.26}$$

Invoking the coercivity estimate from above, we obtain

$$\begin{aligned} C(h_{\star\star})\|u\|^2_{H^1_\epsilon} &\leqslant a(u, u) = (\mathcal{I}_h u, u)_{L^2} = (\phi, u)_{L^2} - (\psi, \sqrt{\varepsilon} u_x)_{L^2} \\ &\leqslant (\|\phi\|_{L^2} + \|\psi\|_{L^2})\|u\|_{H^1_\epsilon}, \end{aligned} \tag{3.27}$$

hence

$$\left\|\sqrt{\epsilon} u\right\|_{H^1} \leqslant \|u\|_{H^1_\epsilon} \leqslant C(h_{\star\star})(\|\phi\|_{L^2} + \|\psi\|_{L^2}). \tag{3.28}$$

Choosing $\phi = 0$, $w = \sqrt{\epsilon} u$, this proves the case $s = 0$ for (3.19).
Next, assume $\phi = 0$ and note

$$hu - \epsilon h^3 u_{xx} = \epsilon (h^3)_x u_x + \sqrt{\varepsilon}\psi_x.$$

Test this against $-u_{xx}$ and integrate by parts. We obtain

$$\begin{aligned} a(u_x, u_x) &= -(u_x, h_x u) - (\sqrt{\varepsilon} u_{xx}, (h^3)_x \sqrt{\varepsilon} u_x + \psi_x) \\ &\leqslant \frac{1}{2} h_{\star\star} \|u_x\|^2_{L^2} + \frac{1}{2} h^3_{\star\star} \left\|\sqrt{\varepsilon} u_{xx}\right\|^2_{L^2} + h^{-3}_{\star\star}\|\psi_x\|^2_{L^2} \\ &\quad + C(h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})\|u\|^2_{H^1_\epsilon}. \end{aligned}$$

By (3.28) we can then infer that

$$\left\|\sqrt{\epsilon} u_x\right\|_{H^1} \leqslant \|u_x\|_{H^1_\epsilon} \leqslant C(h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})\|\psi\|_{H_1}. \tag{3.29}$$

By interpolation, it follows that for every $s \in [0, 1]$,

$$\left\|\sqrt{\varepsilon} u\right\|_{H^{s+1}} + \|u\|_{H^s} \leqslant C(h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})\|\psi\|_{H^s}. \tag{3.30}$$

3. Next, for any $s > 0$, noting $\Lambda^s \mathcal{I}_h u = \sqrt{\epsilon}\partial_x \Lambda^s \psi$, we compute

$$\mathcal{I}_h(\Lambda^s u) = [h, \Lambda^s]u - \varepsilon \partial_x [h^3, \Lambda^s] u_x + \sqrt{\epsilon}\partial_x \Lambda^s \psi, \tag{3.31}$$

so using (3.28) with ϕ and ψ replaced by

$$\tilde{\phi} = [h, \Lambda^s]u, \quad \tilde{\psi} = \Lambda^s \psi - \sqrt{\varepsilon}[h^3, \Lambda^s]u_x \tag{3.32}$$

we find, after using the Kato–Ponce commutator estimate (3.16), that

$$\begin{aligned} \|\Lambda^s u\|_{H^1_\epsilon} &\leqslant C(h_{\star\star})\big(\|[h, \Lambda^s]u\|_{L^2} + \left\|\Lambda^s \psi - \sqrt{\varepsilon}[h^3, \Lambda^s]u_x\right\|_{L^2}\big) \\ &\leqslant C(s, h_{\star\star})\Big(\|h_x\|_{L^\infty}\|u\|_{H^{s-1}} + \|h - h_\star\|_{H^s}\|u\|_{L^\infty} + \|\psi\|_{H^s} \\ &\quad + \left\|(h^3)_x\right\|_{L^\infty}\left\|\sqrt{\varepsilon} u_x\right\|_{H^{s-1}} + \left\|h^3 - h^3_\star\right\|_{H^s}\left\|\sqrt{\varepsilon} u_x\right\|_{L^\infty}\Big). \end{aligned} \tag{3.33}$$

After Taylor-expanding $h^3 - h_\star^3$ in powers of $h - h_\star$ and using the tame product estimate (3.17), we infer

$$\|\Lambda^s u\|_{H^1_\epsilon} \leqslant \hat{C}_1\Big(\big\|\Lambda^{s-1}u\big\|_{H^1_\epsilon} + \|\psi\|_{H^s} + \|h - h_\star\|_{H^s}(\|u\|_{L^\infty} + \big\|\sqrt{\varepsilon}u_x\big\|_{L^\infty})\Big),$$

where $\hat{C}_1$ is a generic constant depending upon s, $h_{\star\star}$, and $\|h - h_\star\|_{W^{1,\infty}}$, independent of ψ, ϵ, α. Note that for $s \leqslant 1$, $\big\|\Lambda^{s-1}u\big\|_{H^1_\epsilon} \leqslant \|u\|_{H^1_\epsilon}$. Hence by induction starting from (3.28), we deduce that for all $s \geqslant 0$,

$$\|\Lambda^s u\|_{H^1_\epsilon} \leqslant \hat{C}_1\Big(\|\psi\|_{H^s} + \|h - h_\star\|_{H^s}(\|u\|_{L^\infty} + \big\|\sqrt{\varepsilon}u_x\big\|_{L^\infty})\Big). \tag{3.34}$$

With $w = \sqrt{\epsilon}u$ as before, since $\|w\|_{H^{s+1}} \leqslant \|\Lambda^s u\|_{H^1_\epsilon}$ we deduce that (3.19) holds.

4. Finally, if $s > \frac{1}{2}$ then due to the embedding $H^s \hookrightarrow L^\infty$, from (3.30) we infer

$$\|u\|_{L^\infty} + \big\|\sqrt{\varepsilon}u_x\big\|_{L^\infty} \leqslant C(s, h_{\star\star}, \|h - h_\star\|_{W^{1,\infty}})\|\psi\|_{H^s}.$$

Using this together with (3.34) proves (3.20), and concludes the proof. □

3.2. Linear analysis

The local-time existence of solutions to the system (3.6) is proved by a standard approach for symmetrizable hyperbolic systems, based on proving convergence of the following iteration scheme: set $W_0(x,t) = W^0(x)$ and inductively determine $W = W_{n+1}$ from $\underline{W} = W_n$ for $n \geqslant 0$ by solving the (linear) initial value problem with coefficients and source term frozen at the (now given) reference state $\underline{W} \in C([0, T/\alpha]; H^s)$:

$$\begin{cases} W_t + B(\underline{W})W_x + F(\underline{W}) = 0, \\ W|_{t=0} = W_0. \end{cases} \tag{3.35}$$

This subsection is devoted to the proof of energy estimates for this linear initial value problem. A symmetrizer for $B(\underline{W})$ is given by

$$A(\underline{W}) = \begin{pmatrix} g & 0 \\ 0 & \underline{h} \end{pmatrix}. \tag{3.36}$$

(Here $\underline{h} = h_\star + \alpha\underline{\eta}$ where $\underline{W} = (\underline{\eta}, \underline{u})^T$.) A natural energy for the IVP (3.35) is

$$E^s(W) \stackrel{\text{def}}{=} (\Lambda^s W, A(\underline{W})\Lambda^s W) = g\|\eta\|^2_{H^s} + (\Lambda^s u, \underline{h}\,\Lambda^s u) \tag{3.37}$$

which is equivalent to $\|W\|^2_{H^s}$ provided that $0 < h_{\star\star} \leqslant |\underline{h}| \leqslant \|\underline{h}\|_{L^\infty} < \infty$.

The following theorem establishes that the iteration scheme is well-defined, and provides an energy estimate that controls the norms of all the solutions in the scheme.

Theorem 3.5 (Energy estimate). *Fix $h_\star > 0$. Let $s > \frac{3}{2}$, $h_{\star\star} \in (0, h_\star)$ and $R > 0$. Then there exists constants T, $K > 0$ depending upon s, $h_{\star\star}$ and R but independent of $\varepsilon, \alpha \in (0, 1]$, with the following property. Assuming that $W_0 = (\eta_0, u_0) \in H^s$ satisfies*

$$h_0 > 2h_{\star\star} \quad \text{and} \quad E^s(W_0) < \frac{R}{2}, \tag{3.38}$$

and that $\underline{W} = (\underline{\eta}, \underline{u}) \in C([0, T/\alpha]; H^s) \cap C^1([0, T/\alpha]; H^{s-1})$ satisfies

$$\underline{h} \geqslant h_{\star\star}, \quad E^s(\underline{W}) \leqslant R, \quad \|\underline{W}_t\|_{H^{s-1}} \leqslant K \quad \text{for all } t \in [0, T/\alpha], \tag{3.39}$$

there exists a unique solution $W = (\eta, u)^T \in C([0, T/\alpha]; H^s)$ *to* (3.35) *satisfying*

$$h \geqslant h_{\star\star}, \quad E^s(W) \leqslant R, \quad \|W_t\|_{H^{s-1}} \leqslant K \quad \text{for all } t \in [0, T/\alpha], \tag{3.40}$$

and furthermore

$$E^s(W(\cdot, t)) \leqslant \mathrm{e}^{C\alpha t} E^s(W_0) + \mathrm{e}^{C\alpha t} - 1 < R \tag{3.41}$$

for all $t \in [0, T/\alpha]$, *for some* $C = C(s, h_{\star\star}, R) > 0$.

