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Pricing of Mortgage-Backed Securities with Option-Adjusted Spread

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Abstract

In this article, the authors developed an alternative methodology to calculate the option-adjusted spread for the mortgage-backed securities using partial differential equation technique. The numerical implementation is discussed in detail, including the convergence and error analysis. This approach provides us a fast and accurate way to pricing MBS.

1. Introduction

The option-adjusted spread (OAS) analytics of the mortgage-backed securities have become increasingly important in today's MBS market. Compared to conventional static pricing method, the OAS has an attractive feature of valuing the MBS's return in excess of U.S. Treasury, at the same time taking the built-in prepayment option into account. The OAS is more and more accepted by many who trade and invest in the MBS as a gauge to measure the securities's response to the change of the interest rate environment.

In order to calculate OAS and related analytics, one often uses Monte Carlo dynamic methodology. However, this method often proves to be slow in convergence. Sometimes large errors result.

In this article, we have developed a new method to calculate OAS and related analytics for MBS, using partial differential equation (PDE) methodology. This would solve the problem of slow convergence and large error resulted from the Monte Carlo method.

One problem we had to overcome in order to develop such method is the path-dependency of the MBS cash flows. That means at any month the principle and interest payment from an MBS depends not only on the current interest rate at the time, but also on the interest rate in the past. We solved the problem by introducing new variables that carry natural meanings. We then carefully designed the numerical algorithm to carry out the computations, so that the convergence is optimized and error is very small.

In the last section, we discussed an example in the framework of a Hull-White interest rate model. We showed how the OAS can be calculated, and the effective duration and convexity also computed.

2. Partial differential equations for mortgage-backed securities

We start with the general form of the N-factor interest term structure model. Suppose that r_1 is the short-term (theoretically, the instantaneous) interest rate, and r_2, $r_3, \ldots, r_N$ are current rates of longer terms. They follow

$$dr_i(t) = \alpha_i(t, r_1(t), \ldots, r_N(t))\,dt + \sum_{j=1}^{N} \sigma_{ij}(t, r_1(t), \ldots, r_N(t))\,dz_j(t),$$
$$(i = 1, 2, \ldots, n) \tag{1}$$

where $z_j(t)$'s are canonical Wiener processes with covariant coefficients

$$\alpha_{kl} = \frac{\operatorname{cov}(dz_k, dz_l)}{|dz_k||dz_l|} \tag{2}$$

Suppose that at a given time t, $r_i(t) = r_i$, and $\Theta(t, T, r_1, \ldots, r_N)$ is the price of the default-riskless zero-coupon bond which matures at T and pays \$1 at the maturity. In this article we will call such bonds zero-coupon treasuries. Using Itô calculus, we have

$$d\Theta = \frac{\partial\Theta}{\partial t}dt + \sum_{i=1}^{N}\frac{\partial\Theta}{\partial r_i}dr^i + \tfrac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2\Theta}{\partial r_i \partial r_j}dr_i dr_j = \left(\frac{\partial\Theta}{\partial t} + \mathcal{L}\Theta\right)dt$$
$$+ \sum_{i,j=1}^{N}\sigma_{ij}\frac{\partial\Theta}{\partial r_i}dz_j\,, \tag{3}$$

where the differential operator

$$\mathcal{L} = \tfrac{1}{2}\sum_{i,j=1}^{N}\eta_{ij}\frac{\partial^2}{\partial r_i \partial r_j} + \sum_{i=1}^{N}\alpha_i\frac{\partial}{\partial r_i}. \tag{4}$$

and

$$\eta_{ij} = \sum_{k,l=1}^{N}\sigma_{ik}\,\sigma_{jl}\,\alpha_{kl}\,. \tag{5}$$

The operator $\mathcal{L}$ is elliptic, but it has degeneracy when $r_i = 0$.

Assume that the market price of risk on change of r_j is $\lambda_j(t, r_1, \ldots, r_N)$. We have the following equation for zero-coupon treasuries.

