Convergence of Vortex Methods for Weak Solutions to the 2-D Euler Equations with Vortex Sheet Data

JIAN-GUO LIU
Temple University

AND

ZHOUPING XIN
Courant Institute

Abstract

We prove the convergence of vortex blob methods to classical weak solutions for the two-dimensional incompressible Euler equations with initial data satisfying the conditions that the vorticity is a finite Radon measure of distinguished sign and the kinetic energy is locally bounded. This includes the important example of vortex sheets. The result is valid as long as the computational grid size h does not exceed the smoothing blob size ε, i.e., $h/\varepsilon \leqq C$. © 1995 John Wiley & Sons, Inc.

1. Introduction and Main Results

Computational vortex methods approximate the vorticity by a finite sum of "blobs," or cores of prescribed shape, which are advected according to the corresponding desingularized velocity field. One important application of these methods is in the numerical simulations of singular flows such as vortex sheets. In particular, Krasny applied such a method in his calculations of the evolution of vortex sheets even past the time when the sheet develops singularities; see [16]and [17]. Even though the convergence of vortex methods is now well documented for smooth solutions to the Euler equations (see [3], [14], [15], and the references therein), there is yet much to be done for the analysis of vortex methods for non-smooth flows. This is particularly so in the case of the vortex sheet problem, in which the vorticity is a measure concentrated on a curve or surface which is initially smooth. This defines a classical Hadamard ill-posed initial value problem. At later time, a singularity in the sheet may develop and the nature of solutions past the singularity formation is unknown; see [7] and [19]. The subtle structure of the singularity has been revealed in several numerical simulations; see Krasny, [16] and [17], and Baker and Shelley, [1]. It is thus important to analyze the structure of the approximate solutions generated by vortex methods.

The aim of this paper is to prove the convergence to a classical weak solution of the vortex methods for the 2-D incompressible Euler equations with a class of "rough" initial data including vortex sheet initial data, in the limit as the grid size and the regularization parameter tend to zero, provided that the initial vorticity has a distinguished sign. Here, vortex sheet initial data is defined to be one for

Communications on Pure and Applied Mathematics, Vol. XLVIII, 611–628 (1995)
 CCC 0010–3640/95/060611-18

which the vorticity is a finite Radon measure and the corresponding velocity field has locally finite kinetic energy; see [10]. This is strongly motivated by the recent result of Delort on the existence of weak solutions to the 2-D incompressible Euler equation with vortex sheet initial data of vorticity with distinguished sign (see [9]), the work of DiPerna and Majda on the structure of weak solutions to the 2-D Euler equations (see [10] and [11]), and the numerical demonstration of convergence for small but fixed blob size by Krasny (see [16] and [17]). Our analysis is closely related to that of [9], [10], and [18], and based on a simple analysis and application of the vorticity maximal function introduced by DiPerna and Majda in [10] and [11].

In the vorticity stream formulation, the incompressible 2-D Euler equations can be expressed in the form (see [10]):

$$\partial_t \omega + \nabla\cdot(\mathbf{u}\,\omega) = 0\,, \tag{1.1}$$

with initial data

$$\omega(\mathbf{x},0) = \omega_0(\mathbf{x})\,, \tag{1.2}$$

where the vorticity $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$ is the curl of the velocity field $\mathbf{u}(\mathbf{x},t)$ and

$$\mathbf{u}(\mathbf{x},t) = (K * \omega)(\mathbf{x},t) \tag{1.3}$$

with the kernel K given by

$$K(\mathbf{x}) = \frac{1}{2\pi|\mathbf{x}|^2}(-x_2,x_1) \equiv \frac{\mathbf{x}^\perp}{2\pi|\mathbf{x}|^2}\,. \tag{1.4}$$

For simplicity in presentation, we will assume in the following that ω_0 has compact support. The vortex method, introduced by Chorin in [8], consists of approximating the vorticity field at each time by a sum of approximate Dirac delta functions centered at particle positions $\mathbf{x}_j(t)$, which are advected according to the regularized velocity field determined by the approximate vorticity. To describe this more precisely, we choose a vortex blob function ϕ with the following properties:

$$\phi(\mathbf{x}) = \phi(|\mathbf{x}|) > 0\,, \quad \int \phi(\mathbf{x})\,d\mathbf{x} = 1\,, \quad \int_{|\mathbf{x}|\leqq 1} \phi(\mathbf{x})\,d\mathbf{x} \geqq \tfrac{1}{2}\,. \tag{1.5}$$

We shall assume also that ϕ is C^2 and decays at infinity at least as fast as $|\mathbf{x}|^{-3}$. Let ε be the smoothing blob size, and set $\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-2}\phi(\mathbf{x}/\varepsilon)$. We cover the support of the initial vorticity by non-overlapping squares, R_j, with side length h and centered at $\alpha_j = jh$ with $j \in Z \times Z$. Denote by ξ_j the initial circulation strength in R_j, i.e.,

$$\xi_j = \int_{R_j} \omega_0(\mathbf{x})\,d\mathbf{x}\,. \tag{1.6}$$

The vortex method is

$$\frac{d}{dt}\mathbf{x}_j(t) = \sum_\ell \xi_\ell K_\varepsilon(\mathbf{x}_j(t) - \mathbf{x}_\ell(t)) , \qquad \mathbf{x}_j(0) = \alpha_j , \tag{1.7}$$

$$\mathbf{u}^\varepsilon(\mathbf{x}, t) \equiv \sum_j \xi_j K_\varepsilon(\mathbf{x} - \mathbf{x}_j(t)) , \tag{1.8}$$

$$\omega_\varepsilon(\mathbf{x}, t) \equiv \sum_j \xi_j \delta(\mathbf{x} - \mathbf{x}_j(t)) . \tag{1.9}$$

Here $\delta(\cdot)$ is the Dirac delta function, and $K_\varepsilon = K * \phi_\varepsilon$ which can be written explicitly as

$$K_\varepsilon(\mathbf{x}) = K(\mathbf{x}) f(|\mathbf{x}|/\varepsilon) , \tag{1.10}$$

where $f(r)$ is the cut-off function given by $f(r) = 2\pi \int_0^r s\phi(s)\, ds$. It is easy to show that the finite system of ordinary differential equations (1.7) has a unique solution globally in time under the assumptions (1.5) on the blob function ϕ. Let

$$\omega^\varepsilon(\mathbf{x}, t) \equiv \sum_j \xi_j \phi_\varepsilon(\mathbf{x} - \mathbf{x}_j(t)) , \qquad \mathbf{u}_\varepsilon(\mathbf{x}, t) \equiv \sum_j \xi_j K(\mathbf{x} - \mathbf{x}_j(t)) . \tag{1.11}$$

It is then easily seen that

$$\omega^\varepsilon(\mathbf{x}, t) = \operatorname{curl} \mathbf{u}^\varepsilon(\mathbf{x}, t) , \qquad \omega_\varepsilon(\mathbf{x}, t) = \operatorname{curl} \mathbf{u}_\varepsilon(\mathbf{x}, t) . \tag{1.12}$$

Note that $(\mathbf{u}^\varepsilon, \omega_\varepsilon)(\mathbf{x}, t)$ satisfies the following equation

$$\partial_t \omega_\varepsilon + \nabla\cdot(\mathbf{u}^\varepsilon \omega_\varepsilon) = 0 \tag{1.13}$$

in the sense of distribution.

