Numer. Math. (2004) 97: 555–594
Digital Object Identifier (DOI) 10.1007/s00211-003-0508-3

Numerische
Mathematik

Analysis of a fourth order finite difference method for the incompressible Boussinesq equations

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Received December 5, 2002 / Revised version received September 5, 2003 /
Published online February 25, 2004 –

Summary. The convergence of a fourth order finite difference method for the 2-D unsteady, viscous incompressible Boussinesq equations, based on the vorticity-stream function formulation, is established in this article. A compact fourth order scheme is used to discretize the momentum equation, and long-stencil fourth order operators are applied to discretize the temperature transport equation. A local vorticity boundary condition is used to enforce the no-slip boundary condition for the velocity. One-sided extrapolation is used near the boundary, dependent on the type of boundary condition for the temperature, to prescribe the temperature at "ghost" points lying outside of the computational domain. Theoretical results of the stability and accuracy of the method are also provided. In numerical experiments the method has been shown to be capable of producing highly resolved solutions at a reasonable computational cost.

Mathematics Subject Classification (1991): 35Q35, 65M06, 76M20

1 Introduction

The 2-D incompressible Navier-Stokes equations under the Boussinesq assumption, in the vorticity-stream function formulation, can be written as

⋆ Supported by NSF grant DMS-0107218
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$$
\begin{cases}
\partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = \nu \Delta \omega + g \partial_x \theta \,, \\
\partial_t \theta + (\boldsymbol{u} \cdot \nabla)\theta = \kappa \Delta \theta \,, \\
\Delta \psi = \omega \,, \\
u = -\partial_y \psi \,, \qquad v = \partial_x \psi \,,
\end{cases}
\tag{1.1}
$$

where ω is the vorticity, ψ the stream function, $\boldsymbol{u} = (u, v)^T$ the velocity field, and θ the temperature. The parameter ν represents the kinematic viscosity, κ the heat conductivity, and g the product of the gravity constant with the thermal expansion coefficient. We consider (1.1) on a domain Ω whose boundary is denoted by Γ.

We assume that the computational domain is simply connected and note that the usual no-flow, no-slip boundary conditions for the velocity field, $\boldsymbol{u} \mid_\Gamma = 0$, can be written in terms of the stream function ψ as

$$
\psi \mid_\Gamma = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial \mathbf{n}} \Big|_\Gamma = 0 \,. \tag{1.2}
$$

For the temperature θ, either a Dirichlet boundary condition

$$
\theta \mid_\Gamma = \theta_b \,, \tag{1.3}
$$

where θ_b is a given distribution for the temperature on the boundary, or a Neumann boundary condition

$$
\frac{\partial \theta}{\partial \boldsymbol{n}} \Big|_\Gamma = \theta_f \,, \tag{1.4}
$$

where θ_f is a given heat flux on the boundary, can be imposed. The latter would apply when an insulated (adiabatic) boundary condition is imposed, in which case $\theta_f = 0$.

This paper presents analysis of a fourth order computational method for the Boussinesq equations (1.1) that was recently proposed by the authors in [16]. A description of the overall scheme is given in section 2, which we briefly outline here. A fourth order compact discretization is used for the momentum equation in (1.1). The no-slip boundary condition $\frac{\partial \psi}{\partial \boldsymbol{n}} \Big|_\Gamma = 0$ is converted into a local vorticity boundary condition, such as Briley's fourth order formula or the new fourth order formula discussed in [16]. The no-flow boundary condition $\psi \mid_\Gamma = 0$ is reserved as a Dirichlet boundary condition in the Poisson equation for ψ. We emphasize that a compact approach is crucial here for it avoids the need of prescribing values of the vorticity at computational points outside of the flow domain ("ghost" points). Generally, such values would be computed using extrapolation, which for the vorticity can be troublesome due to the presence of sharp gradients in this variable at the boundary. This is especially true in the case of large Reynolds number flow. In contrast, a compact approach is not indicated for the temperature transport equation. Indeed, the temperature is generally well behaved near the boundary and the prescribed boundary condition, (1.3) or (1.4), allows for the discretization of the temperature equation to fourth order using long-stencil approximations. Moreover, this avoids the additional computational cost of solving a Poisson-like equation involving an auxiliary temperature variable that would be required by a compact approach. However, we now must prescribe temperature data at "ghost" points outside of the computational domain, which are derived using one-sided extrapolation. Additionally, the number of interior points in these formulas is reduced by applying information obtained from the temperature equation at the boundary. Similar ideas can be found in [10].

Detailed numerical experiments have been performed to show that this approach is indeed very accurate and efficient. Benchmark quality simulations of a differentially-heated cavity problem using this method is presented in [13, 16]. This flow was the focus of a special session at the first MIT conference on Computational Fluid and Solid Mechanics in June 2001 [1]. A detailed description of the problem setup, as well as a summary of the overall results can be found in [6]. Submissions to the session included simulations computed using finite difference, finite element, finite volume, and spectral methods. The reference benchmark simulation was computed using a spectral code, which was used to rank the submissions to the special session. In all there were six composite metrics on which submissions were judged. The simulation computed by our method received *three* first place rankings and *one* second place ranking. In particular, with respect to numerical accuracy and efficiency our method performed extremely well. See [6, 13] for a detailed description.

As noted above, the purpose of this paper is to provide a theoretical analysis for the numerical method presented in [16]. As is generally the case when high order discretizations are used in conjunction with high order one-sided extrapolation, stability of the resulting scheme becomes a crucial issue. In what follows, we demonstrate the stability and full accuracy of the method. To facilitate the description, we choose the computational domain as $\Omega = [0, 1]\times[0, 1]$ with grid size $\Delta x = \Delta y = h = \frac{1}{N}$. The following two theorems are the main results:

Theorem 1.1 *Let $\boldsymbol{u}_e \in L^\infty([0, T]; C^{7,\alpha}(\overline{\Omega}))$, $\theta_e \in L^\infty([0, T]; C^6(\overline{\Omega}))$ be the exact solution of the Boussinesq equations (1.1)–(1.2) with the Dirichlet boundary condition (1.3), and $\boldsymbol{u}_h$, θ_h the approximate solution of the fourth order numerical method, namely (2.7), (2.16), and (2.20) below. Then*

$$\|\boldsymbol{u}_e - \boldsymbol{u}_h\|_{L^\infty([0,T],L^2)} + \|\theta_e - \theta_h\|_{L^\infty([0,T],L^2)} \le C(\boldsymbol{u}_e, \theta_e)h^4 \, , \tag{1.6a}$$

where the constant is determined from the exact solution $\boldsymbol{u}_e$, θ_e by

$$C(\boldsymbol{u}_e,\theta_e) = C\Big(\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5}) + \|\theta_e\|_{C^5}\|\boldsymbol{u}_e\|_{C^5} + \|\theta_e\|_{C^6}\Big) \cdot \exp\left\{\frac{CT}{\nu}(1+\|\boldsymbol{u}_e\|_{C^1})^2 + \frac{CT}{\kappa}(1+\|\boldsymbol{u}_e\|_{C^0})^2\right\}. \tag{1.6b}$$

Theorem 1.2 *Let $\boldsymbol{u}_e \in L^\infty([0,T]; C^{7,\alpha}(\overline{\Omega}))$, $\theta_e \in L^\infty([0,T]; C^{8,\alpha}(\overline{\Omega}))$ be the exact solution of the Boussinesq equations (1.1)–(1.2) with the Neumann boundary condition (1.4), and $\boldsymbol{u}_h$, θ_h the approximate solution of the fourth order numerical method, namely (2.7), (2.16), and (2.26) below. Then*

$$\|\boldsymbol{u}_e - \boldsymbol{u}_h\|_{L^\infty([0,T],L^2)} + \|\theta_e - \theta_h\|_{L^\infty([0,T],L^2)} \le C(\boldsymbol{u}_e,\theta_e)h^4, \tag{1.7a}$$

where the constant is determined from the exact solution $\boldsymbol{u}_e$, θ_e by

$$C(\boldsymbol{u}_e,\theta_e) = C\Big(\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5}) + \|\theta_e\|_{C^{7,\alpha}}\|\boldsymbol{u}_e\|_{C^{6,\alpha}} + \|\theta_e\|_{C^{8,\alpha}}\Big) \cdot \exp\left\{\frac{CT}{\nu}(1+\|\boldsymbol{u}_e\|_{C^1})^2 + \frac{CT}{\kappa}(1+\|\boldsymbol{u}_e\|_{C^0})^2\right\}. \tag{1.7b}$$

Remark 1.3 To simplify the analysis of a numerical method, one usually considers the semi-discrete scheme, with spatial discretization and continuous derivative in time. This is the so called “method of lines” approach, as it is composed of a system of ODEs. If the spatially discrete scheme is proven to be convergent, the full accuracy for the fully discrete scheme can be established as long as the temporal discretization is consistent and stable. For the numerical scheme proposed in this article, we choose a high order Runge-Kutta method, an explicit multi-stage method, to update the dynamic equations in time. Full order convergence analysis is valid for either the forward Euler method or the classical RK4 method. Since the proof of this standard approach is long due to many technical considerations, we choose to omit it.

We note that the constants C appearing above depend on ν and κ. The details of the discrete L^2 norms for different variables will be provided in section 3. For simplicity, we use $\|\cdot\|_{C^{m,\alpha}}$ to denote the $L^\infty([0,T]; C^{m,\alpha})$ norm. It should be noted that the exact solution does not generally satisfy the regularity assumption of the above theorems in a square domain, which is a shortcoming of all convergence proofs for finite difference methods. Nevertheless, in many cases, such as periodic channel flow or Taylor-Couette flow in an annular domain, the solution does possess the required regularity. We note that in the finite difference setting, the regularity assumption in Theorem 1.1 and Theorem 1.2 is almost optimal.

In section 3 we first illustrate the techniques used in proving the theorems above by analyzing the stability of the long stencil operators and one-sided approximations of the temperature near the boundary using a simple one-dimensional heat equation model. The convergence proof of the fourth order method with the Dirichlet or Neumann boundary condition for the temperature is then established in sections 4 and 5, respectively. In both cases, approximate solutions for the velocity, vorticity, and temperature are constructed and shown to be consistent up to $O(h^4)$ with solutions of the finite difference scheme. Fourth order convergence then results from an estimate for the error between the approximate solution and the numerical solution. A crucial point in the stability analysis for the error functions is that both the compact and the long-stencil operators have negative eigenvalues, henceforth are well-posed. In addition, careful treatment of the boundary terms is required to recover an energy estimate. Here, discrete elliptic regularity is applied to control the boundary terms of the vorticity equation, while a cancellation analysis is used to deal with the boundary terms of the temperature equation.

2 Description of the scheme

In this section we describe in detail the fourth order finite difference method for (1.1) proposed by the authors in [16]. First, a fourth order compact approach for the momentum equation is outlined in section 2.1. Then in section 2.2 the temperature transport equation is approximated by long-stencil operators, along with one-sided extrapolation to obtain "ghost" point values for the temperature outside of the computational domain.

In this article, $\widetilde{D}_x$, $\widetilde{D}_y$, D_x^2 and D_y^2 are the standard centered difference operators for ∂_x, ∂_y, ∂_x^2 and ∂_y^2, respectively. Similar definitions can be applied to $\widetilde{D}_y$ and D_y^2.

2.1 Momentum equation

The momentum equation is solved by the Essentially Compact Fourth order scheme (EC4) proposed by E & Liu in [8] for the two-dimensional Navier-Stokes equations. The starting point of the scheme is a compact fourth order approximation of the Laplacian Δ given by

$$\Delta = \frac{\Delta_h + \frac{h^2}{6} D_x^2 D_y^2}{1 + \frac{h^2}{12}\Delta_h} + O(h^4)\,, \tag{2.1}$$

where $\Delta_h = D_x^2 + D_y^2$. Substituting the difference operator in (2.1) for the Laplacian in the momentum equation and then multiplying the result by the denominator of the difference operator (2.1) gives the $O(h^4)$ approximation

$$\begin{aligned} &\Big(1 + \frac{h^2}{12}\Delta_h\Big)\partial_t\omega + \Big(1 + \frac{h^2}{12}\Delta_h\Big)\nabla\cdot(\boldsymbol{u}\omega) - g\Big(1 + \frac{h^2}{12}\Delta_h\Big)\partial_x\theta \\ &\qquad = \Big(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2\Big)\omega\,. \end{aligned} \tag{2.2}$$

The same procedure applied to the kinematic equation results in the $O(h^4)$ approximation

$$\Big(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2\Big)\psi = \Big(1 + \frac{h^2}{12}\Delta_h\Big)\omega\,. \tag{2.3}$$

As in [8], the nonlinear convection term in the momentum equation is fully discretized as

$$\begin{aligned} \Big(1 + \frac{h^2}{12}\Delta_h\Big)\nabla\cdot(\boldsymbol{u}\omega) &= \widetilde{D}_x\Big(1 + \frac{h^2}{6}D_y^2\Big)(u\omega) + \widetilde{D}_y\Big(1 + \frac{h^2}{6}D_x^2\Big)(v\omega) \\ &\quad - \frac{h^2}{12}\Delta_h\Big(u\widetilde{D}_x\omega + v\widetilde{D}_y\omega\Big) + O(h^4)\,. \end{aligned} \tag{2.4}$$

The first and the second terms in (2.4) are compact. The third term is not, yet it does not cause any problem in actual computations since $u^n\widetilde{D}_x\omega^n + v^n\widetilde{D}_y\omega^n$ can be taken as 0 on the boundary. The gravity term $(1 + \frac{h^2}{12}\Delta_h)\partial_x\theta$ is dealt with similarly. A formal Taylor expansion gives

$$\begin{aligned} \Big(1 + \frac{h^2}{12}\Delta_h\Big)\partial_x &= \widetilde{D}_x\Big(1 + \frac{h^2}{12}D_y^2 - \frac{h^2}{12}D_x^2\Big) + O(h^4) \\ &= \widetilde{D}_x + \frac{h^2}{12}\widetilde{D}_x D_y^2 - \frac{h^2}{12}\widetilde{D}_x D_x^2 + O(h^4)\,. \end{aligned} \tag{2.5}$$

Note that at a horizontal computational boundary the third term on the right-hand side of (2.5) requires values of θ at "ghost" points lying outside of the computational domain. The prescription of these will be discussed below. Finally, by the introduction of an intermediate variable $\overline{\omega}$

$$\overline{\omega} = \Big(1 + \frac{h^2}{12}\Delta_h\Big)\omega\,, \tag{2.6}$$

the momentum equation is approximated to $O(h^4)$ by

$$\begin{aligned} &\partial_t\overline{\omega} + \widetilde{D}_x\Big(1 + \frac{h^2}{6}D_y^2\Big)(u\omega) + \widetilde{D}_y\Big(1 + \frac{h^2}{6}D_x^2\Big)(v\omega) \\ &\qquad - \frac{h^2}{12}\Delta_h\Big(u\widetilde{D}_x\omega + v\widetilde{D}_y\omega\Big) - g\widetilde{D}_x\Big(1 + \frac{h^2}{12}(D_y^2 - D_x^2)\Big)\theta \\ &= \nu\Big(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2\Big)\omega\,. \end{aligned} \tag{2.7}$$

The stream function is solved using (2.3) (the right-hand side of which is $\overline{\omega}$) with the Dirichlet boundary condition $\psi\mid_\Gamma = 0$. The velocity $\boldsymbol{u} = \nabla^\top \psi = (-\partial_y \psi, \partial_x \psi)$ is then obtained by long-stencil approximations to ∂_x and ∂_y, namely

$$u = -\widetilde{D}_y\Big(1 - \frac{h^2}{6}D_y^2\Big)\psi\,, \quad v = \widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\psi\,. \tag{2.8}$$

Note that (2.8) requires values of ψ at "ghost" points. This is discussed below, along with the boundary condition for the vorticity, which when given $\overline{\omega}$ is required in order to determine ω from (2.6).