$$\frac{\partial\Theta}{\partial t} + \mathcal{L}\Theta - \sum_{i,j=1}^{N}\lambda_j\sigma_{ij}\frac{\partial\Theta}{\partial r_i} - r_i\Theta = 0 \tag{6}$$

with the terminal condition

$$\Theta(T, T, r_i, \ldots, r_N) = 1,$$

Consider a mortgage-backed security which is supported by a pool which matures at time T. Let c be the coupon rate, w the WAC, and δ the (actual) delay of the coupon distribution. We denote

$$\tilde{\Theta} = \Theta(t, t+\delta, r_1, \ldots, r_N), \tag{7}$$

the discount factor for treasury at time t of a term δ. We further assume that during the period from t and $t+dt$, the underlying pool generate a cash-flow $F(t, r_1, \ldots, r_N, \rho_1, \ldots, \rho_p)$, where $\rho_1, \ldots, \rho_p$ are status variable other than the interest rates $r_1, \ldots, r_N$. These status variables cover a broad category of parameters which affect the pricing of mortgage-backed securities, such as the pool factor, the prepayment speed, and sometimes the lagged yield for a long term treasury usually used to determine the cost of mortgage loans, etc. They follow (or can be approximated by) hidden Markov processes in the following manner.

$$d\rho_k(t) = \gamma_k(t, r_1(t), \ldots, r_N(t)\ \rho_1(t), \ldots, \rho_p(t))dt,\ (k = 1, \ldots, p). \tag{8}$$

For the sake of simplicity, we will restrict our discussion to the pass-throughs and strips, although more complicated collaterals can be discussed with similar methods. We also assume that the instruments in our discussion are supported by a single pool of mortgage. The discussion can be applied to multiple pool supported instruments either by aggregating the pools or to introduce more status variables.

Suppose that $P(t, T, r_1, \ldots, r_N, \rho_1, \ldots, \rho_p)$ is the price for the mortgage-backed security. Using Feynman-Kac path integral, we have

$$P(t, T, r_1, \ldots, r_{N,}\ \rho_1, \ldots, \rho_p) = \text{Expt}$$
$$\left[\int_t^T e^{-a(s-t)}\Theta(t, s, r_1(s), \ldots, r_N(s)) \times F(s, r_1(s), \ldots, r_N(s)\rho_1(s), \ldots, \rho_p(s))\, ds\right], \tag{9}$$

where α is the option adjusted spread, and the expectation is taken with the market condition at time t when $r_1(t) = r_1, \ldots, r_N(t) = r_N$ and $\rho_1(t) = \rho_1, \ldots, \rho_p(t) = \rho_p$.

Using Itô calculus, one gets

$$dP = \frac{\partial P}{\partial t}dt + \sum_{i=1}^{N}\frac{\partial P}{\partial r_i}dr_i + \tfrac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2 P}{\partial r_i \partial r_j}dr_i\, dr_j + \tag{10}$$

$$\sum_{k=1}^{p}\frac{\partial P}{\partial \rho_k}dp_k = \left[\frac{\partial P}{\partial t} + \mathcal{L}P + \sum_{k=1}^{p}\gamma_k\frac{\partial P}{\partial \rho_k}\right]dt + \sum_{i,j=1}^{N}\sigma_{ij}\frac{\partial P}{\partial r_i}dz_j\,.$$

Consider a portfolio consisting of one share of mortgage-backed security and ν_i shares of zero-coupon treasury that matures at T_i, ($i=1, \ldots, n$.) Note that ν_i may be negative, as we may take a short position. We use Θ_i to denote $\Theta(t,T_i,r_1,\ldots,r_N)$. Then by (3),

$$d\Theta_i = \left(\frac{\partial\Theta_i}{\partial t} + \mathcal{L}\Theta_i\right)dt + \sum_{k,j=1}^{N}\sigma_{kj}\frac{\partial\Theta_i}{\partial r_k}dz_j\,. \tag{11}$$

During the time period from t to $t+dt$, the change of value of the portfolio is $dP - \sum_{i=1}^{N}\nu_i d\Theta_i$, and the portfolio generated an income of $F(t,r_1,\ldots,r_N,\rho_1,\ldots,\rho_p)dt$. Thus at the end of the period, the portfolio worths

$$d(P + \sum_{i=1}^{N}\nu_i\Theta_i) + F\tilde{\Theta}dt$$

$$= \left[\frac{\partial P}{\partial t} + \mathcal{L}P + \sum_{k=1}^{p}\gamma_k\frac{\partial P}{\partial\rho_k} + \sum_{i=1}^{N}\nu_i\left(\frac{\partial\Theta_i}{\partial t} + \mathcal{L}\Theta_i\right) + F\tilde{\Theta}\right]dt + \tag{12}$$

$$\sum_{k,j=1}^{N}\sigma_{kj}\frac{\partial(P + \sum_{i=1}^{N}\nu_i\Theta_i)}{\partial r_k}dz_j\,.$$

Choose ν_1, ..., ν_N so that at time t,

$$\sum_{i=1}^{N}\frac{\partial\Theta_i}{\partial r_k}\nu_i = -\frac{\partial P}{\partial r_k}, \qquad (k = 1,\ldots,N). \tag{13}$$