We shall recall the definition (see [10]) that a vector function $\mathbf{u}(\mathbf{x}, t) \in L^\infty([0, T), L^2_{\text{loc}}(\mathbb{R}^2))$ is a classical weak solution of the 2-D Euler equations on $[0, T]$ with initial velocity $\mathbf{u}_0$ if:

(i) for all test function $\theta(\mathbf{x}, t) \in C_0^\infty(\mathbb{R}^2 \times (0, T))$,

$$\iint \Big(\nabla^\perp \theta_t \cdot \mathbf{u} + (\nabla^\perp \otimes \nabla\theta) : (\mathbf{u} \otimes \mathbf{u})\Big)\, d\mathbf{x}\, dt = 0 ; \tag{1.14}$$

(ii) div $\mathbf{u} = 0$ in the sense of distribution; and

(iii) $\mathbf{u}(\mathbf{x}, t) \in \operatorname{Lip}([0, T), H^{-m}_{\text{loc}}(\mathbb{R}^2))$ for some $m > 0$ and $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$.

In (1.14), $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, and $A : B$ denote the matrix product.

We can now state our main result on the convergence of the vortex blob method to a classical weak solution of the 2-D Euler equations.

THEOREM 1.1. *Suppose that the initial vorticity ω_0 is a Radon measure on $\mathbb{R}^2$ with finite total mass, i.e., $\omega_0 \in M(\mathbb{R}^2)$, and ω_0 has a distinguished sign. Assume further that $\omega_0 \in H^{-1}_{\text{loc}}(\mathbb{R}^2)$. Let $\mathbf{u}^\varepsilon$ be the approximate solution generated by the vortex method as described above. Then there is a subsequence of $\mathbf{u}^\varepsilon$ which converges as $\varepsilon \to 0$ to a classical weak solution of the 2-D Euler equations with the given initial data $\mathbf{u}_0(x)$, provided that we choose grid size h so that $h \leqq C\varepsilon$ for any given positive constant C. The convergence is strong in $L^p_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}_+)$ for $1 \leqq p < 2$, and weak in $L^\infty([0,T), L^2_{\text{loc}}(\mathbb{R}^2))$.*

Remarks

(i) The vortex methods associated with the class of blob functions satisfying (1.5) include, in particular, the one used by Krasny; see [16] and [17]. In his simulation,

$$K_\varepsilon(\mathbf{x}) = \frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2 + \varepsilon^2},$$

with the corresponding blob function $\phi(\mathbf{x}) = \frac{1}{\pi}\left(\frac{1}{1+|\mathbf{x}|^2}\right)^2$. It also should be emphasized that in our theorem, there is no initialization procedure required (see (1.6)), in contrast to the previous studies in [2] and [10].

(ii) It is remarkable that the only requirement on the ratio of the grid size h to the blob size ε is that h/ε is bounded. This is mainly due to the distinguished sign of the initial vorticity.

(iii) The analysis in this paper is elementary and essentially self-contained.

(iv) Following DiPerna and Majda (see [10]), one can define the maximal vorticity function for a discrete vorticity field $\omega_\varepsilon(x,t)$ as

$$M_r(\omega_\varepsilon) = \sup_{\substack{\mathbf{x}\in\mathbb{R}^2 \\ 0<t\leqq T}} \sum_{|x_j - x| \leqq r} |\xi_j|, \qquad \text{for } 0 < r \leqq \tfrac{1}{2}. \tag{1.15}$$

One of the key ingredients in our analysis of convergence is the following estimate on the maximal vorticity function:

$$M_r(\omega_\varepsilon) \leqq C\left(\ln \frac{1}{r+\varepsilon}\right)^{-1/2} \tag{1.16}$$

with a positive constant C independent of ε. It is the derivation of estimate (1.16) that we use the assumption that the vorticity field has a distinguished sign. In fact, as follows from our later analysis, we have

COROLLARY 1.1. *Under the same hypothesis as in Theorem* 1.1 *except that the initial vorticity may change sign, the same conclusions hold provided that the estimate* (1.16) *is true.*

We hope that this corollary may provide a useful criterion to test the convergence of a numerical method for the vortex sheet problem when the initial vorticity has both signs.

We note that the similar approach of using vorticity maximal function has been employed by Majda in his recent simplified proof of Delort's result; see [18].

(v) It should be noted here that in both theoretical and numerical analysis and engineering applications (see [4], [7], and [19]), the evolution of the vortex sheets is studied through the well-known Birkhoff-Rott integro-differential equation whose validity after the formation of a singularity is not well understood. Our convergence to a classical weak solution for Krasny's desingularization procedure of the Birkhoff-Rott equation gives an appropriate interpretation of this integro-differential equation even past the formation of a singularity.

(vi) Thus far we have assumed that the initial vorticity field ω_0 has compact support. This is unnecessary. In fact, using an elementary cutoff argument, we have verified that all the conclusions above hold provided that $|\mathbf{x}|^2\omega_0(\mathbf{x}) \in M(\mathbb{R}^2)$ and $\omega_0(\mathbf{x})$ is bounded for large $|\mathbf{x}|$.

There is a large amount of literature on the convergence of vortex methods; see [2], [3], [5], [6], [10], [12], [14], [15], and the references therein. The formulation of the problem and the analysis in this paper are closely related to those of DiPerna and Majda in [10] in which they proved the convergence of the vortex methods with a special initializations to a measure-valued solution of the Euler equations under the assumption that initial energy is finite. For vortex sheet initial data, Beale obtained convergence of a vortex method with a special initialization in the context of measure-valued solutions provided that the grid size h is exponentially smaller than the blob size ε; see [2]. Assuming the sheet is analytic and close to horizontal, Caflisch and Lowengrub show in [6] that the vortex blob approximation to the sheet converges strongly for a time interval before the singularity formation.

We conclude this introduction by outlining the remainder of this paper. In Section 2, we derive the main stability estimate by showing that the approximate velocity field has locally finite energy uniformly with respect to the grid size h and blob size ε, provided that the ratio h/ε is bounded from above. It should be noted that in contrast to the continuous case (see [10] and [18]), a simple energy estimate is not available for the velocity field $\mathbf{u}^\varepsilon$ in (1.8) due to the consistency error. We will take a direct approach to overcome this difficulty. First, we use the Biot-Savart law to express the local kinetic energy as a quadratic form for the vorticity field (1.9) with a singular kernel. This singular kernel is unbounded in general. The key observation is that this kernel has only a logarithmic singularity, which can be verified by a careful potential estimate. This, combined with the estimate (1.16) on the maximal vorticity function, leads to the desired stability bounds. The weak consistency of the vortex method is given in Section 3. The consistency error is analyzed by exploiting the symmetry properties of the kernel (1.4) and the vortex blob function (1.5), and by using decay estimate (1.16) for the maximal vorticity function. Finally, based on the mentioned stability and weak consistency estimate, we can obtain the convergence of the vortex method easily by following Delort's basic idea in [9]. This is given in Section 4.

2. Stability Analysis

We start with the derivation of the estimate (1.16) on the maximal vorticity function $M_r(\omega_\varepsilon)$ defined in (1.15). To this end, we set

$$\begin{aligned} G(\mathbf{x}) &= \tfrac{1}{2\pi}\ln|\mathbf{x}|\,, \qquad G_\varepsilon(\mathbf{x}) = \phi_\varepsilon * G(\mathbf{x})\,, \\ \psi^\varepsilon(\mathbf{x},t) &= G_\varepsilon * \omega_\varepsilon \equiv \sum_j \xi_j\, G_\varepsilon(\mathbf{x}-\mathbf{x}_j(t))\ . \end{aligned} \tag{2.1}$$

It is evident that

$$K_\varepsilon(\mathbf{x}) = \nabla^\perp G_\varepsilon \quad \text{and} \quad \mathbf{u}^\varepsilon = \nabla^\perp \psi^\varepsilon\ . \tag{2.2}$$

We also define

$$I(t) = -\int_{\mathbb{R}^2} \psi^\varepsilon(\mathbf{x},t)\,\omega_\varepsilon(\mathbf{x},t)\,d\mathbf{x} \equiv \sum_{j,\ell} \xi_j\,\xi_\ell\, G_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)\ . \tag{2.3}$$

This is a conserved quantity, i.e.,

LEMMA 2.1.

