# Convergence of a Galerkin Method for 2-D Discontinuous Euler Flows

JIAN-GUO LIU
*University of Maryland*

AND

ZHOUPING XIN
*Courant Institute*
*The Chinese University of Hong Kong*

**Abstract**

We prove the convergence of a discontinuous Galerkin method approximating the 2-D incompressible Euler equations with discontinuous initial vorticity: $\omega_0 \in L^2(\Omega)$. Furthermore, when $\omega_0 \in L^\infty(\Omega)$, the whole sequence is shown to be strongly convergent. This is the first convergence result in numerical approximations of this general class of discontinuous flows. Some important flows such as vortex patches belong to this class. © 2000 John Wiley & Sons, Inc.

## 1 Introduction

Numerical simulations of 2-D discontinuous incompressible flows are of considerable interest in both theoretical analysis and applications. It is believed that the Lagrangian methods such as vortex methods [6, 9] or the ones based on contour dynamics [1, 16] give preferable treatments for such flows, especially for inviscid interfacial flows. However, the convergence of such methods poses great difficulties. Past efforts concentrate on either special flows (see [5, 12, 13]) or require heavy machinery (such as large deviation [14]) and yield much weaker convergence results [2, 3, 14].

However, for more complicated flows (such as a flow mixing), such front-tracking methods are impossible to implement. Thus grid-based methods such as finite difference and finite elements are called for. Yet, the studies of such methods have been carried out only recently [10, 11]. In particular, a discontinuous Galerkin method was proposed in [11], which has the main advantage that the energy is conserved even for upwind-type numerical fluxes, and, amusingly, the numerical enstrophy is nonincreasing in time. The main observation of this paper is that the boundedness of energy and enstrophy is sufficient for strong convergence for a class of discontinuous initial data $\omega_0 \in L^2$ including vortex patches. In particular, our results imply that the discontinuous Galerkin methods in [11] do converge for such flows. It should also be noted that the convergence of a class

Communications on Pure and Applied Mathematics, Vol. LIII, 0786–0798 (2000)

of finite difference methods for some class of 2-D discontinuous Euler flows has recently been announced in [7].

## 2 A Discontinuous Galerkin Method

The 2-D incompressible Euler equation in vorticity stream-function formulation reads

$$\partial_t \omega + (\nabla^\perp \psi \cdot \nabla)\omega = 0, \quad \nabla^\perp = (-\partial_y, \partial_x), \quad \Delta\psi = \omega, \tag{2.1a}$$

with no-flow boundary condition

$$\psi = 0 \quad \text{on } \partial\Omega \tag{2.1b}$$

and initial condition

$$\omega|_{t=0} = \omega_0(x) \in L^2(\Omega) \tag{2.1c}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected domain with a $C^2$ boundary or piecewise $C^2$ boundary with convex corners. Assume that $\Omega$ is equipped with a quasi-uniform triangulation (see (4.4.5) in [4]) $T_h = \{K\}$ consisting of polygons $K$ of maximum size (diameter) $h$. Denote $\Omega_h = \bigcup K$. The vorticity $\omega$ is approximated by $\omega_h$ in a discontinuous finite element space $V_h^k = \{v : v|_K \in P^k(K),\ \forall K \in T_h\}$, while the stream function $\psi$ is approximated by $\psi_h$ in a continuous one $W_{0,h}^k = V_h^k \cap C_0(\Omega_h)$. Here $P^k(K)$ denotes the set of all polynomials of degree at most $k$ on the cell $K$. In the following, we will also use the notation $\langle \cdot \rangle$ to stand for the standard integration over the whole domain $\Omega_h$, while an integral over a subdomain $K$ is denoted by $\langle \cdot \rangle_K$. The semidiscrete discontinuous Galerkin method in [11] can be described by looking for $\omega_h \in V_h^k$ and $\psi_h \in W_{0,h}^k$ such that

$$\langle \partial_t \omega_h v_h \rangle_K - \langle \omega_h \mathbf{u}_h \cdot \nabla v_h \rangle_K + \sum_{e \in \partial K} \langle \mathbf{u}_h \cdot \mathbf{n} \widehat{\omega_h} v_h^- \rangle_e = 0 \tag{2.2a}$$

for all $K \in T_h$ and $v_h \in V_h^k$,

$$-\langle \nabla \psi_h \cdot \nabla \varphi_h \rangle_{\Omega_h} = \langle \omega_h, \varphi_h \rangle_{\Omega_h}, \quad \forall \varphi_h \in W_{0,h}^k, \tag{2.2b}$$

where $e$ is a cell boundary and $\mathbf{n}$ is its unit outward-pointing normal.

We now explain the notation used in (2.2). First, the velocity field is given by $\mathbf{u}_h = \nabla^\perp \psi_h$. Note that even though both $\omega_h$ and the test function $v_h$ may be discontinuous across the cell boundaries, the velocity field possesses continuous normal components across each cell boundary due to the definition of the finite element space $W_{0,h}^k$. Thus the numerical flux in (2.2a) can be defined as follows: Denote by $v_h^-$ ($v_h^+$) the value of $v_h$ from the inside (outside) of the element $K$; then the *upwind flux* is set to be

$$\widehat{\omega_h} = \begin{cases} \omega_h^- & \text{if } \mathbf{u}_h \cdot \mathbf{n} \geq 0, \\ \omega_h^+ & \text{if } \mathbf{u}_h \cdot \mathbf{n} < 0. \end{cases} \tag{2.3}$$

We note that for smooth flows, we could use a central flux defined by

$$\widehat{\omega_h} = \frac{1}{2}\left(\omega_h^+ + \omega_h^-\right). \tag{2.3$'$}$$

However, the upwind fluxes (2.3) are preferred since the main concerns here are discontinuous flows.