Then the portfolio becomes interest-risk-neutral during the period from t to $t+dt$. Note that the interest-risk-neutrality means that the present value of the portfolio would not be affected by the interest environment change during the short period. It does not imply, however, that the prepayment risk of the mortgage in the portfolio is neutralized. Such risk is not hedgible by the treasuries, although the interest rate indirectly plays a role in the prepayment. As the portfolio becomes interest-risk-neutral, it should have a return of riskless short-term interest rate in the period from t to $t+dt$. In addition, the mortgage in the portfolio earns an extra option adjusted spread. (It is for this extra spread the investors are willing to take extra risk of "irrational behavior'' of mortgage holders.) Thus

$$\frac{\partial P}{\partial t} + \mathcal{L}P + \sum_{k=1}^{p}\gamma_k\frac{\partial P}{\partial\rho_k}\sum_{i=1}^{N}\nu_i\left(\frac{\partial\Theta_i}{\partial t} + \mathcal{L}\Theta_i\right) + F\tilde{\Theta} = r_i\left(P + \sum_{i=1}^{N}\nu_i\Theta_i\right) + \alpha P\,.$$

Using (6) and (13), one has

$$\frac{\partial P}{\partial t} + \mathcal{L}P - \sum_{i,j=1}^{N}\alpha_j\sigma_{ij}\frac{\partial P}{\partial r_i} + \sum_{k=1}^{p}\gamma_k\frac{\partial P}{\partial\rho_k} - (r_1 + \alpha)P = -F\tilde{\Theta}. \tag{14}$$

with terminal condition

$$P(T, T, r_i, \ldots, r_N, \rho_1, \ldots, \rho_p) = 0, \tag{15}$$

Naturally, we have that

$$\lim_{r_i \to \infty} P(t, T, r_1, \ldots, r_N, \rho_1, \ldots, \rho_p) = 0, \tag{16}$$

for any $i=1, \ldots, N$.

It is interesting to observe that although we used the zero-coupon treasuries to hedge the mortgage portfolio to get the equation (14), the values of the hedging bonds do not explicitly appear in the equation for the value of mortgage. This is a desirable and conceptually important point. Because we may use different sets of zero-coupon treasuries to hedge the mortgage, and we should get the same equation no matter what set of treasuries we choose to use.

Another interesting point to observe is that, comparing equation (6) and equation (14), one can see that the path dependency in the case of mortgage has been codified into the movement along the characteristic lines $\rho_k = \rho_k(t)$.

The status variables $\rho_1, \ldots, \rho_p$, usually contains the pool factor and the prepayment speed. Let f be the pool factor. Also, we denote g such that $1-e^{-g\Delta t}$ is the single month mortality of the underlying pool. (Here $\Delta t = \frac{1}{12}$ year.) This implies that $1-e^{g}$ is the annual CPR. We have

$$\frac{f(t+\Delta t)}{f(t)} = \frac{f_0(t+\Delta t)}{f_0(t)} e^{-g\Delta t},$$

where

$$f_0(t) = 1 - \frac{e^{wt}-1}{e^{wT}-1} = \frac{e^{wT}-e^{wt}}{e^{wT}-1} \tag{17}$$

is the amortization schedule factor, i.e., the pool factor under zero prepayment assumption. This results in the following stochastic process

$$df = -\left(\frac{w}{e^{w(T-t)}-1} + g\right) f dt. \tag{18}$$

The single month mortality g follows a process specified in a prepayment model. In Section 4, we will give an example of such a process.

In the above equation, F is the cash flow velocity at time t, with interest rates $r_1, \ldots, r_N$ and pool factor f. Denote F_P the cash flow from the principal and F_I the cash flow from the interest. Suppose the pass-through has coupon rate c, then we have

$$F_p = \left(\frac{w}{e^{w(T-t)} - 1} + g \right) f \tag{19}$$

$$F_I = cf\,.$$

For pass-throughs, $F = F_p + F_I$

For POs $F = F_P\,.$

For IOs $F = F_I$

Equation (14) can be called *the general equation for mortgage-backed securities*. In the following section, we will see that its coefficients must satisfy some conditions in order to guarantee its well-posedness. These conditions are in general satisfied by most econometric models used in the industry. In section 4, we will solve the equation for a particular case.