Proof. Since all coefficients of the initial value problem (3.35) are independent of unknowns, by a standard Friedrichs mollification approach we have the well-posedness of the symmetrizable hyperbolic system. We will focus on the proof of the (*a priori*) energy estimate.

For simplicity, we use underlines to denote the dependence on $\underline{W}$:

$$\underline{A} := A(\underline{W}), \quad \underline{B} := B(\underline{W}), \quad \underline{F} := F(\underline{W}), \quad \underline{f} := f(\underline{W}).$$

We compute that

$$\begin{aligned} \partial_t E^s(W) &= \partial_t(\Lambda^s W, \underline{A}\Lambda^s W) \\ &= (\underline{h}_t \Lambda^s u, \Lambda^s u) + 2(\Lambda^s W_t, \underline{A}\Lambda^s W). \end{aligned} \tag{3.42}$$

Using equation (3.35) and integrating by parts, we obtain

$$\begin{aligned} \partial_t E^s(W) &= (\underline{h}_t \Lambda^s u, \Lambda^s u) - 2(\underline{AB}\Lambda^s W_x, \Lambda^s W) \\ &\quad + 2([\underline{B}, \Lambda^s] W_x, \underline{A}\Lambda^s W) - 2(\Lambda^s \underline{F}, \underline{A}\Lambda^s W). \end{aligned} \tag{3.43}$$

Now we turn to bound each of the four terms on the right-hand side of (3.43) in turn. In the estimates below, various constants denoted by C may change from line to line without changing the notation.

(1) Since $\|\underline{h}_t\|_{L^\infty} = \alpha \|\underline{\eta}_t\|_{L^\infty} \leqslant \alpha C(s) K$ due to the embedding $H^{s-1} \hookrightarrow L^\infty$,

$$|(\underline{h}_t \Lambda^s u, \Lambda^s u)| \leqslant \|\underline{h}_t\|_{L^\infty} \|u\|_{H^s}^2 \leqslant \alpha C(s, h_{\star\star}) K E^s(W). \tag{3.44}$$

(2) For the second term, note that

$$\underline{AB} = \begin{pmatrix} \alpha g \underline{u} & g \underline{h} \\ g \underline{h} & \alpha \underline{hu} \end{pmatrix} \tag{3.45}$$

is symmetric, so we can take advantage of this symmetry and move the derivative from W terms to the $\underline{AB}$ term. We use

$$|\underline{h}_x| \leqslant \alpha C(s) \big\|\underline{\eta}\big\|_{H^s} \leqslant \alpha C(s, h_{\star\star}) E^s(\underline{W})^{1/2} \leqslant \alpha C(s, h_{\star\star}, R) \tag{3.46}$$

together with the analogous bound $|\underline{u}_x| \leqslant C(s, h_{\star\star}, R)$ and obtain

$$\begin{aligned} 2|(\underline{AB}\Lambda^s W_x, \Lambda^s W)| &= |((\underline{AB})_x \Lambda^s W, \Lambda^s W)| \\ &\leqslant |(\alpha g \underline{u}_x \Lambda^s \eta, \Lambda^s \eta)| + |2(g \underline{h}_x \Lambda^s \eta, \Lambda^s u)| + |(\alpha (\underline{hu})_x \Lambda^s u, \Lambda^s u)| \\ &\leqslant \alpha C(s, h_{\star\star}, R) E^s(W). \end{aligned} \tag{3.47}$$

(3) For the third term, it is crucial to use the Kato–Ponce commutator estimate (3.16) together with the embedding $H^{s-1} \hookrightarrow L^\infty$:

$$\begin{aligned}|([\underline{B},\Lambda^s]W_x,\underline{A}\Lambda^s W)| &= \left|\left(\alpha[\underline{u},\Lambda^s]\eta_x+\alpha[\underline{\eta},\Lambda^s]u_x, g\Lambda^s\eta\right)+(\alpha[\underline{u},\Lambda^s]u_x,\underline{h}\Lambda^s u)\right| \\ &\leqslant \alpha C(s)\left(\|\underline{u}\|_{H^s}\|\eta_x\|_{H^{s-1}}+\|\underline{\eta}\|_{H^s}\|u_x\|_{H^{s-1}}\right)\|\eta\|_{H^s} \\ &+\ \alpha C(s)\|\underline{u}\|_{H^s}\|u_x\|_{H^{s-1}}\|\underline{h}\|_{L^\infty}\|u\|_{H^s} \\ &\leqslant \alpha C(s,h_{\star\star},R)\left(\|\eta\|^2_{H^s}+\|\eta\|_{H^s}\|u\|_{H^s}+\|u\|^2_{H^s}\right) \\ &\leqslant \alpha C(s,h_{\star\star},R)E^s(W).\end{aligned} \tag{3.48}$$

(4) For the fourth term (the nonlocal term), we exploit part 3) of lemma 3.3 (replacing $1+s$ by s and using the bound $\|\underline{h}-h_\star\|_{W^{1,\infty}} \leqslant CR$) to get

$$\begin{aligned}\|\underline{f}\|_{H^s} &= \left\|\varepsilon\alpha\mathcal{I}_{\underline{h}}^{-1}\partial_x(2\underline{h}^3\underline{u}_x^2-\frac{1}{2}g\underline{h}^2\underline{\eta}_x^2)\right\|_{H^s} \\ &\leqslant \alpha C(s,h_{\star\star},R)\left\|2\underline{h}^3\underline{u}_x^2-\frac{1}{2}g\underline{h}^2\underline{\eta}_x^2\right\|_{H^{s-1}} \leqslant \alpha C(s,h_{\star\star},R),\end{aligned} \tag{3.49}$$

where the last inequality is obtained by expanding $\underline{h}=h_\star+\alpha\underline{\eta}$ and using the fact that $H^{s-1}(\mathbb{R})$ is a Banach algebra. Then we deduce

$$\begin{aligned}|(\Lambda^s\underline{F},\underline{A}\Lambda^s W)| &= \left|\left(\Lambda^s\underline{f},\underline{h}\Lambda^s u\right)\right| \leqslant \|\underline{h}\|_{L^\infty}\|\underline{f}\|_{H^s}\|u\|_{H^s} \\ &\leqslant \alpha C(s,h_{\star\star},R)(1+E^s(W)).\end{aligned} \tag{3.50}$$

(5) Before proceeding further, we bound W_t using (3.35), obtaining

$$\begin{aligned}\|W_t\|_{H^{s-1}} &= \|B(\underline{W})W_x+\underline{F}\|_{H^{s-1}} \\ &\leqslant C(s,h_{\star\star},R)(E^s(W)+1).\end{aligned} \tag{3.51}$$

We fix the choice of $K=K(s,h_{\star\star},R)$ at this point, requiring that

$$C(s,h_{\star\star},R)(R+1)<K.$$

(6) Now, substituting all estimates back into (3.43), we find the Gronwall-type differential inequality

$$\partial_t E^s(W) \leqslant \alpha C(s,h_{\star\star},R)(E^s(W)+1). \tag{3.52}$$

This in turn gives the energy inequality

$$E^s(W) \leqslant \mathrm{e}^{C\alpha t}E^s(W_0)+\mathrm{e}^{C\alpha t}-1. \tag{3.53}$$

We choose $T=T(s,h_{\star\star},R)>0$ small enough so that

$$\mathrm{e}^{CT}E^s(W_0)+\mathrm{e}^{CT}-1<R.$$

(7) Now pointwise, we have the bound

$$|h_t|=\alpha|\eta_t|=\alpha|\underline{u}h_x+\underline{h}u_x| \leqslant \alpha C(s,h_{\star\star})(E^s(\underline{W})+E^s(W))$$

hence from the hypotheses on the initial data,

$$h(x,t)=h_0(x)+\int_0^t h_t(x,\tau)\,\mathrm{d}\tau > 2h_{\star\star}-\alpha t C(s,h_{\star\star})(R+E^s(W)). \tag{3.54}$$

Making T smaller if necessary, we can ensure that

$$h_{\star\star} > C(s, h_{\star\star})2RT. \tag{3.55}$$

Now, considering (3.53), (3.51) and (3.54) in turn, we conclude that the inequalities in (3.40) and (3.41) all hold as desired. ☐

3.3. Proof of theorem 3.1

The rSV system has a structure highly resembling that of the classical shallow-water system. With the energy estimate, proofs of existence and uniqueness are standard, so we omit details.

The proof that the relative energy $\tilde{E}_\star$ is conserved relies on a few basic facts: provided $s \geqslant 2$, we have

$$\eta, u \in C([0, T], H^2(\mathbb{R})) \cap C^1([0, T], H^1(\mathbb{R})).$$

Also, for any $v, w \in H^1(\mathbb{R})$, $\int_{\mathbb{R}} vw_x = -\int_{\mathbb{R}} v_x w$ and $\mathcal{I}_h^{-1}(v_x) \in H^2(\mathbb{R})$, with

$$\int_{\mathbb{R}} (hu - \varepsilon(h^3 u_x)_x)\mathcal{I}_h^{-1}(v_x) = \int_{\mathbb{R}} uv_x.$$

Using these facts, the details of checking that $\partial_t \tilde{E}_\star = 0$ from (3.6) are rather tedious but straightforward, so we omit them. For general initial data $W_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, we can approximate by initial data $\tilde{W}_0 \in H^2(\mathbb{R})$ and infer from (3.11) that relative energy $\tilde{E}_\star$ is constant in time for the H^s solution W.