It is well-known that 2-D smooth Euler flows preserve energy and enstrophy. It is remarkable that the same holds true for the discontinuous Galerkin method mentioned above. Indeed, the first important property of this scheme is the conservation of (no numerical dissipation in) energy

$$\|\nabla^\perp \psi_h(\cdot,t)\|_{L^2(\Omega_h)} = \|\nabla^\perp \psi_h(\cdot,0)\|_{L^2(\Omega_h)} \tag{2.4}$$

for the upwind flux (2.3), which can be verified directly by taking $v_h = \psi_h$ in (2.2a), using the fact that $\mathbf{u}_h \cdot \nabla \psi_h = 0$, summing up the resulting equations over all $K$ in the triangulation, and using (2.2b) and the continuity of the normal velocity across the cell boundaries. Next, taking $v_h = \omega_h$, integrating by parts for the second term in (2.2a), summing up for all $K$, and estimating the terms involving cell boundaries by using (2.3) and the continuity of the normal component of the velocity field across the cell boundaries, one can show that the enstrophy decays in the sense that

$$\|\omega_h(\cdot,t)\|_{L^2(\Omega_h)} \le \|\omega_h(\cdot,0)\|_{L^2(\Omega_h)} \le \|\omega_0\|_{L^2(\Omega_h)} \tag{2.5}$$

where the initial data $\omega_h(\cdot,0)$ is taken as the $L^2$ projection of $\omega_0$ and hence is uniformly bounded in $L^2$. Furthermore, taking $\varphi_h = \psi_h$ in (2.2b), one derives the fact that

$$\|\nabla \psi_h\|^2_{L^2(\Omega_h)} = -\langle \omega_h, \psi_h \rangle_{\Omega_h}. \tag{2.6}$$

Our main observation in this paper is the fact that these three simple properties, (2.4)–(2.6), yield a strong convergence. To prove and state such a result, one needs some estimate of time regularity first.

## 3 Time Regularity Estimate

In this section, we will prove a lemma about the time regularity for the approximate solutions constructed by the discontinuous Galerkin method, which is needed for the argument of convergence. Although it is rather routine in the continuous case to obtain time regularity through the Euler equations given the spatial regularity, such regularity estimates in the discrete case are much more involved and are the main technical parts of this paper. We need a convention before stating the main lemma. Note that on a boundary element, the approximate solutions $\omega_h$ and $\psi_h$ are polynomials, and we can use them to extend/restrict these functions from $\Omega_h$ into the domain $\Omega$; we will refer to such extensions as the natural extensions. It is noted that one may use the zero extensions, but the arguments in the proof of the following lemma cannot be simplified much since one needs to estimate the restriction anyway (see $(3.18')$).

LEMMA 3.1 (Time Regularity) *There is a constant $q_o$ such that*

$$\|\partial_t \omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} + \|\partial_t \psi_h\|_{L^\infty([0,T),L^q(\Omega))} \leq C \tag{3.1}$$

*for any $q_0 \leq q < 2$, where $\omega_h$ and $\psi_h$ denote, respectively, their natural extensions or restrictions from $\Omega_h$ to $\Omega$.*

PROOF: We first show that

$$\|\partial_t \omega_h\|_{L^\infty([0,T),L^2(\Omega_h))} \leq \frac{C}{h^2}\,. \tag{3.2}$$

For any smooth function $v \in C_0^\infty(\Omega)$ with zero extension to the outside, let $v_h$ be the $L^2$ projection of $v$ in the space of $V_h^k$. Now taking $v_h$ as a test function in (2.2a), one gets

$$\begin{aligned}\langle \partial_t \omega_h v\rangle_{\Omega_h} &= \langle \partial_t \omega_h v_h\rangle_{\Omega_h} \\ &= \sum_K \langle \omega_h \mathbf{u}_h \cdot \nabla v_h\rangle_K - \sum_K \sum_{e\in\partial K} \langle \mathbf{u}_h \cdot \mathbf{n}\widehat{\omega_h} v_h^-\rangle_e \equiv I_1 + I_2\,.\end{aligned} \tag{3.3}$$

$I_1$ can be estimated directly as

$$|I_1| \leq \|\omega_h\|_{L^\infty} \sum_k \|\mathbf{u}_h\|_{L^2(K)} \|\nabla v_h\|_{L^2(K)} \leq C\|\omega_h\|_{L^\infty(\Omega_h)} \|\mathbf{u}_h\|_{L^2(\Omega_h)} \|\nabla v_h\|_{L^2(\Omega_h)}\,.$$

Using the inverse inequality (cf. (4.5.12) in [4]),

$$\|\omega_h\|_{L^\infty} \leq \frac{C}{h}\|\omega_h\|_{L^2}\,, \qquad \|\nabla v_h\|_{L^2} \leq \frac{C}{h}\|v_h\|_{L^2}\,,$$

the $L^2$ estimate for $\mathbf{u}_h$ and $\omega_h$ in (2.4) and (2.5), and the fact that $v_h$ is the $L^2$ projection of $v$, one has

$$|I_1| \leq \frac{C}{h^2}\|\omega_h\|_{L^2(\Omega_h)} \|\mathbf{u}_h\|_{L^2(\Omega_h)} \|v_h\|_{L^2(\Omega_h)} \leq \frac{C}{h^2}\|v\|_{L^2(\Omega_h)}\,. \tag{3.4}$$