(i) $I(t) \equiv I(0)$ *for all* $t > 0$.

(ii) *If* $h \leqq C\varepsilon$ *for any given positive constant* C, *then* $I(0)$ *is bounded independent of* ε *and* h.

Proof: Direct computation using (1.7) yields

$$\begin{aligned} \frac{d}{dt}I(t) &= -\sum_{j,\ell} \xi_j\,\xi_\ell\,\nabla G_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)\cdot(\dot{\mathbf{x}}_j-\dot{\mathbf{x}}_\ell) \\ &= -\sum_{j,\ell,k} \xi_j\,\xi_\ell\,\xi_k\,\nabla G_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)\cdot\big[K_\varepsilon(\mathbf{x}_j-\mathbf{x}_k)-K_\varepsilon(\mathbf{x}_\ell-\mathbf{x}_k)\big] \\ &= -2\sum_{j,\ell,k} \xi_j\,\xi_\ell\,\xi_k\,\nabla G_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)\cdot K_\varepsilon(\mathbf{x}_j-\mathbf{x}_k) \\ &= -2\sum_{j,\ell,k} \xi_j\,\xi_\ell\,\xi_k\,\nabla G_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)\cdot\nabla^\perp G_\varepsilon(\mathbf{x}_j-\mathbf{x}_k) = 0\,, \end{aligned}$$

where we have used the symmetry properties of ∇G_ε and K_ε, formula (2.2), and the trivial identity that $\nabla a \cdot \nabla^\perp b = -\nabla^\perp a \cdot \nabla b$ for any smooth scalar functions a and b. This proves (i). To prove (ii), we first bound the following related integral over $\mathbb{R}^2 \times \mathbb{R}^2$,

$$J_0 = -\iint G_\varepsilon(\mathbf{x}-\mathbf{y})\,\omega_0(\mathbf{x})\,\omega_0(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}\ . \tag{2.4}$$

Since the initial vorticity is assumed to have compact support and $\omega_0 \in H^{-1}_{\mathrm{loc}}$, one has

$$\begin{aligned} J_0 &= -\int (G_\varepsilon * \omega_0)(\mathbf{x})\,\omega_0(\mathbf{x})\,d\mathbf{x} \leqq C\|\omega_0\|_{H^{-1}_{\mathrm{loc}}}\,\|G_\varepsilon * \omega_0\|_{H^1_{\mathrm{loc}}} \\ &\leqq C_1\|\nabla^\perp(G_\varepsilon * \omega_0)\|_{L^2_{\mathrm{loc}}} + C_1\|G_\varepsilon * \omega_0\|_{L^2_{\mathrm{loc}}} \leqq C_2\ . \end{aligned} \tag{2.5}$$

In the last step above, we have used the following simple estimates

$$\|G_\varepsilon * \omega_0\|_{L^2_{\mathrm{loc}}} = \|G * (\phi_\varepsilon * \omega_0)\|_{L^2_{\mathrm{loc}}} \leqq \|\phi_\varepsilon * \omega_0\|_{L^1}\,\|G\|_{L^2_{\mathrm{loc}}} \leqq C\ ,$$

where and from now on, we use C to denote any generic positive constant independent of ε and h.

Next, note that for any $(\mathbf{x},\mathbf{y}) \in R_j \times R_\ell$, it holds that

$$|G_\varepsilon(\mathbf{x}-\mathbf{y}) - G_\varepsilon(\alpha_j - \alpha_\ell)| \leqq |\nabla G_\varepsilon(\cdot)|\ |(\mathbf{x},\mathbf{y}) - (\alpha_j,\alpha_\ell)| \leqq C\frac{h}{\varepsilon} \leqq C$$

where we have used the bound $|\nabla G_\varepsilon(\mathbf{x})| = |K_\varepsilon(\mathbf{x})| \leqq C/\varepsilon$ which can be checked directly. It follows that

$$\begin{aligned} |I(0) - J_0| &= \Big|\sum_{j,\ell} \xi_j\,\xi_\ell\,G_\varepsilon(\alpha_j - \alpha_\ell) - J_0\Big| \\ &= \Big|\sum_{j,\ell}\int_{R_j\times R_\ell} G_\varepsilon(\alpha_j - \alpha_\ell)\,\omega_0(\mathbf{x})\,\omega_0(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} - J_0\Big| \\ &= \Big|\sum_{j,\ell}\int_{R_j\times R_\ell} [G_\varepsilon(\alpha_j - \alpha_\ell) - G_\varepsilon(\mathbf{x}-\mathbf{y})]\,\omega_0(\mathbf{x})\,\omega_0(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}\Big| \\ &\leqq C_3\sum_{j,\ell}\xi_j\,\xi_\ell \leqq C_4\ . \end{aligned}$$

Consequently

$$|I(0)| \leqq C_5$$

which proves (ii). The proof of Lemma 2.1 is complete.

As a consequence of Lemma 2.1, one has

$$I(t) = I(0) < C .$$

This implies that

$$\begin{aligned} - \sum_{|\mathbf{x}_j - \mathbf{x}_\ell| \leqq 1/2} \xi_j \xi_\ell G_\varepsilon(\mathbf{x}_j - \mathbf{x}_\ell) &= I(t) + \sum_{|\mathbf{x}_j - \mathbf{x}_\ell| > 1/2} \xi_j \xi_\ell G_\varepsilon(\mathbf{x}_j - \mathbf{x}_\ell) \\ &\leqq C + \sum_{j,\ell} \xi_j \xi_\ell \left(1 + |\mathbf{x}_j|^2 + |\mathbf{x}_\ell|^2\right) \leqq C_6 . \end{aligned} \tag{2.6}$$

Here we have used the conservation of the second moment for the discrete vorticity field, i.e., $\sum_j \xi_j \, |\mathbf{x}_j(t)|^2 = \sum_j \xi_j \, |\alpha_j|^2$, which can be verified directly. Using the properties of the vortex blob functions, one can derive the following estimate by a simple calculation

$$\ln \frac{1}{|\mathbf{x}| + \varepsilon^{1/3}} \leqq -2G_\varepsilon(\mathbf{x}) + C , \qquad \text{for } |\mathbf{x}| \leqq \tfrac{1}{2} . \tag{2.7}$$

Noting that $\ln 1/(|\mathbf{x}| + \varepsilon) \leqq C \ln 1/(|\mathbf{x}| + \varepsilon^{1/3})$, one can combine (2.7) with (2.6) to obtain

$$\sum_{|\mathbf{x}_j - \mathbf{x}_\ell| \leqq 1/2} \xi_j \xi_\ell \ln \frac{1}{|\mathbf{x}_j - \mathbf{x}_\ell| + \varepsilon} \leqq C , \tag{2.8}$$

which implies the desired decay estimate (1.16) on the maximal vorticity function.

With this preparation, we now turn to the main stability analysis. The main result of this section is the following proposition.