We now turn to the fourth order boundary condition for the vorticity, focusing our discussion on the boundary Γ_x where $j = 0$. The main point in deriving a boundary condition for the vorticity is to convert the boundary condition $\frac{\partial\psi}{\partial \boldsymbol{n}} = 0$ into a boundary condition for ω using the kinematic relation $\Delta\psi = \omega$. One possibility is Briley's formula

$$\omega_{i,0} = \frac{1}{18h^2}(108\psi_{i,1} - 27\psi_{i,2} + 4\psi_{i,3})\,, \tag{2.9}$$

which results from a centered fourth order discretization of $\Delta\psi = \omega$ at the boundary along with the one-sided Taylor expansions of the stream function

$$\psi_{i,-1} = 6\psi_{i,1} - 2\psi_{i,2} + \frac{1}{3}\psi_{i,3} - 4h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O(h^5)\,, \tag{2.10}$$

and

$$\psi_{i,-2} = 40\psi_{i,1} - 15\psi_{i,2} + \frac{8}{3}\psi_{i,3} - 12h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O(h^5)\,. \tag{2.11}$$

Alternatively, we can use a new fourth order formula for the vorticity,

$$\omega_{i,0} = \frac{1}{h^2}\Big(8\psi_{i,1} - 3\psi_{i,2} + \frac{8}{9}\psi_{i,3} - \frac{1}{8}\psi_{i,4}\Big) + O(h^4)\,, \tag{2.12}$$

which is derived in the same manner as (2.9), but instead of (2.10)–(2.11) we now estimate the stream function at the "ghost" points using

$$\psi_{i,-1} = 10\psi_{i,1} - 5\psi_{i,2} + \frac{5}{3}\psi_{i,3} - \frac{1}{4}\psi_{i,4} - 5h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O(h^6)\,, \tag{2.13}$$

and

$$\psi_{i,-2} = 80\psi_{i,1} - 45\psi_{i,2} + 16\psi_{i,3} - \frac{5}{2}\psi_{i,4} - 30h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O\left(h^6\right)\,. \tag{2.14}$$

The latter boundary formula (2.12) gives fourth order accuracy for the vorticity on the boundary, while the Briley's formula indicates a third order accuracy, by formal Taylor expansion. Yet, the numerical evidence shows that both (2.9) and (2.12) result in full fourth order accuracy for the two-dimensional Navier-Stokes equations, with compact difference operators applied at the interior points. See a relevant discussion in [16]. For computational convenience, we suggest using Briley's formula along with (2.10)–(2.11). However, for conciseness of the analysis of the Boussinesq equations in the present article we use (2.12). We note that the philosophy of local vorticity boundary conditions can been extended, in particular, to derive local pressure boundary conditions for the velocity-pressure formulation of the Navier-Stokes equations. Moreover, unlike the vorticity-stream function formulation, the local pressure boundary condition approach is easily extended to three-dimensional flows; see [14].

2.2 Temperature transport equation

To solve the temperature transport equation $\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta = \kappa\Delta\theta$, we discretize ∂_x, ∂_y, and Δ using standard fourth order long-stencil operators:

(2.15)
$$\partial_x = \widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big) + O(h^4)\,, \qquad \partial_y = \widetilde{D}_y\Big(1 - \frac{h^2}{6}D_y^2\Big) + O(h^4)\,,$$
$$\Delta = \Delta_h - \frac{h^2}{12}\Big(D_x^4 + D_y^4\Big) + O(h^4)\,.$$

Thus an $O(h^4)$ approximation for the temperature equation is given by

(2.16)
$$\partial_t\theta + u\widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\theta + v\widetilde{D}_y\Big(1 - \frac{h^2}{6}D_y^2\Big)\theta = \kappa\Big(\Delta_h - \frac{h^2}{12}(D_x^4 + D_y^4)\Big)\theta.$$

Because of the use of long-stencil operators in (2.16) we must prescribe θ at "ghost" points lying outside of the computational domain. We discuss this issue next for the two boundary conditions considered herein, namely Dirichlet and Neumann.

2.2.1 Dirichlet boundary condition for temperature. In the case of a Dirichlet boundary condition θ is given on the boundary by θ_b (see (1.3)), hence we only need to update (2.16) at the interior grid points (x_i, y_j), $1 \le i, j \le N-1$. Thus, only one "ghost" point value must be prescribed, e.g. $\theta_{i,-1}$ along the boundary Γ_x. Local Taylor expansion at the boundary gives

$$\theta_{i,-1} = \frac{20}{11}\theta_{i,0} - \frac{6}{11}\theta_{i,1} - \frac{4}{11}\theta_{i,2} + \frac{1}{11}\theta_{i,3} + \frac{12}{11}h^2\partial_y^2\theta_{i,0} + O(h^5). \tag{2.17}$$

Using standard finite difference stencils, approximation of $h^2\partial_y^2\theta_{i,0}$ to high order would necessarily increase the size of the stencil in (2.17). Alternatively, we will use the PDE and its derivatives (see the detailed discussion in [16]). Since the velocity $\boldsymbol{u}$ vanishes on the boundary, the temperature transport equation along Γ_x reads

$$\partial_t\theta\,|_{\Gamma_x} = \kappa\Delta\theta\,|_{\Gamma_x} = \kappa(\partial_x^2+\partial_y^2)\theta\,|_{\Gamma_x} = \kappa(\partial_x^2\theta_b + \partial_y^2\theta\,|_{\Gamma_x})\,. \tag{2.18}$$

The above evaluation leads to

$$\partial_y^2\theta\,|_{\Gamma_x} = \frac{1}{\kappa}\partial_t\theta_b - \partial_x^2\theta_b\,, \tag{2.19}$$

where the right hand side is a known function since θ is given by θ_b on the boundary. The combination of (2.19) and (2.17) gives

(2.20)

$$\theta_{i,-1} = \frac{20}{11}\theta_{i,0} - \frac{6}{11}\theta_{i,1} - \frac{4}{11}\theta_{i,2} + \frac{1}{11}\theta_{i,3} + \frac{12}{11}h^2\Big(\frac{1}{\kappa}\partial_t\theta_b - \partial_x^2\theta_b\Big) + O(h^5).$$

Similar arguments follow along the other three boundaries of Ω. It will be shown in later sections that this formula gives full 4-th order accuracy.

Alternatively, a fourth order Taylor expansion near the boundary results in only one interior point in the formula for $\theta_{i,-1}$, namely

$$\theta_{i,-1} = 2\theta_{i,0} - \theta_{i,1} + h^2\partial_y^2\theta_{i,0} + O(h^4)\,, \tag{2.21}$$

which along with (2.19) gives

$$\theta_{i,-1} = 2\theta_{i,0} - \theta_{i,1} + h^2\Big(\frac{1}{\kappa}\partial_t\theta_b - \partial_x^2\theta_b\Big) + O(h^4)\,. \tag{2.22}$$

This is a $O(h^4)$ formula analogous to (2.20). Our numerical experiments indicate that both (2.20) and (2.22) are stable and full accuracy is achieved. Since (2.22) only requires one interior point, we suggest its use in actual computations.

2.2.2 Neumann boundary condition for temperature. For the Neumann boundary condition (1.4) the temperature on the boundary is not known explicitly, only its normal derivative. Thus, (2.16) is applied at every computational point (x_i, y_j), $0 \le i, j \le N$ requiring us to determine two "ghost" point values, e.g. $\theta_{i,-1}$ and $\theta_{i,-2}$ along Γ_x. As in the Dirichlet case above we begin by deriving one-sided approximations. Local Taylor expansion near the boundary gives

$$\begin{aligned}\theta_{i,-1} &= \theta_{i,1} - 2h\partial_y\theta_{i,0} - \frac{h^3}{3}\partial_y^3\theta_{i,0} + O(h^5)\,,\\ \theta_{i,-2} &= \theta_{i,2} - 4h\partial_y\theta_{i,0} - \frac{8h^3}{3}\partial_y^3\theta_{i,0} + O(h^5)\,.\end{aligned} \tag{2.23}$$

The term $\partial_y\theta_{i,0}$ in (2.23) is known from the flux boundary condition (1.4). It remains to determine $\partial_y^3\theta_{i,0}$, for which we again use information from the PDE and its derivatives. Applying ∂_y to the temperature equation along Γ_x gives

$$\theta_{yt} + u_y\theta_x + u\theta_{xy} + v_y\theta_y + v\theta_{yy} = \kappa(\theta_{yxx} + \partial_y^3\theta)\,. \tag{2.24}$$

Since θ_y is given along Γ_x, the first term on the left-hand side as well as the first term on the right-hand side of (2.24) are known functions, θ_{ft} and θ_{fxx}, respectively. The third and fifth terms on the left-hand side are zero since $\boldsymbol{u}\mid_\Gamma = 0$. The fourth term on the left-hand side is also zero due to the no-slip boundary condition and incompressibility, i.e. $v_y = -u_x = 0$ on Γ_x. It remains to evaluate the second term on the left-hand side. Since $v_x = 0$ along Γ_x it follows that $u_y = -(v_x - u_y) = -\omega$ along Γ_x. Moreover, since in the Neumann case (2.16) is updated at all grid points including the boundary points, θ_x on Γ_x can be calculated by the standard fourth order long-stencil formula (2.15). Combining these arguments, $\partial_y^3\theta$ is approximated along Γ_x by

$$\partial_y^3\theta_{i,0} = \frac{1}{\kappa}\Big(\theta_{ft} - \omega_{i,0}\widetilde{D}_x(1 - \frac{h^2}{6}D_x^2)\theta_{i,0}\Big) - \theta_{fxx}\,. \tag{2.25}$$

Substitution of (2.25) in (2.23) gives

$$\begin{aligned}
\theta_{i,-1} &= \theta_{i,1} - 2h\theta_f - \frac{h^3}{3}\Big(\frac{1}{\kappa}\theta_{ft} - \frac{1}{\kappa}\omega_{i,0}\widetilde{D}_x(1 - \frac{h^2}{6}D_x^2)\theta_{i,0} - \theta_{fxx}\Big)\,,\\
\theta_{i,-2} &= \theta_{i,2} - 4h\theta_f - \frac{8h^3}{3}\Big(\frac{1}{\kappa}\theta_{ft} - \frac{1}{\kappa}\omega_{i,0}\widetilde{D}_x(1 - \frac{h^2}{6}D_x^2)\theta_{i,0} - \theta_{fxx}\Big)\,.
\end{aligned} \tag{2.26}$$

We note that in the no-flux (or fixed-flux) case we have $\theta_{ft} = \theta_{fxx} = 0$, and (2.26) reduces to

$$\begin{aligned}
\theta_{i,-1} &= \theta_{i,1} + \frac{h^3}{3}\frac{\omega_{i,0}}{\kappa}\widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\theta_{i,0}\,,\\
\theta_{i,-2} &= \theta_{i,2} + \frac{8h^3}{3}\frac{\omega_{i,0}}{\kappa}\widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\theta_{i,0}\,.
\end{aligned} \tag{2.27}$$

Analogous formulas follow for the remaining three boundaries.

3 Stability of long-stencil operators and one-sided approximation

In this section we study a simple model, the one-dimensional heat equation, to explain why long-stencil operators coupled with one-sided approximation are stable. The approach used here will be applied to the convergence proof of the full nonlinear two-dimensional equations in sections 4 and 5. The one-dimensional heat equation is given by

$$\partial_t \theta = \kappa \partial_x^2 \theta \,. \tag{3.1}$$

Applying the fourth order spatial approximation (2.15) to (3.1) gives

$$\partial_t \theta = \kappa \Big(D_x^2 - \frac{h^2}{12} D_x^4 \Big) \theta \,. \tag{3.2}$$

Note that both the second and fourth order difference operators that appear on the right-hand side of (3.2) are well-posed. It is this very important fact that allows us to prove stability.

3.1 Dirichlet boundary condition for θ

For conciseness of presentation, we take $\theta_b = 0$ in (1.3). In this case, we have $\theta_0 = \theta_N = 0$ and the one-sided approximation for θ_{-1} analogous to (2.20) can be written as

$$\theta_{-1} = \frac{20}{11}\theta_0 - \frac{6}{11}\theta_1 - \frac{4}{11}\theta_2 + \frac{1}{11}\theta_3 + O(h^5) \,. \tag{3.3}$$

We use the discrete L^2-norm and the discrete L^2-inner product defined by

$$\|u\|_1 = \langle u \,,\, u \rangle_1^{1/2} \,, \qquad \langle u \,,\, v \rangle_1 = h \sum_{1 \le i \le N-1} u_i \, v_i \,, \tag{3.4}$$

and introduce $\|\nabla_h u\|_2$ defined by

$$\|\nabla_h u\|_2^2 = \sum_{0 \le i \le N-1} (D_x^+ u_i)^2 \; h \,, \quad \text{where} \quad D_x^+ u_i = \frac{u_{i+1} - u_i}{h} \,. \tag{3.5}$$

We note that a two-dimensional version of the corresponding inner product and L^2 norm can be defined in a straightforward way.