Next we proceed to estimate $I_2$. Since $\mathbf{u}_h \cdot \mathbf{n}$ is continuous and $\widehat{\omega_h}$ takes the same value from both sides of a cell boundary, the contribution of the two sides will give the jump of $v_h$. Therefore, one can obtain

$$|I_2| \leq \sum_K \sum_{e\in\partial K} \left\langle |\mathbf{u}_h \cdot \mathbf{n}\widehat{\omega_h}(v_h^+ - v_h^-)|\right\rangle_e\,. \tag{3.5}$$

Thanks to the quasi-uniform regularity in the triangulation [4], one can map each triangle back and forth to the reference triangle. Using the fact that all norms in a finite-dimensional space are equivalent in the reference triangle, one can show that

$$\sum_K \sum_{e\in\partial K} \|\omega_h\|_{L^2(e)} \|v_h\|_{L^2(e)} \leq \frac{C}{h} \sum_K \|\omega_h\|_{L^2(K)} \|v_h\|_{L^2(K)}\,. \tag{3.6}$$

Hence

$$|I_2| \leq \frac{C}{h^2}\|v_h\|_{L^2(\Omega_h)} \leq \frac{C}{h^2}\|v\|_{L^2(\Omega_h)}\,. \tag{3.7}$$

Combining (3.4) with (3.7) shows

$$|\langle \partial_t \omega_h v \rangle_{\Omega_h}| \le \frac{C}{h^2} \|v\|_{L^2(\Omega_h)}, \tag{3.8}$$

which yields the desired estimate (3.2), since the natural extension of $\partial_t \omega_h$ to $\Omega$ from $\Omega_h$ gives equivalent norms. Hence we have also shown

$$\|\partial_t \omega_h\|_{L^\infty([0,T),L^2(\Omega))} \le \frac{C}{h^2}. \tag{3.2$'$}$$

Next, let $I_h v$ be the piecewise linear interpolation of $v$ in $V_h^k$. Then one can decompose $I_1$ in (3.3) as

$$I_1 = \sum_K \langle \omega_h \mathbf{u}_h \cdot \nabla I_h v \rangle_K + \sum_K \langle \omega_h \mathbf{u}_h \cdot \nabla (v_h - I_h v) \rangle_K \equiv I_{11} + I_{12}. \tag{3.9}$$

$I_{11}$ is bounded by

$$\begin{aligned} |I_{11}| &\le \sum_K \|\omega_h\|_{L^2(K)} \|\mathbf{u}_h\|_{L^2(K)} \|\nabla I_h v\|_{L^\infty} \\ &\le \|\omega_h\|_{L^2(\Omega_h)} \|\mathbf{u}_h\|_{L^2(\Omega_h)} \|\nabla I_h v\|_{L^\infty(\Omega_h)}. \end{aligned} \tag{3.10}$$

Due to the inequality

$$\|\nabla I_h v\|_{L^\infty(\Omega_h)} \le C \|v\|_{W^{1,\infty}(\Omega)},$$

one obtains from (2.4), (2.5), and (3.10) that

$$|I_{11}| \le C \|v\|_{W^{1,\infty}(\Omega)}. \tag{3.11}$$

One can estimate $I_{12}$ similarly. Indeed,

$$|I_{12}| \le \sum_K \|\omega_h\|_{L^2(K)} \|\mathbf{u}_h\|_{L^2(K)} \|\nabla (v_h - I_h v)\|_{L^\infty(\Omega_h)} \le C \|\nabla (v_h - I_h v)\|_{L^\infty(\Omega_h)}.$$

Using the inverse inequality [4]

$$\|\nabla (v_h - I_h v)\|_{L^\infty(\Omega_h)} \le \frac{C}{h^2} \|v_h - I_h v\|_{L^2(\Omega_h)}$$

and

$$\|v_h - I_h v\|_{L^2(\Omega_h)} \le \|v - I_h v\|_{L^2(\Omega_h)}, \tag{3.12}$$

which holds true since $v_h$ is the $L^2$ projection of $v$, one gets

$$|I_{12}| \le \frac{C}{h^2} \|v - I_h v\|_{L^2(\Omega_h)}. \tag{3.13}$$

This together with the standard estimate for the interpolation (cf. (4.4.21) in [4]) leads to

$$|I_{12}| \le C \|v\|_{H^2(\Omega)}. \tag{3.14}$$

It follows from (3.11) and (3.14) that we have obtained an estimate on $I_1$ in (3.3) independent of $h$. Now we can also derive an $h$-independent estimate on $I_2$ in (3.3) as follows: Noting that $I_h v$ is continuous at the cell boundary, one can insert it into the right-hand side of (3.5) to obtain

$$[|I_2| \le \sum_K \sum_{e\in\partial K} \langle |\mathbf{u}_h\cdot\mathbf{n}\widehat{\omega_h}(v_h - I_h v)|\rangle_e \le \\ C\|\mathbf{u}_h\|_{L^\infty} \sum_K \sum_{e\in\partial K} \|\omega_h\|_{L^2(e)}\|v_h - I_h v\|_{L^2(e)}\,. \tag{3.15}$$

As in (3.6), one has

$$|I_2| \le \frac{C}{h^2}\|v_h - I_h v\|_{L^2(\Omega_h)} \le \frac{C}{h^2}\|v - I_h v\|_{L^2(\Omega_h)} \le C\|v\|_{H^2(\Omega)}\,. \tag{3.16}$$

Collecting estimates (3.11), (3.14), and (3.16), we arrive at

$$|\langle \partial_t\omega_h, v\rangle_{\Omega_h}| \le C(\|v\|_{W^{1,\infty}(\Omega)} + \|v\|_{H^2}(\Omega)) \le C\|v\|_{W^{2,p}(\Omega)} \tag{3.17}$$

for any $p > 2$.

