Numer. Math. (2002) 91: 543–576
Digital Object Identifier (DOI) 10.1007/s002110100311

Numerische Mathematik

Analysis of finite difference schemes for unsteady Navier-Stokes equations in vorticity formulation

Cheng Wang[1,⋆], **Jian-Guo Liu**[2,⋆⋆]

[1] Institute for Scientific Computing and Applied Mathematics and Department of Mathematics, Indiana University, Bloomington, IN 47405, USA; e-mail: cwang@indiana.edu
[2] Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, MD 20742, USA; e-mail: jliu@math.umd.edu

Received December 10, 1999 / Revised version received November 5, 2000 /
Published online August 17, 2001 –

Summary. In this paper, we provide stability and convergence analysis for a class of finite difference schemes for unsteady incompressible Navier-Stokes equations in vorticity-stream function formulation. The no-slip boundary condition for the velocity is converted into local vorticity boundary conditions. Thom's formula, Wilkes' formula, or other local formulas in the earlier literature can be used in the second order method; while high order formulas, such as Briley's formula, can be used in the fourth order compact difference scheme proposed by E and Liu. The stability analysis of these long-stencil formulas cannot be directly derived from straightforward manipulations since more than one interior point is involved in the formula. The main idea of the stability analysis is to control local terms by global quantities via discrete elliptic regularity for stream function. We choose to analyze the second order scheme with Wilkes' formula in detail. In this case, we can avoid the complicated technique necessitated by the Strang-type high order expansions. As a consequence, our analysis results in almost optimal regularity assumption for the exact solution. The above methodology is very general. We also give a detailed analysis for the fourth order scheme using a 1-D Stokes model.

Mathematics Subject Classification (1991): 65M06, 76M20

⋆ Supported by NSF grant DMS-9505275
⋆⋆ Supported by NSF grant DMS-9505275
Correspondence to: C. Wang

1. Introduction

The 2-D Navier-Stokes equations in vorticity-stream function formulation read

$$
(1.1)\qquad \begin{cases} \partial_t \omega + \nabla\cdot(\boldsymbol{u}\omega) = \nu \Delta\omega\,, \\ \Delta\psi = \omega\,, \\ u = -\partial_y \psi\,, \qquad v = \partial_x \psi\,, \end{cases}
$$

where $\boldsymbol{u} = (u, v)$ denotes the velocity field, $\omega = \nabla \times \boldsymbol{u} = -\partial_y u + \partial_x v$ denotes the vorticity. The no-penetration, no-slip boundary condition can be written in terms of the stream function ψ

$$
(1.2)\qquad \psi = C_i, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0\,, \qquad \text{at each boundary section } \ \Gamma_i\,,
$$

where C_i is a constant for each Γ_i. In the simply-connected domain, C_0 can be taken as 0.

In 2-D incompressible flow, the above formulation has the advantage that it not only eliminates the pressure variable, but also enforces the incompressibility automatically. Thus it makes computation very convenient. Yet, the main difficulties in the numerical simulation of (1.1), (1.2) are the boundary conditions:

1. The implementation of the two boundary conditions for the stream function in (1.2).

2. When the vorticity is updated in time (in the momentum equation), there is no definite boundary condition for vorticity (see the detail in [15]).

3. Determining the constants C_i at each boundary of “holes” Γ_i if the computational domain is multi-connected.

The methodology to overcome the first two difficulties is to solve for the stream function using Dirichlet boundary condition $\psi = 0$ on Γ, and then to calculate the vorticity at the boundary by a local formula, which is derived from the kinematic relation $\Delta\psi = \omega$, combined with no-slip boundary condition $\frac{\partial \psi}{\partial \boldsymbol{n}} = 0$. In other words, the vorticity boundary condition enforces no-slip boundary condition. Some related work can be found in [15], [4], [5]. The issue related to the third difficulty will be discussed in the forthcoming paper [19].