Multiplying (3.2) by 2θ at interior grid points $1 \le i \le N-1$, and applying standard energy estimates gives

(3.6)
$$\partial_t \|\theta\|_1^2 + 2\kappa \|\nabla_h \theta\|_2^2 + \frac{\kappa h^2}{6} \Big(\|D_x^2 \theta\|_1^2 + \frac{1}{h} (\theta_1 D_x^2 \theta_0 + \theta_{N-1} D_x^2 \theta_N) \Big) = 0 \,.$$

An estimate of the boundary term $\theta_1 D_x^2 \theta_0$ (and $\theta_{N-1} D_x^2 \theta_N$) requires some subtlety since the term $D_x^2 \theta_0$ involves the one-sided "ghost" boundary condition (3.3), namely

$$D_x^2 \theta_0 = \frac{1}{h^2} \left(\frac{5}{11}\theta_1 - \frac{4}{11}\theta_2 + \frac{1}{11}\theta_3 \right) . \tag{3.7}$$

Since we have assumed that θ vanishes on the boundary, (3.7) can be rewritten as

$$D_x^2\theta_0 = -\frac{2}{11}D_x^2\theta_1 + \frac{1}{11}D_x^2\theta_2\,. \tag{3.8}$$

As we will see below, the purpose of the form of (3.8) is to control local terms by global terms. Application of Cauchy's inequality gives the estimate

$$\begin{aligned}\frac{1}{h}\theta_1 D_x^2\theta_0 &\geq -\frac{2^2}{4\cdot 11^2\cdot h^3}\theta_1^2 - h(D_x^2\theta_1)^2 - \frac{1^2}{4\cdot 11^2\cdot h^3}\theta_1^2 - h(D_x^2\theta_2)^2\\ &= -\frac{5}{4\cdot 11^2\cdot h^3}\theta_1^2 - h(D_x^2\theta_1)^2 - h(D_x^2\theta_2)^2\,.\end{aligned} \tag{3.9}$$

The first term on the right-hand side of (3.9) can be controlled by one of the terms in $2\kappa\|\nabla_h\theta\|_2^2$ appearing in (3.6), and the last two terms controlled by $\|D_x^2\theta\|_1^2$ appearing in (3.7). The term $\theta_{N-1}D_x^2\theta_N$ is handled in a similar fashion. Combing these estimates gives

$$\partial_t\|\theta\|_1^2 + \kappa\|\nabla_h\theta\|_2^2 \leq 0\,. \tag{3.10}$$

This proves stability of the fourth order long-stencil operator together with one-sided approximations near the boundary.

Remark 3.1 Alternatively, we can couple (3.2) with one-dimensional fourth order extrapolation corresponding to (2.22), namely

$$\theta_{-1} = 2\theta_0 - \theta_1\,. \tag{3.11}$$

Stability of (3.2) with (3.11) is more direct. Indeed, $D_x^2\theta_0$ is in fact 0 by (3.11). Therefore, (3.10) can be obtained immediately. Thus the fourth order scheme with either (3.3) or (3.11) is stable.

3.2 Neumann boundary condition for θ

We assume $\theta_f = 0$ in (1.4). In this case equation (3.2) is updated at all grid points $0 \leq i \leq N$. The corresponding one-sided approximations for θ_{-1} and θ_{-2}, analogous to (2.26), are given by

$$\theta_{-1} = \theta_1\,, \qquad \theta_{-2} = \theta_2\,, \tag{3.12}$$

since $\partial_x^3\theta(0) = \partial_x^3\theta(1) = 0$, which follows from derivations similar to (2.24)–(2.25).

Since θ does not necessarily vanish on the boundary, we introduce the following discrete L^2-norm and L^2-inner product

(3.13)
$$\|u\|_3 = \langle u\,,\,u\rangle_3^{1/2}\,, \qquad \langle u\,,\,v\rangle_3 = h\left(\frac{1}{2}u_0v_0 + \sum_{1\le i\le N-1} u_i\,v_i + \frac{1}{2}u_Nv_N\right).$$

The two-dimensional versions can be similarly defined.

An energy estimate is accomplished by taking the $\langle\,,\,\rangle_3$ inner product of the equation (3.2) with 2θ. It is straightforward to verify

$$\langle \theta\,,\,D_x^2\theta\rangle_3 = -\|\nabla_h\theta\|_2^2\,, \qquad \langle \theta\,,\,D_x^4\theta\rangle_3 = \|D_x^2\theta\|_3^2\,, \tag{3.14}$$

assuming the "ghost" point prescription (3.12). Moreover, observe that (3.14) is a discrete version of integration by parts in the case of the symmetric prescription (3.12). This is a crucial reason for the choice of symmetric extrapolation for the temperature as presented in section 2 when a Neumann boundary condition is imposed. As a result of (3.14), we have

$$\partial_t\|\theta\|_3^2 + 2\kappa\|\nabla_h\theta\|_2^2 + \frac{\kappa h^2}{6}\|D_x^2\theta\|_3^2 = 0\,, \tag{3.15}$$

which indicates stability of the fourth order long-stencil operator and one-sided approximation (3.12) near the boundary.

4 Convergence proof of Theorem 1.1

The convergence proof of the fourth order method for (1.1) proposed by the authors in [16] is composed of technical consistency analysis for the approximated solutions and the corresponding error estimate. A typical difficulty that arises in the analysis of finite difference methods is that if a direct truncation error estimate is performed, an apparent loss of accuracy near the boundary results, as can be seen by formal observation; see [11,12,21]. Instead, we construct an approximate velocity field and vorticity from the exact stream function. An approximate temperature can then be chosen as either an exact solution or the one which includes an $O(h^4)$ correction term, depending on the boundary condition for the temperature. The constructed velocity field, vorticity, and temperature are then proven to satisfy the momentum equation up to an $O(h^4)$ truncation error, including the vorticity boundary condition. Similarly, the temperature transport equation is also shown to be satisfied up to an $O(h^4)$ truncation error. This gives the consistency of our discretizations of the Boussinesq equations (1.1). The error analysis is based on energy estimates. In the error estimate of the temperature transport equation, we apply the stability analysis of the long-stencil operators and one-sided approximations near the boundary, which was outlined in section 3.

The fourth order method with Dirichlet boundary condition (1.3) for the temperature is considered in this section. The corresponding analysis with the Neumann boundary condition (1.4) is provided in section 5. For simplicity of presentation we assume $\theta_b = 0$.

4.1 Consistency analysis

Denote by ψ_e, $\boldsymbol{u}_e$, ω_e, and θ_e the exact solutions of (1.1)-(1.3), and extend ψ_e, θ_e smoothly to $[-\delta, 1+\delta]^2$, and let $\Psi_{i,j} = \psi_e(x_i, y_j)$, $\Theta_{i,j} = \theta_e(x_i, y_j)$ for $-2 \le i, j \le N+2$. Approximates for U and V are constructed via

(4.1)

$$U_{i,j} = -\widetilde{D}_y\Big(1 - \frac{h^2}{6}D_y^2\Big)\Psi\,, \;\; V_{i,j} = \widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\Psi\,, \;\text{ for } 0 \le i, j \le N\,.$$

We next construct an approximate vorticity. First define

$$\overline{\Omega}_{i,j} = \Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\Psi\,, \qquad \text{for} \quad 1 \le i, j \le N-1\,. \tag{4.2}$$

Then Ω is recovered by solving the system

$$\Big(1 + \frac{h^2}{12}\Delta_h\Big)\Omega_{i,j} = \overline{\Omega}_{i,j}\,, \tag{4.3}$$

with boundary condition (say on Γ_x, $j = 0$)

$$\Omega_{i,0} = (\omega_e)_{i,0} + h^4\widehat{\omega}_{i,0}\,, \qquad 0 \le i \le N\,, \tag{4.4}$$

where the function $\widehat{\omega}$ is defined by

$$\widehat{\omega} = \Big(-\frac{1}{240}\partial_x^4 - \frac{1}{240}\partial_y^4 + \frac{1}{90}\partial_x^2\partial_y^2\Big)\omega_e\,. \tag{4.5}$$

Note that $h^4\widehat{\omega}$ is exactly the $O(h^4)$ truncation error of $\Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\psi_e - \Big(1 + \frac{h^2}{12}\Delta_h\Big)\omega_e$. The purpose of the introduction of $h^4\widehat{\omega}$ is to maintain higher order consistency needed in the truncation error estimate for the discrete derivatives of the constructed vorticity, as we will see in the following lemma.

Lemma 4.1 *For grid points* $0 \le i, j \le N$ *we have that*

$$\Omega = \omega_e + h^4\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8}\,. \tag{4.6}$$

Proof. The construction of Ω and Ψ, and a Taylor expansion of ψ_e and ω_e shows that at each grid point (x_i, y_j), $1 \le i, j \le N-1$,

$$\begin{aligned} \Big(1+\frac{h^2}{12}\Delta_h\Big)\Omega &= \Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\psi_e \\ &= \Big(1+\frac{h^2}{12}\Delta_h\Big)\omega_e + h^4\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8}\,, \end{aligned} \tag{4.7}$$

where $\widehat{\omega}$ was introduced in (4.5). The approximation (4.7) gives

$$\Big(1+\frac{h^2}{12}\Delta_h\Big)(\Omega-\omega_e-h^4\widehat{\omega}) = -\frac{h^6}{12}\Delta_h\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8} = O(h^6)\|\psi_e\|_{C^8}, \tag{4.8}$$

since the second order differences of $\widehat{\omega}$ is bounded by $\|\psi_e\|_{C^8}$. The combination of (4.8) and (4.4), and the property that the matrix $I+\frac{h^2}{12}\Delta_h$ is uniformly diagonally dominant, results in (4.6). □

The analysis of the approximate velocities U and V is more straightforward. From the definitions of U and V, and a Taylor expansion of ψ_e, we have at grid points (x_i, y_j), $0 \le i, j \le N$,

$$\begin{aligned} U &= u_e + \frac{1}{30}h^4\partial_y^5\psi_e + O(h^5)\|\psi_e\|_{C^6}\,, \\ V &= v_e - \frac{1}{30}h^4\partial_x^5\psi_e + O(h^5)\|\psi_e\|_{C^6}\,. \end{aligned} \tag{4.9}$$

(4.6) and (4.9) provide estimates of the differences between the approximate U, V, and Ω and the exact solution. We must now carry out an analysis of the finite difference operators applied to U, V, and Ω. The results for the convection and diffusion terms of the momentum equation are stated in the following lemma, for which we only provide a brief description of the analysis.

Lemma 4.2 *For interior grid points (x_i, y_j), $1 \le i, j \le N-1$, we have that*

$$\widetilde{D}_x(1+\frac{h^2}{6}D_y^2)(U\Omega) = \Big(1+\frac{h^2}{6}\Delta\Big)\partial_x(u_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,. \tag{4.10}$$

$$\widetilde{D}_y\Big(1+\frac{h^2}{6}D_x^2\Big)(V\Omega) = \Big(1+\frac{h^2}{6}\Delta\Big)\partial_y(v_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,, \tag{4.11}$$

$$\frac{h^2}{12}\Delta_h(U\widetilde{D}_x\Omega + V\widetilde{D}_y\Omega) = \frac{h^2}{12}\Delta(u_e\partial_x\omega_e + v_e\partial_y\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,, \tag{4.12}$$

$$\left(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\right)\Omega = \left(1 + \frac{h^2}{12}\Delta\right)\Delta\omega_e + O(h^4)\|\psi_e\|_{C^8}\,. \tag{4.13}$$

Proof. The verification of the above lemma relies on the estimates (4.6) and (4.9). (4.13) is a direct consequence of (4.6) along with a Taylor expansion of ω_e. (4.10) results from the combination of (4.9) and (4.6), along with a Taylor expansion of $u_e\omega_e$. The derivation of (4.11)–(4.12) is similar. $\square$

Next we examine the time marching term. At the interior grid points (x_i, y_j), $1 \le i, j \le N-1$,

(4.15)

$$\partial_t\left(1 + \frac{h^2}{12}\Delta_h\right)\Omega = \left(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\right)\partial_t\psi_e = \left(\Delta + \frac{h^2}{12}(\partial_x^4 + \partial_y^4) + \frac{h^2}{6}\partial_x^2\partial_y^2\right)\partial_t\psi_e + O(h^4)\|\partial_t\psi_e\|_{C^6}\,.$$

The first term on the right-hand side is exactly $(1 + \frac{h^2}{12}\Delta)\partial_t\omega_e$. For an estimate of the second term consider the following Poisson equation satisfied by $\partial_t\psi_e$:

$$\begin{cases} \Delta(\partial_t\psi_e) = \partial_t\omega_e\,, & \text{in} \quad \Omega\,, \\ \partial_t\psi_e = 0\,, & \text{on} \quad \Gamma\,. \end{cases} \tag{4.16}$$

A Schauder estimate of (4.16) gives

(4.17)

$$\|\partial_t\psi_e\|_{C^{6,\alpha}} \le C\|\partial_t\omega_e\|_{C^{4,\alpha}} \le C(\|\psi_e\|_{C^{8,\alpha}} + \|\psi_e\|_{C^{7,\alpha}}\|\psi_e\|_{C^{5,\alpha}} + \|\theta_e\|_{C^{5,\alpha}})\,,$$

where C depends on ν and κ, and in the second step we have applied the original momentum equation. Therefore

$$\partial_t\left(1 + \frac{h^2}{12}\Delta_h\right)\Omega = \left(1 + \frac{h^2}{12}\Delta\right)\partial_t\omega_e + O(h^4)(\|\psi_e\|_{C^{8,\alpha}} + \|\psi_e\|_{C^{7,\alpha}}\|\psi_e\|_{C^{5,\alpha}})\,. \tag{4.18}$$

Next, a Taylor expansion of θ_e shows that for the gravity term we have

$$\widetilde{D}_x\left(1 + \frac{h^2}{12}(D_y^2 - D_x^2)\right)\Theta = (1 + \frac{h^2}{12}\Delta)\partial_x\theta_e + O(h^4)\|\theta_e\|_{C^5}\,. \tag{4.19}$$

The combination of (4.18)–(4.19) and Lemma 4.2, along with the original PDE which implies that $\Big(1+\dfrac{h^2}{12}\Delta\Big)(\partial_t\omega_e+\boldsymbol{u}_e\cdot\nabla\omega_e-g\partial_x\theta_e-\nu\Delta\omega_e)=0$, results in

$$
\begin{aligned}
\partial_t\overline{\Omega}&+\widetilde{D}_x\Big(1+\frac{h^2}{6}D_y^2\Big)(U\Omega)+\widetilde{D}_y\Big(1+\frac{h^2}{6}D_x^2\Big)(V\Omega)\\
&-\frac{h^2}{12}\Delta_h(U\widetilde{D}_x\Omega+V\widetilde{D}_y\Omega)-g\widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\Theta\\
&=\nu\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\Omega+\boldsymbol{f}\,,
\end{aligned}
\tag{4.20}
$$

where $|\boldsymbol{f}|\le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5})+Ch^4\|\theta_e\|_{C^{5,\alpha}}$.