To complete the proof of the first part of (3.1), we need to estimate the difference for the left-hand term between $\Omega$ and $\Omega_h$. First,

$$|\langle \partial_t\omega_h, v\rangle_{\Omega\setminus\Omega_h}| \le \|\partial_t\omega_h\|_{L^2(\Omega\setminus\Omega_h)}\|v\|_{L^\infty(\Omega\setminus\Omega_h)}\sqrt{|\Omega\setminus\Omega_h|}$$

since $v$ vanishes on the boundary $\partial\Omega$ and because of the simple estimate $|\Omega\setminus\Omega_h| \le Ch^2$. This estimate is due to the fact that the distance from $\partial\Omega$ to $\partial\Omega_h$ is $O(h^2)$, which, in turn, is a consequence of the standard but unstated assumption that all the vertices lying on the boundary of $\partial\Omega_h$ also lie on the boundary of $\Omega$. One then has

$$\|v\|_{L^\infty(\Omega\setminus\Omega_h)} \le Ch^2\|v\|_{W^{1,\infty}(\Omega)}\,.$$

We can obtain from these and $(3.2')$ that

$$|\langle \partial_t\omega_h v\rangle_{\Omega\setminus\Omega_h}| \le Ch\|v\|_{W^{1,\infty}(\Omega)} \le Ch\|v\|_{W^{2,q}(\Omega)} \tag{3.18}$$

for any $1 \le q < 2$. Similarly, one can obtain

$$|\langle \partial_t\omega_h v\rangle_{\Omega_h\setminus\Omega}| \le Ch\|v\|_{W^{2,q}(\Omega)}\,. \tag{3.18$'$}$$

Consequently, from (3.17), (3.18), and (3.18′),

$$\|\partial_t\omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} \le C \quad \text{for any } 1 \le q < 2\,. \tag{3.19}$$

This proves the first part of (3.1).

Next we prove the second part of the inequality in (3.1). For $f \in L^p(\Omega)$ with zero extension to the outside, we let $\phi$ solve the following problem:

$$-\Delta\phi = f \ \text{ in } \Omega\,, \quad \phi|_{\partial\Omega} = 0\,, \tag{3.20}$$

and let $\phi_h \in W^k_{0,h}$ be the finite element solution

$$\langle \nabla\phi_h, \nabla\varphi_h\rangle_{\Omega_h} = \langle f, \varphi_h\rangle_{\Omega_h} \quad \text{for any } \varphi_h \in W^k_{0,h}\,. \tag{3.21}$$

Since the domain is either $C^2$ or piecewise $C^2$ with convex corners, we can choose $p_0 > 2$ such that for any $p_0 > p > 2$, the following elliptic regularity estimate holds:

$$\|\phi\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}\,, \tag{3.22}$$

and the standard finite element estimate for the Poisson equation (cf. (5.4.8) in [4]) yields

$$\|\phi_h - \phi\|_{L^2(\Omega_h)} \le Ch^2 \|\phi\|_{H^2(\Omega)}\,. \tag{3.23}$$

Since $\partial_t \psi_h \in W^k_{0,h}$, we have

$$\langle \partial_t \psi_h, f\rangle_{\Omega_h} = \langle \nabla\partial_t \psi_h, \nabla\phi_h\rangle_{\Omega_h} = -\langle \partial_t \omega_h, \phi_h\rangle_{\Omega_h} \tag{3.24}$$

where we have used (3.21) and (2.2b) after taking a time derivative. Rewrite (3.24) as

$$\langle \partial_t \psi_h, f\rangle_{\Omega_h} = -\langle \partial_t \omega_h, \phi\rangle_{\Omega_h} - \langle \partial_t \omega_h, (\phi_h - \phi)\rangle_{\Omega_h}\,. \tag{3.25}$$

Let $q$ be the dual number of $p : 1/p + 1/q = 1$. One gets

$$\begin{aligned} |\langle \partial_t \psi_h, f\rangle_{\Omega_h}| &\le \|\partial_t \omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} \|\phi\|_{W^{2,p}(\Omega)} \\ &\quad + \|\partial_t \omega_h\|_{L^\infty([0,T),L^2(\Omega_h))} \|\phi_h - \phi\|_{L^2(\Omega_h)} \\ &\le C\|\phi\|_{W^{2,p}(\Omega)} + \frac{C}{h^2}\|\phi_h - \phi\|_{L^2(\Omega_h)}\,. \end{aligned} \tag{3.26}$$

Thus, (3.22) and (3.23) imply that

$$|\langle \partial_t \psi_h, f\rangle_{\Omega_h}| \le C\|f\|_{L^p(\Omega)}\,.$$

Let $q_0$ be the dual number of $p_0$; then

$$\|\partial_t \psi_h\|_{L^\infty([0,T),L^q(\Omega_h))} \le C \quad \text{for any } q_0 \le q < 2\,. \tag{3.27}$$