4. Blow-up criteria

In this section, our aim is to establish the following criteria for finite-time breakdown of regular solutions.

Theorem 4.1. *Let $[0, T_{\max})$ be the maximal interval of existence of the solution from theorem 3.1. Then if $T_{\max} < \infty$ we have either*

$$\limsup_{t \to T_{\max}^-} \|\partial_x W(\cdot, t)\|_{L^\infty} = \infty, \tag{4.1}$$

or

$$\liminf_{t \to T_{\max}^-} \inf_{x \in \mathbb{R}} h(x, t) = 0. \tag{4.2}$$

From the uniform positivity criterion in proposition 2.1 and the change of variables in (3.1) (which implies $\alpha^2 \tilde{E}_\star = E_\star$), we deduce that for small-energy perturbations of constant states, a finite maximal existence time implies derivative blow-up.

Corollary 4.2. *If $\alpha^2 \tilde{E}_\star < \frac{1}{3} g \sqrt{\varepsilon} h_\star^3$ and the maximal time of existence $T_{\max} < \infty$, then*

$$\limsup_{t \to T_{\max}^-} \|\partial_x W(\cdot, t)\|_{L^\infty} = \infty. \tag{4.3}$$

This result follows from the fact that under the given hypothesis,

$$\inf_{x\in\mathbb{R}} h(x,t) \geqslant h_E > 0 \tag{4.4}$$

for all $t \in [0, T_{\max})$, with h_E given by proposition 2.1, so (4.2) cannot occur.

We will use a Gronwall-type inequality to derive the blow-up criterion of theorem 4.1. For this argument, it is necessary to improve the estimate on the nonlocal operator $\mathcal{I}_h^{-1}$ from (3.19) by controlling the L^∞ norms that appear explicitly on the right-hand side. Toward this aim, the following classical Landau–Kolmogorov interpolation inequality is crucial.

Lemma 4.3. *Let $\phi \in C^2(\mathbb{R};\mathbb{R})$ be such that*

$$\|\phi\|_{L^\infty} < \infty, \quad \|\phi''\|_{L^\infty} < \infty.$$

Then

$$\|\phi'\|_{L^\infty}^2 \leqslant 2\|\phi\|_{L^\infty}\|\phi''\|_{L^\infty}. \tag{4.5}$$

Proof. Without loss we may assume $\|\phi''\|_{L^\infty} = 1$ and $\phi'(0) = a > 0$. Then

$$\begin{aligned}\phi(a) &= \phi(0) + \int_0^a \phi'(x)\,\mathrm{d}x = \phi(0) + \int_0^a \left(\phi'(0) + \int_0^x \phi''(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &\geqslant \phi(0) + \int_0^a \left(a + \int_0^x (-1)\,\mathrm{d}y\right)\mathrm{d}x = \phi(0) + \frac{1}{2}a^2.\end{aligned} \tag{4.6}$$

Similarly, $\phi(-a) \leqslant \phi(0) - \frac{1}{2}a^2$. So

$$2\|\phi\|_{L^\infty} \geqslant \phi(a) - \phi(-a) \geqslant a^2 = |\phi'(0)|^2. \tag{4.7}$$

□

The nonlocal operator $\mathcal{I}_h^{-1}$ will be bounded using the following lemma. Below, the Banach space of continuous functions $\phi\colon \mathbb{R} \to \mathbb{R}$ having finite limits $\phi(\pm\infty)$ at $\pm\infty$ is denoted by $C_{\lim}(\mathbb{R})$, or just $C_{\lim}$. We let C_0 denote the subspace of continuous functions on $\mathbb{R}$ that vanish at $\pm\infty$, having limits $\phi(\pm\infty) = 0$.

Lemma 4.4. *Let $s > \frac{3}{2}$ and let $h - h_\star \in H^s(\mathbb{R})$ with $0 < h_{\min} \leqslant h \leqslant h_{\max} < \infty$. Then $\mathcal{I}_h^{-1}$ is well-defined on $C_{\lim}$. If $\phi \in C_{\lim}$ then $h\phi \in C_{\lim}$, and with $v = \mathcal{I}_h^{-1}(h\phi)$, it holds that $v \in C_{\lim}$ and v_x, $v_{xx} \in C_0$, with*

$$\|v\|_{L^\infty} \leqslant \|\phi\|_{L^\infty} \quad \text{and} \quad \|v_x\|_{L^\infty} \leqslant \frac{2}{\sqrt{\varepsilon}}\frac{h_{\max}^2}{h_{\min}^3}\|\phi\|_{L^\infty}. \tag{4.8}$$

Proof.

1. We first show that the L^∞ estimates hold for $\phi \in L^2(\mathbb{R}) \cap C_0$. In this case, $v \in H^2(\mathbb{R})$, since $\mathcal{I}_h^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is well-defined by lemma 3.3. Moreover, v is C^2 since h is C^1 and $hv - \epsilon(h^3 v_x)_x = h\phi$.
   We introduce a new variable z on $\mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}z} = h^3\frac{\mathrm{d}}{\mathrm{d}x}.$$

Then in terms of the new variable we have

$$v - \varepsilon h^{-4} v_{zz} = \phi. \tag{4.9}$$

By the maximum principle it follows $\|v\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$. Therefore

$$\varepsilon \|v_{zz}\|_{L^\infty} = \left\|h^4(v-\phi)\right\|_{L^\infty} \leqslant 2h_{\max}^4 \|\phi\|_{L^\infty}, \tag{4.10}$$

hence the Landau–Kolmogorov interpolation inequality (4.5) implies

$$(h_{\min})^3 \|v_x\|_{L^\infty} \leqslant \|v_z\|_{L^\infty} \leqslant (2\|v\|_{L^\infty}\|v_{zz}\|_{L^\infty})^{1/2} \leqslant \frac{2h_{\max}^2}{\sqrt{\varepsilon}} \|\phi\|_{L^\infty}.$$

Since $L^2(\mathbb{R}) \cap C_0$ is dense in C_0, it follows $\mathcal{I}_h^{-1}$ is well-defined on C_0, and the estimates (4.8) hold for all $\phi \in C_0$.

2. Now consider an arbitrary $\phi \in C_{\lim}$, and define $w_\phi \in C^\infty(\mathbb{R})$ by

$$w_\phi(x) = \frac{\phi(-\infty)}{h_\star} + \frac{\phi(+\infty) - \phi(-\infty)}{h_\star} \frac{e^x}{1+e^x}. \tag{4.11}$$

This function has the property that all its derivatives vanish at $\pm\infty$, and

$$hw_\phi(-\infty) = \phi(-\infty), \quad hw_\phi(+\infty) = \phi(+\infty). \tag{4.12}$$

Then $\phi - \mathcal{I}_h w_\phi \in C_0$. Due to step 1, we may then define

$$u = \mathcal{I}_h^{-1}\phi \stackrel{\text{def}}{=} w_\phi + \mathcal{I}_h^{-1}(\phi - \mathcal{I}_h w_\phi). \tag{4.13}$$

Note that u is C^2 and

$$\mathcal{I}_h u = \phi. \tag{4.14}$$

The estimates (4.8) on $v = \mathcal{I}_h^{-1}(h\phi)$ follow similarly as in step 1. □

Next, we observe that $\partial_x \mathcal{I}_h^{-1} \partial_x \circ h^3$ is a nonlocal operator of order zero. We can extract the local part of this operator, as follows. Since $\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x$,

$$-\varepsilon \partial_x \mathcal{I}_h^{-1} \partial_x \circ h^3 \circ \partial_x = \partial_x - \partial_x \mathcal{I}_h^{-1} \circ h. \tag{4.15}$$

This motivates us to write, for any nice enough function ϕ,

$$-\varepsilon \partial_x \mathcal{I}_h^{-1} \partial_x (h^3 \phi) = \phi - \partial_x \mathcal{I}_h^{-1} \left(h \int_{-\infty}^x \phi \right). \tag{4.16}$$

In light of this, we have the following L^∞ estimates, which in particular improve the estimate (3.19) in part (2) of lemma 3.3.

Lemma 4.5. *Let $s > \frac{3}{2}$ and suppose $h - h_\star \in H^s$ with $0 < h_{\min} \leqslant h \leqslant h_{\max} < \infty$.*

1) If $w = \epsilon \mathcal{I}_h^{-1} \partial_x \psi$ with $\psi \in L^1 \cap C_{\lim}$, then

$$\|w\|_{L^\infty} + \|w_x\|_{L^\infty} \leqslant C(\varepsilon, h_{\min}, h_{\max})(\|\psi\|_{L^\infty} + \|\psi\|_{L^1}). \tag{4.17}$$

2) If furthermore $\psi \in H^{s-1}$, then

$$\|w\|_{H^s} \leqslant \hat{C}_3 \left(\|\psi\|_{H^{s-1}} + \|h - h_\star\|_{H^{s-1}} (\|\psi\|_{L^\infty} + \|\psi\|_{L^1}) \right), \tag{4.18}$$

where $\hat{C}_3 = C(s, \epsilon, h_{\min}, \|h - h_\star\|_{W^{1,\infty}})$.