3. Discussion on Underline Mathematical Theory}

We denote

$$p_i = \alpha_i - \sum_{j=1}^{N} \lambda_j \sigma_{ij} - \tfrac{1}{2} \sum_{j=1}^{N} \frac{\partial \eta_{ij}}{\partial r_j}\,.$$

Then (14) can be written as

$$\frac{\partial P}{\partial t} + \tfrac{1}{2} \sum_{i,j=1}^{N} \frac{\partial}{\partial r_i} \left(\eta_{ij} \frac{\partial P}{\partial r_j} \right) + \sum_{i=1}^{N} p_i \frac{\partial P}{\partial r_i} + \sum_{k=1}^{p} \gamma_k \frac{\partial P}{\partial \rho_k} - (r_i + a) P = -F\tilde{\Theta}. \tag{20}$$

with terminal condition

$$P(T, T, r_i, \ldots, r_N, \rho_1, \ldots, \rho_p) \equiv 0 \tag{21}$$

and boundary condition

$$\lim_{r_i \to \infty} P(T, T, r_1, \ldots, r_N, \rho_1, \ldots, \rho_p) \equiv 0\,, \tag{22}$$

Note that the diffusion coefficient σ_{ij} is degenerate when $r_i = 0$, i.e.,

$$\sigma_{ij}\,|_{ri=0} = 0. \tag{23}$$

This implies that

$$\eta_{ij}\,|_{r_i=0} = 0.$$

In order to have a unique solution to (20) with boundary condition (21) and (22) but no other additional condition for variables $r_1, \ldots, r_N$, we need the following conditions

$$a_i\,|_{ri=0} \geq \tfrac{1}{2}\lim_{r_i\to 0}\sum_{j=1}^{N}\frac{\partial\eta_{ij}}{\partial r_j}$$

or equivalently

$$a_i\,|_{ri=0} \geq \tfrac{1}{2}\lim_{r_i\to 0}\sum_{j,k,l=1}^{N} a_{kl}\sigma_{jl}\frac{\partial\sigma_{ik}}{\partial r_j}. \tag{24}$$

Clearly, we have

$$\eta_{ij}\,|_{r_i=0} \equiv 0,\; p_i\,|_{r_i=0} \geq 0.$$

For state variables $\rho_1, \ldots, \rho_p$, we assume the following condition is always satisfied: For each k, the ρ_k varies in interval $[a_k, b_k]$, (a_k may be $-\infty$, b_k may be ∞.) Furthermore,

$$\begin{array}{lll} \gamma_k\,|_{\rho_k=a_k} \geq 0, \text{ or} & \gamma_k\,|_{\rho_k=a_k} < 0, & P|_{\rho_k=a_k} = 0, \\ \gamma_k\,|_{\rho_k=b_k} \leq 0, \text{ or} & \gamma_k\,|_{\rho_k=b_k} > 0, & P|_{\rho_k=b_k} = 0. \end{array} \tag{25}$$

The coefficients $\eta_{ij} p_i$ are assumed to be differentiable, and γ_k are assumed to piecewise differentiable. Suppose the coefficients γ_k is discontinuous, and the hypersurface $f(t, \vec{r}, \vec{\rho}) = 0$ is the front of discontinuity, we assume the following condition of "market nationality'':

$$(\vec{v}_+ . \vec{n})(\vec{v}_- . \vec{n}) > 0 \tag{26}$$

where $\vec{n}$ is the normal vector at the hypersurface, $\vec{v}_+$ and $\vec{v}_-$ are the limit vectors of $(1, \gamma_1, \ldots, \gamma_p)$ at the surface from both sides.

The conditions (23), (25) and (26) are naturally posed. However, condition (24) may be violated in some circumstances as we use interest rate models such as

Cox-Ingersoll-Ross model when interest rate yield curve are flat at a low level and volatility is very high. Nevertheless, if the model is one-factor CIR model, such circumstance will not affect the general outcome of our analysis. This was proved by Feller [F] for the case of absence of variables ρ_k. His proof can be easily extended to the case when ρ_k's are in presence..

Without loss of generality, we assume that $0 \le \rho_k \le 1$, and there is no discontinuity in coefficients, since the "market rationality'' condition will ensure the terms arise out of the existence of the front surface in the following equation would not affect our estimation.