Since a natural extension of $\psi_h$ to $\Omega$ from $\Omega_h$ gives equivalent norms, we therefore have

$$\|\partial_t \psi_h\|_{L^\infty([0,T),L^q(\Omega))} \le C \quad \text{for any } q_0 \le q < 2\,. \tag{3.28}$$

This gives the second part of (3.1). □

## 4 A Uniqueness Theorem

In this section we generalize Judovič's uniqueness theorem [8] to show that weak solutions with corresponding vorticity in $L^\infty$ are unique in the wider class where the vorticity are in $L^2$. This generalization will be used in the next section to obtain a stronger convergence theorem. The precise statement is as follows:

THEOREM 4.1 *Assume that $\omega_0 \in L^\infty(\Omega)$ and $\partial\Omega$ is $C^2$. Then the weak solution*

$$\omega \in L^\infty\big([0,T), L^\infty(\Omega)\big) \cap \mathrm{Lip}\big([0,T), W^{-2,r}(\Omega)\big) \tag{4.1}$$

*to* (2.1) *is unique in the space $L^\infty([0,T), L^q(\Omega)) \cap \mathrm{Lip}([0,T), W^{-2,r}(\Omega))$ where $q > \frac{4}{3}$ and $1 \le r < 2$.*

PROOF: Suppose that the initial boundary value problem (2.1) for the 2-D Euler equations has two weak solutions with the same initial vorticity $\omega_0 \in L^\infty$ and the following regularities:

$$\omega_1 \in L^\infty\big([0,T),L^\infty(\Omega)\big)\cap \mathrm{Lip}\left([0,T),W^{-2,r}(\Omega)\right) \tag{4.2}$$

and

$$\omega_2 \in L^\infty\big([0,T),L^q(\Omega)\big)\cap \mathrm{Lip}\left([0,T),W^{-2,r}(\Omega)\right), \tag{4.3}$$

$q > \frac{4}{3}$ and $1 \le r < 2$. It suffices to show that $\omega_1 = \omega_2$.

Denote by $\psi_1$ and $\psi_2$ the stream functions in (2.1) corresponding to $\omega_1$ and $\omega_2$, respectively. Then by the theory of elliptic regularity, we have

$$\psi_1 \in L^\infty\big([0,T),W^{2,p}(\Omega)\big)\cap \mathrm{Lip}\big([0,T),L^r(\Omega)\big) \tag{4.4}$$

and

$$\psi_2 \in L^\infty\big([0,T),W^{2,q}(\Omega)\big)\cap \mathrm{Lip}\big([0,T),L^r(\Omega)\big). \tag{4.5}$$

We denote the corresponding velocities by $\mathbf{u}_1 = \nabla^\perp\psi_1$ and $\mathbf{u}_2 = \nabla^\perp\psi_2$. Then one can rewrite the Euler equations (2.1a) in a distribution sense as

$$\nabla^\perp(\partial_t\mathbf{u}_1 + \mathbf{u}_1\nabla\mathbf{u}_1) = 0 \quad \text{in } D' \tag{4.6}$$

and

$$\nabla^\perp(\partial_t\mathbf{u}_2 + \mathbf{u}_2\nabla\mathbf{u}_2)* = 0 \quad \text{in } D'. \tag{4.7}$$

Set

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \qquad \psi = \psi_1 - \psi_2, \tag{4.8}$$

and subtract (4.7) from (4.6) to get

$$\nabla^\perp(\partial_t\mathbf{u} + \mathbf{u}_2\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{u}_1) = 0 \quad \text{in } D'. \tag{4.9}$$

It follows from (4.4) and (4.5) that

$$\psi \in L^\infty\big([0,T),W^{2,q}(\Omega)\big)\cap \mathrm{Lip}\big([0,T),L^r(\Omega)\big). \tag{4.10}$$

Therefore, one can take $\psi$ as a test function in (4.9) to obtain

$$\int_\Omega \mathbf{u}(\partial_t\mathbf{u} + \mathbf{u}_2\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{u}_1)\,d\mathbf{x} = 0 \tag{4.11}$$

where one has used the fact that $\mathbf{u} = \nabla^\perp\psi$ and $\psi$ vanishes on the boundary. Define

$$E(t) = \int_\Omega |\mathbf{u}|^2\,d\mathbf{x}. \tag{4.12}$$

Due to the regularity assumption (4.2) and (4.3), one can show that

$$\frac{d}{dt}E(t) = 2\int_\Omega \mathbf{u}\cdot\mathbf{u}_t\,d\mathbf{x}. \tag{4.13}$$

Using the fact that

$$\int_\Omega \mathbf{u}\cdot\mathbf{u}_2\nabla\mathbf{u}\,d\mathbf{x} = 0 \tag{4.14}$$

and using equation (4.9), we have

$$\frac{d}{dt}E(t) \le 2\int_\Omega |\mathbf{u}|^2|\nabla\mathbf{u}_1|\,d\mathbf{x}. \tag{4.15}$$