Proof. From (4.16), we have

$$-\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\psi = h^{-3}\psi - \partial_x\mathcal{I}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\psi\right). \tag{4.19}$$

Due to lemma 4.4,

$$\begin{aligned}\left\|\partial_x\mathcal{I}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\psi\right)\right\|_{L^\infty} &\leqslant C(\varepsilon,h_{\min},h_{\max})\left\|\int_{-\infty}^{x} h^{-3}\psi\right\|_{L^\infty}\\ &\leqslant C(\varepsilon,h_{\min},h_{\max})\|\psi\|_{L^1}.\end{aligned} \tag{4.20}$$

Then it follows

$$\|w_x\|_{L^\infty} = \left\|\epsilon\partial_x\mathcal{I}_h^{-1}\partial_x\psi\right\|_{L^\infty} \leqslant C(\varepsilon,h_{\min},h_{\max})(\|\psi\|_{L^\infty}+\|\psi\|_{L^1}). \tag{4.21}$$

From definition of $\mathcal{I}_h^{-1}$, we also have

$$-\varepsilon\mathcal{I}_h^{-1}\partial_x\circ h^3\circ\partial_x = \mathrm{Id} - \mathcal{I}_h^{-1}\circ h \tag{4.22}$$

whence, again due to lemma 4.4,

$$\begin{aligned}\|w\|_{L^\infty} = \left\|\epsilon\mathcal{I}_h^{-1}\partial_x\psi\right\|_{L^\infty} &= \left\|\epsilon\mathcal{I}_h^{-1}\partial_x\circ h^3\circ\partial_x\left(\int_{-\infty}^{x} h^{-3}\psi\right)\right\|_{L^\infty}\\ &\leqslant C(\varepsilon,h_{\min},h_{\max})\|\psi\|_{L^1}.\end{aligned} \tag{4.23}$$

This proves part (1). To deduce part (2), simply use the result of part (1) together with part (2) of lemma 3.3. □

Next we apply these results to provide a tame estimate on the nonlocal term in the system (3.6).

Corollary 4.6. *Let $s>\frac{3}{2}$, let $W=(\eta,u)\in H^s$, and suppose $h=h_\star+\alpha\eta$ satisfies $0<h_{\min}\leqslant h\leqslant h_{\max}$. Let $E=\tilde{E}_\star$ be the energy given by (3.10). Then with $f(W)$ given by (3.8), we have*

$$\|f(W)\|_{H^s}\leqslant\hat{C}\|W\|_{H^s},\qquad\text{where}\quad \hat{C}=C(s,\epsilon,h_{\min},E,\|W\|_{W^{1,\infty}}). \tag{4.24}$$

Proof. Let $\psi=2h^3u_x^2-\frac{1}{2}gh^2\eta_x^2$, so $w=\epsilon\mathcal{I}_h^{-1}\partial_x\psi=f(W)$. Then clearly

$$\|\psi\|_{L^\infty}\leqslant C(\|W\|_{W^{1,\infty}})\quad\text{and}\quad\|\psi\|_{L^1}\leqslant C(h_{\min},h_{\max})E. \tag{4.25}$$

Combined with (4.18), this implies

$$\|f(W)\|_{H^s}\leqslant\hat{C}\left(\|\psi\|_{H^{s-1}}+\|\eta\|_{H^{s-1}}\right) \tag{4.26}$$

where $\hat{C}=C(s,\varepsilon,h_{\min},E,\|W\|_{W^{1,\infty}})$. By applying the tame product estimate (3.17) several times to the terms of ψ (expanding h in powers of $h-h_\star$), we deduce

$$\|\psi\|_{H^{s-1}}\leqslant C(s,\|W\|_{W^{1,\infty}})\|W\|_{H^s}.$$

Combining this with (4.26), we obtain (4.24). □

Now we are ready to present the proof of the blow-up criterion.

Proof of theorem 4.1. Suppose $W \in C([0, T_{\max}); H^s)$ is the solution of (3.6) from theorem 3.1 on a maximal time interval with $T_{\max} < \infty$. We claim that it is impossible that

$$\sup_{t\in[0,T_{\max})} \|W_x(\cdot,t)\|_{L^\infty} < \infty \quad \text{and} \quad h_{\min} = \inf_{t\in[0,T_{\max})} \inf_{\mathbb{R}} h(\cdot,t) > 0. \tag{4.27}$$

Suppose on the contrary that (4.27) holds. Then because the energy $\tilde{E}_\star$ is conserved, we have

$$\|W(\cdot,t)\|_{L^\infty} \leqslant C\|W(\cdot,t)\|_{H^1} \leqslant C(h_{\min},\varepsilon)\|W^0\|_{H^1}, \tag{4.28}$$

hence $\|W(\cdot,t)\|_{W^{1,\infty}}$ remains bounded on $[0, T_{\max})$. We claim that $\|W(\cdot,t)\|_{H^s}$ also remains bounded. By part (i) of theorem 3.1 it then follows we can continue the solution to a larger time interval, contradicting maximality of $T_{\max}$.

To bound $\|W(\cdot,t)\|_{H^s}$ we modify the previous energy estimates as follows. We define a new energy by

$$\tilde{E}^s(W) \stackrel{\text{def}}{=} (\Lambda^s W, A(W)\Lambda^s W) = g\|\eta\|_{H^s}^2 + (\Lambda^s u, h\Lambda^s u), \tag{4.29}$$

replacing $\underline{W}$ in (3.37) by W. Due to the positive depth condition in (4.27) we have

$$\tilde{E}^s(W) \geqslant C(s, h_{\min})\|W\|_{H^s}^2. \tag{4.30}$$

Similarly to (3.42) but with $\underline{W} = W$ we compute

$$\begin{aligned} \partial_t(\Lambda^s W, A\Lambda^s W) = {} & (h_t\Lambda^s W, \Lambda^s W) + ((AB)_x\Lambda^s W, \Lambda^s W) \\ & + 2([B,\Lambda^s]W_x, A\Lambda^s W) - 2(\Lambda^s F, A\Lambda^s W). \end{aligned} \tag{4.31}$$

We now revise the previous estimates of the four terms as follows.

(1) For the first term, $|h_t| = \alpha|(hu)_x| \leqslant C(\alpha, \|W\|_{W^{1,\infty}})$, so

$$|(h_t\Lambda^s u, \Lambda^s u)| \leqslant C\|W\|_{H^s}^2. \tag{4.32}$$

(2) For the second term,

$$\begin{aligned} & |((AB)_x\Lambda^s W_x, \Lambda^s W)| \\ & \leqslant |(\alpha g u_x\Lambda^s\eta, \Lambda^s\eta)| + |2(gh_x\Lambda^s\eta, \Lambda^s u)| + |(\alpha(hu)_x\Lambda^s u, \Lambda^s u)| \\ & \leqslant C(s, \|W\|_{W^{1,\infty}})\|W\|_{H^s}^2. \end{aligned} \tag{4.33}$$

(3) For the third term, using the Kato–Ponce commutator estimate (3.16) and the tame product estimate (3.17), we get

$$\begin{aligned} & |([B,\Lambda^s]W_x, A\Lambda^s W)| \\ & \leqslant |(\alpha[u,\Lambda^s]\eta_x, g\Lambda^s\eta)| + |(\alpha[\eta,\Lambda^s]u_x, g\Lambda^s\eta)| + |(\alpha[u,\Lambda^s]u_x, h\Lambda^s u)| \\ & \leqslant C(s, \|W\|_{W^{1,\infty}})\|W\|_{H^s}^2. \end{aligned} \tag{4.34}$$

(4) For the fourth term, the estimate in corollary 4.6 yields

$$\|f(W)\|_{H^s} \leqslant C(s, \epsilon, h_{\min}, E, \|W\|_{W^{1,\infty}})\|W\|_{H^s}, \tag{4.35}$$

where $E = \tilde{E}_\star(W^0)$ is the constant energy of the solution. Hence

$$|(\Lambda^s F, A\Lambda^s W)| = |\Lambda^s f, h\Lambda^s u| \leqslant C(s, \varepsilon, \|W\|_{W^{1,\infty}})\|W\|_{H^s}^2. \tag{4.36}$$

Collecting everything yields

$$\partial_t \tilde{E}^s(W) \leqslant C\|W\|_{H^s}^2 \leqslant C\tilde{E}^s(W), \tag{4.37}$$

due to (4.30). By Gronwall's inequality it follows $\tilde{E}^s(W(\cdot,t))$ remains bounded on $[0, T_{\max})$, hence the same is true for $\|W(\cdot,t)\|_{H^s}$.

This completes the proof of theorem 4.1. □

5. Derivative blow-up in finite time

In this section, the main goal is to show that solutions to (1.1), (1.12) with certain initial conditions do exhibit derivative blow-up. The general strategy is to show that derivatives of the classical Riemann invariants satisfy coupled Ricatti-type equations that must exhibit blow-up.