PROPOSITION 2.1. *Let $\mathbf{u}^\varepsilon(\mathbf{x}, t)$ be the approximate velocity field given in* (1.8). *Then for any given $R > 0$, there exists a positive constant $C = C(R)$ independent of ε and h such that*

$$\int_{|\mathbf{x}| \leqq R} |\mathbf{u}^\varepsilon(\mathbf{x}, t)|^2 \, d\mathbf{x} \leqq C . \tag{2.9}$$

Proof: First, choosing a nonnegative function $\chi \in C_0^\infty(\mathbb{R}^2)$ which is equal to one on the ball $\{\mathbf{x} : |\mathbf{x}| \leqq R\}$, we may bound the local kinetic energy as

$$\begin{aligned} \int_{|\mathbf{x}| \leqq R} |\mathbf{u}^\varepsilon(\mathbf{x}, t)|^2 \, d\mathbf{x} &\leqq \int \chi(\mathbf{x}) \, |\mathbf{u}^\varepsilon(\mathbf{x}, t)|^2 \, d\mathbf{x} \\ &= \sum_{j,\ell} \xi_j \xi_\ell \int \chi(\mathbf{x}) K_\varepsilon(\mathbf{x} - \mathbf{x}_j(t)) \cdot K_\varepsilon(\mathbf{x} - \mathbf{x}_\ell(t)) \, d\mathbf{x} . \end{aligned} \tag{2.10}$$

The key step in deriving the stability estimate (2.9) is the following lemma.

LEMMA 2.2. *Assume* $\theta(\mathbf{x}) \in C_0^\infty(\mathbb{R}^2)$. *Let the matrix-valued function* $\mathscr{K}^\varepsilon(\mathbf{y},\mathbf{z};\theta)$ *be defined as*

$$\mathscr{K}^\varepsilon(\mathbf{y},\mathbf{z};\theta) = \int \theta(\mathbf{x}) K_\varepsilon(\mathbf{x}-\mathbf{y}) \otimes K_\varepsilon(\mathbf{x}-\mathbf{z})\, d\mathbf{x}\,. \tag{2.11}$$

Then the following estimates hold for $|\mathbf{y}-\mathbf{z}| \leqq 1/2$,

$$\left|\mathscr{K}^\varepsilon_{11}(\mathbf{y},\mathbf{z};\theta)\right| + \left|\mathscr{K}^\varepsilon_{22}(\mathbf{y},\mathbf{z};\theta)\right| \leqq C \ln \frac{1}{|\mathbf{y}-\mathbf{z}|+\varepsilon} \tag{2.12}$$

$$\left|\mathscr{K}^\varepsilon_{12}(\mathbf{y},\mathbf{z};\theta)\right| + \left|\mathscr{K}^\varepsilon_{21}(\mathbf{y},\mathbf{z};\theta)\right| \leqq C \tag{2.13}$$

$$\left|\mathscr{K}^\varepsilon_{11}(\mathbf{y},\mathbf{z};\theta) - \mathscr{K}^\varepsilon_{22}(\mathbf{y},\mathbf{z};\theta)\right| \leqq C \tag{2.14}$$

where C *depends only on* θ.

Assuming Lemma 2.2 for a moment, we now complete the proof of Proposition 2.1. Using (2.12) in (2.10), one obtains that

$$\begin{aligned}
&\sum_{j,\ell} \xi_j \xi_\ell \int \chi(\mathbf{x}) K_\varepsilon(\mathbf{x}-\mathbf{x}_j)\cdot K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell)\, d\mathbf{x} \\
&\quad \leqq \sum_{|\mathbf{x}_j-\mathbf{x}_\ell|\leqq 1/2} \xi_j \xi_\ell \int \chi(\mathbf{x}) K_\varepsilon(\mathbf{x}-\mathbf{x}_j)\cdot K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell)\, d\mathbf{x} \\
&\quad\quad + \sum_{|\mathbf{x}_j-\mathbf{x}_\ell|> 1/2} \xi_j \xi_\ell \int \chi(\mathbf{x}) K_\varepsilon(\mathbf{x}-\mathbf{x}_j)\cdot K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell)\, d\mathbf{x} \equiv I_1 + I_2\,.
\end{aligned}$$

For I_2, since there is no singularity in the kernel for $|\mathbf{x}_j - \mathbf{x}_\ell| \geqq 1/2$, it is evident that $I_2 \leqq C$. As for I_1, we may use Lemma 2.2 to obtain

$$I_1 \leqq C \sum_{|\mathbf{x}_j-\mathbf{x}_\ell|\leqq 1/2} \xi_j \xi_\ell \ln \frac{1}{|\mathbf{x}_j-\mathbf{x}_\ell|+\varepsilon} \leqq C\,,$$

where the last step follows from (2.8). This proves Proposition 2.1.

The remainder of this section is devoted to the proof of Lemma 2.2. It should be noted that the boundness of the inequality (2.13) in the case $\varepsilon = 0$ is one of the key observations of Delort (see [9]), in which case a simplified proof is given in [13].

Proof of Lemma 2.2: We first derive (2.12). Set $a = \frac{1}{4}|\mathbf{y}-\mathbf{z}|$. Since $\theta(\mathbf{x})$ has compact support, it suffices to bound the integral

$$C\int_{|\mathbf{x}|\leqq R} \frac{1}{|\mathbf{x}-\mathbf{y}|+\varepsilon}\,\frac{1}{|\mathbf{x}-\mathbf{z}|+\varepsilon}\,d\mathbf{x} \tag{2.15}$$

for any given constant $R > 0$. We decompose the integral (2.15) over three regions $B_a(\mathbf{y}) \equiv \{\mathbf{x} : |\mathbf{x}-\mathbf{y}| \leqq a\}$, $B_a(\mathbf{z})$, and $\Omega = \{\mathbf{x} : |\mathbf{x}-\mathbf{y}| > a, |\mathbf{x}-\mathbf{z}| > a, |\mathbf{x}| \leqq R\}$, and denote the integrals as J_1, J_2, and J_3, respectively. We first estimate J_1, while J_2 can be estimated in a similar way. For J_1, it follows that

$$J_1 \leqq \frac{1}{a+\varepsilon}\int_{B_a(\mathbf{y})} \frac{1}{|\mathbf{x}-\mathbf{y}|+\varepsilon}\,d\mathbf{x} \leqq C\,.$$

Note that for J_3, one has $|\mathbf{x}-\mathbf{y}| \leqq 5|\mathbf{x}-\mathbf{z}|$, choosing $b \geqq a$ so that Ω is contained in the annulus $\{\mathbf{x} : b \leqq |\mathbf{x}-\mathbf{y}| \leqq b+2R\}$. We then obtain

$$\begin{aligned} J_3 &\leqq 5\int_{b\leqq|\mathbf{x}-\mathbf{y}|\leqq b+2R} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|+\varepsilon}\right)^2 d\mathbf{x} \\ &\leqq 10\pi\int_b^{b+2R} \frac{r\,dr}{(r+\varepsilon)^2} \leqq 10\pi \ln\frac{b+2R+\varepsilon}{b+\varepsilon} \leqq C\ln\frac{1}{a+\varepsilon}\,. \end{aligned}$$