Using the classical potential estimate

$$\|\nabla u_1(\cdot,t)\|_{L^p} \le Cp\|\omega_1(\cdot,t)\|_{L^\infty} \tag{4.16}$$

for all $1 < p < \infty$, where $C$ is an constant independent of $p$, $u_1$, and $\omega_1$, together with the fact that

$$\|\omega_1(\cdot,t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}, \tag{4.17}$$

one shows by the Hölder inequality that

$$\frac{d}{dt}E(t) \le \left(\int_\Omega |\mathbf{u}|^{2p/(p-1)}\,d\mathbf{x}\right)^{\frac{p-1}{p}} \|\nabla\mathbf{u}_1\|_{L^p} \le Cp\left(\int_\Omega |\mathbf{u}|^{2p/(p-1)}\,d\mathbf{x}\right)^{\frac{p-1}{p}}$$

where $C$ is independent of $p$. The right-hand side above can be further estimated by

$$\int_\Omega |\mathbf{u}|^{2p/(p-1)}\,d\mathbf{x} = \int_\Omega (|\mathbf{u}|^2)^{(p-2)/(p-1)}(|\mathbf{u}|^4)^{1/(p-1)} = \|\mathbf{u}\|_{L^2}^{2(p-2)/(p-1)}\|\mathbf{u}\|_{L^4}^{4/(p-1)}.$$

On the other hand,

$$\|\mathbf{u}\|_{L^4} \le C\|\mathbf{u}\|_{W^{1,q}}^{q/(4(q-1))}\|\mathbf{u}\|_{L^2}^{(3q-4)/(4(q-1))}.$$

Since $\|\mathbf{u}\|_{W^{1,q}}$ is bounded, we thus have shown that

$$\frac{d}{dt}E(t) \le CpE(t)^{1-(5q-4)/(4p(q-1))}; \tag{4.18}$$

therefore

$$\frac{d}{dt}\left(E(t)^{(5q-4)/(4p(q-1))}\right) \le C. \tag{4.19}$$

Now one can conclude that $E(t) \equiv 0$. Indeed, taking an interval $[0,T^*]$ with the property that $CT^* \le \frac{1}{2}$, one obtains from (4.20) and $E(0) = 0$ that

$$E(t) \le \left(\frac{1}{2}\right)^{\frac{4p(q-1)}{5q-4}} \to 0 \quad \text{as } p \to \infty, \tag{4.20}$$

so $E(t) \equiv 0$ for $t \in [0,T^*]$. Repeating these arguments, we conclude that $E(t) = 0$ for all $t < T$. This completes the proof. □

## 5 Main Convergence Theorem

Finally, we are able to state and prove our main convergence theorem.

THEOREM 5.1 *Let $\Omega \subset R^2$ be a simply connected domain with $C^2$ boundary (or piecewise smooth $C^2$ boundary with convex corners) and equipped with a quasi-uniform triangulation. Suppose that the initial vorticity $\omega_0$ belongs to $L^2(\Omega)$. Let $(\omega_h, \psi_h) \in V_h^k \times W_{0,h}^k$ be the approximate solutions generated by the discontinuous Galerkin method* (2.2)*. Then there exists a convergent subsequence of $(\omega_h, \psi_h)$ (for which we will still use the same notation for simplicity) such that*

$$\omega_h \rightharpoonup \omega \quad (\text{star weakly}) \text{ in } L^\infty\big([0,T), L^2(\Omega)\big) \cap \mathrm{Lip}\big([0,T), W^{-2,q}(\Omega)\big) \tag{5.1}$$

*for any $q_0 \le q < 2$ and*

$$\psi_h \to \psi \quad (\text{strongly}) \text{ in } L^2\big([0,T), H^1(\Omega)\big), \tag{5.2}$$

*and the limiting functions $\omega$ and $\psi$ have the properties that*

$$\omega \in L^\infty\big([0,T), L^2(\Omega)\big) \cap \mathrm{Lip}\big([0,T), W^{-2,q}(\Omega)\big) \tag{5.3}$$

*and*

$$\psi \in L^2\big([0,T), H_0^1(\Omega_0)\big) \cap \mathrm{Lip}\big([0,T), L^q(\Omega)\big) \tag{5.4}$$

*for any $q_0 \le q < 2$, and for any $\phi \in C_0^\infty(\Omega \times [0,t))$*

$$\int_0^T \int_\Omega \big(\omega\phi_t + \omega\nabla^\perp\psi \cdot \nabla\phi\big)\, d\mathbf{x}\, dt + \int_\Omega \omega_0 \phi(\,\cdot\,, 0)\, d\mathbf{x} = 0, \tag{5.5}$$

$$\Delta\psi = \omega \quad \text{in } D'. \tag{5.6}$$

*In other words, $(\omega, \psi)$ is a weak solution of the Euler equation* (2.1) *with initial data $\omega_0$. Furthermore, if the initial data $\omega_0 \in L^\infty(\Omega)$ and $\partial\Omega$ is $C^2$, then the whole sequence of $(\omega_h, \psi_h)$ will converge to the unique solution of the Euler equations, and the limiting vorticity $\omega$ is bounded in $L^\infty([0,T), L^\infty(\Omega))$.*

PROOF: As in Lemma 3.1, we extend $\omega_h$ and $\psi_h$ to $\Omega$ from $\Omega_h$ naturally. First, (2.5) and (3.1) show that there is a subsequence of $\omega_h$ (for which we still use the same notation) such that

$$\omega_h \rightharpoonup \omega \quad (\text{star weakly}) \text{ in } L^\infty\big([0,T), L^2(\Omega)\big) \cap \mathrm{Lip}\big([0,T), W^{-2,q}(\Omega)\big), \tag{5.7}$$

and the limiting function $\omega$ satisfies (5.3).