5.1. Ricatti-type equations for derivatives of Riemann invariants

Write the Riemann invariants $R_\pm$ of the classical shallow water system and the two corresponding characteristic speeds $\lambda_\pm$ as

$$\begin{aligned} R_+ &= u + 2\sqrt{gh}, \qquad & \lambda_+ &= u + \sqrt{gh}, \\ R_- &= u - 2\sqrt{gh}, \qquad & \lambda_- &= u - \sqrt{gh}. \end{aligned} \tag{5.1}$$

These quantities satisfy

$$\lambda_+ = \frac{1}{4}(3R_+ + R_-), \quad \lambda_- = \frac{1}{4}(R_+ + 3R_-). \tag{5.2}$$

Next, note that the function inside the nonlocal term in (1.12) takes the form

$$2h^3u_x^2 - \frac{1}{2}gh^2h_x^2 = 2h^3(u_x^2 - \frac{1}{4}gh^{-1}h_x^2) = 2h^3(\lambda_+)_x(\lambda_-)_x. \tag{5.3}$$

From this one finds that the evolution equations for $R_\pm$ along characteristic curves take the form

$$\frac{\mathrm{d}^+}{\mathrm{d}t}R_+ := (R_+)_t + \lambda_+(R_+)_x = -2\varepsilon\mathcal{I}_h^{-1}\partial_x\big(h^3(\lambda_+)_x(\lambda_-)_x\big), \tag{5.4}$$

$$\frac{\mathrm{d}^-}{\mathrm{d}t}R_- := (R_-)_t + \lambda_-(R_-)_x = -2\varepsilon\mathcal{I}_h^{-1}\partial_x\big(h^3(\lambda_+)_x(\lambda_-)_x\big). \tag{5.5}$$

Here $\frac{\mathrm{d}^+}{\mathrm{d}t}$, $\frac{\mathrm{d}^-}{\mathrm{d}t}$ indicate the derivatives along '+' and '−' characteristic curves, respectively.

Next, we derive evolution equations for the derivatives of these classical Riemann invariants, writing

$$P_+ = (R_+)_x = u_x + \sqrt{\frac{g}{h}}h_x, \quad P_- = (R_-)_x = u_x - \sqrt{\frac{g}{h}}h_x. \tag{5.6}$$

Clearly

$$(\lambda_+)_x = \frac{1}{4}(3P_+ + P_-), \quad (\lambda_-)_x = \frac{1}{4}(P_+ + 3P_-). \tag{5.7}$$

Differentiating (5.4)–(5.5), one obtains that P_+ and P_- satisfy a system of Riccati-type equations containing a nonlocal term:

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ = -\frac{1}{4}(3P_+ + P_-)P_+ - 2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\big(h^3(\lambda_+)_x(\lambda_-)_x\big), \tag{5.8}$$

$$\frac{\mathrm{d}^-}{\mathrm{d}t}P_- = -\frac{1}{4}(P_+ + 3P_-)P_- - 2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\big(h^3(\lambda_+)_x(\lambda_-)_x\big). \tag{5.9}$$

In this system, the nonlocal operator $\partial_x\mathcal{I}_h^{-1}\partial_x \circ h^3$ has a local part which we extract as in the previous section, using the formula (4.16). This motivates us to introduce a primitive for the product $(\lambda_+)_x(\lambda_-)_x$, writing

$$G(y,t) = \int_{-\infty}^{y} (\lambda_+)_x(\lambda_-)_x\,\mathrm{d}x. \tag{5.10}$$

In terms of this quantity we can write

$$-2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\big(h^3(\lambda_+)_x(\lambda_-)_x\big) = 2(\lambda_+)_x(\lambda_-)_x - Q, \tag{5.11}$$

where

$$Q \stackrel{\text{def}}{=} 2\partial_x\mathcal{I}_h^{-1}(hG). \tag{5.12}$$

Using (5.7) we find that the Ricatti-type evolution equations for $P_\pm$ take the form

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ = -\frac{3}{8}P_+^2 + P_+P_- + \frac{3}{8}P_-^2 - Q, \tag{5.13}$$

$$\frac{\mathrm{d}^-}{\mathrm{d}t}P_- = \frac{3}{8}P_+^2 + P_+P_- - \frac{3}{8}P_-^2 - Q. \tag{5.14}$$

These two equations are of central importance because the nonlocal term Q that appears here is essentially a constant, and after that (5.13)–(5.14) is a system whose behaviors are governed by the quadratic terms in P_+ and P_-.

To see that the integral in (5.10) is well-defined, note that

$$\begin{aligned} |(\lambda_+)_x(\lambda_-)_x| &= \frac{1}{16}|(3P_+ + P_-)(P_+ + 3P_-)| \\ &\leqslant \frac{1}{2}(P_+^2 + P_-^2) \leqslant 2\left(u_x^2 + \frac{g}{h}h_x^2\right). \end{aligned} \tag{5.15}$$

It then follows from conservation of the relative energy $E_\star$ in (2.1) that as long as $h \geqslant h_{\min} > 0$ we have the estimate

$$\|G(\cdot,t)\|_{L^\infty} \leqslant \int_{\mathbb{R}} |(\lambda_+)_x(\lambda_-)_x|\,\mathrm{d}x \leqslant \frac{2}{h_{\min}^3}\int_{\mathbb{R}} \big(h^3u_x^2 + gh^2h_x^2\big)\,\mathrm{d}x \leqslant \frac{4E_\star}{\varepsilon h_{\min}^3}. \tag{5.16}$$

To handle the nonlocal term Q in the Ricatti-type system (5.13)–(5.14), we exploit lemma 4.4 to address the L^∞ bound on the solution of the elliptic operator $\mathcal{I}_h$. Lemma 4.4 together with the estimate in (5.16) immediately yields the following uniform bound for the nonlocal term Q given by (5.12).

Proposition 5.1. *For any classical solution W of (3.6) satisfying $0 < h_{\min} \leqslant h \leqslant h_{\max} < \infty$ on a time interval $[0, T_\star)$, we have*

$$\|Q(\cdot,t)\|_{L^\infty} \leqslant \frac{4}{\sqrt{\varepsilon}}\frac{h_{\max}^2}{h_{\min}^3}\|G(\cdot,t)\|_{L^\infty} \leqslant \frac{16}{\varepsilon^{3/2}}\frac{h_{\max}^2}{h_{\min}^6}E_\star. \tag{5.17}$$

Now we are ready to state our main theorem.

Theorem 5.2. *Fix $\epsilon, \alpha > 0$. Then there exists compactly supported smooth initial data $W^0 = (\eta^0, u^0)$ of the IVP (3.6), having arbitrarily small relative energy $\tilde{E}_\star$, for which the derivatives of the solution will blow up in finite time. The precise meaning of this is that there exists $T \in (0, \infty)$ such that the solution exists and stays smooth for all $(x, t) \in \mathbb{R} \times [0, T)$, and*

$$\sup_{\mathbb{R}\times[0,T)} P_+(x,t) + |P_-(x,t)| < \infty, \tag{5.18}$$

but

$$\inf_{x\in\mathbb{R}} P_+(x,t) \to -\infty \quad \text{as } t \uparrow T. \tag{5.19}$$

Remark 5.3.

(i) The blow-up behavior described in this theorem implies that

$$\inf_{x\in\mathbb{R}} u_x(x,t) \to -\infty \quad \text{and} \quad \inf_{x\in\mathbb{R}} h_x(x,t) \to -\infty \qquad \text{as } t \uparrow T,$$

see (5.36) and (5.25) below.

(ii) We will show that blow-up as described in the theorem occurs for any initial data that satisfy certain explicit upper bounds on relative energy $E_\star$, $|P_-|$, and P_+, such that $\inf P_+(\cdot, 0)$ is sufficiently negative; see lemma 5.6 below.

5.2. Proof of derivative blow-up

Next, we will sketch some of the fundamental ideas of the main proof. Our goal is to construct initial data such that P_+ blows up while P_- stays bounded. If indeed P_- stays bounded then it is rather easy to infer from (5.13) P_+ blows up quickly if P_+ is initially large on some individual characteristic. However, to show P_- stays bounded everywhere while P_+ blows up somewhere, (5.14) requires us to show that the integral of P_+^2 along all the '$-$' characteristics has to remain bounded. A principal difficulty is that the characteristic speeds depend (nonlocally) on the solution itself. Moreover, due to (5.7) we expect both $(\lambda_+)_x$ and $(\lambda_-)_x$ to blow up to $-\infty$, as P_+ does. This indicates that characteristics curves are concentrating in the vicinity of the singularity.

Let us introduce the flow maps $X_+, X_- : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ along the '$+$' and '$-$' characteristic curves, defined through

$$\begin{cases} \dfrac{\partial X_+}{\partial t}(\xi,t) = \lambda_+(X_+(\xi,t),t) \\ X_+(\xi,0) = \xi \end{cases}, \quad \begin{cases} \dfrac{\partial X_-}{\partial t}(\zeta,t) = \lambda_-(X_-(\zeta,t),t) \\ X_-(\zeta,0) = \zeta \end{cases}, \tag{5.20}$$

where ξ, ζ are Lagrangian variables. Differentiating the first set of equations in ξ, one obtains

$$\left(\frac{\partial X_+}{\partial \xi}\right)_t = (\lambda_+)_x \frac{\partial X_+}{\partial \xi}, \quad \frac{\partial X_+}{\partial \xi}(\xi,0) = 1. \tag{5.21}$$

So, if some constant $L \geqslant (\lambda_+)_x$ everywhere along a certain '+' characteristic curve for time in $[0,t]$, it follows that $\frac{\partial X_+}{\partial \xi} \leqslant \mathrm{e}^{Lt}$ everywhere on the curve in this time interval. The same holds true for the '$-$' characteristic curves. When it happens that $L \leqslant 0$, nearby characteristics curves focus towards each other and concentrate as time increases.