Collecting these estimates proves (2.12). The inequality (2.13) can be derived as follows. Consider the integral

$$\int \theta(\mathbf{x})\frac{(x_1-y_1)(x_2-z_2)}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2}\, f\left(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\right) f\left(\frac{|\mathbf{x}-\mathbf{z}|}{\varepsilon}\right) d\mathbf{x}\,,$$

which may be rewritten as

$$\begin{aligned} &\int \theta(\mathbf{x})\frac{(x_1-y_1)(x_2-y_2)}{|\mathbf{x}-\mathbf{y}|^4}\, f\left(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x} \\ &\quad + \int \theta(\mathbf{x})\frac{x_1-y_1}{|\mathbf{x}-\mathbf{y}|^2}\, f\left(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\right) \left[\frac{x_2-z_2}{|\mathbf{x}-\mathbf{z}|^2}\, f\left(\frac{|\mathbf{x}-\mathbf{z}|}{\varepsilon}\right)\right. \\ &\quad \left. - \frac{x_2-y_2}{|\mathbf{x}-\mathbf{y}|^2}\, f\left(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\right)\right] d\mathbf{x} \\ &\equiv J_4 + J_5\,. \end{aligned}$$

We now estimate J_4 and J_5 separately. We begin with J_4. Assume supp $\theta \subset B_R(0)$. If $|\mathbf{y}| \geqq R$, then $\theta(\mathbf{y}) = 0$, and so

$$\begin{aligned} |J_4| &= \left|\int [\theta(\mathbf{x})-\theta(\mathbf{y})]\frac{(x_1-y_1)(x_2-y_2)}{|\mathbf{x}-\mathbf{y}|^4}\, f\left(\frac{|\mathbf{x}-\mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x}\right| \\ &\leqq M\int_{B_R(0)} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{x} \leqq C\,. \end{aligned}$$

For $|\mathbf{y}| \leqq R$, one has

$$\begin{aligned}
|J_4| &= \left| \int_{B_{2R}(\mathbf{y})} \theta(\mathbf{x}) \frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^4} f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x} \right| \\
&= \left| \int_{B_{2R}(\mathbf{y})} [\theta(\mathbf{x}) - \theta(\mathbf{y})] \frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^4} f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x} \right. \\
&\quad \left. + \int_{B_{2R}(\mathbf{y})} \theta(\mathbf{y}) \frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^4} f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x} \right| \\
&= \left| \int_{B_{2R}(\mathbf{y})} [\theta(\mathbf{x}) - \theta(\mathbf{y})] \frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^4} f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right)^2 d\mathbf{x} \right| \leqq C\,.
\end{aligned}$$

To estimate J_5, we decompose the integral over three regions $B_a(\mathbf{y})$, $B_a(\mathbf{z})$, and $\Omega = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}| \geqq a, |\mathbf{x} - \mathbf{z}| \geqq a, |\mathbf{x}| \leqq R\}$ as for (2.12). The integrals over the two balls are estimated as for J_1 and J_2 before. To estimate the integral on Ω, we note that for $\mathbf{x} \in \Omega$, it can be computed easily that

$$\begin{aligned}
&\left| \frac{x_2 - z_2}{|\mathbf{x} - \mathbf{z}|^2} f\left(\frac{|\mathbf{x} - \mathbf{z}|}{\varepsilon}\right) - \frac{x_2 - y_2}{|\mathbf{x} - \mathbf{y}|^2} f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right) \right| \\
&\quad \leqq C \frac{|\mathbf{y} - \mathbf{z}|}{|\mathbf{x} - \mathbf{y}|^2} + \frac{C}{|\mathbf{x} - \mathbf{y}|} \left| f\left(\frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon}\right) - f\left(\frac{|\mathbf{x} - \mathbf{z}|}{\varepsilon}\right) \right| \leqq C \frac{|\mathbf{y} - \mathbf{z}|}{|\mathbf{x} - \mathbf{y}|^2}\,.
\end{aligned}$$

It follows that

$$|J_5| \leqq C + C\|\theta\|_{L^\infty} \int_\Omega \frac{|\mathbf{y} - \mathbf{z}|}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{x} \leqq C\,.$$

In a similar way, one proves (2.14). This completes the proof of Lemma 2.2.

3. Weak Consistency

In order to check the weak consistency of the approximation generated by the vortex method with the Euler equations as the blob size ε tends to zero, we rewrite the equation (1.13) for $(\mathbf{u}^\varepsilon, \omega_\varepsilon)$ as

$$\langle \theta_t, \omega_\varepsilon \rangle + \langle \nabla\theta, \mathbf{u}^\varepsilon \omega^\varepsilon \rangle = \langle \nabla\theta, \mathbf{u}^\varepsilon(\omega^\varepsilon - \omega_\varepsilon) \rangle \tag{3.1}$$

for any given test function $\theta \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$. Setting

$$E_\theta = \langle \nabla\theta, \mathbf{u}^\varepsilon(\omega^\varepsilon - \omega_\varepsilon) \rangle\,, \tag{3.2}$$

we need to show that

$$E_\theta \to 0\,, \qquad \text{as } \varepsilon \to 0^+\,. \tag{3.3}$$

Using (1.8), (1.9), (1.12), and the properties of the blob function, we have from (3.2) that

$$
\begin{aligned}
E_\theta &= \langle \nabla\theta, \mathbf{u}^\varepsilon \omega^\varepsilon \rangle - \langle \nabla\theta, \mathbf{u}^\varepsilon \omega_\varepsilon \rangle ,\\
&= \sum_j \xi_j \left[\int \nabla\theta(\mathbf{x})\, \mathbf{u}^\varepsilon(\mathbf{x})\, \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\, d\mathbf{x} - \nabla\theta(\mathbf{x}_j)\, \mathbf{u}^\varepsilon(\mathbf{x}_j) \right]\\
&= \sum_j \xi_j \int \left[\nabla\theta(\mathbf{x})\, \mathbf{u}^\varepsilon(\mathbf{x}) - \nabla\theta(\mathbf{x}_j)\, \mathbf{u}^\varepsilon(\mathbf{x}_j) \right] \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\, d\mathbf{x}\\
&= \sum_j \xi_j \int \left[\nabla\theta(\mathbf{x}) - \nabla\theta(\mathbf{x}_j) \right] \mathbf{u}^\varepsilon(\mathbf{x})\, \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\, d\mathbf{x}\\
&\quad + \sum_j \xi_j \int \nabla\theta(\mathbf{x}_j) \left[\mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{u}^\varepsilon(\mathbf{x}_j) \right] \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\, d\mathbf{x} \equiv J_6 + J_7 .
\end{aligned}
$$

For J_6, one has

$$
\begin{aligned}
|J_6| &= \left| \sum_j \xi_j \int \left[\nabla\theta(\mathbf{x}) - \nabla\theta(\mathbf{x}_j) \right] \mathbf{u}^\varepsilon(\mathbf{x})\, \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\, d\mathbf{x} \right|\\
&\leqq M \sum_{j,\ell} \xi_j \xi_\ell \int \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j) \frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon} \left| \varepsilon K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell) \right| d\mathbf{x}
\end{aligned}
$$

which can be decomposed as

$$
\begin{aligned}
&M \sum_{j,\ell} \xi_j \xi_\ell \int_{|\mathbf{x}-\mathbf{x}_\ell|<\sqrt{\varepsilon}} \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j) \frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon} \left| \varepsilon K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell) \right| d\mathbf{x}\\
&\quad + M \sum_{j,\ell} \xi_j \xi_\ell \int_{|\mathbf{x}-\mathbf{x}_\ell|\geqq\sqrt{\varepsilon}} \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j) \frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon} |\varepsilon K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell)|\, d\mathbf{x} .
\end{aligned}
$$