We note that the constructed vorticity Ω satisfies the fourth order formula (2.12) up to $O(h^4)$ on the boundary. To see this, consider the following one-sided Taylor expansion of ψ_e on the boundary applied to the kinematic equation relating ω_e and ψ_e,

$$
(\omega_e)_{i,0}=\frac{1}{h^2}\Big(8\Psi_{i,1}-3\Psi_{i,2}+\frac{8}{9}\Psi_{i,3}-\frac{1}{8}\Psi_{i,4}\Big)+O(h^4)\|\psi_e\|_{C^6}\,,
\tag{4.21}
$$

which in combination with the definition of $\Omega_{i,0}$ in (4.4) and the fact that $|\widehat{\omega}_{i,0}|\le C\|\psi_e\|_{C^6}$, show that the vorticity boundary condition is satisfied up to $O(h^4)$. In particular,

$$
\Omega_{i,0}=\frac{1}{h^2}\Big(8\Psi_{i,1}-3\Psi_{i,2}+\frac{8}{9}\Psi_{i,3}-\frac{1}{8}\Psi_{i,4}\Big)+O(h^4)\|\psi_e\|_{C^6}\,.
\tag{4.22}
$$

The truncation error analysis for the temperature equation is more direct. A local Taylor expansion of θ_e gives

$$
\begin{aligned}
\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta&=\partial_x\theta_e+O(h^4)\|\theta_e\|_{C^5}\,,\\
\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta&=\partial_y\theta_e+O(h^4)\|\theta_e\|_{C^5}\,,\\
\Big(\Delta_h-\frac{h^2}{12}(D_x^4+D_y^4)\Big)\Theta&=\Delta\theta_e+O(h^4)\|\theta_e\|_{C^6}\,.
\end{aligned}
\tag{4.23}
$$

It can be seen from (4.9) and (4.23) that

(4.24)
$$
U\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta=u_e\partial_x\theta_e+O(h^4)(\|\boldsymbol{u}_e\|_{C^0}\|\theta_e\|_{C^5}+\|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^1})\,,
$$

and a similar result follows for $V\widetilde{D}_y\Big(1-\dfrac{h^2}{6}D_y^2\Big)\Theta$. An estimate for the convection term in the temperature equation is then given by

$$U\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta + V\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta = \boldsymbol{u}_e\cdot\nabla\theta_e + O(h^4)(\|\boldsymbol{u}_e\|_{C^0}\|\theta_e\|_{C^5}+\|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^1})\,. \tag{4.25}$$

Finally, from (4.23) and (4.25), along with the original temperature equation $\partial_t\theta_e + \boldsymbol{u}_e\cdot\nabla\theta_e = \kappa\Delta\theta_e$, we have that

$$\begin{aligned}\partial_t\Theta + U\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta + V\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta \\ = \kappa\Big(\Delta_h - \frac{h^2}{12}(D_x^4+D_y^4)\Big)\Theta + \boldsymbol{g}\,,\end{aligned} \tag{4.26}$$

where $|\boldsymbol{g}| \le Ch^4(\|\theta_e\|_{C^5}\|\boldsymbol{u}_e\|_{C^0} + \|\theta_e\|_{C^1}\|\boldsymbol{u}_e\|_{C^5} + \|\theta_e\|_{C^6})$.

In addition, it should be mentioned that

$$\begin{aligned}\Theta_{i,-1} = \frac{20}{11}\Theta_{i,0} - \frac{6}{11}\Theta_{i,1} - \frac{4}{11}\Theta_{i,2} + \frac{1}{11}\Theta_{i,3} \\ +\frac{12}{11}h^2\Big(\frac{1}{\kappa}\partial_t\theta_b - \partial_x^2\theta_b\Big) + \boldsymbol{e}_{i,0}\,,\end{aligned} \tag{4.27}$$

where $|\boldsymbol{e}_{i,0}| \le Ch^5\|\theta_e\|_{C^5}$, as discussed in section 2. The approximation (4.27) will be used in the estimates of error functions in the next subsection. This completes the consistency analysis.

4.2 Proof of Theorem 1.1

We now prove Theorem 1.1, and begin by defining the following error functions at all grid points (x_i, y_j), $0 \le i, j \le N$,

$$\begin{aligned}&\widetilde{\psi} = \psi - \Psi\,, \quad \widetilde{\omega} = \omega - \Omega\,, \quad \widetilde{u} = u - U\,, \\ &\widetilde{v} = v - V\,, \quad \widetilde{\theta} = \theta - \Theta\,, \quad \widetilde{\overline{\omega}} = \overline{\omega} - \overline{\Omega}\,.\end{aligned} \tag{4.28}$$

Subtracting (4.20) and (4.26) from the numerical scheme (2.7), (2.16), and (2.8) we have

(4.29)

$$\begin{cases}\partial_t\widetilde{\theta} + \mathcal{L}_1 = \kappa\Big(\Delta_h - \frac{h^2}{12}(D_x^4+D_y^4)\Big)\widetilde{\theta} - \boldsymbol{g}\,, \qquad \widetilde{\theta}\,|_\Gamma = 0\,, \\ \Big(1+\frac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega} + \mathcal{L}_2 - g\widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\widetilde{\theta} = \nu\Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega} - \boldsymbol{f}\,, \\ \Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\psi} = \Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\omega}\,, \qquad \widetilde{\psi}\,|_\Gamma = 0\,, \\ \widetilde{u} = -\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\widetilde{\psi}\,, \quad \widetilde{v} = \widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}\,,\end{cases}$$

where the linearized convection error terms $\mathcal{L}_1$ and $\mathcal{L}_2$ appearing in the temperature and vorticity equation, respectively, can be represented as

$$
\begin{aligned}
\mathcal{L}_1 &= \widetilde{u}\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta + u\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\widetilde{\theta} \\
&\quad +\widetilde{v}\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta + v\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\widetilde{\theta}\,, \\
\mathcal{L}_2 &= \widetilde{D}_x\Big(1+\frac{h^2}{6}D_y^2\Big)(\widetilde{u}\Omega + u\widetilde{\omega}) + \widetilde{D}_y\Big(1+\frac{h^2}{6}D_x^2\Big)(\widetilde{v}\Omega + v\widetilde{\omega}) \\
&\quad -\frac{h^2}{12}\Delta_h(u\widetilde{D}_x\widetilde{\omega} + v\widetilde{D}_y\widetilde{\omega} + \widetilde{u}\widetilde{D}_x\Omega + \widetilde{v}\widetilde{D}_y\Omega)\,.
\end{aligned}
\tag{4.30}
$$

The local truncation error terms satisfy $|\boldsymbol{g}| \le Ch^4(\|\theta_e\|_{C^6} + \|\theta_e\|_{C^5}\|\boldsymbol{u}_e\|_{C^0} + \|\theta_e\|_{C^1}\|\boldsymbol{u}_e\|_{C^5})$ and $|\boldsymbol{f}| \le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5}) + Ch^4\|\theta_e\|_{C^5}$. Along the boundary (say on Γ_x, $j=0$) we have

$$
\begin{aligned}
\widetilde{\psi}_{i,-1} &= 10\widetilde{\psi}_{i,1} - 5\widetilde{\psi}_{i,2} + \frac{5}{3}\widetilde{\psi}_{i,3} - \frac{1}{4}\widetilde{\psi}_{i,4}\,, \\
\widetilde{\omega}_{i,0} &= \frac{1}{h^2}\Big(8\widetilde{\psi}_{i,1} - 3\widetilde{\psi}_{i,2} + \frac{8}{9}\widetilde{\psi}_{i,3} - \frac{1}{8}\widetilde{\psi}_{i,4}\Big) + \boldsymbol{h}_{i,0}\,, \\
\widetilde{\theta}_{i,-1} &= -\frac{6}{11}\widetilde{\theta}_{i,1} - \frac{4}{11}\widetilde{\theta}_{i,2} + \frac{1}{11}\widetilde{\theta}_{i,3} + \boldsymbol{e}_{i,0}\,,
\end{aligned}
\tag{4.31}
$$

where $|\boldsymbol{h}_{i,0}| \le Ch^4\|\boldsymbol{u}_e\|_{C^5}$ and $|\boldsymbol{e}_{i,0}| \le Ch^5\|\theta_e\|_{C^5}$.

We now derive estimates of the error functions for the closed system (4.29) along with the boundary conditions (4.31). Multiplying the vorticity error equation by $-(1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}$, and the temperature error equation by $\widetilde{\theta}$ at the interior grid points $1 \le i,\, j \le N-1$, we have

$$
\begin{aligned}
&-\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\Big(1+\frac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega}\Big\rangle_1 \\
&\quad +\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega}\Big\rangle_1 \\
&= \Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\mathcal{L}_2\Big\rangle_1 - g\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\ \widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\widetilde{\theta}\Big\rangle_1 \\
&\quad +\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\boldsymbol{f}\Big\rangle_1\,,
\end{aligned}
\tag{4.32}
$$

and

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_1^2 + \langle\widetilde{\theta}, \mathcal{L}_1\rangle_1 = \kappa\Big\langle\widetilde{\theta}, \Big(\Delta_h - \frac{h^2}{12}(D_x^4 + D_y^4)\Big)\widetilde{\theta}\Big\rangle_1 - \langle\widetilde{\theta}, \boldsymbol{g}\rangle_1 \,. \tag{4.33}$$

First we focus on the vorticity equation (4.32). Summing by parts and using the discrete kinematic relationship between $\widetilde{\psi}$ and $\widetilde{\omega}$ gives

$$\begin{aligned} &- \Big\langle\Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}, \Big(1 + \frac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega}\Big\rangle_1 \\ &= -\Big\langle\Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}, \Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\partial_t\widetilde{\psi}\Big\rangle_1 = \frac{1}{2}\frac{d\widetilde{E}}{dt} \,, \end{aligned} \tag{4.34}$$

with

$$\begin{aligned} \widetilde{E} &= \|\nabla_h\widetilde{\psi}\|_2^2 - \frac{h^2}{12}\|\Delta_h\widetilde{\psi}\|_1^2 - \frac{h^2}{6}\|D_xD_y\widetilde{\psi}\|_1^2 \\ &+ \frac{h^4}{72}\Big(\|D_xD_y^2\widetilde{\psi}\|_1^2 + \|D_yD_x^2\widetilde{\psi}\|_1^2\Big) \,, \end{aligned} \tag{4.35}$$

in which the vanishing boundary condition for $\widetilde{\psi}$ was utilized.

For the diffusion term in (4.32) we have the following estimate.

Proposition 4.3 *The following inequality holds*

$$\Big\langle\Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}, \Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega}\Big\rangle_1 \geq \frac{1}{8}\|\widetilde{\widetilde{\omega}}\|_1^2 - h^8 \,. \tag{4.36}$$

Proof. Summing by parts and keeping in mind that $\widetilde{\psi}\ |_\Gamma = 0$, we have

$$\begin{aligned} &\Big\langle\Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\ \Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega}\Big\rangle_1 \\ &= \Big\langle\Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\psi}\,,\ \Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\omega}\Big\rangle_1 + \mathcal{B} \,. \end{aligned} \tag{4.37}$$

The first term on the right-hand side of (4.37) is exactly $\|\widetilde{\widetilde{\omega}}\|^2$ since

$$\Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\psi} = \Big(1 + \frac{h^2}{12}\Delta_h\Big)\widetilde{\omega} = \widetilde{\widetilde{\omega}} \,,$$

and the boundary term $\mathcal{B}$ can be decomposed as $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$ where

(4.38)
$$\begin{aligned}
\mathcal{B}_1 &= \sum_{i=1}^{N-1}\Big(\Big(1+\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}_{i,1}\cdot\widetilde{\omega}_{i,0} + \Big(1+\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}_{i,N-1}\cdot\widetilde{\omega}_{i,N})\Big) \\
&\quad + \sum_{j=1}^{N-1}\Big(\Big(1+\frac{h^2}{6}D_y^2\Big)\widetilde{\psi}_{1,j}\cdot\widetilde{\omega}_{0,j} + \Big(1+\frac{h^2}{6}D_y^2\Big)\widetilde{\psi}_{N-1,j}\cdot\widetilde{\omega}_{N,j})\Big) \\
\mathcal{B}_2 &= \frac{h^4}{72}\sum_{i=1}^{N-1}(D_x^2\widetilde{\psi}_{i,1}D_x^2\widetilde{\omega}_{i,0} + D_x^2\widetilde{\psi}_{i,N-1}D_x^2\widetilde{\omega}_{i,N}) \\
&\quad + \frac{h^4}{72}\sum_{j=1}^{N-1} D_y^2\widetilde{\psi}_{1,j}D_y^2\widetilde{\omega}_{0,j} + D_y^2\widetilde{\psi}_{N-1,j}D_y^2\widetilde{\omega}_{N,j}) \\
\mathcal{B}_3 &= \frac{1}{6}(\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0} + \widetilde{\psi}_{1,N-1}\widetilde{\omega}_{0,N} + \widetilde{\psi}_{N-1,1}\widetilde{\omega}_{N,0} + \widetilde{\psi}_{N-1,N-1}\widetilde{\omega}_{N,N})\,.
\end{aligned}$$

To complete the proof we estimate the three boundary terms separately in the following Lemmas.

Lemma 4.4 *We have the estimate*

(4.39)
$$\mathcal{B}_1 \geq \frac{1}{3h^2}\mathcal{B}_\psi - \frac{3}{4}\Big(\|\Big(1+\frac{h^2}{6}D_y^2\Big)D_x^2\widetilde{\psi}\|_1^2 + \|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}\|_1^2\Big) - Ch^9\,,$$

where $\mathcal{B}_\psi$ is given by

$$\mathcal{B}_\psi = \sum_{i=1}^{N-1}(\widetilde{\psi}_{i,1}^2 + \widetilde{\psi}_{i,N-1}^2) + \sum_{j=1}^{N-1}(\widetilde{\psi}_{1,j}^2 + \widetilde{\psi}_{N-1,j}^2)\,. \tag{4.40}$$

Proof. The boundary condition (4.31) for $\widetilde{\omega}$ implies that $\sum_{i=1}^{N-1}\Big(1+\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}_{i,1}$ $\widetilde{\omega}_{i,0}$ can be written in two parts, I_1 and I_2, where

$$\begin{aligned}
I_1 &= \frac{1}{h^2}\sum_{i=1}^{N-1}(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}\Big(8\widetilde{\psi}_{i,1} - 3\widetilde{\psi}_{i,2} + \frac{8}{9}\widetilde{\psi}_{i,3} - \frac{1}{8}\widetilde{\psi}_{i,4}\Big)\,, \\
I_2 &= \sum_{i=1}^{N-1}(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}\cdot \boldsymbol{h}_{i,0}\,.
\end{aligned} \tag{4.41}$$

The term I_2 can be controlled by Cauchy's inequality directly. First, recall the definition of $\boldsymbol{h}_{i,0}$ in (4.31). Then summing by parts gives I_2

$$= \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} (1 + \frac{h^2}{6} D_x^2) \boldsymbol{h}_{i,0}, \text{ and we have}$$