The key to the proof of blow-up will be to establish two things:

(a) $\frac{\partial X_+}{\partial \xi} P_+^2$ is close to constant along the '+' characteristic curves; i.e. the concentrating effect of the '+' characteristic curves and the blow-up effect of P_+^2 offset and exactly balance each other.
(b) The integrals of P_+^2 along the '$-$' characteristic curves are bounded.

We shall use (a) to derive (b). The exact meaning of (a) rests on the fact that, temporarily fixing ξ and abusing notation to write $P_+ = P_+(X_+(\xi,t),t)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X_+}{\partial \xi} P_+^2\right) &= \left(\frac{\partial}{\partial t}\frac{\partial X_+}{\partial \xi}\right) P_+^2 + \frac{\partial X_+}{\partial \xi} 2P_+ \frac{\mathrm{d}}{\mathrm{d}t}(P_+) \\ &= (\lambda_+)_x \frac{\partial X_+}{\partial \xi} P_+^2 + \frac{\partial X_+}{\partial \xi} 2P_+ \left(\frac{1}{2}(\lambda_+)_x (3P_- - P_+) - Q\right) \\ &= \frac{\partial X_+}{\partial \xi} P_+ (3P_-(\lambda_+)_x - 2Q). \end{aligned} \tag{5.22}$$

Here the important point is that the highest order terms (cubic in P_+) match exactly and go away.

Proof of theorem 5.2. We are now ready to begin the main argument, proceeding in several steps.

Step zero. We will work with smooth initial data with relative energy sufficiently small so that we can apply proposition 2.1(b) and corollary 4.2. We introduce several explicit constants in this proof chosen as follows: the initial depth at infinity $h_\star > 0$ is arbitrary. We define positive constants $C_1(\varepsilon), C_2(\varepsilon, h_\star), C_3(\varepsilon, h_\star)$ explicitly by

$$C_1(\varepsilon) \stackrel{\text{def}}{=} \frac{6g}{\sqrt{\varepsilon}}, \quad C_2(\varepsilon, h_\star) \stackrel{\text{def}}{=} \frac{72C_1}{\sqrt{\varepsilon}h_\star}, \quad C_3(\varepsilon, h_\star) \stackrel{\text{def}}{=} \frac{16C_1}{\sqrt{2gh_\star}}. \tag{5.23}$$

We let $T_{\max}$ denote the maximal time of existence of the smooth solution, and establish some preliminary bounds for solutions whose relative energy from (2.1) satisfies

$$E_\star \leqslant \frac{1}{6} g\sqrt{\varepsilon} h_\star^3. \tag{5.24}$$

Using this bound in proposition 2.1(b) we get $h_E \geqslant \frac{1}{2} h_\star$, and from this and remark 2.2(iii) it follows that up to the maximal time of existence the depth satisfies the bounds

$$h_{\min} \leqslant h \leqslant h_{\max}, \quad \text{with} \quad h_{\min} = \frac{1}{2}h_\star, \quad h_{\max} = \frac{3}{2}h_\star. \tag{5.25}$$

Next, we can bound the fluid velocity by a Sobolev-like inequality, writing

$$\|u\|_{L^\infty}^2 \leqslant \int_{\mathbb{R}} 2|uu_x|\,\mathrm{d}x \leqslant \int_{\mathbb{R}} (hu^2 + \epsilon h^3 u_x^2)\frac{\mathrm{d}x}{\sqrt{\varepsilon h^2}} \leqslant \frac{2E_\star}{\sqrt{\varepsilon h_{\min}^2}} \leqslant g h_{\max}. \tag{5.26}$$

Hence the characteristic speeds from (5.1) are bounded by

$$\|\lambda_\pm\|_{L^\infty} \leqslant 2\sqrt{g h_{\max}}. \tag{5.27}$$

Next, as in (5.15) and (5.16), we find that

$$\|P_\pm(\cdot,t)\|_{L^2}^2 \leqslant \frac{4E_\star}{\varepsilon h_{\min}^3} < C_1. \tag{5.28}$$

Finally, applying proposition 5.1, we deduce that

$$\|Q(\cdot,t)\|_{L^\infty} \leqslant \frac{4h_{\max}^2}{\sqrt{\varepsilon} h_{\min}^3} C_1 = \frac{72 C_1}{\sqrt{\varepsilon} h_\star} \leqslant C_2. \tag{5.29}$$

Step one. The key to the proof will involve obtaining bounds on the quantity

$$M(t) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}, s\in[0,t]} P_+(x,s) + \sup_{x\in\mathbb{R}, s\in[0,t]} |P_-(x,s)| \tag{5.30}$$

that are valid on a fixed time interval independent of any lower bounds on P_+.

Lemma 5.4. *There exists $T_\star = T_\star(\epsilon, h_*) > 0$ independent of the initial data, such that if (5.24) holds and also*

$$M(0) \leqslant \frac{1}{4}C_3, \tag{5.31}$$

then

$$M(t) \leqslant C_3 \quad \text{for all } t \in [0, T_{\max} \wedge T_\star].$$

Taking this result for granted for the moment, let us complete the proof of theorem 5.2. We study solutions with smooth initial data that satisfy the relative energy bound (5.24) and the (one-sided) sup bound (5.31).

We first claim that under a further condition on initial data, necessarily $T_{\max} \leqslant T_\star$. We argue as follows. From (5.6) we infer that if $T_{\max} > T_\star$, then on the time interval $[0, T_\star]$ the norm $\|W(\cdot,t)\|_{H^2}$ is bounded and hence so is $\|P_\pm\|_{L^\infty}$.

From (5.13), however, using the inequality $P_+P_- \leqslant \frac{1}{8}P_+^2 + 2P_-^2$ we find that along any '+' characteristic $x = X_+(\xi, t)$,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ \leqslant -\frac{3}{8}P_+^2 + P_+P_- + \frac{3}{8}P_-^2 + C_2 \leqslant -\frac{1}{4}P_+^2 + 3C_3^2 + C_2 \leqslant -\frac{1}{8}P_+^2, \tag{5.32}$$

provided $\frac{1}{8}P_+(\xi,t)^2 \geqslant 3C_3^2 + C_2$ at $t = 0$ (for then P_+^2 is increasing). Choose $\kappa_0 < 0$ so large that

$$\frac{1}{8}\kappa_0^2 \geqslant 3C_3^2 + C_2 \quad \text{and} \quad \kappa_0 < -\frac{8}{T_\star}, \tag{5.33}$$

and set $\kappa(t) = \kappa_0(1 + \frac{1}{8}\kappa_0 t)^{-1}$. Since $\kappa' = -\frac{1}{8}\kappa^2$, it follows that if $P_+(\xi,0) \leqslant \kappa_0$ then

$$P_+(\xi,t) \leqslant \kappa(t) \to -\infty \quad \text{as} \quad t \uparrow -8\kappa_0^{-1} < T_\star. \tag{5.34}$$

This proves the following.

Lemma 5.5. *Necessarily $T_{\max} \leqslant T_\star$, if initially (5.24) and (5.31) hold, and*

$$\inf_{\xi\in\mathbb{R}} P_+(\xi,0) < \kappa_0. \tag{5.35}$$

Now, the essential point is that it is straightforward to construct smooth initial data that satisfy the required bounds to this point. We omit the proof of the following.

Lemma 5.6. *There exist smooth initial data W^0 of compact support in $\mathbb{R}$ such that $E_\star$ is arbitrarily small, $M(0)$ satisfies (5.31), and $P_+(\cdot,0)$ satisfies (5.35).*

With any such initial data, it then follows further from corollary 4.2, the formulas

$$u_x = \frac{1}{2}(P_+ + P_-), \quad h_x = \sqrt{\frac{h}{g}}(P_+ - P_-), \tag{5.36}$$

and lemma 5.4, that P_+ cannot remain bounded below and must satisfy

$$\liminf_{t\uparrow T_{\max}} \inf_{x\in\mathbb{R}} P_+(x,t) = -\infty. \tag{5.37}$$

We claim that actually (5.19) holds, meaning that the '$\liminf$' here can be replaced by '$\lim$'. The reason for this is that from (5.13) we have that along any '$+$' characteristic,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ \geqslant -\frac{3}{8}P_+^2 + P_+P_- + \frac{3}{8}P_-^2 - C_2 \geqslant -\frac{1}{2}P_+^2 - 3C_3^2 - C_2 \geqslant -\frac{1}{2}(P_+ + C_4)^2, \tag{5.38}$$

where $\frac{1}{2}C_4^2 = 3C_3^2 + C_2$. By consequence, if we suppose that (5.19) is false, and instead $\inf_x P(x,t_k) \geqslant \kappa_1 > -\infty$ for some sequence $t_k \to T_{\max}$ in $[0,T_{\max})$, then for k so large that $1 + \frac{1}{2}\kappa_1(T_{\max} - t_k) \geqslant \frac{1}{2}$, we find by solving the Ricatti inequality above that

$$\inf_{x\in\mathbb{R}} P_+(x,t) + C_4 \geqslant \frac{\kappa_1}{1 + \frac{1}{2}\kappa_1(t - t_k)} \geqslant 2\kappa_1 \quad \text{for all } t \in [t_k, T_{\max}). \tag{5.39}$$

This contradicts (5.37) and proves (5.19).