The subject of vorticity boundary condition has a long history, going back to Thom’s formula in 1933. See [6], [14], [17], [11], [15], [4]. In this paper, we concentrate on the case of local vorticity boundary condition, thus avoid biharmonic equation and coupling between the kinematic equation and the vorticity boundary condition. This approach dramatically simplifies the computation. In the second order scheme, Thom’s formula, Wilkes’ formula, or some other local formulas can be selected and coupled with centered difference scheme at the interior points, as discussed in detail in

[4]. The advantage of Thom's formula lies in its simplicity: only one interior point of stream function is involved in it. Thus the stability of it is very straightforward, as argued in Sect. 3. Yet, it was always very confusing why formulas like Thom's, which seems hopelessly first order by formal Taylor expansion on the boundary, is actually second order accurate. This mystery can be explained by Strang-type high order expansions. It was proven in [16] in 1964 that for nonlinear hyperbolic or parabolic equations, the L^2-stability for the linearized problem and the smoothness of the exact solution implies that the scheme has full accuracy in L^∞ norm. The main idea in the proof is the construction of high order expansions with respect to the scheme. In his thesis [10], Meth proved the stability for the linearized problem. The theoretical convergence analysis of the second order scheme with Thom's formula on the boundary was given by Hou and Wetton in [8]. It relied upon Strang-type high order expansions, which resulted in much more regularity assumption of the exact solution than needed. A technical assumption of one-sided physical, one-sided periodic boundary condition was imposed.

To emphasize the main point of this paper, we should mention that the stability of Thom's formula cannot be applied to long-stencil formulas automatically, as Wilkes' formula and other local formulas. It was a doubtful question for a long time: are these long-stencil formulas stable? In this paper, we answer this question by performing a simple, clean analysis of the second order scheme using Wilkes' formula to determine the vorticity on the boundary. In fact, direct calculations and standard local estimates cannot work it out. We overcome this difficulty, which comes from the boundary term, by adopting a new technique: recasting the form of the formula in terms of the second derivative of ψ near the boundary, e.g. $D_x^2\psi$, $D_y^2\psi$, whose L^2 norms can be controlled by the L^2 norm of $(D_x^2 + D_y^2)\psi$, which is just ω. Such bound can be viewed as elliptic regularity in the discrete level. In other words, the stability of Wilkes' formula is guaranteed by controlling local terms by global terms, then applying elliptic regularity to control the global terms by the diffusion term. Another crucial point is that Wilkes' formula gives second order accuracy for the vorticity on the boundary by formal Taylor expansion. Thus, the consistency analysis is easier than that of Thom's formula; no Strang-type expansion is needed. This fact leads to almost optimal regularity assumption for the exact solution in the main theorem.

The method used in the analysis of the second order scheme and the corresponding vorticity boundary formulas is quite general. It can be applied to fourth order scheme in a similar way. For example, E and Liu proposed their Essentially Compact fourth order scheme (EC4) in [5], and proved the fourth order convergence of the method. We will give an analysis of their proposed fourth order scheme, using a 1-D model for the Stokes equations.

The boundary condition for vorticity, such as Briley's formula in [2], will also be analyzed in detail. The advantage of this 1-D model is its simplicity of illustration, both for stability and consistency analysis. Since Briley's formula indicates only third order accuracy on the boundary, by formal Taylor expansion, the consistency analysis of it is accomplished by making expansions which are implemented by a third order polynomial. That is a Strang-type analysis. In addition, we should note that the consistency analysis of the fourth order scheme is much more tricky than that of the second order method, since the fourth order scheme involves an intermediate variable for vorticity. The technique used in the stability analysis is similar to that of Wilkes' formula: applying elliptic regularity for the stream function at the discrete level, and then controlling these local terms induced from the boundary by global terms. This shows that our analysis gives a general methodology to deal with long-stencil formulas. The convergence analysis of the EC4 scheme with Briley's formula is given in this paper, illustrating why fourth order scheme combined with third order boundary formula still works. The complete analysis of the corresponding scheme for full nonlinear NSE in 2-D will appear in [18].

In Sect. 2 we describe the second order scheme for 2-D NSE with a $[0,1] \times [0,1]$ box as the domain. Then, the detailed stability analysis of Wilkes' formula in the case of linear Stokes equations is given in Sect. 3. In Sect. 4, we show the convergence of the second order scheme combined with Wilkes' formula. In Sect. 5, we analyze the EC4 scheme using the 1-D model for Stokes equations. Stability and convergence analysis of the method using Briley's formula is presented, by which we hope to explain the ideas in our consistency analysis clearly, including Strang-type expansion and the construction of the approximate profiles. In Sect. 6, the numerical accuracy checks for both the second and fourth order schemes are presented.