(4.42)
$$I_2 \geq -\frac{1}{36} \sum_{i=1}^{N-1} \frac{\widetilde{\psi}_{i,1}^2}{h^2} - 9 \sum_{i=1}^{N-1} h^2 \Big((1 + \frac{h^2}{6} D_x^2) \boldsymbol{h}_{i,0} \Big)^2 \geq -\frac{1}{36} \sum_{i=1}^{N-1} \frac{\widetilde{\psi}_{i,1}^2}{h^2} - C h^9 \, ,$$

since $|\boldsymbol{h}_{i,0}| \leq C h^4 \|\boldsymbol{u}_e\|_{C^5}$. As for I_1, since $\widetilde{\psi}$ vanishes on the boundary, the term $8\widetilde{\psi}_{i,1} - 3\widetilde{\psi}_{i,2} + \frac{8}{9}\widetilde{\psi}_{i,3} - \frac{1}{8}\widetilde{\psi}_{i,4}$ can be rewritten as

$$\begin{aligned} 8\widetilde{\psi}_{i,1} - 3\widetilde{\psi}_{i,2} + \frac{8}{9}\widetilde{\psi}_{i,3} - \frac{1}{8}\widetilde{\psi}_{i,4} &= \frac{25}{6}\widetilde{\psi}_{i,1} - \frac{115}{72} h^2 (D_y^2 \widetilde{\psi})_{i,1} \\ &+ \frac{23}{36} h^2 (D_y^2 \widetilde{\psi})_{i,2} - \frac{1}{8} h^2 (D_y^2 \widetilde{\psi})_{i,3} \, , \end{aligned} \tag{4.43}$$

which implies that I_1 can be, after summing by parts, expressed as

(4.44)
$$\begin{aligned} I_1 = \frac{25}{6h^2} \sum_{i=1}^{N-1} (\widetilde{\psi}_{i,1}^2 + \frac{h^2}{6} \widetilde{\psi}_{i,1} D_x^2 \widetilde{\psi}_{i,1}) - \frac{115}{72} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,1} \\ + \frac{23}{36} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,2} - \frac{1}{8} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,3}. \end{aligned}$$

The first term on the right-hand side of (4.44) is estimated directly, while for the remaining terms we apply Cauchy's inequality, giving

$$\frac{25}{6h^2} \sum_{i=1}^{N-1} (\widetilde{\psi}_{i,1}^2 + \frac{h^2}{6} \widetilde{\psi}_{i,1} D_x^2 \widetilde{\psi}_{i,1}) \geq \frac{1}{3} \frac{25}{6h^2} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1}^2 \, , \tag{4.45}$$

$$\begin{aligned} & -\frac{115}{72} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,1} \\ & \geq - \sum_{i=1}^{N-1} \Big(\frac{1}{3h^2} \frac{115^2}{72^2} |\widetilde{\psi}_{i,1}|^2 + \frac{3}{4} h^2 \Big| \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,1} \Big|^2 \Big), \\ & \frac{23}{36} \sum_{i=1}^{N-1} \widetilde{\psi}_{i,1} \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,2} \\ & \geq - \sum_{i=1}^{N-1} \Big(\frac{1}{3h^2} \frac{23^2}{36^2} |\widetilde{\psi}_{i,1}|^2 + \frac{3}{4} h^2 \Big| \Big(1 + \frac{h^2}{6} D_x^2\Big) (D_y^2 \widetilde{\psi})_{i,2} \Big|^2 \Big) \, , \end{aligned}$$

$$-\frac{1}{8}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,3}$$
$$\geq -\sum_{i=1}^{N-1}\Big(\frac{1}{3h^2}\frac{1^2}{8^2}|\widetilde{\psi}_{i,1}|^2-\frac{3}{4}h^2\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,3}\Big|^2\Big).$$

Since $\frac{1}{3}\cdot\frac{25}{6}-\frac{1}{3}\left(\frac{115^2}{72^2}+\frac{23^2}{36^2}+\frac{1^2}{8^2}\right)\geq\frac{13}{36}$, we have

$$I_1\geq\frac{13}{36h^2}\sum_{i=1}^{N-1}|\widetilde{\psi}_{i,1}|^2-\frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1,2,3}\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}\Big|^2. \tag{4.46}$$

The combination of I_1 and I_2 then gives

$$\sum_{i=1}^{N-1}\Big(1+\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}_{i,1}\widetilde{\omega}_{i,0}$$
$$\geq\frac{1}{3h^2}\sum_{i=1}^{N-1}|\widetilde{\psi}_{i,1}|^2-\frac{3}{4}\sum_{i=1}^{N-1}\sum_{j=1,2,3}\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}\Big|^2h^2-Ch^9. \tag{4.47}$$

Finally, we obtain

$$\mathcal{B}_1\geq\frac{1}{3h^2}\mathcal{B}_\psi-\frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\Big|\Big(1+\frac{h^2}{6}D_y^2\Big)D_x^2\widetilde{\psi}_{i,j}\Big|^2$$
$$-\frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}\Big|^2-Ch^9, \tag{4.48}$$

where $\mathcal{B}_\psi$ was defined in (4.40). Moreover, (4.39) is a direct consequence of (4.48). The treatment of the other three boundary terms is exactly the same. This completes the proof of Lemma 4.4. □

To complete the estimate of $\mathcal{B}_1$ we need to control $\|\Big(1+\frac{h^2}{6}D_y^2\Big)D_x^2\widetilde{\psi}\|_1$ and $\|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}\|_1$. However, standard local estimates do not work in this case. The methodology we adopt here is similar to that used in [21], i.e., control the local terms by global terms via elliptic regularity.

Lemma 4.5 *For any $\widetilde{\psi}$ that vanishes on the boundary, we have*

$$\|D_x^2\widetilde{\psi}\|_1^2 + \|D_y^2\widetilde{\psi}\|_1^2 \le \frac{9}{8}\|\widetilde{\widetilde{\omega}}\|_1^2\,, \tag{4.49}$$

$$\|\Big(1+\frac{h^2}{6}D_y^2\Big)D_x^2\widetilde{\psi}\|_1^2 + \|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}\|_1^2 \le \|\widetilde{\widetilde{\omega}}\|_1^2\,. \tag{4.50}$$

Proof. Given the homogeneous boundary condition $\widetilde{\psi}_{i,j}\;|_\Gamma = 0$, we perform a Sine transformation of $\{\widetilde{\psi}_{i,j}\}$ in both the x and y directions, i.e.,

$$\widetilde{\psi}_{i,j} = \sum_{k,\ell=1}^{N-1}\Big(\frac{2}{\sqrt{2N}}\Big)^2\widehat{\widetilde{\psi}}_{k,\ell}\,\sin(k\pi x_i)\,\sin(\ell\pi y_j)\,. \tag{4.51}$$

Parseval's equality gives

$$\sum_{i,j=1}^{N-1}(\widetilde{\psi}_{i,j})^2 = \sum_{k,\ell=1}^{N-1}\Big|\widehat{\widetilde{\psi}}_{k,\ell}\Big|^2\,. \tag{4.52}$$

If we introduce

$$f_k = -\frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2}\Big), \qquad g_\ell = -\frac{4}{h^2}\sin^2\Big(\frac{\ell\pi h}{2}\Big), \tag{4.53}$$

then the Fourier Sine expansions of $D_x^2\widetilde{\psi}$ and $D_y^2\widetilde{\psi}$ are given by, respectively,

$$\begin{aligned} D_x^2\widetilde{\psi}_{i,j} &= \sum_{k,l=1}^{N-1}\Big(\frac{2}{\sqrt{2N}}\Big)^2 f_k\,\widehat{\widetilde{\psi}}_{k,l}\,\sin(k\pi x_i)\,\sin(\ell\pi y_j)\,,\\ D_y^2\widetilde{\psi}_{i,j} &= \sum_{k,l=1}^{N-1}\Big(\frac{2}{\sqrt{2N}}\Big)^2 g_\ell\,\widehat{\widetilde{\psi}}_{k,l}\,\sin(k\pi x_i)\,\sin(\ell\pi y_j)\,, \end{aligned} \tag{4.54}$$

which in turn implies that

$$\begin{aligned} \sum_{i,j=1}^{N-1}|\widetilde{\widetilde{\omega}}_{i,j}|^2 &= \sum_{i,j=1}^{N-1}\Big|\Big(\Delta_h + \frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\psi}_{i,j}\Big|^2\\ &= \sum_{k,\ell=1}^{N-1}\Big(f_k + g_\ell + \frac{h^2}{6}f_kg_\ell\Big)^2\Big|\widehat{\widetilde{\psi}}_{k,\ell}\Big|^2\,. \end{aligned} \tag{4.55}$$

Similarly, we have

$$\begin{aligned} \sum_{i,j=1}^{N-1}|\Big(1+\frac{h^2}{6}D_y^2\Big)D_x^2\widetilde{\psi}_{i,j}|^2 &= \sum_{k,\ell=1}^{N-1}|\Big(1+\frac{h^2}{6}g_\ell\Big)f_k|^2\Big|\widehat{\widetilde{\psi}}_{k,\ell}\Big|^2\,,\\ \sum_{i,j=1}^{N-1}|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}|^2 &= \sum_{k,\ell=1}^{N-1}|\Big(1+\frac{h^2}{6}f_k\Big)g_\ell|^2\Big|\widehat{\widetilde{\psi}}_{k,\ell}\Big|^2\,. \end{aligned} \tag{4.56}$$

On the other hand, direct calculation along with the fact that $-\dfrac{4}{h^2} \le f_k, g_\ell \le 0$ shows that

$$|f_k + g_\ell + \frac{h^2}{6} f_k g_\ell|^2 \ge |\Big(1 + \frac{h^2}{6} g_\ell\Big) f_k|^2 + |\Big(1 + \frac{h^2}{6} f_k\Big) g_\ell|^2 , \tag{4.57}$$

$$|f_k + g_\ell + \frac{h^2}{6} f_k g_\ell|^2 \ge \Big(f_k^2 + g_\ell^2 - \frac{2}{9} f_k g_\ell\Big) \ge \frac{8}{9}(f_k^2 + g_\ell^2) . \tag{4.58}$$

Combining (4.55)–(4.57) gives (4.50). Estimate (4.49) can be argued in a similar fashion. Lemma 4.5 is proven. □

The combination of Lemmas 4.4 and 4.5 results in the estimate

$$\mathcal{B}_1 \ge \frac{1}{3h^2} \mathcal{B}_\psi - \frac{3}{4} \|\widetilde{\widetilde{\omega}}\|_1^2 - Ch^9 . \tag{4.59}$$

An estimate for $\mathcal{B}_2$ can be derived in a similar fashion, which we only briefly outline. Consider first the expression $\sum_i D_x^2 \widetilde{\psi}_{i,1} D_x^2 \widetilde{\omega}_{i,0}$ in $\mathcal{B}_2$. Once again the boundary condition (4.31) for $\widetilde{\omega}$ leads to examining $\sum_i D_x^2 \widetilde{\psi}_{i,1}$ $D_x^2 \widetilde{\omega}_{i,0}$ in two parts, I_3 and I_4, given by

$$\begin{aligned} I_3 &= \frac{1}{h^2} \sum_{i=1}^{N-1} D_x^2 \widetilde{\psi}_{i,1} \Big(8 D_x^2 \widetilde{\psi}_{i,1} - 3 D_x^2 \widetilde{\psi}_{i,2} + \frac{8}{9} D_x^2 \widetilde{\psi}_{i,3} - \frac{1}{8} D_x^2 \widetilde{\psi}_{i,4}\Big) , \\ I_4 &= \sum_{i=1}^{N-1} D_x^2 \widetilde{\psi}_{i,1} D_x^2 \boldsymbol{h}_{i,0} . \end{aligned} \tag{4.60}$$

The estimate of I_3 and I_4 is similar to that of I_1 and I_2, respectively. Repeating the arguments in the proof of Lemma 4.4, we arrive at (omitting the details)

$$\mathcal{B}_2 \ge -\frac{1}{144} h^4 \|D_y^2 D_x^2 \widetilde{\psi}\|_1^2 - Ch^9 . \tag{4.61}$$

On the other hand, the fact that $\|D_y^2 D_x^2 \widetilde{\psi}\|_1 \le \dfrac{4}{h^2} \|D_x^2 \widetilde{\psi}\|_1$ and $\|D_y^2 D_x^2 \widetilde{\psi}\|_1 \le \dfrac{4}{h^2} \|D_y^2 \widetilde{\psi}\|_1$ implies

$$\begin{aligned} \|D_y^2 D_x^2 \widetilde{\psi}\|_1^2 &= \frac{1}{2}(\|D_y^2 D_x^2 \widetilde{\psi}\|_1^2 + \|D_y^2 D_x^2 \widetilde{\psi}\|_1^2) \le \frac{8}{h^4} \|D_x^2 \widetilde{\psi}\|_1^2 + \frac{8}{h^4} \|D_y^2 \widetilde{\psi}\|_1^2 \\ &\le \frac{9}{h^4} \|\widetilde{\widetilde{\omega}}\|_1^2 , \end{aligned} \tag{4.62}$$

where in the last step we applied (4.49) in Lemma 4.5. Substituting (4.62) into (4.61), we arrive at

$$\mathcal{B}_2 \geq -\frac{1}{16}\|\widetilde{\omega}\|_1^2 - Ch^9 . \tag{4.63}$$

Finally, $\mathcal{B}_3$ can be controlled by applying Cauchy's inequality (we only consider here the term $\frac{1}{6}\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0}$)

$$\frac{1}{6}\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0} \geq -\frac{1}{12}\frac{\widetilde{\psi}_{1,1}^2}{h^2} - \frac{1}{12}h^2\widetilde{\omega}_{0,0}^2 \geq -\frac{1}{12}\frac{\widetilde{\psi}_{1,1}^2}{h^2} - Ch^{10}\|\psi_e\|_{C^8}^2 , \tag{4.64}$$

where in the last step we used the fact that $|\widetilde{\omega}_{0,0}| \leq Ch^4\|\psi_e\|_{C^8}$ by our construction of Ω in section 3.1. The first term on the right-hand side of (4.64) can be absorbed into the $\mathcal{B}_\psi$ term, giving

$$\mathcal{B}_3 \geq -\frac{1}{12h^2}\mathcal{B}_\psi - Ch^9 . \tag{4.65}$$

The combination of (4.63), (4.65), and Lemma 4.4 shows that $\mathcal{B} \geq -\frac{13}{16}\|\widetilde{\omega}\|_1^2 - h^8$, whose substitution into (4.37) is exactly (4.36). This completes the proof of Proposition 4.3. □

The estimates for the linearized convection terms in (4.32) are given in the following proposition. The proof is similar to that of Proposition 4.3 and the details are left to interested readers.

Proposition 4.6 *Assume a-priori that the error functions for the velocity field and temperature satisfy*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leq h^2 , \qquad \|\widetilde{\theta}\|_{L^\infty} \leq h^2 . \tag{4.66}$$

Then we have

$$\left\langle \left(1 + \frac{h^2}{12}\Delta_h\right)\widetilde{\psi} \, , \, \mathcal{L}_2 \right\rangle_1 \leq \widetilde{C}_1\|\nabla_h\widetilde{\psi}\|_2^2 + \frac{\nu}{8}\|\widetilde{\omega}\|_1^2 + h^8 , \tag{4.67}$$

where $\widetilde{C}_1 = \dfrac{C(1+\|\boldsymbol{u}_e\|_{C^0})^2}{\nu} + C(2+\|\boldsymbol{u}_e\|_{C^1})^2 + C\|\boldsymbol{u}_e\|_{C^5}$.