Next, it follows from (2.5), (2.6), and the Poincaré inequality that

$$\|\psi_h\|_{L^\infty([0,T),H^1)} \le C. \tag{5.8}$$

This, together with (3.1), shows that there is a subsequence of $\psi_h$ such that

$$\psi_h \rightharpoonup \psi \quad (\text{star weakly}) \text{ in } L^\infty\big([0,T), H^1(\Omega)\big) \cap \mathrm{Lip}\big([0,T), L^q(\Omega)\big) \tag{5.9}$$

and

$$\psi \in L^\infty\big([0,T), H^1(\Omega)\big) \cap \mathrm{Lip}\big([0,T), L^q(\Omega)\big).$$

Since the distance between $\partial\Omega_h$ to $\partial\Omega$ is of order $O(h^2)$ and $\psi_h$ vanishes on $\partial\Omega_h$, we have

$$\|\psi_h\|_{L^\infty(\partial\Omega)} \le Ch^2\|\nabla\psi_h\|_{L^\infty(\Omega_h)} \le Ch\|\nabla\psi_h\|_{L^2(\Omega_h)} \le Ch \to 0\,. \tag{5.10}$$

In the above we have used the inverse inequality

$$\|\nabla\psi_h\|_{L^\infty(\Omega_h)} \le \frac{C}{h}\|\nabla\psi_h\|_{L^2(\Omega_h)}$$

and energy bound (2.4). Hence $\psi$ satisfies (5.4).

We now show that $\psi_h$ converges strongly. First, it follows from (5.8), (3.1), and the Lions-Aubin lemma that

$$\psi_h \to \psi \quad \text{strongly in } L^2([0,T)\times\Omega)\,, \tag{5.11}$$

which, together with (2.6) and (5.7), yields

$$\int_0^T \|\nabla\psi_h\|_{L^2}^2\,dt = -\int_0^T \langle\omega_h\,\psi_h\rangle dt \to -\int_0^T \langle\omega\,\psi\rangle dt\,. \tag{5.12}$$

To obtain the strong convergence that

$$\nabla\psi_h \to \nabla\psi \quad \text{strongly in } L^2([0,T)\times\Omega)\,, \tag{5.13}$$

it suffices to show that

$$\int_0^T \|\nabla\psi_h\|_{L^2}^2\,dt \to \int_0^T \|\nabla\psi\|_{L^2}^2\,dt\,,$$

which is a direct consequence of

$$-\int_0^T \langle\omega\psi\rangle dt = \int_0^T \|\nabla\psi\|_{L^2}^2\,dt\,, \tag{5.14}$$

which will be verified below. Indeed, for any $\phi \in C_0^\infty(\Omega\times[0,T))$, taking $\varphi_h = I_h\phi$ in (2.2b) yields

$$-\int_0^T \langle\nabla\psi_h\cdot\nabla I_h\phi\rangle dt = \int_0^T \langle\omega_h I_h\phi\rangle dt \tag{5.15}$$

where $I_h$ is the interpolation operator in $W_{0,h}^k$. Using the strong convergence (cf. (4.4.21) in [4]),

$$I_h\phi \to \phi \quad \text{strongly in } L^\infty([0,T),L^2(\Omega)) \text{ and } L^\infty([0,T),H^1(\Omega))\,, \tag{5.16}$$

and weak convergences (5.7) and (5.9), one shows from (5.15) that

$$-\int_0^T \langle\nabla\psi\cdot\nabla\phi\rangle dt = \int_0^T \langle\omega\phi\rangle dt\,. \tag{5.17}$$

This immediately gives (5.14) since $C_0^\infty((0,T)\times\Omega)$ is dense in $L^2([0,T),H_0^1(\Omega))$.

As a consequence of (5.13), one concludes that

$$\mathbf{u}_h \to \mathbf{u} \quad \text{strongly in } L^2([0,T)\times\Omega) \tag{5.18}$$

where $\mathbf{u} = \nabla^{\perp}\psi$.

Now we show that the limit functions $(\omega,\psi)$ are indeed a weak solution to the Euler equations. To this end, one can take $v_h$ to be $I_h\phi$ for any $\phi \in C_0^\infty(\Omega\times[0,t))$ in (2.2a), sum over all the cells, and integrate in time to obtain

$$\int_0^T \big(\langle \partial_t \omega_h I_h\phi\rangle - \langle \omega_h \mathbf{u}_h \cdot \nabla I_h\phi\rangle\big)dt = 0\,, \tag{5.19}$$

where we have used the facts that the upwind fluxes $\widehat{\omega_h}$ are the same for the adjacent elements, both $I_h\phi$ and the normal component of the velocity field are continuous across the interior cell boundaries, and $\mathbf{u}\cdot\mathbf{n} = 0$ on the exterior cell boundaries. Since $\phi$ is compactly supported and $\partial_t I_h\phi = I_h\partial_t\phi$, we can integrate by parts to get

$$\int_0^T \big(\langle \omega_h I_h \partial_t\phi\rangle + \langle \omega_h \mathbf{u}_h \cdot \nabla I_h\phi\rangle\big)dt + \langle \omega_h(\,\cdot\,,0) I_h\phi(\,\cdot\,,0)\rangle = 0\,. \tag{5.20}$$