2. Description of the second order scheme

Navier-Stokes equations in 2-D with no-slip boundary condition imposed on both sides are taken into consideration. For simplicity of presentation, we take the computation domain as $\Omega = [0,1] \times [0,1]$ with grid size $\Delta x = \Delta y = h$. The no-slip boundary conditions are imposed at $\{y = 0, 1\}$ and $\{x = 0, 1\}$, denoted by Γ_x and Γ_y, respectively. The associated numerical grids are denoted by $\{x_i = i/N, y_j = j/N,\ i, j = 0, 1, \cdots, N\}$. At these grid points, NSE can be discretized by standard centered difference formulas:

(2.1)
$$\begin{cases} \partial_t \omega + \widetilde{D}_x(u\omega) + \widetilde{D}_y(v\omega) = \nu \Delta_h \omega \,, \\ \Delta_h \psi = \omega \,, \qquad \psi \mid_\Gamma = 0 \,, \\ u = -\widetilde{D}_y \psi \,, \qquad v = \widetilde{D}_x \psi \,, \end{cases}$$

where $\widetilde{D}_x$, $\widetilde{D}_y$ are the centered difference operators

(2.2)
$$\widetilde{D}_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \,, \qquad \widetilde{D}_y u_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h} \,,$$

and Δ_h is the standard 5-point Laplacian $\Delta_h = D_x^2 + D_y^2$, where

(2.3)
$$D_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \,, \qquad D_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \,.$$

As pointed out in the introduction, there are two boundary conditions for ψ. The Dirichlet boundary condition $\psi = 0$ on Γ is implemented to solve the stream function via the vorticity as in (2.1). Yet the normal boundary condition, $\frac{\partial \psi}{\partial \boldsymbol{n}} = 0$, cannot be enforced directly. The way to overcome this difficulty is to convert it into the boundary condition for vorticity. As can be seen by the fact that $\psi \mid_\Gamma = 0$, we have the approximation for the vorticity on the boundary (say on Γ_x, $j = 0$)

(2.4)
$$\omega_{i,0} = D_y^2 \psi_{i,0} = \frac{1}{h^2}(\psi_{i,1} + \psi_{i,-1}) = \frac{2\psi_{i,1}}{h^2} - \frac{2}{h} \frac{\psi_{i,1} - \psi_{i,-1}}{2h} \,,$$

where $(i, -1)$ refers to the "ghost" grid point outside of the computational domain. Taking the approximation identity $\frac{\psi_{i,1} - \psi_{i,-1}}{2h} = 0$, which implies that $\psi_{i-1} = \psi_{i,1}$, as a second order normal boundary condition for $(\partial_y \psi)_{i,0} = 0$, we arrive at **Thom's formula**

(2.5)
$$\omega_{i,0} = \frac{2\psi_{i,1}}{h^2} \,.$$

We should mention here that by formal Taylor expansion, Thom's formula is only first order accurate for ω on the boundary. More sophisticated consistency analysis can guarantee that the scheme is indeed second order accurate, which was first proven in [8].

The vorticity on the boundary can also be determined by other approximations of $\psi_{i,-1}$. For example, using a third order one-sided approximation for the normal boundary condition $\frac{\partial \psi}{\partial \boldsymbol{n}} = 0$

(2.6)
$$(\partial_y \psi)_{i,0} = \frac{-\psi_{i,-1} + 3\psi_{i,1} - \frac{1}{2}\psi_{i,2}}{3h} = 0 \,,$$

which leads to

$$\psi_{i,-1} = 3\psi_{i,1} - \frac{1}{2}\psi_{i,2}\,,$$

and then plugging it back into the difference vorticity formula $\omega_{i,0} = \frac{1}{h^2}(\psi_{i,1} + \psi_{i,-1})$ as in (2.4), we have **Wilkes-Pearson's formula**

$$\omega_{i,0} = \frac{1}{h^2}\left(4\psi_{i,1} - \frac{1}{2}\psi_{i,2}\right)\,. \tag{2.7}$$

See [13] for more details. This formula gives us second order accuracy for the vorticity on the boundary.

2.1. Time discretization

The implementation of the time discretization of the scheme (2.1), along with the vorticity boundary condition we mentioned above, either (2.5) or (2.7), or other local formulas used in the earlier literature, was discussed in detail in [4]. There are two main points: first, the vorticity profile ω in the interior points (x_i, y_j) $1 \le i, j \le N-1$ is updated by the momentum equation, which is enough to solve for the stream function, since Dirichlet boundary condition is imposed for ψ; then, local boundary formulas for vorticity, either (2.5) or (2.7), can be applied, since the stream function values at the interior points have been obtained in the last step. Thus the momentum equation can be updated again.