Using the properties that $|\varepsilon K_\varepsilon(\mathbf{x})| \leqq C$, and $|\varepsilon K_\varepsilon(\mathbf{x})| \leqq C\sqrt{\varepsilon}$ for $|\mathbf{x}| \geqq \sqrt{\varepsilon}$, we can bound the above integrals by

$$
\begin{aligned}
&C \sum_{j,\ell} \xi_j \xi_\ell \int_{|\mathbf{x}-\mathbf{x}_\ell|<\sqrt{\varepsilon}} \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j) \frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon}\, d\mathbf{x}\\
&\quad + C\sqrt{\varepsilon} \sum_{j,\ell} \xi_j \xi_\ell \int_{|\mathbf{x}-\mathbf{x}_\ell|\geqq\sqrt{\varepsilon}} \phi_\varepsilon(\mathbf{x}-\mathbf{x}_j) \frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon}\, d\mathbf{x}
\end{aligned}
$$

which can be further estimated as follows, thanks to the decay properties of the blob function

$$
\begin{aligned}
&C\sqrt{\varepsilon}\sum_{j,\ell}\xi_j\xi_\ell + C\sum_{|\mathbf{x}_j-\mathbf{x}_\ell|\leqq\varepsilon^{1/3}}\xi_j\xi_\ell\int_{|\mathbf{x}-\mathbf{x}_\ell|<\sqrt{\varepsilon}}\phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon}\,d\mathbf{x}\\
&\quad + C\sum_{|\mathbf{x}_j-\mathbf{x}_\ell|>\varepsilon^{1/3}}\xi_j\xi_\ell\int_{|\mathbf{x}-\mathbf{x}_\ell|<\sqrt{\varepsilon}}\phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\frac{|\mathbf{x}-\mathbf{x}_j|}{\varepsilon}\,d\mathbf{x}\\
&\leqq C(\varepsilon^{1/2}+\varepsilon^{1/3})\sum_{j,\ell}\xi_j\xi_\ell + CM_{\varepsilon^{1/3}}(\omega_\varepsilon)\sum_j\xi_j\,.
\end{aligned}
$$

The last term goes to zero as $\varepsilon \to 0$ due to the estimate (1.16) on the maximal vorticity function. Thus J_6 tends to zero as $\varepsilon \to 0$. We now turn to J_7. Employing the symmetry properties of the blob function and the kernel K, we have

$$
\begin{aligned}
|J_7| &= \left|\sum_{j,\ell}\xi_j\xi_\ell\int\nabla\theta(\mathbf{x}_j)[K_\varepsilon(\mathbf{x}-\mathbf{x}_\ell)-K_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)]\phi_\varepsilon(\mathbf{x}-\mathbf{x}_j)\,d\mathbf{x}\right|\\
&= \left|\sum_{j,\ell}\xi_j\xi_\ell\nabla\theta(\mathbf{x}_j)[(\phi_\varepsilon * K_\varepsilon)(\mathbf{x}_j-\mathbf{x}_\ell)-K_\varepsilon(\mathbf{x}_j-\mathbf{x}_\ell)]\right|\\
&\leqq \tfrac{1}{2}\sum_{j,\ell}\xi_j\xi_\ell\,|\nabla\theta(\mathbf{x}_j)-\nabla\theta(\mathbf{x}_\ell)|\;|\phi_\varepsilon * (K_\varepsilon-K)(\mathbf{x}_j-\mathbf{x}_\ell)|\,.
\end{aligned}
$$

To apply the estimate (1.16), we decompose the above sum as follows

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{|\mathbf{x}_j-\mathbf{x}_\ell|\geqq\varepsilon^{1/3}}\xi_j\xi_\ell\,|\nabla\theta(\mathbf{x}_j)-\nabla\theta(\mathbf{x}_\ell)|\;|\phi_\varepsilon * (K_\varepsilon-K)(\mathbf{x}_j-\mathbf{x}_\ell)|\\
&\quad + C\sum_{|\mathbf{x}_j-\mathbf{x}_\ell|<\varepsilon^{1/3}}\xi_j\xi_\ell\,|\mathbf{x}_j-\mathbf{x}_\ell|\;|\phi_\varepsilon * (K_\varepsilon-K)(\mathbf{x}_j-\mathbf{x}_\ell)| \equiv J_8 + J_9\,.
\end{aligned}
$$

We first estimate J_8. By (1.10), one has

$$\phi_\varepsilon * (K_\varepsilon - K)(\mathbf{x}) = K(\mathbf{x})\,g(|\mathbf{x}|/\varepsilon)$$

where

$$g(r) = 2\pi\int_0^r s\,[(\phi * \phi)(s) - \phi(s)]\,ds$$

satisfying $|g(r)| \leqq C/(1 + r)$, which follows from $s^3(\phi * \phi)(s) \leqq C$, as easily checked by using the decay assumption on $\phi(s)$. It follows that

$$|\phi_\varepsilon * (K_\varepsilon - K)(\mathbf{x})| \leqq C\frac{\varepsilon}{|\mathbf{x}| + \varepsilon}\frac{1}{|\mathbf{x}|}\,. \tag{3.4}$$

Applying this in J_8, we obtain

$$J_8 \leqq C \sum_{|\mathbf{x}_j - \mathbf{x}_\ell| \geqq \varepsilon^{1/3}} \xi_j \xi_\ell \frac{\varepsilon}{|\mathbf{x}_j - \mathbf{x}_\ell| + \varepsilon} \frac{1}{|\mathbf{x}_j - \mathbf{x}_\ell|} \leqq C\varepsilon^{1/3} .$$

Finally, (3.4) implies that the summand in J_9 is bounded uniformly, and so

$$J_9 \leqq C \sum_{|\mathbf{x}_j - \mathbf{x}_\ell| \leqq \varepsilon^{1/3}} \xi_j \xi_\ell \leqq C M_{\varepsilon^{1/3}}(\omega_\varepsilon) \sum_j \xi_j$$

which converges to zero as $\varepsilon \to 0$ by (1.16). Collecting all the estimates we have obtained thus far and noting that

$$\langle \nabla\theta, \mathbf{u}^\varepsilon \omega^\varepsilon \rangle = \iint \nabla^\perp \otimes \nabla\theta : \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon \, d\mathbf{x}\, dt ,$$

we arrive at

PROPOSITION 3.1. *The vortex blob approximation* (1.7)–(1.11) *is weakly consistent with the 2-D Euler equations in the sense that for any* $\theta \in C_0^2(\mathbb{R}^2 \times \mathbb{R}_+)$,

$$(3.5) \qquad \iint (\nabla\theta_t \cdot \mathbf{u}^\varepsilon + \nabla^\perp \otimes \nabla\theta : \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) \, d\mathbf{x}\, dt \to 0 , \qquad \text{as } \varepsilon \to 0^+ .$$

Proof: Note that equation (3.1) can be rewritten as

$$(3.6) \quad \iint (\nabla\theta_t \cdot \mathbf{u}^\varepsilon + \nabla^\perp \otimes \nabla\theta : \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) \, d\mathbf{x}\, dt = E_\theta + \iint \nabla\theta_t \cdot (\mathbf{u}^\varepsilon - \mathbf{u}_\varepsilon) d\mathbf{x}\, dt .$$

From the preceding discussion, we need only show that the last integral in (3.6) goes to zero as $\varepsilon \to 0$. This will be a consequence of the following estimate

$$(3.7) \qquad \int_{|\mathbf{x}| \leqq R} |\mathbf{u}^\varepsilon - \mathbf{u}_\varepsilon| \, d\mathbf{x} \leqq C(R) \varepsilon \ln \tfrac{1}{\varepsilon} .$$

To prove (3.7), one checks easily by using (1.5) and (1.10) that

$$|K_\varepsilon(\mathbf{x}) - K(\mathbf{x})| \leqq C \frac{\varepsilon}{|\mathbf{x}| + \varepsilon} \frac{1}{|\mathbf{x}|} .$$

It follows that

$$\begin{aligned} \int_{|\mathbf{x}| \leqq R} |\mathbf{u}^\varepsilon - \mathbf{u}_\varepsilon| \, d\mathbf{x} &\leqq \sum_j \xi_j \int_{|\mathbf{x}| \leqq R} |(K_\varepsilon - K)(\mathbf{x} - \mathbf{x}_j)| \, d\mathbf{x} \\ &\leqq C \sum_j \xi_j \int_{|\mathbf{x}| \leqq R} \frac{\varepsilon}{|\mathbf{x}| + \varepsilon} \frac{1}{|\mathbf{x}|} \, d\mathbf{x} \leqq C(R) \varepsilon \ln \tfrac{1}{\varepsilon} \sum_j \xi_j \end{aligned}$$

which verifies (3.7). Thus Proposition 3.1 is proved.