Using the weak convergence of $\omega_h$ in (5.7) and the strong convergence (5.16) of $I_h\partial_t\phi$, one shows that

$$\int_0^T \langle \omega_h I_h \partial_t\phi\rangle dt \to \int_0^T \langle \omega\, \partial_t\phi\rangle dt \quad \text{and} \quad \langle \omega_h(\,\cdot\,,0) I_h\phi(\,\cdot\,,0)\rangle \to \langle \omega_0\phi(\,\cdot\,,0)\rangle\,.$$

Similarly, it follows from the weak convergence of $\omega_h$ in (5.7), the strong convergence of $\mathbf{u}_h$ in (5.18), and the strong convergence (5.16) of $\nabla I_h\phi$ that

$$\int_0^T \langle \omega_h \mathbf{u}_h \cdot \nabla I_h\phi\rangle dt \to \int_0^T \langle \omega\, \mathbf{u}\cdot\nabla\phi\rangle dt\,.$$

Hence

$$\int_0^T \langle \omega \partial_t\phi\rangle dt + \int_0^T \langle \omega \mathbf{u}\cdot\nabla\phi\rangle dt + \langle \omega_0\phi(\,\cdot\,,0)\rangle = 0\,. \tag{5.21}$$

This gives (5.5). Finally, (5.6) follows from (2.2b) by taking $\varphi_h = I_h\phi$. Thus we have proved that $(\psi,\omega)$ is a weak solution to the Euler equations (2.1).

In the case that the initial data $\omega_0 \in L^\infty(\Omega)$, then the Cauchy problem for the Euler equations has a solution $\omega \in L^\infty([0,T)\times\Omega)$, and from Theorem 4.1 we know that this solution is unique in the class of (5.3) and (5.4). Therefore every convergent subsequence has the same limit. As a consequence, the whole sequence of $(\omega_h,\psi_h)$ converges to the unique solution. This completes the proof of the theorem. ☐

**Acknowledgments.** The authors would like to thank Professors Qiang Du and Jun Zou for their careful reading and many useful comments on the first draft of this paper, and Professor John Osborn for some helpful discussions. The research of Liu was supported in part by NSF Grant DMS-9805621. The research of Xin was supported in part by the Zheng Ge Ru Foundation, HK/RGC Grant 4219/99P, NSF Grants DMS-96-00013 and DMS-9971978, and DOE Grant De-FG02-88ER-25053.

## Bibliography


[1] Baker, G. R.; Shelley, M. J. On the connection between thin vortex layers and vortex sheets. *J. Fluid Mech.* **215** (1990), 161–194.

[2] Beale, J. T. The approximation of weak solutions to the 2-D Euler equations by vortex elements. *Multidimensional hyperbolic problems and computations (Minneapolis, MN, 1989)*, 23–37. IMA Volumes in Mathematics and Its Applications, 29. Springer, New York, 1991.

[3] Brenier, Y.; Cottet, G. H. Convergence of particle methods with random rezoning for the two-dimensional Euler and Navier-Stokes equations. *SIAM J. Numer. Anal.* **32** (1995), no. 4, 1080–1097.

[4] Brenner, S.; Scott, L. R. *The mathematical theory of finite element methods*. Texts in Applied Mathematics, 15. Springer, New York, 1994.

[5] Caflisch, R. E.; Lowengrub, J. S. Convergence of the vortex method for vortex sheets. *SIAM J. Numer. Anal.* **26** (1989), no. 5, 1060–1080.

[6] Chorin, A. J. Numerical study of slightly viscous flow. *J. Fluid Mech.* **57** (1973), no. 4, 785–796.

[7] Filho, M. C. L.; Lopes, H. N.; Tadmor, E. Approximate solutions of the incompressible Euler equations with no concentrations. Preprint, 1998.

[8] Judovič, V. I. Non-stationary flow of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.* **3** (1963) 1032–1066.

[9] Krasny, R. Computation of vortex sheet roll-up in the Trefftz plane. *J. Fluid Mech.* **184** (1987), 123–155.

[10] Levy, D.; Tadmor, E. Non-oscillatory central schemes for the incompressible 2-D Euler equations. *Math Research Lett.* **4** (1997), no. 2-3, 321–340.

[11] Liu, J.-G.; Shu, C.-W. A high order discontinuous Galerkin method for incompressible flows. *J. Comput. Phys.* (2000), to appear.

[12] Liu, J.-G.; Xin, Z. P. Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheets data. *Comm. Pure Appl. Math.* **48** (1995), no. 6, 611–628.

[13] Schochet, S. The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. *Comm. Partial Differential Equations* **20** (1995), no. 5-6, 1077–1104.

[14] Schochet, S. The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math.* **49** (1996), no. 9, 911–965.

[15] Temam, R. *Navier-Stokes equations. Theory and numerical analysis.* Studies in Mathematics and Its Applications, Vol. 2. North-Holland, Amsterdam–New York–Oxford, 1977.

[16] Zabusky, N. J.; Hughes, M. H.; Roberts, K. V. Contour dynamics for the Euler equations in two dimensions. *J. Comput. Phys.* **135** (1997), no. 2, 217–226.



JIAN-GUO LIU
University of Maryland
Institute for Physical Science and Technology
Department of Mathematics
College Park, MD 20742
E-mail: jliu@math.umd.edu

ZHOUPING XIN
Courant Institute
251 Mercer Street
New York, NY 10012
E-mail: xinz@cims.nyu.edu, zpxin@ims.cuhk.edu.hk


Received May 1999.