All the above procedure can be implemented very efficiently via explicit treatment of the diffusion term. That avoids the coupling between the momentum equation and the kinematic equation, and therefore makes the whole scheme very robust. This approach differs markedly from the global boundary condition for vorticity described in earlier literature. To overcome cell-Reynolds number constraint, high order Runge-Kutta time-stepping, such as Rk3 or RK4, was suggested, as discussed in detail in [4]. We refer to [9] for a finite element version of the above mentioned scheme to handle the general domain.

3. Stability of Wilkes' formula for Stokes equations

One of the main concerns in the computation of Navier-Stokes equations is numerical stability. For simplicity, we only consider Stokes equations in this section, where nonlinear terms are neglected. The second order scheme applied to Stokes equations corresponding to (2.1) turns out to be

$$\begin{cases} \partial_t \omega = \nu \Delta_h \omega\,, \\ \Delta_h \psi = \omega\,, \qquad \psi\,|_{\Gamma} = 0\,, \end{cases} \tag{3.1}$$

and either Thom's formula (2.5) or Wilkes' formula (2.7) can be chosen to implement the system (3.1).

Now we introduce some notations.

Definition 3.1 The discrete L^2-norm and L^2-inner product are defined as

$$\|u\| = \langle u\,,\,u\rangle^{1/2}\,, \qquad \langle u\,,\,v\rangle = \sum_{1\le i,j\le N-1} u_{i,j}\,v_{i,j}\,h^2\,. \tag{3.2}$$

In the case that $u\,|_\Gamma = 0$, the notation $\|\nabla_h u\|$ is introduced by

$$\|\nabla_h u\|^2 = \sum_{j=1}^{N-1}\sum_{i=0}^{N-1}(D_x^+ u_{i,j})^2 h^2 + \sum_{i=1}^{N-1}\sum_{j=0}^{N-1}(D_y^+ u_{i,j})^2 h^2\,, \tag{3.3}$$

where $D_x^+ u$, $D_y^+ u$ are defined as

$$D_x^+ u_{i,j} = \frac{u_{i+1,j}-u_{i,j}}{h}\,, \quad D_y^+ u_{i,j} = \frac{u_{i,j+1}-u_{i,j}}{h}\,. \tag{3.4}$$

Similar notations of L^2 norms and inner products, one-sided difference norms in 1-D analogous to (3.2)-(3.4) can also be introduced. These 1-D notations will be used in Sect. 5.

First, we look at the stability argument of Thom's formula, which is straightforward. For Stokes equations, the stability analysis can be described as following: multiplying the momentum equation in (3.1) by $-\psi$, we have $-\langle\psi,\partial_t\omega\rangle + \langle\psi,\Delta_h\omega\rangle = 0$. The first term is exactly

$$-\langle\psi,\partial_t\omega\rangle = -\langle\psi,\partial_t\Delta_h\psi\rangle = \frac{1}{2}\frac{d}{dt}\|\nabla_h\psi\|^2\,, \tag{3.5}$$

where in the second step we used the fact that ψ vanishes on the boundary. The second term can be estimated via summing by parts

$$\langle\psi,\Delta_h\omega\rangle = \langle\Delta_h\psi,\omega\rangle + \mathcal{B} = \|\omega\|^2 + \mathcal{B}\,, \tag{3.6}$$

where we used the fact that $\Delta_h\psi = \omega$. The boundary term $\mathcal{B}$ is decomposed into four parts: $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$

$$\begin{aligned} \mathcal{B}_1 &= \sum_{i=1}^{N-1}\psi_{i,1}\omega_{i,0}\,, \mathcal{B}_2 = \sum_{i=1}^{N-1}\psi_{i,N-1}\omega_{i,N}\,, \\ \mathcal{B}_3 &= \sum_{j=1}^{N-1}\psi_{1,j}\omega_{0,j}\,, \mathcal{B}_4 = \sum_{j=1}^{N-1}\psi_{N-1,j}\omega_{N,j}\,. \end{aligned} \tag{3.7}$$

As can be seen, to ensure the stability of the scheme, an estimate to control the boundary term $\mathcal{B}$ is required. For simplicity of presentation, we only consider $\mathcal{B}_1$ here. Thom's boundary condition (2.5) is applied to recover $\mathcal{B}_1$

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \psi_{i,1} \cdot \frac{2\psi_{i,1}}{h^2} = \sum_{i=1}^{N-1} \frac{2\psi_{i,1}^2}{h^2} \geq 0 \,. \tag{3.8}$$

The estimate for other three boundary terms can be obtained in the same way. Then we have $\mathcal{B} \geq 0$, whose substitution into (3.6), along with (3.5) gives us the stability of the second order scheme (3.1) with Thom's formula (2.5). This observation was first made by Meth in [10]. It was also used in [8] to prove the convergence of Thom's formula.