In addition, by Cauchy's inequality and the boundary condition for the temperature error function $\widetilde{\theta}$ in (4.31), we have the estimate of the gravity term

$$\begin{aligned} &\left| \left\langle \left(1 + \frac{h^2}{12}\Delta_h\right)\widetilde{\psi} \, , \, \widetilde{D}_x\left(1 - \frac{h^2}{12}(D_y^2 - D_x^2)\right)\widetilde{\theta} \right\rangle_1 \right| \\ &\quad \leq C(\|\widetilde{\psi}\|_1^2 + \|\nabla_h\widetilde{\theta}\|_2^2) + Ch^{10} , \end{aligned} \tag{4.68}$$

In (4.33), the local truncation error term $-\langle\widetilde{\theta}, \boldsymbol{g}\rangle_1$ can be controlled by Cauchy's inequality. The technique used in Lemma 4.6 can be applied here for the estimate of the linearized temperature convection term, i.e., the assumed a-priori assumption (4.66) leads to

$$\left|\langle\widetilde{\theta}, \mathcal{L}_1\rangle_1\right| \le \widetilde{C}_2\|\widetilde{\theta}\|_1^2 + \frac{1}{2}\kappa\|\nabla_h\widetilde{\theta}\|_2^2 + h^8\,, \tag{4.69}$$

where $\widetilde{C}_2 = \dfrac{C(1+\|\boldsymbol{u}_e\|_{C^0})^2}{\kappa}$.

Next, we apply the technique demonstrated in section 3.1 for the stability analysis of the long-stencil discretization of the one-dimensional heat equation to the temperature diffusion term.

Proposition 4.7 *We have*

$$-\left\langle\widetilde{\theta}, \left(\Delta_h - \frac{h^2}{12}(D_x^4 + D_y^4)\right)\widetilde{\theta}\right\rangle_1 \ge \frac{1}{2}\|\nabla_h\widetilde{\theta}\|_2^2 - h^8\,. \tag{4.70}$$

Proof. The proof of (4.70) is just the two-dimensional version of the stability analysis in section 3. Since $\widetilde{\theta}$ vanishes on the boundary, we have

$$\begin{aligned} &-\left\langle\widetilde{\theta}, \left(\Delta_h - \frac{h^2}{12}(D_x^4 + D_y^4)\right)\widetilde{\theta}\right\rangle_1 \\ &\quad = \|\nabla_h\widetilde{\theta}\|_2^2 + \frac{h^2}{12}\|D_x^2\widetilde{\theta}\|_1^2 + \frac{h^2}{12}\|D_y^2\widetilde{\theta}\|_1^2 + \frac{h^2}{12}\mathcal{B}\,, \end{aligned} \tag{4.71}$$

where $\mathcal{B}$ arises from the boundary terms, which after summation by parts, can be written as

$$\begin{aligned} \mathcal{B} = &\sum_{i=1}^{N-1}\widetilde{\theta}_{i,1}(D_y^2\widetilde{\theta})_{i,0} + \sum_{i=1}^{N-1}\widetilde{\theta}_{i,N-1}(D_y^2\widetilde{\theta})_{i,N} + \sum_{j=1}^{N-1}\widetilde{\theta}_{1,j}(D_x^2\widetilde{\theta})_{0,j} \\ &+ \sum_{j=1}^{N-1}\widetilde{\theta}_{N-1,j}(D_x^2\widetilde{\theta})_{N,j}\,. \end{aligned} \tag{4.72}$$

We focus on the first term appearing on the right-hand side of (4.72); the other three boundary terms can be treated similarly. Applying the boundary condition for $\widetilde{\theta}$ at the "ghost points" as in (4.31), $(D_y^2\widetilde{\theta})_{i,0}$ can be written as

$$(D_y^2\widetilde{\theta})_{i,0} = \frac{1}{h^2}\left(\frac{5}{11}\widetilde{\theta}_{i,1} - \frac{4}{11}\widetilde{\theta}_{i,2} + \frac{3}{11}\widetilde{\theta}_{i,3} + \boldsymbol{e}_{i,0}\right), \tag{4.73}$$

which is analogous to (3.7) except for the local error term $\boldsymbol{e}_{i,0}$ (defined in (4.31)), whose product with $\widetilde{\theta}_{i,1}$ can be controlled by Cauchy's inequality.

Alternatively, we can rewrite the right-hand side of (4.73), as we did in section 3, as

$$(D_y^2\widetilde{\theta})_{i,0} = -\frac{2}{11}(D_y^2\widetilde{\theta})_{i,1} + \frac{1}{11}(D_y^2\widetilde{\theta})_{i,2} + \frac{\boldsymbol{e}_{i,0}}{h^2}\,. \tag{4.74}$$

The aim here is the control of local terms by global terms. Applying Cauchy's inequality to each term in (4.74) leads to

$$\begin{aligned}\widetilde{\theta}_{i,1}(D_y^2\widetilde{\theta})_{i,0} \geq & -\frac{2^2}{4\cdot 11^2\cdot h^2}\widetilde{\theta}_{i,1}^2 - h^2(D_y^2\widetilde{\theta}_{i,1})^2 - \frac{1^2}{4\cdot 11^2\cdot h^2}\widetilde{\theta}_{i,1}^2 - h^2(D_y^2\widetilde{\theta}_{i,2})^2 \\ & -\frac{1}{4h^2}\widetilde{\theta}_{i,1}^2 - \frac{\boldsymbol{e}_{i,0}^2}{h^2} \\ \geq & -\frac{1}{2h^2}\widetilde{\theta}_{i,1}^2 - h^2(D_y^2\widetilde{\theta}_{i,1})^2 - h^2(D_y^2\widetilde{\theta}_{i,2})^2 - \frac{\boldsymbol{e}_{i,0}^2}{h^2}\,.\end{aligned} \tag{4.75}$$

Here the arguments in section 4 can be repeated: the first term appearing above can be controlled by $\|\nabla_h\widetilde{\theta}\|_2^2$, since it will be multiplied by $\frac{h^2}{12}$, resulting in a term greater than $\frac{1}{2}\|\nabla_h\widetilde{\theta}\|_2^2$; the second and third terms can be controlled by $\|D_y^2\widetilde{\theta}\|_2^2$; while the last term can be controlled by

$$\frac{h^2}{12}\sum_{i=1}^{N-1}\frac{\boldsymbol{e}_{i,0}^2}{h^2} \leq Nh^2\cdot Ch^{10}\|\theta_e\|_{C^6}^2\cdot\frac{1}{h^2} \leq Ch^9\|\theta_e\|_{C^6}^2\,, \tag{4.76}$$

where we used the fact that $h = \frac{1}{N}$. (4.70) then follows. □

Finally, the combination of (4.32)–(4.37), and (4.67)–(4.70) gives us

$$\begin{aligned}\frac{1}{2}\frac{d\widetilde{E}}{dt} + \frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_1^2 \leq & \; Ch^8 + C\|\boldsymbol{f}\|_1^2 + C\|\boldsymbol{g}\|_1^2 + \widetilde{C}_1\|\nabla_h\widetilde{\psi}\|_2^2 \\ & + \widetilde{C}_2\|\widetilde{\theta}\|_1^2 - \frac{\kappa}{2}\|\nabla_h\widetilde{\theta}\|_2^2\,.\end{aligned} \tag{4.77}$$

Integrating in time results in

$$\begin{aligned}\widetilde{E} + \|\widetilde{\theta}\|_1^2 \leq & \; C\int_0^T(\|\boldsymbol{f}\|_1^2 + \|\boldsymbol{g}\|_1^2)\,dt + 2\widetilde{C}_2\int_0^T\|\widetilde{\theta}\|_1^2\,dt \\ & + 2\widetilde{C}_1\int_0^T\|\nabla_h\widetilde{\psi}\|_2^2\,dt + CTh^8\,.\end{aligned} \tag{4.78}$$

It can be seen that

$$\|\nabla_h\widetilde{\psi}\|_2^2 \leq 3(\|\nabla_h\widetilde{\psi}\|_2^2 - \frac{h^2}{12}\|\Delta_h\widetilde{\psi}\|_1^2 - \frac{h^2}{6}\|D_xD_y\widetilde{\psi}\|_1^2)\,, \tag{4.79}$$

since $\widetilde{\psi}$ vanishes on the boundary, which along with (4.78) implies that

$$\begin{aligned}\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_1^2 &\le C \int_0^T (\|\boldsymbol{f}\|_1^2 + \|\boldsymbol{g}\|_1^2)\, dt + C\widetilde{C}_1 \int_0^T \|\nabla_h \widetilde{\psi}\|_2^2\, dt \\ &\quad + C\widetilde{C}_2 \int_0^T \|\widetilde{\theta}\|_1^2\, dt + CTh^8 .\end{aligned} \tag{4.80}$$

By Gronwall's inequality we then have

$$\begin{aligned}\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_1^2 &\le C \exp\left\{C\widetilde{C}_1 T + C\widetilde{C}_2 T\right\} \\ &\quad \left(\int_0^T \|\boldsymbol{f}(\cdot,s)\|_1^2 + \|\boldsymbol{g}(\cdot,s)\|_1^2\, ds \; + CTh^8\right) \\ &\le CTh^8 \exp\left\{C\widetilde{C}_1 T + C\widetilde{C}_2 T\right\} \\ &\quad \Big(\|\boldsymbol{u}_e\|_{C^{7,\alpha}}^2 (1 + \|\boldsymbol{u}_e\|_{C^5})^2 + \|\theta_e\|_{C^5}^2 \|\boldsymbol{u}_e\|_{C^5}^2 \\ &\quad + \|\theta_e\|_{C^6}^2 + CTh^8\Big) .\end{aligned} \tag{4.81}$$

Thus, we have proven

$$\begin{aligned}&\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_e(t)\|_{L^2} + \|\theta(\cdot,t) - \theta_e(t)\|_{L^2} \\ &\le Ch^4 \Big(\|\boldsymbol{u}_e\|_{C^{7,\alpha}} (1 + \|\boldsymbol{u}_e\|_{C^5}) + \|\theta_e\|_{C^5} \|\boldsymbol{u}_e\|_{C^5} + \|\theta_e\|_{C^6}\Big) \\ &\quad \cdot \exp\left\{\frac{CT}{\nu}(1 + \|\boldsymbol{u}_e\|_{C^1})^2 + \frac{CT}{\kappa}(1 + \|\boldsymbol{u}_e\|_{C^0})^2\right\} .\end{aligned} \tag{4.82}$$

Using the inverse inequality, we have

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \le Ch^3 . \tag{4.83}$$

At this point, we can introduce a standard concept which asserts that (4.66) will never be violated if h is small enough, and Theorem 1.1 is proven. □

5 Convergence proof of Theorem 1.2

The numerical scheme with the Neumann boundary condition (1.4), namely (2.7), (2.16), and (2.26), is analyzed in this section. For simplicity of presentation we set $\theta_f = 0$ in which case the one-sided extrapolation of the temperature at the boundary is given by (2.27).

The consistency analysis of the momentum equation is the same as that presented in section 4. We denote by ψ_e, $\boldsymbol{u}_e$, and ω_e the exact solutions of (1.1)–(1.2), and (1.4), and extend ψ_e smoothly to $[-\delta, 1+\delta]^2$. Then let $\Psi_{i,j} = \psi_e(x_i, y_j)$ for $-2 \le i, j \le N+2$. The approximated velocity profiles U and V, and the vorticity profile Ω are given by (4.1) and (4.2)–(4.5), respectively. Lemmas 4.1 and 4.2, along with the estimate for the time marching term in (4.18), remain valid. The fourth order approximation (4.22) for the constructed vorticity Ω on the boundary is also preserved.

Regarding the temperature variable, instead of substituting the exact solution into the numerical scheme, a careful construction of an approximated temperature profile is performed by adding an $O(h^4)$ correction term to θ_e to satisfy the truncation error fully to fourth order. The reason for this procedure is to avoid the loss of accuracy near the boundary which would result from a direct truncation error estimate. To be more precise, we construct the approximate temperature field Θ as

$$\Theta = \theta_e + h^4 \widehat{\theta}\,, \tag{5.1}$$

in which the correction function $\widehat{\theta}$ satisfies the Poisson equation

$$\Delta \widehat{\theta} = C^1\,, \tag{5.2a}$$

with the Neumann boundary condition

$$\begin{aligned} \partial_y \widehat{\theta}(x,0) &= \frac{1}{80}\,\partial_y^5 \theta_e(x,0)\,, \quad & \partial_y \widehat{\theta}(x,1) &= \frac{1}{80}\,\partial_y^5 \theta_e(x,1)\,, \\ \partial_x \widehat{\theta}(0,y) &= \frac{1}{80}\,\partial_x^5 \theta_e(0,y)\,, \quad & \partial_x \widehat{\theta}(1,y) &= \frac{1}{80}\,\partial_x^5 \theta_e(1,y)\,. \end{aligned} \tag{5.2b}$$

The scalar C^1 (a function of time t) is chosen as $\int_\Omega C^1\, d\boldsymbol{x} = \int_{\partial\Omega} \frac{\partial \widehat{\theta}}{\partial \boldsymbol{n}}\, d\boldsymbol{n}$ in order to maintain consistency with the Neumann boundary condition, i.e.,

$$\begin{aligned} C^1 = \frac{1}{|\Omega|}\Big(\int_0^1 & - \frac{1}{80}\partial_y^5 \theta_e(x,0) + \frac{1}{80}\partial_y^5 \theta_e(x,1)\, dx \\ & + \int_0^1 -\frac{1}{80}\,\partial_x^5 \theta_e(0,y) + \frac{1}{80}\,\partial_x^5 \theta_e(1,y)\, dy \Big)\,. \end{aligned} \tag{5.3}$$

A Schauder estimate applied to the Poisson equation (5.2) gives

$$\|\widehat{\theta}\|_{C^{m,\alpha}} \le C \|\theta_e\|_{C^{m+4,\alpha}}\,, \qquad \text{for} \ \ m \ge 2\,. \tag{5.4}$$

The reason for taking the boundary condition for $\widehat{\theta}$ in (5.2b) will become apparent later. A local Taylor expansion for the exact temperature field θ_e at points near the boundary $y = 0$ gives