4. Convergence

With the weak consistency and stability estimates at hand, it is easy to obtain our convergence theorem. For completeness, we will sketch the proof here. We shall show that the sequence $\{\mathbf{u}^\varepsilon\}$ contains a subsequence converging to a L^2_{loc} function which is a weak solution to the 2-D Euler equations. Since $\mathbf{u}^\varepsilon$ and $\omega^\varepsilon = \text{curl}\,\mathbf{u}^\varepsilon$ admit the uniform bounds

$$\max_{0\leqq t\leqq T}\int_{\mathbb{R}^2}\omega^\varepsilon(\mathbf{x},t)\,d\mathbf{x}\leqq C\,,\qquad \max_{0\leqq t\leqq T}\int_{|\mathbf{x}|\leqq R}|\mathbf{u}^\varepsilon(\mathbf{x},t)|^2\,d\mathbf{x}\leqq C\,,\tag{4.1}$$

thus there exist a $\omega\in L^\infty\left([0,T],M(\mathbb{R}^2)\cap H^{-1}_{\text{loc}}(\mathbb{R}^2)\right)$ and $\mathbf{u}\in L^\infty\left([0,T],L^2_{\text{loc}}(\mathbb{R}^2)\right)$ such that passing to a subsequence which we still denote as $\{\omega^\varepsilon,\mathbf{u}^\varepsilon\}$, one has

$$\begin{aligned}\mathbf{u}^\varepsilon &\rightharpoonup \mathbf{u} \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}_+)\,,\\ \omega^\varepsilon &\rightharpoonup \omega \quad \text{in } M(\mathbb{R}^2\times\mathbb{R}_+)\,,\end{aligned}\tag{4.2}$$

furthermore,

$$\omega=\text{curl}\,\mathbf{u}\,,\qquad \text{div}\,\mathbf{u}=0\,,\tag{4.3}$$

in the sense of distribution. A simple compactness argument (see [10], [20]) shows

$$\mathbf{u}^\varepsilon\to\mathbf{u}\quad\text{in } L^p_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}_+)\,,\qquad 1\leqq p<2\,.\tag{4.4}$$

To show the function $\mathbf{u}$ is a desired weak solution, we first verify (1.13) by following Delort's basic idea. From (3.6), one needs to show that

$$\lim_{\varepsilon\to0^+}\iint\nabla^\perp\otimes\nabla\theta:\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon\,d\mathbf{x}\,dt=\iint\nabla^\perp\otimes\nabla\theta:\mathbf{u}\otimes\mathbf{u}\,d\mathbf{x}\,dt\tag{4.5}$$

for any given test function $\theta\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}_+)$. Since

$$\begin{aligned}&\iint\nabla^\perp\otimes\nabla\theta:\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon\,d\mathbf{x}\,dt\\ &\quad=\iint\partial_{x_1x_2}\theta\,[(u_2^\varepsilon)^2-(u_1^\varepsilon)^2]\,d\mathbf{x}\,dt+\iint[\partial_{x_1x_1}\theta-\partial_{x_2x_2}\theta]u_1^\varepsilon u_2^\varepsilon\,d\mathbf{x}\,dt\end{aligned}$$

it suffices to show that

$$\lim_{\varepsilon\to0^+}\iint\partial_{x_1x_2}\theta\,[(u_2^\varepsilon)^2-(u_1^\varepsilon)^2]\,d\mathbf{x}\,dt=\iint\partial_{x_1x_2}\theta\,[u_2^2-u_1^2]\,d\mathbf{x}\,dt\tag{4.6}$$

and

$$\lim_{\varepsilon\to 0^+}\iint[\partial_{x_1x_1}\theta-\partial_{x_2x_2}\theta]u_1^\varepsilon u_2^\varepsilon\,d\mathbf{x}\,dt=\iint[\partial_{x_1x_1}\theta-\partial_{x_2x_2}\theta]u_1u_2\,d\mathbf{x}\,dt\,. \tag{4.7}$$

We now sketch the proof of (4.6); similar analysis applies to (4.7). Denoting $\partial_{x_1x_2}\theta$ by η, one has

$$\iint\eta\left((u_1^\varepsilon)^2-(u_2^\varepsilon)^2\right)d\mathbf{x}\,dt=\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\omega^\varepsilon(\mathbf{y})\,\omega^\varepsilon(\mathbf{z})\,\mathscr{H}_\eta(\mathbf{y},\mathbf{z})\,d\mathbf{y}\,d\mathbf{z}\,dt \tag{4.8}$$

where

$$\mathscr{H}_\eta(\mathbf{y},\mathbf{z})=\int_{\mathbb{R}^2}\eta(\mathbf{x})\left(\frac{(y_1-x_1)(x_1-z_1)}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2}-\frac{(y_2-x_2)(x_2-z_2)}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2}\right)d\mathbf{x}\,. \tag{4.9}$$

For any given $0<r\leqq 1/2$, $\mathscr{H}_\eta(\mathbf{y},\mathbf{z})$ is a continuous bounded function on the region $|\mathbf{y}-\mathbf{z}|\geqq r$, as can be checked easily. $\mathscr{H}_\eta(\mathbf{y},\mathbf{z})$ is bounded for $|\mathbf{y}-\mathbf{z}|\leqq r$, which follows from (2.14) in Lemma 2.2 with $\varepsilon=0$. This and (4.2) imply that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\iint\eta\left((u_1^\varepsilon)^2-(u_2^\varepsilon)^2\right)d\mathbf{x}\,dt\\&=\int_0^T\iint_{|\mathbf{y}-\mathbf{z}|\geqq r}\omega(\mathbf{y})\,\omega(\mathbf{z})\,\mathscr{H}_\eta(\mathbf{y},\mathbf{z})\,d\mathbf{y}\,d\mathbf{z}\,dt\\&\quad+\lim_{\varepsilon\to 0}\int_0^T\iint_{|\mathbf{y}-\mathbf{z}|<r}\omega^\varepsilon(\mathbf{y})\,\omega^\varepsilon(\mathbf{z})\,\mathscr{H}_\eta(\mathbf{y},\mathbf{z})\,d\mathbf{y}\,d\mathbf{z}\,dt\,.\end{aligned} \tag{4.10}$$

The last term on the right-hand side of (4.10) can be estimated as follows. One has

$$\begin{aligned}\left|\iint_{|\mathbf{y}-\mathbf{z}|<r}\omega^\varepsilon(\mathbf{y})\,\omega^\varepsilon(\mathbf{z})\,\mathscr{H}_\eta(\mathbf{y},\mathbf{z})\,d\mathbf{y}\,d\mathbf{z}\right|&\leqq M\iint_{|\mathbf{y}-\mathbf{z}|<r}\omega^\varepsilon(\mathbf{y})\,\omega^\varepsilon(\mathbf{z})\,d\mathbf{y}\,d\mathbf{z}\\&\leqq C\|\omega^\varepsilon(\cdot)\|_{L^1}\int_{|\mathbf{x}|<2r}\omega^\varepsilon(\mathbf{x})\,d\mathbf{x}\leqq C\left(M_{r+\varepsilon^{1/3}}+\varepsilon^{1/3}\right)\end{aligned}$$

which goes to zero due to the estimate (1.16) on the maximal vorticity function. This proves (4.6), and the identity (1.14) is verified.