It can be seen that Wilkes' formula (2.7) involves more interior points than Thom's formula (2.5). A natural question arises: is Wilkes' formula stable?

We follow the same procedure as above. The identities (3.5), (3.6) are still valid. The only difference is the boundary term $\mathcal{B}$, which is still represented as in (3.7). Similarly, we only consider $\mathcal{B}_1$ here. Wilkes' boundary condition (2.7) is applied to recover $\mathcal{B}_1$

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \frac{\psi_{i,1}}{h^2} \left(4\psi_{i,1} - \frac{1}{2}\psi_{i,2} \right) \,. \tag{3.9}$$

However, a direct calculation cannot control $\mathcal{B}_1$, since two interior points $\psi_{i,1}$ and $\psi_{i,2}$ of stream function are involved in Wilkes' formula. Therefore, the straightforward argument (3.8) does not work here. To overcome this difficulty, we can apply the property that ψ vanishes on the boundary and then rewrite the term $4\psi_{i,1} - \frac{1}{2}\psi_{i,2}$ as

$$4\psi_{i,1} - \frac{1}{2}\psi_{i,2} = 3\psi_{i,1} - \frac{1}{2} h^2 D_y^2 \psi_{i,1} \,. \tag{3.10}$$

This transformation is to control local terms by global terms, the purpose of which can be seen later. Now $\mathcal{B}_1$ can be estimated via applying Cauchy inequality for $\psi_{i,1} \cdot D_y^2 \psi_{i,1}$

$$\begin{aligned} \mathcal{B}_1 &= \sum_{i=1}^{N-1} \frac{\psi_{i,1}}{h^2} \left(3\psi_{i,1} - \frac{1}{2} h^2 D_y^2 \psi_{i,1} \right) \\ &\geq \sum_{i=1}^{N-1} \left(\frac{3\psi_{i,1}^2}{h^2} - \frac{1}{2h^2} \frac{1^2}{2^2} \psi_{i,1}^2 - \frac{1}{2} |D_y^2 \psi_{i,1}|^2 h^2 \right) \\ &\geq - \sum_{i=1}^{N-1} \frac{1}{2} |D_y^2 \psi_{i,1}|^2 h^2 \,, \end{aligned} \tag{3.11}$$

where we used the fact that $3 - \frac{1}{2} \cdot \frac{1^2}{2^2} \geq 0$ in the last step. Repeating the same argument for $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, we arrive at

$$
\begin{aligned}
\mathcal{B} &\geq -\frac{1}{2} \sum_{i=1}^{N-1} (|D_y^2 \psi_{i,1}|^2 + |D_y^2 \psi_{i,N-1}|^2) h^2 \\
&\quad -\frac{1}{2} \sum_{j=1}^{N-1} (|D_x^2 \psi_{1,j}|^2 + |D_x^2 \psi_{N-1,j}|^2) h^2 \\
&\geq -\frac{1}{2} \|D_x^2 \psi\|^2 - \frac{1}{2} \|D_y^2 \psi\|^2 .
\end{aligned}
\tag{3.12}
$$

As can be seen, the transformation (3.10) helps us to bound the boundary term, which is a local term, by global terms $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$. Our next aim is to control the terms $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$ by the diffusion term $\|\omega\|^2$. The following lemma is necessary.