Step two. It remains to prove lemma 5.4, using a continuation argument. Set

$$T_3 \stackrel{\text{def}}{=} \sup\{t \in [0,T_{\max}) : M(t) \leqslant C_3\}. \tag{5.40}$$

Then for $t \in [0,T_3)$ we have the following estimates. First, as in (5.32) we have

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ \leqslant -\frac{3}{8}P_+^2 + P_+P_- + \frac{3}{8}P_-^2 + C_2 \leqslant 3C_3^2 + C_2,$$

whence

$$\sup_x P_+(x,t) \leqslant \frac{1}{4}C_3 + t(3C_3^2 + C_2). \tag{5.41}$$

Similarly, we find that along '$-$' characteristics,

$$\frac{\mathrm{d}^-}{\mathrm{d}t}P_- \geqslant \frac{3}{8}P_+^2 + P_+P_- - \frac{3}{8}P_-^2 - C_2 \geqslant -3C_3^2 - C_2,$$

whence

$$\inf_x P_-(x,t) \geqslant -\frac{1}{4}C_3 - t(3C_3^2 + C_2). \tag{5.42}$$

Finally, in a similar way we find

$$\frac{\mathrm{d}^-}{\mathrm{d}t}P_- \leqslant 3P_+^2 + C_2,$$

whence

$$\sup_x P_-(x,t) \leqslant \frac{1}{4}C_3 + tC_2 + 3\sup_{\zeta\in\mathbb{R}} \int_0^t P_+^2(X_-(\zeta,s),s)\,\mathrm{d}s. \tag{5.43}$$

The following estimate on this last integral is the key to the proof.

Lemma 5.7. *There exists $T_{\star\star} = T_{\star\star}(\epsilon, h_\star) > 0$ independent of the initial data, such that if $T_3 < T_{\max} \wedge T_{\star\star}$ then*

$$\sup_{\zeta\in\mathbb{R}} \int_0^t P_+^2(X_-(\zeta,s),s)\,\mathrm{d}s \leqslant \frac{1}{8}C_3 \quad \text{for all } t \in [0, T_3].$$

Taking this result for granted for the moment, we complete the proof of lemma 5.4.

Proof of lemma 5.4 We choose $T_\star > 0$ such that

$$T_\star \leqslant T_{\star\star} \quad \text{and} \quad 2T_\star(3C_3^2 + C_2) < \frac{1}{8}C_3. \tag{5.44}$$

We claim that then $T_3 \geqslant T_{\max} \wedge T_\star$. Indeed, if not, then by combining the result of lemma 5.7 with the estimates in (5.41)–(5.43), we infer that

$$M(t) \leqslant \frac{7}{8}C_3 + 2t(3C_3^2 + C_2) < C_3 \quad \text{for all } t \in [0, T_3]. \tag{5.45}$$

But then by continuity, $M(t) < C_3$ on a larger time interval, contradicting the definition of T_3 in (5.40). □

Step three. Now it remains only to prove lemma 5.7.

Proof of lemma 5.7. We first note that due to (5.25) the difference between characteristic speeds at the same point satisfies

$$\lambda_+ - \lambda_- = 2\sqrt{gh} \in [\sqrt{2gh_\star}, \sqrt{6gh_\star}]. \tag{5.46}$$

Now, suppose $T_3 < T_{\max}$, and fix any $\zeta_0 \in \mathbb{R}$. For each $t \in [0, T_3]$, due to (5.46) there is a unique $\xi = \xi_0(t) \leqslant \zeta_0$ such that the '$+$' characteristic starting from ξ and the '$-$' characteristic starting from ζ_0 intersect at time t, i.e.

$$X_+(\xi_0(t), t) = X_-(\zeta_0, t). \tag{5.47}$$

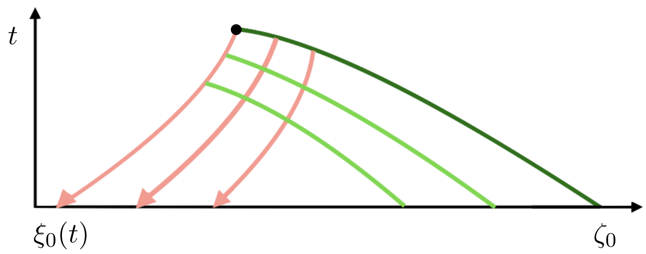

Figure 1. Pullback from $-$ characteristics along $+$ characteristics. '$\bullet$' marks the point (x, t) where $x = X_+(\xi_0(t), t) = X_-(\zeta_0, t)$.

(See figure 1 for a sketch of the situation.) Note that due to (5.20) and the bound on characteristic speeds in (5.27) we can say that

$$\begin{aligned}\zeta_0 - \xi_0(t) &\leqslant |\zeta_0 - X_-(\zeta_0, t)| + |X_+(\xi_0(t), t) - \xi_0(t)| \\ &\leqslant (\|\lambda_+\|_{L^\infty} + \|\lambda_-\|_{L^\infty})t \leqslant 4\sqrt{gh_{\max}}t.\end{aligned} \tag{5.48}$$

Differentiating (5.47) in t, we find

$$\frac{\partial X_+}{\partial \xi}(\xi_0(t), t)\frac{\mathrm{d}\xi_0}{\mathrm{d}t} + \lambda_+(X_+(\xi_0(t), t), t) = \lambda_-(X_-(\zeta_0, t), t).$$

Due to (5.47) and (5.46) it follows

$$-\frac{\partial X_+}{\partial \xi}(\xi_0(t), t)\frac{\mathrm{d}\xi_0}{\mathrm{d}t} = \lambda_+ - \lambda_- = 2\sqrt{gh}.$$

Now by changing variables $s = s_0(\xi)$ using the inverse function $s_0 = \xi_0^{-1}$, we get, writing $(P_+^2 \circ X_+)(\xi, \tau) = P_+(X_+(\xi, \tau), \tau)^2$,

$$\begin{aligned}&\int_0^t P_+^2(X_-(\zeta_0, s), s)\,\mathrm{d}s \\ &= \int_{\xi_0(t)}^{\zeta_0}\left(\frac{P_+^2}{2\sqrt{gh}}\right)(X_+(\xi, s_0(\xi)), s_0(\xi))\frac{\partial X_+}{\partial \xi}(\xi, s_0(\xi))\,\mathrm{d}\xi \\ &\leqslant \frac{1}{\sqrt{2gh_\star}}\int_{\xi_0(t)}^{\zeta_0} P_+^2(X_+(\xi, s_0(\xi)), s_0(\xi))\frac{\partial X_+}{\partial \xi}(\xi, s_0(\xi))\,\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2gh_\star}}\int_{\xi_0(t)}^{\zeta_0}\left(P_+^2(\xi, 0) + \int_0^{s_0(\xi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(P_+^2 \circ X_+\frac{\partial X_+}{\partial \xi}\right)(\xi, \tau)\,\mathrm{d}\tau\right)\mathrm{d}\xi \\ &\leqslant \frac{1}{\sqrt{2gh_\star}}\|P_+(\cdot, 0)\|_{L^2}^2 + \frac{1}{\sqrt{2gh_\star}}\int_{\xi_0(t)}^{\zeta_0}\int_0^{s_0(\xi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(P_+^2 \circ X_+\frac{\partial X_+}{\partial \xi}\right)\mathrm{d}\tau\,\mathrm{d}\xi. \\ &\leqslant \frac{1}{16}C_3 + \frac{1}{\sqrt{2gh_\star}}\mathcal{A}\end{aligned} \tag{5.49}$$

where

$$\mathcal{A} \stackrel{\text{def}}{=} \int_{\xi_0(t)}^{\zeta_0} \int_0^{s_0(\xi)} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(P_+^2 \circ X_+ \frac{\partial X_+}{\partial \xi}\right)(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi. \tag{5.50}$$

To get the fourth line in (5.49) we use the fundamental theorem of calculus along the '+' characteristic starting from $(\xi, 0)$, and to get the last line we use (5.28) and the definition of C_3 from (5.23).

Now, by the crucial derivative computation (5.22), since $(\lambda_+)_x = \frac{1}{4}(3P_+ + P_-)$ and $|P_-| \leqslant M(t) \leqslant C_3$, we can bound the the integrand of $\mathcal{A}$ by a quadratic polynomial in P_+ times $\frac{\partial X_+}{\partial \xi}$, in which the linear term can be bounded by the quadratic term and the constant term:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(P_+^2 \circ X_+ \frac{\partial X_+}{\partial \xi}\right)(\xi,\tau) &\leqslant \left|\frac{\partial X_+}{\partial \xi} P_+ (3(\lambda_+)_x P_- - 2Q)\right| \\ &\leqslant \frac{\partial X_+}{\partial \xi}\left(\frac{3}{4}|P_+P_-|\,|3P_+ + P_-| + 2|P_+|C_2\right) \\ &\leqslant \frac{\partial X_+}{\partial \xi}\left(3C_3P_+^2 + C_3^3 + C_2^2\right).\end{aligned}$$

Due to the bound (5.48), by using Fubini's theorem we obtain

$$\begin{aligned}\int_{\xi_0(t)}^{\zeta_0}\int_0^{s_0(\xi)} \frac{\partial X_+}{\partial \xi}(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi &= \int_0^t (\xi_0(\tau) - \xi_0(t))\,\mathrm{d}\tau \\ &\leqslant (\zeta_0 - \xi_0(t))t \leqslant 4\sqrt{gh_{\max}}t^2 < 5\sqrt{gh_\star}t^2.\end{aligned}$$