It remains to prove that $\mathbf{u}(\mathbf{x},t)\in\mathrm{Lip}\left([0,T),H^{-3}_{\mathrm{loc}}(\mathbb{R}^2)\right)$. It suffices to show that $\mathbf{u}_\varepsilon(\mathbf{x},t)\in\mathrm{Lip}\left([0,T),H^{-3}_{\mathrm{loc}}(\mathbb{R}^2)\right)$ uniformly; see [10]. Let $\theta=\theta(\mathbf{x})\in C_0^\infty(\mathbb{R}^2)$. One then has

$$\int\nabla^\perp\theta\,\partial_t\mathbf{u}_\varepsilon\,d\mathbf{x}=-\int\nabla^\perp\otimes\nabla\theta:\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon\,d\mathbf{x}+\int\nabla\theta\cdot\mathbf{u}^\varepsilon(\omega^\varepsilon-\omega_\varepsilon)\,d\mathbf{x}\,.$$

It follows from the proof of Proposition 3.1 and Sobolev's embedding lemma that

$$\left| \int \nabla^{\perp}\theta \, \partial_t \mathbf{u}_\varepsilon \, d\mathbf{x} \right| \leqq C \|\nabla^{\perp}\theta\|_{W^{1,\infty}(\mathbb{R}^2)} \leqq C \|\nabla^{\perp}\theta\|_{H^3(\mathbb{R}^2)} . \tag{4.11}$$

Since $\mathbf{u}_\varepsilon$ is divergence free, (4.11) implies that

$$\|\partial_t \mathbf{u}_\varepsilon\|_{L^\infty\left([0,T), H^{-3}_{\mathrm{loc}}(\mathbb{R}^2)\right)} \leqq C ,$$

which yields immediately

$$\|\partial_t \mathbf{u}^\varepsilon\|_{L^\infty\left([0,T), H^{-3}_{\mathrm{loc}}(\mathbb{R}^2)\right)} \leqq C .$$

Consequently, one has $\mathbf{u}(\mathbf{x},t) \in \mathrm{Lip}\left([0,T), H^{-3}_{\mathrm{loc}}(\mathbb{R}^2)\right)$, and so $\mathbf{u}(\mathbf{x},t)$ is a desired classical weak solution. The proof of Theorem 1.1 is completed.

Acknowledgments. This work was done while the authors were visiting the Institute for Advanced Study. The hospitality of the faculty and staff at the IAS was greatly appreciated. The authors wish to thank Professors Weinan E and R. Krasny for many useful discussions and their suggestions. The authors would like also to thank Professors J. Goodman and P. Lax for their interest in this work.

The research of the first author was supported in part by National Science Foundation Grant No. DMS-9304580 and a summer research fellowship from Temple University. The research of the second author was supported in part by a Sloan Foundation Fellowship, National Science Foundation Grants DMS-9304580 and DMS-93-03887, and Department of Energy Grant DE-FG02-88ER-25053.

Bibliography

[1] Baker, G. R., and Shelley, M. J., *On the connection between thin vortex layers and vortex sheets,* J. Fluid Mech. 215, 1990, pp. 161–194.

[2] Beale, J. T., *The approximation of weak solutions to the 2-D Euler equations by vortex elements,* pp. 23–37 in: *Multidimensional Hyperbolic Problems and Computations,* J. Glimm and A. Majda, eds., IMA, 1991.

[3] Beale, J. T., and Majda, A., *Vortex methods. II: Higher order accuracy in two and three dimensions,* Math. Comp. 39, 1982, pp. 29–52.

[4] Birkhoff, G., *Helmholtz and Taylor instability,* Proc. Sympos. Appl. Math. 13, 1962, pp. 55–76.

[5] Brenier, Y., and Cottet, G. H., *Convergence of particle methods with random rezoning for the 2-D Euler and Navier-Stokes equations,* SIAM J. Numer. Anal., to appear.

[6] Caflisch, R., and Lowengrub, J., *Convergence of the vortex method for vortex sheets,* SIAM J. Numer Anal. 26, 1989, pp. 1060–1080.

[7] Caflisch, R., and Orellana, O., *Singularity formulation and ill-posedness of vortex sheets,* SIAM J. Math. Anal. 20, 1988, pp. 293–307.

[8] Chorin, A. J., *Numerical study of slightly viscous flow,* J. Fluid Mech. 57, 1973, pp. 785–796.

[9] Delort, J.-M., *Existence de nappes de tourbillon en dimension deux,* J. Amer. Math. Soc. 4, 1991, pp. 553–586.

[10] DiPerna, R. J., and Majda, A. J., *Concentrations in regularizations for 2-D incompressible flow,* Comm. Pure Appl. Math. 40, 1987, pp. 301–345.
[11] DiPerna, R. J., and Majda, A., *Reduced Hausdorff dimension and concentration cancellation for two-dimensional incompressible flow,* J. Amer. Math. Soc. 1, 1988, pp. 59–95.
[12] E, W., and Hou, T. Y., *Homogenization and convergence of the vortex method for 2-D Euler equations with oscillatory vorticity fields,* Comm. Pure Appl. Math. 43, 1990, pp. 821–855.
[13] Evans, L. C., and S. Müller, S., *Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity,* J. Amer. Math. Soc. 7, 1994, pp. 199–220.
[14] Goodman, J., Hou, T. Y., and Lowengrub, J., *Convergence of the point vortex method for 2-D Euler equations,* Comm. Pure Appl. Math. 43, 1990, pp. 415–430.
[15] Hald, D. H., *Convergence of vortex methods for Euler's equations, II,* SIAM J. Numer. Anal. 16, 1979, pp. 726–755.
[16] Krasny, R., *Computation of vortex sheet roll-up in the Trefftz plane,* J. Fluid Mech. 184, 1987, pp. 123–155.
[17] Krasny, R., *Computing vortex sheet motion,* pp. 1573–1583 in: *Proceedings of the International Congress of Mathematicians, August 21–29, 1990, Kyoto, Japan,* Vol. II, I. Satake, ed., Springer-Verlag, Tokyo, 1991.
[18] Majda, A., *Remarks on weak solutions for vortex sheets with a distinguished sign,* Indiana Univ. Math. J. 42, 1993, pp. 921–939.
[19] Moore, D. W., *The spontaneous appearance of a singularity in the shape of an evolving vortex sheet,* Proc. Roy. Soc. London Ser. A 365, 1979, pp. 105–119.
[20] Sulem, P., Sulem, C., Bardos, C., and Frisch, U., *Finite time analyticity for the two- three-dimensional Kevin-Helmholtz instability,* Comm. Math. Phys. 80, 1981, pp. 485–516.

Received March 1994.