(5.5)

$$\begin{aligned}
(\theta_e)_{i,-1} &= (\theta_e)_{i,1} - \frac{h^3}{3}\,\partial_y^3\theta_e(x_i,0) - \frac{h^5}{60}\,\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^7} \\
&= (\theta_e)_{i,1} + \frac{h^3}{3\kappa}\,(\omega_e)_{i,0}(\partial_x\theta_e)_{i,0} - \frac{h^5}{60}\,\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^7}\,, \\
(\theta_e)_{i,-2} &= (\theta_e)_{i,2} - \frac{8h^3}{3}\,\partial_y^3\theta_e(x_i,0) - \frac{32h^5}{60}\,\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^7} \\
&= (\theta_e)_{i,2} + \frac{8h^3}{3\kappa}(\omega_e)_{i,0}(\partial_x\theta_e)_{i,0} - \frac{32h^5}{60}\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^7},
\end{aligned}$$

due to the no-flux boundary condition for θ_e and the derivation for $\partial_y^3\theta_e$ as given in (2.25) by applying the original PDE on the boundary. The insertion of the boundary conditions given by (5.2b) into a Taylor expansion of $\widehat{\theta}$, along with the Schauder estimate $\|\widehat{\theta}\|_{C^3} \le C\|\theta_e\|_{C^{7,\alpha}}$ given by (5.4), gives

$$\begin{aligned}
\widehat{\theta}_{i,-1} &= \widehat{\theta}_{i,1} - 2h\partial_y\widehat{\theta}_{i,0} + O(h^3)\partial_y^3\widehat{\theta}_{i,0} \\
&= \widehat{\theta}_{i,1} - \tfrac{h}{40}\,\partial_y^5\theta_e(x_i,0) + O(h^3)\|\theta_e\|_{C^{7,\alpha}}\,, \\
\widehat{\theta}_{i,-2} &= \widehat{\theta}_{i,2} - 4h\partial_y\widehat{\theta}_{i,0} + O(h^3)\partial_y^3\widehat{\theta}_{i,0} \\
&= \widehat{\theta}_{i,2} - \tfrac{h}{20}\,\partial_y^5\theta_e(x_i,0) + O(h^3)\|\theta_e\|_{C^{7,\alpha}}\,.
\end{aligned} \tag{5.6}$$

The combination of (5.5) and (5.6) results in an estimate for $\Theta = \theta_e + h^4\widehat{\theta}$ given by

(5.7)

$$\begin{aligned}
\Theta_{i,-1} &= \Theta_{i,1} + \frac{h^3}{3\kappa}\,(\omega_e)_{i,0}(\partial_x\theta_e)_{i,0} - \frac{h^5}{24}\,\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^{7,\alpha}}\,, \\
\Theta_{i,-2} &= \Theta_{i,2} + \frac{8h^3}{3\kappa}\,(\omega_e)_{i,0}(\partial_x\theta_e)_{i,0} - \frac{7h^5}{12}\,\partial_y^5\theta_e(x_i,0) + O(h^7)\|\theta_e\|_{C^{7,\alpha}}\,.
\end{aligned}$$

Similar results can be obtained at the other three boundary segments, namely $y = 1$, $x = 0$, and $x = 1$. Note that the $O(h^5)$ coefficients of $\Theta_{i,-1}$ and $\Theta_{i,-2}$ have the ratio $1 : 14$. This will be needed for the error analysis of the inner product of the temperature with the diffusion term in the temperature equation. This crucial point is the reason for the choice of the boundary condition for $\widehat{\theta}$ in (5.2b).

A direct consequence of the Schauder estimate (5.4) is given by

$$\|\widehat{\theta}\|_{W^{2,\infty}(\Omega)} \le C\|\widehat{\theta}\|_{C^{2,\alpha}} \le C\|\theta_e\|_{C^{6,\alpha}}\,, \tag{5.8}$$

in which $\|\cdot\|_{W^{m,\infty}(\Omega)}$ represents the maximum value, at grids points, of the given function up to m-th order finite-difference, over the domain Ω. As a result, we have

$$\|\Theta - \theta_e\|_{W^{2,\infty}(\Omega)} = h^4\|\widehat{\theta}\|_{W^{2,\infty}(\Omega)} \le Ch^4\|\theta_e\|_{C^{6,\alpha}} . \tag{5.9}$$

The combination of (5.9) and a local Taylor expansion for θ_e gives the estimates

$$\begin{aligned} \widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\Theta &= \widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\theta_e + O(h^4)\|\theta_e\|_{C^{6,\alpha}} \\ &= \Big(1+\frac{h^2}{12}\Delta\Big)\partial_x\theta_e + O(h^4)\|\theta_e\|_{C^{6,\alpha}} , \end{aligned} \tag{5.10}$$

$$\begin{aligned} \widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta &= \partial_x\theta_e + O(h^4)\|\theta_e\|_{C^{6,\alpha}} , \\ \widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta &= \partial_y\theta_e + O(h^4)\|\theta_e\|_{C^{6,\alpha}} , \\ \Big(\Delta_h-\frac{h^2}{12}(D_x^4+D_y^4)\Big)\Theta &= \Delta\theta_e + O(h^4)\|\theta_e\|_{C^{6,\alpha}} , \end{aligned} \tag{5.11}$$

$$U\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta = u_e\partial_x\theta_e + O(h^4)\|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^{6,\alpha}} , \tag{5.12}$$

$$V\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta = v_e\partial_y\theta_e + O(h^4)\|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^{6,\alpha}} . \tag{5.13}$$

Moreover, taking the time derivative of (5.2) leads to a Poisson equation for $\partial_t\widehat{\theta}$, namely

$$\Delta(\partial_t\widehat{\theta}) = \partial_t C^1 , \tag{5.14a}$$

with the Neumann boundary conditions

$$\begin{aligned} \partial_y(\partial_t\widehat{\theta})(x,0) &= \frac{1}{80}\,(\partial_t\partial_y^5\theta_e)(x,0) , \quad & \partial_y(\partial_t\widehat{\theta})(x,1) &= \frac{1}{80}\,(\partial_t\partial_y^5\theta_e)(x,1) , \\ \partial_x(\partial_t\widehat{\theta})(0,y) &= \frac{1}{80}\,(\partial_t\partial_x^5\theta_e)(0,y) , \quad & \partial_x(\partial_t\widehat{\theta})(1,y) &= \frac{1}{80}\,(\partial_t\partial_x^5\theta_e)(1,y) . \end{aligned} \tag{5.14b}$$

Similarly, the value of $\partial_t C^1$ is given by

$$\begin{aligned}\partial_t C^1 = \frac{1}{|\Omega|}\Big(&-\int_0^1 \frac{1}{80}(\partial_t\partial_y^5\theta_e)(x,0) + \frac{1}{80}(\partial_t\partial_y^5\theta_e)(x,1)\,dx \\ &+\int_0^1 -\frac{1}{80}\,(\partial_t\partial_x^5\theta_e)(0,y) + \frac{1}{80}\,(\partial_t\partial_x^5\theta_e)(1,y)\,dy\Big)\,.\end{aligned} \tag{5.15}$$

A Schauder estimate applied to the Poisson equation (5.14) reads

$$\begin{aligned}\|\partial_t\widehat{\theta}\|_{C^{m,\alpha}} \le C\|\partial_t\theta_e\|_{C^{m+4,\alpha}} &\le C\Big(\|\boldsymbol{u}_e\|_{C^{m+4,\alpha}}\|\theta_e\|_{C^{m+5,\alpha}} + \|\theta_e\|_{C^{m+6,\alpha}}\Big)\,, \\ &\text{for } \ m\ge 2\,,\end{aligned} \tag{5.16}$$

in which the original temperature transport equation $\partial_t\theta_e+(\boldsymbol{u}_e\cdot\nabla)\theta_e=\kappa\Delta\theta_e$ was used. It can be seen that (5.16) amounts to

$$\partial_t\Theta = \partial_t\theta_e + O(h^4)\Big(\|\boldsymbol{u}_e\|_{C^{6,\alpha}}\|\theta_e\|_{C^{7,\alpha}} + \|\theta_e\|_{C^{8,\alpha}}\Big)\,. \tag{5.17}$$

The combination of (4.10)–(4.13), (4.18)–(4.19), (5.10), and the original momentum equation results in

$$\begin{aligned}&\partial_t\overline{\Omega} + \widetilde{D}_x\Big(1+\frac{h^2}{6}D_y^2\Big)(U\Omega) + \widetilde{D}_y\Big(1+\frac{h^2}{6}D_x^2\Big)(V\Omega) \\ &\quad -\frac{h^2}{12}\Delta_h(U\widetilde{D}_x\Omega + V\widetilde{D}_y\Omega) - g\widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\Theta \\ &\quad = \nu\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\Omega + \boldsymbol{f}\,,\end{aligned} \tag{5.18}$$

where $|\boldsymbol{f}| \le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5}) + Ch^4\|\theta_e\|_{C^{6,\alpha}}$. Similarly, the combination of (5.11)–(5.13), (5.17), and the original temperature equation gives

$$\begin{aligned}&\partial_t\Theta + U\widetilde{D}_x\Big(1-\frac{h^2}{6}D_x^2\Big)\Theta + V\widetilde{D}_y\Big(1-\frac{h^2}{6}D_y^2\Big)\Theta \\ &\quad = \kappa\Big(\Delta_h - \frac{h^2}{12}(D_x^4+D_y^4)\Big)\Theta + \boldsymbol{g}\,,\end{aligned} \tag{5.19}$$

where $|\boldsymbol{g}| \le Ch^4(\|\boldsymbol{u}_e\|_{C^{6,\alpha}}\|\theta_e\|_{C^{7,\alpha}} + \|\theta_e\|_{C^{8,\alpha}})$.

The error functions at the computational grid points $(x_i, y_j), 0\le i, j\le N$ are defined in the same way as in (4.28). Subtracting (5.18)–(5.19) from the numerical scheme (2.7), (2.16), and (2.27), gives

$$
\begin{cases}
\partial_t\widetilde{\theta} + \mathcal{L}_1 = \kappa\Big(\Delta_h - \dfrac{h^2}{12}(D_x^4 + D_y^4)\Big)\widetilde{\theta} - \boldsymbol{g}\,, \qquad \widetilde{\theta}\mid_\Gamma = 0\,,\\
\Big(1 + \dfrac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega} + \mathcal{L}_2 - g\widetilde{D}_x\Big(1 + \dfrac{h^2}{12}(D_y^2 - D_x^2)\Big)\widetilde{\theta} = \nu\Big(\Delta_h + \dfrac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega} - \boldsymbol{f}\,,\\
\Big(\Delta_h + \dfrac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\psi} = \Big(1 + \dfrac{h^2}{12}\Delta_h\Big)\widetilde{\omega}\,, \qquad \widetilde{\psi}\mid_\Gamma = 0\,,\\
\widetilde{u} = -\widetilde{D}_y\Big(1 - \dfrac{h^2}{6}D_y^2\Big)\widetilde{\psi}\,, \quad \widetilde{v} = \widetilde{D}_x\Big(1 - \dfrac{h^2}{6}D_x^2\Big)\widetilde{\psi}\,,
\end{cases}
\tag{5.20}
$$

in which the linearized convection error terms $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by (4.31), and the local truncation error terms satisfy

$$
\begin{aligned}
|\boldsymbol{g}| &\le Ch^4(\|\boldsymbol{u}_e\|_{C^{6,\alpha}}\|\theta_e\|_{C^{7,\alpha}} + \|\theta_e\|_{C^{8,\alpha}})\,,\\
|\boldsymbol{f}| &\le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1 + \|\boldsymbol{u}_e\|_{C^5}) + Ch^4\|\theta_e\|_{C^{6,\alpha}}\,.
\end{aligned}
\tag{5.21}
$$

On the boundary, (say on Γ_x, $j = 0$), we have

$$
\begin{aligned}
\widetilde{\psi}_{i,-1} &= 10\widetilde{\psi}_{i,1} - 5\widetilde{\psi}_{i,2} + \frac{5}{3}\widetilde{\psi}_{i,3} - \frac{1}{4}\widetilde{\psi}_{i,4}\,,\\
\widetilde{\omega}_{i,0} &= \frac{1}{h^2}\Big(8\widetilde{\psi}_{i,1} - 3\widetilde{\psi}_{i,2} + \frac{8}{9}\widetilde{\psi}_{i,3} - \frac{1}{8}\widetilde{\psi}_{i,4}\Big) + \boldsymbol{h}_{i,0}\,,
\end{aligned}
\tag{5.22}
$$

with $|\boldsymbol{h}_{i,0}| \le Ch^4\|\boldsymbol{u}_e\|_{C^5}$, which is the same as in (4.31). We then conclude from (5.7), and using the approximations (4.22) and (5.11), that for the temperature field

$$
\begin{aligned}
\Theta_{i,-1} &= \Theta_{i,1} + \frac{h^3}{3\kappa}\,\Omega_{i,0}\widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\Theta_{i,0} - \frac{h^5}{24}\,\partial_y^5\theta_e(x_i,0)\\
&\quad + O(h^7)\Big(\|\theta_e\|_{C^{7,\alpha}} + \|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^{6,\alpha}}\Big)\,,\\
\Theta_{i,-2} &= \Theta_{i,2} + \frac{8h^3}{3\kappa}\,\Omega_{i,0}\widetilde{D}_x\Big(1 - \frac{h^2}{6}D_x^2\Big)\Theta_{i,0} - \frac{7h^5}{12}\,\partial_y^5\theta_e(x_i,0)\\
&\quad + O(h^7)\Big(\|\theta_e\|_{C^{7,\alpha}} + \|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^{6,\alpha}}\Big)\,.
\end{aligned}
\tag{5.23}
$$

Subtracting (5.23) from (2.27), we arrive at

$$
\begin{aligned}
\widetilde{\theta}_{i,-1} &= \widetilde{\theta}_{i,1} + \frac{h^3}{3\kappa}\,\boldsymbol{q}_i^b - \frac{h^5}{24}\,\boldsymbol{r}_i^b + \boldsymbol{e}_i^{b1}\,,\\
\widetilde{\theta}_{i,-2} &= \widetilde{\theta}_{i,2} + \frac{8h^3}{3\kappa}\,\boldsymbol{q}_i - \frac{7h^5}{12}\,\boldsymbol{r}_i + \boldsymbol{e}_i^{b2}\,,
\end{aligned}
\tag{5.24a}
$$

where

(5.24b)

$$\boldsymbol{q}_i^b = \widetilde{\omega}_{i,0}\widetilde{D}_x(1-\frac{h^2}{6}D_x^2)\theta_{i,0} + \Omega_{i,0}\widetilde{D}_x(1-\frac{h^2}{6}D_x^2)\widetilde{\theta}_{i,0}\,, \quad \boldsymbol{r}_i^b = \partial_y^5\theta_e(x_i,0)\,,$$
$$|\boldsymbol{e}_i^{b1}|\,,|\boldsymbol{e}_i^{b2}| \le Ch^7\Big(\|\theta_e\|_{C^{7,\alpha}} + \|\boldsymbol{u}_e\|_{C^5}\|\theta_e\|_{C^{6,\alpha}}\Big)\,.$$

Once again, we observe that the $O(h^5)$ coefficients of $\widetilde{\theta}_{i,-1}$ and $\widetilde{\theta}_{i,-2}$ have the ratio $1 : 14$. Such a ratio is a crucial point in the error analysis of the temperature diffusion term in (5.20), which will be established in detail in Proposition 5.1.