To bound the integral of $\frac{\partial X_+}{\partial \xi}P_+^2$, one uses the inequality

$$\frac{1}{s}\int_0^s f(\tau)\,\mathrm{d}\tau \leqslant f(0) + \int_0^s |f'(\tau)|\,\mathrm{d}\tau \quad \forall f \in C^1(\mathbb{R}) \tag{5.51}$$

to obtain

$$\begin{aligned}&\int_{\xi_0}^{\zeta_0}\int_0^{s_0(\xi)} \frac{\partial X_+}{\partial \xi}P_+^2\,\mathrm{d}\tau\,\mathrm{d}\xi \\ &\leqslant \int_{\xi_0}^{\zeta_0} s_0(\xi)\left(P_+^2(\xi,0) + \int_0^{s_0(\xi)}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(P_+^2 \circ X_+ \frac{\partial X_+}{\partial \xi}\right)(\xi,\tau)\right|\mathrm{d}\tau\right)\mathrm{d}\xi \\ &\leqslant t(\|P_+(\cdot,0)\|_{L^2}^2 + \mathcal{A}) \leqslant t(C_1 + \mathcal{A}).\end{aligned} \tag{5.52}$$

Putting these bounds into (5.50), one obtains

$$\mathcal{A} \leqslant 3C_3t(C_1 + \mathcal{A}) + (C_3^3 + C_2^2)5\sqrt{gh_\star}t^2.$$

We now choose $T_{\star\star} > 0$ to be so small that

$$3C_3T_{\star\star} < \frac{1}{2} \quad \text{and} \quad 5(C_3^3 + C_2^2)T_{\star\star} < \frac{1}{2}. \tag{5.53}$$

Then if $T_3 < T_{\star\star}$, it follows that for all $t \in [0, T_3]$,

$$\mathcal{A} \leqslant 6C_3C_1t + \sqrt{gh_*}t.$$

Further restricting $T_{\star\star}$ to be so small that

$$\frac{1}{\sqrt{2gh_\star}}\left(6C_3C_1 + \sqrt{gh_\star}\right)T_{\star\star} \leqslant \frac{1}{16}C_3,$$

we can conclude from (5.49) that for all $t \in [0, T_3]$,

$$\int_0^t P_+^2(X_-(\zeta_0, s), s)\, \mathrm{d}s \leqslant \frac{1}{16}C_3 + \frac{1}{\sqrt{2gh_\star}}\mathcal{A} \leqslant \frac{1}{8}C_3. \tag{5.54}$$

This finishes the proof of lemma 5.7. □

With this, the proof of theorem 5.2 is complete.

6. Asymptotic blow-up profile

We recall that for the classical shallow water equations ($\varepsilon = 0$), the system (5.4) and (5.5) admits simple wave solutions with $R_- \equiv 0$ and R_+ satisfying an inviscid Burgers equation. Namely, (5.4) with $\varepsilon = 0$ yields

$$(R_+)_t + \lambda_+(R_+)_x = 0, \qquad \lambda_+ = \frac{3}{4}R_+. \tag{6.1}$$

As is well known (and briefly discussed below) smooth solutions of this equation with $(R_+)_x < 0$ somewhere must break down in finite time, and typically develop a profile with a cube-root singularity at the blow-up point, with

$$R_+ \sim a_0 - b_0(x - x_0)^{1/3}. \tag{6.2}$$

Then after blow-up, the singularity changes type as a shock discontinuity develops.

For the rSV system with $\varepsilon > 0$, the coefficients of the quadratic terms in the Ricatti-type system (5.13) and (5.14) differ from their values in the classical system (5.8) and (5.9) with $\varepsilon = 0$, due to an ε-independent contribution of the local part of the nonlocal term. As we discuss heuristically in this section, this difference appears to change the nature of the typical solution profile at the time of blow-up. For the blowing-up solutions from section 5 above, we will argue that one should expect that the profile near a blow-up point should typically have a $\frac{3}{5}$-root singularity instead:

$$R_+ \sim a_\varepsilon - b_\varepsilon(x - x_0)^{3/5}. \tag{6.3}$$

What happens after the blow-up time is not known, but we may conjecture that solutions develop $\frac{2}{3}$-root singularities, like the weakly singular traveling waves described in [25].

6.1. Blow-up profile for the rSV equations

Let us describe heuristically why we may expect the blow-up profile in (6.3). Suppose we start close to the blow-up time, taking $P_0(\xi)$ to be initial data for P_+ like that described in the proof of theorem 5.2, with a large negative minimum at $\xi = 0$, say. In the vicinity of $\xi = 0$ we then typically expect quadratic behavior near the minimum, with

$$P_0(\xi) \approx P_0(0) + c_0\xi^2\ . \tag{6.4}$$

Since Q and P_- are bounded before P_+ blows up, we assume

$$|Q| + |P_-| \ll |P_+|, \tag{6.5}$$

and neglect these terms, rewriting (5.13) and (5.21) as

$$\frac{\mathrm{d}^+}{\mathrm{d}t}P_+ = \frac{\mathrm{d}}{\mathrm{d}t}\left(P_+(X_+(\xi,t),t)\right) = -\frac{3}{8}P_+^2, \tag{6.6}$$

$$\left(\frac{\partial X_+}{\partial \xi}\right)_t = \frac{3}{4}P_+\frac{\partial X_+}{\partial \xi}. \tag{6.7}$$

With the initial data P_0 we solve (6.6) along the '+' characteristic curves to get

$$P_+(X_+(\xi,t),t) = \frac{P_0(\xi)}{1+\frac{3}{8}tP_0(\xi)}. \tag{6.8}$$

Following the '+' characteristic curve emitting from the global minimum point 0 of P_0 we expect blow-up to happen first at $\xi = 0$ at the time $T = \left(\frac{3}{8}|P_0(0)|\right)^{-1} \ll 1$, and we find

$$P_+(X_+(\xi,T),T) \approx -\frac{c_1}{\xi^2}, \qquad c_1 = \frac{|P_0(0)|}{\frac{3}{8}Tc_0}. \tag{6.9}$$

Now, from (6.7) and (6.6) one can compute that

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\left(\frac{\partial X_+}{\partial \xi}P_+^2\right) = 0. \tag{6.10}$$

(This balancing effect agrees with the rigorous computation (5.22).) Integrating this equation along characteristics up to the blow-up time T, we find for ξ close to 0 that

$$\frac{\partial X_+}{\partial \xi}(\xi,T) = \frac{P_0^2(\xi)}{P_+^2(X_+(\xi,T),T)} = \left(1+\frac{3}{8}TP_0(\xi)\right)^2 = \left(\frac{3}{8}Tc_0\right)^2\xi^4. \tag{6.11}$$

Integrating in ξ we get

$$X_+(\xi,T) - X_+(0,T) = c_2\xi^5.$$

Solving for ξ and using this in (6.9) we find, for x near $x_0 = X_+(0,T)$,

$$P_+(x,T) \approx -c_3(x-x_0)^{-2/5}.$$

Integrating in x now yields (6.3).

We remark that these heuristics lead us to expect that $P_+(\cdot,T)$ belongs to $L^p(\mathbb{R})$ for $p < \frac{5}{2}$. However, if we repeat the calculations with P_0 having a degenerate minimum $\sim P_0 + c_0\xi^{2n}$ for arbitrary $n \in \mathbb{N}$ we find that in general $P_+(\cdot,T)$ need not remain in L^p for any $p > 2$.

6.2. Comparison with the inviscid Burgers equation

We briefly indicate how the calculations above differ with the situation when $\varepsilon = 0$. In this case, the characteristic speed $u = \lambda_+$ in (6.1) satisfies the inviscid Burgers equation

$$u_t + uu_x = 0. \tag{6.12}$$

From Burgers equation $v \stackrel{\text{def}}{=} u_x = \frac{3}{4}(R_+)_x$ satisfies $v_t + uv_x = -v^2$, which implies that along characteristic curves $X(\xi, \cdot)$ with $\frac{\partial X}{\partial t}(\xi, t) = u(X(\xi, t), t)$ we have, analogous to (6.7) and (6.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(v(X(\xi,t),t)\big) = -v^2, \qquad \left(\frac{\partial X}{\partial \xi}\right)_t = v\frac{\partial X}{\partial \xi}(\xi,t).$$

Then similar to (6.9) it follows that $v(X(\xi,t),t) \approx -c/\xi^2$ at the time of blow-up.

Differing from (6.10), however, we have instead

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X}{\partial \xi}v\right) = 0, \tag{6.13}$$

where v appears here and not v^2. Now instead of (6.11) one finds $\frac{\partial X}{\partial \xi} \sim c\xi^2$ and $X - x_0 \sim c\xi^3$. From this one deduces that at blow-up,

$$v(x,t) \sim -c(x-x_0)^{-2/3}, \tag{6.14}$$

whence (6.2) follows.

Acknowledgments

The authors are grateful to an anonymous referee for indicating how to fix a serious mistake in the original proof of the blow-up criterion. RLP thanks Robin Ming Chen for discussions regarding the Green–Naghdi equations. This material is based upon work supported by the National Science Foundation under grants DMS 1514826 and 1812573 (JGL) and DMS 1515400 and 1812609 (RLP), partially supported by the Simons Foundation under grant 395796, by the Center for Nonlinear Analysis (CNA) under National Science Foundation PIRE Grant No. OISE-0967140, and by the NSF Research Network Grant No. RNMS11-07444 (KI-Net).

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