We have the following energy estimates.

$$-\frac{\partial}{\partial t}\int_{R_+^N}\int_{[0,1]^p} P^2\, dV_\rho\, dV_r + \sum_{i,j=1}^{N}\int_{R_+^N}\int_{[0,1]^p} \eta_{ij}\frac{\partial P}{\partial r_j}\frac{\partial P}{\partial r_i}\, dV_\rho\, dV_r +$$

$$\sum_{i=1}^{N}\int_{R_+^N}\int_{[0,1]^p}\frac{\partial P_i}{\partial r_i}P^2\, dV_\rho\, dV_r + \sum_{k=1}^{p}\int_{R_+^N}\int_{[0,1]^p}\frac{\partial P}{\partial \rho_k}\gamma_k P^2\, dv_\rho\, dV_r +$$

$$\int_{R_+^N}\int_{[0,1]^p} 2(r_1 + a)P^2\, dV_\rho\, dV_{r\,=}$$

$$2\int_{R_+^N}\int_{[0,1]^p} PF\tilde{\Theta}\, dV_\rho\, dV_r - \sum_{i,j=1}^{N}\int_{R_+^{N-1}}\int_{[0,1]^p}(p_i P^2)|r_i = 0\, dV_\rho\, d\hat{V}_{r_i} -$$

$$-\sum_{k=1}^{p}\int_{R_+^N}\int_{[0,1]^{p-1}}(\gamma_k P^2)|_{\rho k=0}\, d\hat{V}_{\rho k}\, dV_r\, .$$

Here we use the notation

$$\begin{aligned}
dV_r &= dr_1\, dr_2 \ldots dr_N\ ,\\
dV_\rho &= d\rho_1\, d\rho_2 \ldots d\rho_p\ ,\\
d\hat{V}_{ri} &= dr_1 \ldots dr_{i-1}\, dr_{i+1} \ldots dr_N\ ,\\
d\hat{V}_{\rho k} &= d_{\rho_1} \ldots d\rho_{k-1}\, d\rho_{k+1} \ldots d\rho_p\ .
\end{aligned}$$

Assume that

$$\left|\frac{\partial p_i}{\partial r_i}\right| + \left|\frac{\partial \gamma_j}{\partial \rho_j}\right| \le C\,, \qquad |\tilde{\Theta}| \le C\,.$$

Then we have

$$\int_{R_+^N}\int_{[0,1]^p} P^2\, dV_\rho\, dV_r + \sum_{i,j=1}^{N}\int_t^T\int_{R_+^N}\int_{[0,1]^p} e^{C_1(r-t)}\,\eta_{ij}\frac{\partial P}{\partial r_j}\frac{\partial P}{\partial r_i}\, dV_\rho\, dV_r\, d\tau$$

$$\leq C_2\int_t^T\int_{R_+^N}\int_{[0,1]^p} e^{C_i(\tau-t)}F^2\, dV_\rho\, dV_r\, d\tau. \tag{27}$$

Similarly, we have estimate of the high order derivatives

$$\|P\|^2_{H^l} \leq C_4\int_t^T e^{C_3(\tau-t)}\,\|F^2\|_{H^l}\, d_\tau \tag{28}$$

This proves the following theorem.

Theorem: Suppose $F \in L_t^2(H^l_{r,\rho})$ for some integer $l \geq 0$ and (23), (24), (25) and (26) holds true, then there is a unique solution $P \in L_t^\infty(L^l_{r,\rho})$ to equation (20) with (21), (22) and

$$\|P\|_{L_t^\infty(H^l_{r,\rho})} \leq C\,\|F\|_{L_t^2(H^l_{r,\rho})}\,. \tag{29}$$

Now, we describe the numerical method and its convergence theory for the equation (14). Without loss of generality, we use a one-factor model:

$$\frac{\partial P}{\partial t} + a(t,r)\frac{\partial^2 P}{\partial r^2} + b(t,r)\frac{\partial P}{\partial r} + c(t,r,\rho)\frac{\partial P}{\partial \rho} +$$

$$d(t,r,\rho)P + F(t,r,\rho) = 0. \tag{30}$$

The generalization to the multifactor-models is straightforward.

We first make the following transformation

$$x = \frac{1}{1+\nu r},$$

Then (30) becomes

$$\frac{\partial P}{\partial t} + \tilde{a}(t,x)\frac{\partial^2 P}{\partial x^2} + \tilde{b}(t,x)\frac{\partial P}{\partial x} + \tilde{c}(t,x,\rho)\frac{\partial P}{\partial \rho} +$$

$$\tilde{d}(t,x,\rho)P + \tilde{F}(t,x,\rho) = 0. \tag{31}$$

where

$$\tilde{a} = \lambda^2 x^4 a(t, \frac{1-x}{vx}),$$

$$\tilde{b} = 2\lambda^2 x^3 a(t, \frac{1-x}{vx}) - vx^2 b(t, \frac{1-x}{vx}).$$

Some other kind of transformation is needed for other variables ρ in some applications in order to reduce the large or infinite range to a small one, or sometimes to reduce singularities. Such transformations can only be dealt case by case. We will explain in detail by an example in section 4.