Lemma 3.2 *For any ψ such that $\psi \mid_\Gamma = 0$, we have*

$$
\|D_x^2 \psi\|^2 + \|D_y^2 \psi\|^2 \leq \|(D_x^2 + D_y^2)\psi\|^2 = \|\omega\|^2 . \tag{3.13}
$$

Proof. Since $\psi_{i,j}$ is zero on Γ, we can take Sine transforms for $\{\psi_{i,j}\}$ in both i-direction and j-direction, i.e.,

$$
\psi_{i,j} = \sum_{k,\ell} \widehat{\psi}_{k,\ell} \sin(k\pi x_i) \sin(\ell \pi y_j) . \tag{3.14}
$$

Then Parserval equality gives

$$
\sum_{i,j} (\psi_{i,j})^2 = \sum_{k,\ell} \left| \widehat{\psi}_{k,\ell} \right|^2 . \tag{3.15}
$$

If we introduce

$$
f_k = -\frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2} \right) , \qquad g_\ell = -\frac{4}{h^2} \sin^2 \left(\frac{\ell \pi h}{2} \right) , \tag{3.16}
$$

we obtain the Fourier expansion of $D_x^2\psi$ and $D_y^2\psi$

$$
\begin{aligned}
D_x^2 \psi_{i,j} &= \sum_{k,l} f_k \, \widehat{\psi}_{k,l} \sin(k\pi x_i) \sin(\ell \pi y_j) , \\
D_y^2 \psi_{i,j} &= \sum_{k,l} g_\ell \, \widehat{\psi}_{k,l} \sin(k\pi x_i) \sin(\ell \pi y_j) ,
\end{aligned}
\tag{3.17}
$$

which implies that

$$
\sum_{i,j} |\omega_{i,j}|^2 = \sum_{i,j} |\Delta_h \psi_{i,j}|^2 = \sum_{k,\ell} |g_\ell + f_k|^2 \, |\widehat{\psi}_{k,\ell}|^2 . \tag{3.18}
$$

Since $f_k \le 0$, $g_\ell \le 0$, which indicates that $(f_k + g_\ell)^2 \ge f_k^2 + g_\ell^2$, we arrive at

$$\sum_{i,j} |\omega_{i,j}|^2 \ge \sum_{k,\ell} (f_k^2 + g_\ell^2) |\widehat{\psi}_{k,\ell}|^2 = \sum_{i,j} (|D_x^2 \psi_{i,j}|^2 + |D_y^2 \psi_{i,j}|^2) \, , \tag{3.20}$$

which shows exactly (3.13). Lemma 3.2 is proven. □

The combination of Lemma 3.2 and the inequality (3.12) gives us that $\mathcal{B} \ge -\frac{1}{2}\|\omega\|^2$. Plugging back into (3.6), along with (3.5), we have the stability estimate of the second order scheme with Wilkes' boundary condition

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h \psi\|^2 + \frac{1}{2}\nu\|\omega\|^2 \le 0 \, . \tag{3.20}$$

Remark 3.3 The purpose of Lemma 3.2 is to control L^2 norms of $D_x^2\psi$ and $D_y^2\psi$ by the discrete Laplacian of ψ, which enables us to control local terms by the global diffusion term. In fact, it is a discrete version of the elliptic regularity for (discrete) Poisson equation. This type of estimate was first used in [5].

Remark 3.4 Let's review the stability analysis for Wilkes' formula. The main difficulty comes from the boundary term. Our trick is to rewrite it via the boundary condition for vorticity, therefore to convert it into an expression in terms of ψ near the boundary. Next, we apply Cauchy inequality to bound it by $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$. Then we can apply an estimate like (3.13), to control $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$ by $\|\omega\|^2$, which leads to the bound of the boundary term by the diffusion term.

4. Analysis of second order scheme for 2-D NSE

We state our main theorem in this paper.

Theorem 4.1 *Let $\boldsymbol{u}_e \in L^\infty([0,T]; C^{5,\alpha}(\overline{\Omega}))$, ψ_e, ω_e be the exact solution of the Navier-Stokes equations (1.1), (1.2) and $\boldsymbol{u}_h$, ω_h be the approximate solution of the second order scheme (2.1) with Pearson-Wilkes formula (2.7), then we have*

$$\begin{aligned} &\|\boldsymbol{u}_e - \boldsymbol{u}_h\|_{L^\infty([0,T],L^2)} + \sqrt{\nu}\|\omega_e - \omega_h\|_{L^2([0,T],L^2)} \\ &\quad \le Ch^2 \|\boldsymbol{u}_e\|_{L^\infty([0,T],C^{5,\alpha})} \left(1 + \|\boldsymbol{u}_e\|_{L^\infty([0,T],C^3)}\right) \\ &\qquad \cdot \mathit{exp}\left\{ \frac{CT}{\nu}\left(1 + \|\boldsymbol{u}_e\|^2_{L^\infty([0,T],C^1)}\right) \right\} \, . \end{aligned} \tag{4.1}$$