The estimate of the error functions in the system (5.20)–(5.22) and (5.24) is similar to that in section 4. Multiplying the vorticity error equation by $-\Big(1+\dfrac{h^2}{12}\Delta_h\Big)\widetilde{\psi}$ at interior grid points $1 \le i, j \le N-1$ gives the same identity as in (4.32)

(5.25)

$$\begin{aligned}
&-\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\Big(1+\frac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega}\Big\rangle_1\\
&\quad+\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega}\Big\rangle_1\\
&\quad=\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\mathcal{L}_2\Big\rangle_1 - g\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\ \widetilde{D}_x\Big(1+\frac{h^2}{12}(D_y^2-D_x^2)\Big)\widetilde{\theta}\Big\rangle_1\\
&\qquad+\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\boldsymbol{f}\Big\rangle_1\,.
\end{aligned}$$

The energy estimate of the temperature error is different from the Dirichlet boundary condition case since the temperature field is updated at every grid point $0 \le i, j \le N$. Taking the $\langle\ ,\ \rangle_3$ inner product (see the definition in (3.13)) of the temperature error equation with $\widetilde{\theta}$ gives

(5.26)

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_3^2 + \langle\widetilde{\theta},\mathcal{L}_1\rangle_3 = \kappa\Big\langle\widetilde{\theta},\Big(\Delta_h - \frac{h^2}{12}(D_x^4+D_y^4)\Big)\widetilde{\theta}\Big\rangle_3 - \langle\widetilde{\theta},\boldsymbol{g}\rangle_3\,,$$

which is also the same as (4.33) except for the difference of inner product and L^2 norms. Again, this is due to the fact that the temperature field is updated at every grid points $0 \le i, j \le N$.

An estimate for (5.25) is the same as that for (4.32). The identities (4.34)–(4.35), Propositions 4.3 and 4.6, and (4.68) are still valid. More precisely,

$$
\begin{aligned}
&-\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\,\Big(1+\frac{h^2}{12}\Delta_h\Big)\partial_t\widetilde{\omega}\Big\rangle \\
&\quad = -\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\,\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\partial_t\widetilde{\psi}\Big\rangle = \frac{1}{2}\frac{d\widetilde{E}}{dt}\,,
\end{aligned} \tag{5.27}
$$

with

$$
\begin{aligned}
\widetilde{E} &= \|\nabla_h\widetilde{\psi}\|_2^2 - \frac{h^2}{12}\|\Delta_h\widetilde{\psi}\|_1^2 - \frac{h^2}{6}\|D_xD_y\widetilde{\psi}\|_1^2 \\
&\quad + \frac{h^4}{72}\Big(\|D_xD_y^2\widetilde{\psi}\|_1^2 + \|D_yD_x^2\widetilde{\psi}\|_1^2\Big)\,,
\end{aligned}
$$

$$
\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi},\,\Big(\Delta_h+\frac{h^2}{6}D_x^2D_y^2\Big)\widetilde{\omega}\Big\rangle_1 \ge \frac{1}{8}\|\widetilde{\widetilde{\omega}}\|_1^2 - h^8\,, \tag{5.28}
$$

$$
\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\,\mathcal{L}_2\Big\rangle_1 \le \widetilde{C}_1\|\nabla_h\widetilde{\psi}\|_2^2 + \frac{\nu}{8}\|\widetilde{\widetilde{\omega}}\|_1^2 + h^8\,, \tag{5.29}
$$

$$
\begin{aligned}
&\left|\Big\langle\Big(1+\frac{h^2}{12}\Delta_h\Big)\widetilde{\psi}\,,\,\widetilde{D}_x\Big(1-\frac{h^2}{12}(D_y^2-D_x^2\Big)\widetilde{\theta}\Big\rangle_1\right| \le C(\|\widetilde{\psi}\|_1^2 + \|\nabla_h\widetilde{\theta}\|_2^2) \\
&\qquad\qquad + Ch^{10}\,,
\end{aligned} \tag{5.30}
$$

provided the a-priori assumption (4.66) is satisfied, along with $\widetilde{C}_1$ as introduced after (4.67).

Similar to (4.69), the linearized temperature convection term can be controlled by

$$
\left|\langle\widetilde{\theta},\mathcal{L}_1\rangle_3\right| \le \widetilde{C}_2\|\widetilde{\theta}\|_3^2 + \frac{1}{2}\kappa\|\nabla_h\widetilde{\theta}\|_2^2 + h^8\,, \tag{5.31}
$$

with $\widetilde{C}_2 = \frac{C(1+\|\boldsymbol{u}_e\|_{C^0})^2}{\kappa}$.

An estimate of the temperature diffusion term in (4.69) is outlined below. Its proof relies on the stability analysis given in section 3.2 and some error estimates.

Proposition 5.1 *We have*

$$
\begin{aligned}
&-\Big\langle\widetilde{\theta},\Big(\Delta_h-\frac{h^2}{12}(D_x^4+D_y^4)\Big)\widetilde{\theta}\Big\rangle_3 \ge \frac{3}{4}\|\nabla_h\widetilde{\theta}\|_2^2 - C(\|\theta_e\|_{C^{6,\alpha}}+1) \\
&\qquad\qquad \times(\|\nabla_h\widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_3^2) - \frac{1}{2}h^8\,.
\end{aligned} \tag{5.32}
$$

Proof. Summing by parts under the inner product $\langle\ ,\ \rangle_3$ and using the boundary extrapolation (5.24) gives

(5.33)
$$\left\langle \widetilde{\theta}, \left(\Delta_h \frac{h^2}{12}(D_x^4 + D_y^4)\right)\widetilde{\theta} \right\rangle_3 = -\|\nabla_h \widetilde{\theta}\|_2^2 - \frac{h^2}{12}\|D_x^2\widetilde{\theta}\|_3^2 - \frac{h^2}{12}\|D_y^2\widetilde{\theta}\|_3^2 + \mathcal{B}\,.$$

The boundary term $\mathcal{B}$ can be decomposed as

(5.34a)
$$\mathcal{B} = \mathcal{B}^1 + \mathcal{B}^2 + \mathcal{B}^3 + \mathcal{B}^4\,,$$

in which $\mathcal{B}^1$, corresponding to the boundary term along $y = 0$, reads

(5.34b)
$$\begin{aligned}
\mathcal{B}^1 &= \frac{1}{12}\sum_{i=1}^{N-1}\left(\widetilde{\theta}_{i,1}(\widetilde{\theta}_{i,-1} - \widetilde{\theta}_{i,1}) + \frac{1}{2}\widetilde{\theta}_{i,0}\Big[(\widetilde{\theta}_{i,-2} - \widetilde{\theta}_{i,2}) - 16(\widetilde{\theta}_{i,-1} - \widetilde{\theta}_{i,1})\Big]\right)\\
&= \frac{1}{12}\sum_{i=1}^{N-1}\left(\widetilde{\theta}_{i,1}\Big[\frac{h^3}{3\kappa}\boldsymbol{q}_i^b - \frac{h^5}{24}\boldsymbol{r}_i^b + \boldsymbol{e}_i^{b1}\Big]\right.\\
&\quad \left. + \frac{1}{2}\widetilde{\theta}_{i,0}\Big[\frac{8h^3}{3\kappa}\boldsymbol{q}_i^b - \frac{7h^5}{12}\boldsymbol{r}_i^b + \boldsymbol{e}_i^{b2} - 16\Big(\frac{h^3}{3\kappa}\boldsymbol{q}_i^b - \frac{h^5}{24}\boldsymbol{r}_i^b + \boldsymbol{e}_i^{b1}\Big)\Big]\right).
\end{aligned}$$

It should be noted that the derivation of (5.34b) comes from the formula for $\widetilde{\theta}_{i,-1}$ and $\widetilde{\theta}_{i,-2}$ in (5.24a). The definitions of $\boldsymbol{q}_i^b$, $\boldsymbol{r}_i^b$, $\boldsymbol{e}_i^{b1}$, $\boldsymbol{e}_i^{b2}$ were given in (5.24b). The boundary terms along $y = 1$, $x = 0$, and $x = 1$ can be similarly presented. In more detail, $\mathcal{B}^1$ can be simplified as

$$\begin{aligned}
\mathcal{B}^1 &= \frac{h^3}{36\kappa}\sum_{i=1}^{N-1}\boldsymbol{q}_i^b(\widetilde{\theta}_{i,1} - 4\widetilde{\theta}_{i,0}) + \frac{1}{12}\sum_{i=1}^{N-1}\left(\boldsymbol{e}_i^{b1}(\widetilde{\theta}_{i,1} - 8\widetilde{\theta}_{i,0}) + \frac{1}{2}\boldsymbol{e}_i^{b2}\widetilde{\theta}_{i,0}\right)\\
&\quad + \frac{h^5}{288}\sum_{i=1}^{N-1}\boldsymbol{r}_i^b(\widetilde{\theta}_{i,0} - \widetilde{\theta}_{i,1})\\
&\equiv I_1^b + I_2^b + I_3^b\,.
\end{aligned}$$
(5.35)

The first term I_1^b can be controlled by using the form of $\boldsymbol{q}_i^b$ as in (5.24b), namely

$$\begin{aligned}
h^3 \sum_{i=1}^{N-1} \boldsymbol{q}_i^b \widetilde{\theta}_{i,1} &= h^3 \sum_{i=1}^{N-1} \widetilde{\theta}_{i,1} \widetilde{\omega}_{i,0} \widetilde{D}_x (1 - \frac{h^2}{6} D_x^2) \widetilde{\theta}_{i,0} \\
&\quad + h^3 \sum_{i=1}^{N-1} \widetilde{\theta}_{i,1} \Omega_{i,0} \widetilde{D}_x (1 - \frac{h^2}{6} D_x^2) \widetilde{\theta}_{i,0} \\
&\le C \|\theta\|_{W^{1,\infty}} \|\nabla_h \widetilde{\psi}\|_2 \|\widetilde{\theta}\|_3 + Ch \|\Omega\|_{L^\infty} \|\widetilde{\theta}\|_3 \|\nabla_h \widetilde{\theta}\|_2 + Ch^{10} \\
&\le C(\|\theta_e\|_{C^{6,\alpha}} + 1)(\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_3^2) + Ch(\|\boldsymbol{u}_e\|_{C^5} + 1) \\
&\quad (\|\widetilde{\theta}\|_3^2\| + \|\nabla_h \widetilde{\theta}\|_2^2) + Ch^{10} ,
\end{aligned} \tag{5.36}$$

in which the first inequality comes from the boundary formula for $\widetilde{\omega}_{i,0}$ in (5.22). The second inequality results from the estimate (5.9), (4.6), and the a-priori assumption (4.66). A similar result can be obtained for $h^3 \sum_{i=1}^{N-1} \boldsymbol{q}_i^b \widetilde{\theta}_{i,0}$. Then we arrive at

$$\begin{aligned}
I_1^b &\le C(\|\theta_e\|_{C^{6,\alpha}} + 1)(\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_3^2) + Ch(\|\boldsymbol{u}_e\|_{C^5} + 1) \\
&\quad (\|\widetilde{\theta}\|_3^2\| + \|\nabla_h \widetilde{\theta}\|_2^2) + Ch^{10} .
\end{aligned} \tag{5.37}$$

The term I_2^b can be controlled by Cauchy's inequality and the estimate (5.24b), giving

$$I_2^b \le C \left| \sum_{i=1}^{N-1} \boldsymbol{e}_i^{b1} \widetilde{\theta}_{i,1} \right| + C \left| \sum_{i=1}^{N-1} \boldsymbol{e}_i^{b1} \widetilde{\theta}_{i,0} \right| \le C \|\widetilde{\theta}\|_3^2 + Ch^{10} . \tag{5.38}$$

What remains is the estimate of I_3^b. As can be seen, the detailed estimates for $\widetilde{\theta}_{i,-1}$ and $\widetilde{\theta}_{i,-2}$ in (5.24), which shows that the $O(h^5)$ coefficients of $\widetilde{\theta}_{i,-1}$, $\widetilde{\theta}_{i,-2}$ have the ratio 1 : 14, allows the term I_3^b to be written as $\frac{h^5}{288} \sum_{i=1}^{N-1} \boldsymbol{r}_i^b (\widetilde{\theta}_{i,0} - \widetilde{\theta}_{i,1})$. That is crucial to implement the error analysis below. An application of Cauchy's inequality shows that

$$\begin{aligned}
I_3^b = \frac{h^5}{288} \sum_{i=1}^{N-1} \boldsymbol{r}_i^b (\widetilde{\theta}_{i,0} - \widetilde{\theta}_{i,1}) &\le \frac{1}{288} h^2 \sum_{i=1}^{N-1} \frac{(\widetilde{\theta}_{i,0} - \widetilde{\theta}_{i,1})^2}{h^2} \\
&\quad + \frac{1}{288} h^{10} \sum_{i=1}^{N-1} \boldsymbol{r}_i^2 .
\end{aligned} \tag{5.39}$$

It is observed that the first term appearing above can be absorbed into $\|\nabla_h \widetilde{\theta}\|_2^2$. Meanwhile, we note that $\boldsymbol{r}_i^b = \partial_y^5 \theta_e(x_i, 0)$, which is a bounded quantity on $y = 0$. Then we get

$$I_3^b = \frac{h^5}{288} \sum_{i=1}^{N-1} \boldsymbol{r}_i^b (\widetilde{\theta}_{i,0} - \widetilde{\theta}_{i,1}) \le \frac{1}{288} \|\nabla_h \widetilde{\theta}\|_2^2 + Ch^9 . \tag{5.40}$$

The combination of (5.37)–(5.38) and (5.40) leads to

$$\mathcal{B}^1 \le C(\|\theta_e\|_{C^{6,\alpha}} + 1)(\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_3^2) + \frac{1}{288}\|\nabla_h \widetilde{\theta}\|_2^2 + \frac{1}{8}h^8 . \tag{5.41}$$

The other three boundary terms $\mathcal{B}^2$, $\mathcal{B}^3$, $\mathcal{B}^4$ can be treated similarly. As a result, we arrive at

$$\mathcal{B} \le C(\|\theta_e\|_{C^{6,\alpha}} + 1)(\|\nabla_h \widetilde{\psi}\|_2^2 + \|\widetilde{\theta}\|_3^2) + \frac{1}{72}\|\nabla_h \widetilde{\theta}\|_2^2 + \frac{1}{2}h^8 . \tag{5.42}$$

The insertion of (5.42) into (5.33) implies (5.32). Proposition 5.1 is proven. □

By the combination of (5.25)–(5.32) we have the following inequality

$$\begin{aligned}\frac{1}{2}\frac{d\widetilde{E}}{dt} + \frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_3^2 \le{}& Ch^8 + C(\|\boldsymbol{f}\|_3^2 + \|\boldsymbol{g}\|_3^2) + \widetilde{C}_1\|\nabla_h \widetilde{\psi}\|_2^2 + \widetilde{C}_2\|\widetilde{\theta}\|_3^2 \\ & -\frac{\kappa}{8}\|\nabla_h \widetilde{\theta}\|_2^2 . \end{aligned} \tag{5.43}$$

The proof of Theorem 1.2 can be carried out by using a similar argument as in (4.78)–(4.83). The details are left to the interested reader. □

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