The time discretization for the second order term is implemented by stable implicit schemes such as backward Euler, Crank-Nicholson, or implicit Runge-Kutta methods. The spatial discretization is implemented by standard center-difference. Since there is no boundary condition for $x = 1$, i.e. $r = 0$, we need a numerical boundary condition at this point. We choose no-flux as the numerical boundary condition. It is known that this numerical boundary condition will not effect the accuracy at interior. (cf. [KL])

The time-discretization for the first order terms is done by explicit scheme. The spatial discretization is done by up-winding method. The following Courant-Friedrichs-Lewy (CFL) stability condition must be satisfied

$$\frac{\Delta t}{\Delta x}\sup|\tilde{b}(t,x)| + \frac{\Delta t}{\Delta \rho}\sup|\tilde{c}(t,x,\rho)| \leq 1 \tag{32}$$

where Δt, Δx and Δρ be the grid sizes for *t, x* and ρ, respectively.

In more detail, we use $P^n_{i,j}$ to approximate $P(t,x,\rho)$ at points $t_n = n\Delta t$, $x_i = i\Delta x$ and $\rho_j = j\Delta\rho$. The above scheme can be carried out as follows

$$\begin{aligned}
&\frac{P^{n+1}_{i,j} - P^n_{i,j}}{\Delta t} + \tilde{a}(t_n, x_i\, \rho_j)\frac{P^n_{i+1,j} - 2P^n_{i+1,j} + P^n_{i-1,j}}{\Delta x^2} + \\
&\tilde{b}+(t_{n+1}, x_i, \rho_j)\frac{P^{n+1}_{i+1,j} - P^{n+1}_{i,j}}{\Delta x} + \tilde{b}-(t_{n+1}, x_i, \rho_j)\frac{P^{n+1}_{i,j} - P^{n+1}_{i-1,j}}{\Delta x} + \\
&\tilde{c}+(t_{n+1}, x_i, \rho_j)\frac{P^{n+1}_{i+1,j} - P^{n+1}_{i,j}}{\Delta \rho} + \tilde{c}-(t_{n+1}, x_i, \rho_j)\frac{P^{n+1}_{i,j} - P^{n+1}_{i,j-1}}{\Delta \rho} + \\
&\tilde{d}(t_{n+1}, x_i, \rho_j)\rho^{n+1}_{i,j} + F(t_{n+1}, x_i, \rho_j) = 0.
\end{aligned} \tag{33}$$

Scheme (33) is first order in space and time. Following standard technique, we can prove the following theorem.

Theorem: Suppose $P \in H_t^2(H_{x,\rho}^3)$ is the solution to equation (20) with (21) and (22), $P_{i,j}^n$ is the approximate solution given by (33) with all appropriate boundary condition satisfied. Assume the CFL condition (32) is satisfied, then we have

$$\sup_{0 \le t_n \le T} \left(\sum_{i,j} | P_{i,j}^n - P(t_n, x_i, \rho_j |^2 \Delta x \Delta \rho \right)^{\frac{1}{2}} \le C(\Delta t + \Delta x + \Delta \rho).$$

In the next section, we will discuss a simple example at length on the numerical computation of our theory. This model, because of its simplicity, illustrates our main idea. And is sufficient in most cases. However, one may call for a more sophisticated model which require higher dimension in computation. After a series of experiments on the models, we discovered that a number of tools which can be applied in our theory to achieve higher efficiency. We observed that there are critical paths (regions) in the state spaces where the behavior of the system exerts most influence on the final outcome.

This region is closely related to the current market environment. By using adaptive mesh refinement method in this critical region, one may substantially lower the cost of computation. The method we use will be adaptive to the change of the environment. Thus is very practical in the ever changing environment of the world of financial industry.

Another important tool which is extremely useful in multi-factor models in artifical decoupling of fasting using alternating iteration. This idea is similar to Schwartz method in domain decomposition computation method. The reason that it can be used is because in most of the place in the state space, factors are often not very closely coupled.

In region where one or more factors do not have significant effect on the equation. The system may be projected onto a space of lower dimension. One may find extensive use of asymptotic analysis in this aspect.

4. A One-Factor Example

Consider the process of Cox-Ingersoll-Ross model of interest rate

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma\sqrt{r(t)}\, dz(t), \tag{34}$$

where κ and σ are positive constants, $z(t)$ is the canonical Wiener process, $\theta(t)$ is a deterministic function of t. The zero-coupon treasuries then have value

$$\Theta(t, T, r) = A(t, T)e^{-B(t,T)r}, \tag{35}$$

where

$$A(t,T) = \exp\left(-\int_t^T \kappa\theta(\tau)B(\tau,T)d\tau\right),$$

$$B(t,T) = \frac{2(e^{\phi(T-t)}-1)}{(\kappa+\lambda+\phi)(e^{\phi(T-t)}-1)+2\phi}, \tag{36}$$

and $\phi^2 = (\kappa+\lambda)^2 = 2\sigma^2$, λ is related to the market price of risk.

Recall that the pool factor variable f follows the process

$$df = -\left(\frac{w}{e^{w(T-t)}-1} + g\right) f dt. \tag{37}$$

We consider a mortgage with the following prepayment model.

$$g = min(1,.4t)\left(s_2(f)(s_3(w-y-m)-.06+h(t))+.06\right);$$

$$s_2(f) = \begin{cases} 1.4-4.*f & \text{if} \quad f<0.1 \\ 1 & \text{if} \quad 0.1<f<0.8 \\ f+0.2 & \text{if} \quad 0.8<f<1 \end{cases}$$

$$s_3(x) = \begin{cases} \dfrac{.0601-.04x}{1.0025-x} & \text{if} \quad x<.0025 \\ 19.42857x+0.011429 & \text{if} \quad .0025<x<.02 \\ \dfrac{0.394+0.3x}{0.98+x} & \text{if} \quad .02<x \end{cases}$$

and $h(t)$ can be used to represent the seasonality factor of prepayment, y is the ten-year rate, m is the spread of cost of mortgage over the ten-year rate. The leading factor min(1,.4t) may be chosen in a more sophistical way “to fit the historical data better’’. The two terms with long-term yield y reflect the incentive to prepay when the interest rate are sufficiently below the WAC. The last term reflects the speed-up of the prepayment as the pool-factor decreases. The seasonality function $h(t)$ is given in the following table:

Figure 1 shows the CPR obtained by the above model, with pool factor set to 1. Please note that we have chosen a simplistic example of prepayment function g here to illustrate our method. In real applications, one can choose far more sophisticated prepayment models. The implementation of these more sophisticated models would be similar to the one shown below.

Recall that we are using CIR term structure model, we can easily compute the ten year yield from current short-term rate r, using (35) and (36).

month t	$h(t)$	month t	$h(t)$
0	-0.0195	6	-0.0337
1	0.0438	7	0.0560
2	0.0275	8	0.0418
3	0.0092	9	0.0435
4	0.0163	10	0.0382
5	0.0105	11	0.0242

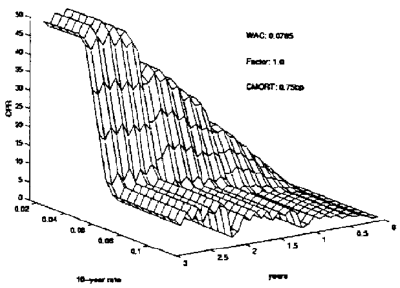

Figure 1: CPR in first three years

$$y(t) = -\tfrac{1}{10}\int_t^{t+10} \kappa\theta(\tau)B(\tau, t+10)d\tau + B(t, t+10). \tag{38}$$

When θ is a constant

$$y(t) = -\frac{\kappa\theta}{5\sigma^2}\ln\left[\frac{2\phi e^{5(\kappa+\lambda+\phi)}}{\kappa+\lambda+\phi)(e^{10\phi}-1)+2\phi}\right]+$$

$$\left[\frac{(e^{10\phi}-1)r}{5[(\kappa+\lambda+\phi)(e^{10\phi}-1)+2\phi]}\right]. \tag{39}$$

Suppose that $P(t,T,r,f)$ is the price for a mortgage-backed security with cash flow $F(t,r,f)$. We have the following equation † for P.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} + (\kappa\theta - (\kappa+\lambda)r)\frac{\partial P}{\partial r} -$$

$$\left(\frac{w}{e^{w(T-t)}-1}g\right)f\frac{\partial P}{\partial f} - (r+a)P = -F\tilde{\Theta} \tag{40}$$

with terminal condition

$$P(T,t,f) \equiv 0. \qquad (41)$$

Footnote

† The example shown here has no lagged long-term yield. If the lagged long-term yield is involved, one can use the following process to approximate the lagged interest rate movement.

$$\begin{cases} dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz, \\ du = 12(r-u)dt. \end{cases}$$

Then the equation (40) is changed to the following form.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + (\kappa\theta - (\kappa+\lambda)r\frac{\partial P}{\partial r} - +12(r-u)\frac{\partial P}{\partial u} - \left(\frac{w}{e^{w(T-t)}-1} + g\right)f$$

$$\frac{\partial P}{\partial f} - (r+a)P = -F\tilde{\Theta}.$$

In the above equation, F is the cash flow speed at time t, with interest rate r and pool factor f. Denote F_P the cash flow from the principal and F_I the cash flow from the interest. Suppose the pass-through has coupon rate c, then we have

$$F_P = \left(\frac{w}{e^{w(T-t)}-1} + g\right)f,$$

$$F_I = cf.$$

For pass-throughs, $F = F_P = F_I$

For POs $F = F_P$.

For IOs $F = F_I$

By equations (7), (35) and (36), we have

$$\tilde{\Theta} = e^{-\int_t^{t+\delta} \kappa\theta(\tau)B(\tau,\tau+\delta)\,\delta\tau - \mathrm{B}(\tau,\tau+\delta)\tau}$$

$$= \left(\frac{2\phi e^{\frac{\delta}{2}(\kappa+\lambda+\phi)}}{\kappa+\lambda+\phi)(e^{\delta\phi}-1)+2\phi}\right)^{\frac{2\kappa b}{\sigma^2}} \exp\left(\frac{-2(e^{\delta\phi}-1)r}{(\kappa+\lambda+\phi)(e^{\delta\phi}-1)+2\phi}\right). \qquad (42)$$

Now we are going to solve equation (40) numerically. We choose the following variable transformation.

$$x = \frac{1}{1+vr}, \quad \rho = \frac{f}{1-e^{w(t-T)}}.$$

Then equation (40) becomes

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 vx^3(1-x)\frac{\partial^2 P}{\partial x^2} + ((\kappa+\lambda)(1-x)x - vx^2\kappa\theta)\frac{\partial P}{\partial x} -$$

$$-\rho g\frac{\partial P}{\partial \rho} - \left(a + \frac{1-x}{vx}\right)P = -F\tilde{\Theta}, \tag{43}$$

where

$$F_P = (g + (w-g)e^{w(t-T)})\rho,$$

$$F_I = c(1-e^{w(t-T)})\rho, \tag{44}$$

$$F = F_P + F_I.$$

Numerically, we use the following boundary conditions:

$$P|_{x=0} = 0, \qquad P_x|_{x=1} = 0, \qquad P|_{\rho=0} = 0.$$

There is no need for boundary conditions at $\rho = 1$, since the up-winding scheme will naturally "flow the data at the boundary out''.

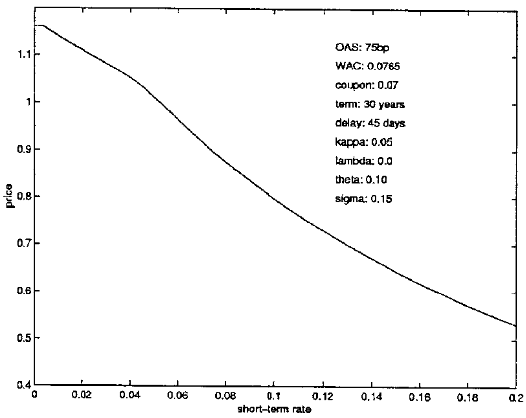

Figure 3: Current price of mortgage as function of interest rate

The following data is used in the numerical demonstration:

κ	λ	θ	σ
0.05	0.00	0.10	0.15

We compute for a 30 year term pass-through with the prepayment data shown above. The coupon for the mortgage is 7%, and the option-adjusted spread is assumed 75 basis points. We ran the program on a SPARC-10 machine, within 2 seconds, we obtained the current and all future price for the pass-through as the function of interest rate r, the pool factor f, and time t. The whole curve of price versus short rate is shown as in Figure 2. Another advantage of this method to calculate OAS, aside of the speediness, is that the effective duration and convexity arrive naturally with the solution. In this case if we assume that the short rate is 5.2%, we have the price is 99.67%. The effective duration is -4.7190 and the effective convexity is -0.5359.

It is important to note that because the program give us all price in the future, the data can be used to do the horizon analysis and the risk management. We will deal with this in our other article([LX]). This makes the PDE approach immensely attractive compared to the now commonly used Monte Carlo method, whose one point a time approach make the horizon analysis very expensive, sometimes impossible. Another advantage of the PDE approach is its reliable accuracy, which the Monte Carlo method often lacks when using too few paths in the trading off for performance.

As one may have expected, the parpayment model plays an important role in the valuation. In the frame of our computation theory, the prepayment model acquired one more important role —- form of the model will directly affects the efficiency of the method.

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