In the convergence proof, we follow the standard procedure of consistency, stability and error analysis. Difficulty in the consistency analysis arises from the fact that centered difference is used at the interior points, while a one-sided formula is used for the vorticity on the boundary. This difficulty is overcome by our construction of an approximate vorticity through finite differences of the exact stream function. All of the truncation errors are then lumped into the momentum equation. Since Wilkes' formula is second order accurate on the boundary, Strang-type expansion can be avoided. This results in an easy consistency analysis near the boundary, which shows that the error function for the vorticity on the boundary is of order $O(h^2)$. The stability of Wilkes' formula has already been established in Sect. 3, thus the validity of the error analysis is guaranteed.

4.1. Consistency analysis

Let $\Psi_{i,j} = \psi_e(x_i, y_j)$ for $-1 \le i, j \le N+1$, (here we extend ψ_e smoothly to $[-\delta, 1+\delta]^2$), and construct U, V, Ω through the finite difference of Ψ to maintain the consistency, especially near the boundary,

$$U_{i,j} = -\widetilde{D}_y \Psi \,, \quad V_{i,j} = \widetilde{D}_x \Psi \,, \quad \Omega_{i,j} = \Delta_h \Psi \,, \qquad \text{for } 0 \le i, j \le N \,. \tag{4.2}$$

Then direct Taylor expansion for ψ_e up to the boundary gives us that at grid points (x_i, y_j), $0 \le i, j \le N$,

$$\begin{aligned} U &= u_e - \frac{h^2}{6}\partial_y^3 \psi_e + O(h^3)\|\psi_e\|_{C^4} \,, \\ V &= v_e + \frac{h^2}{6}\partial_x^3 \psi_e + O(h^3)\|\psi_e\|_{C^4} \,, \\ \Omega &= \omega_e + \frac{h^2}{12}(\partial_x^4 + \partial_y^4)\psi_e + O(h^4)\|\psi_e\|_{C^6} \,. \end{aligned} \tag{4.3}$$

It is obvious that at these grid points, (including boundary points),

$$|U - u_e| + |V - v_e| + |\Omega - \omega_e| \le C h^2 \|\psi_e(\cdot, t)\|_{C^4} \,. \tag{4.4}$$

Now we look at the local truncation errors. We will show that the constructed U, V, Ω satisfy the numerical scheme (2.1), (2.7) up to $O(h^2)$ error. First we look at the diffusion term. The estimate (4.3) indicates

$$\Delta_h \Omega = \Delta_h \omega_e + O(h^2)\|\psi_e\|_{C^6} \,, \tag{4.5}$$

which along with Taylor expansion of ω_e that

$$\Delta_h \omega_e = \Delta \omega_e + O(h^2)\|\omega_e\|_{C^4} = \Delta \omega_e + O(h^2)\|\psi_e\|_{C^6} \,, \tag{4.6}$$

leads to the estimate of the diffusion term: at grid points (x_i, y_j), $1 \le i, j \le N-1$,

$$\Delta_h \Omega = \Delta \omega_e + O(h^2)\|\psi_e(\cdot, t)\|_{C^6} \,. \tag{4.7}$$

The nonlinear convection terms can be treated in a similar fashion. It is implied by (4.3) that, at grid points (x_i, y_j), $0 \le i, j \le N$,

$$\begin{aligned} U\Omega = {} & u_e \omega_e - \frac{h^2}{6} \omega_e \partial_y^3 \psi_e + \frac{h^2}{12} u_e (\partial_x^4 + \partial_y^4) \psi_e \\ & + O(h^3) \|\psi_e\|_{C^6} \|\psi_e\|_{C^4} \,, \end{aligned} \tag{4.8}$$

which leads to the estimate at interior grid points (x_i, y_j), $1 \le i, j \le N-1$,

$$\begin{aligned} \widetilde{D}_x(U\Omega) &= \widetilde{D}_x(u_e\omega_e) - \frac{h^2}{6}\widetilde{D}_x(\omega_e \partial_y^3 \psi_e) + \frac{h^2}{12}\widetilde{D}_x \left(u_e(\partial_x^4 + \partial_y^4)\psi_e \right) \\ &\quad + O(h^2)\|\psi_e\|_{C^4}\|\psi_e\|_{C^6} \\ &= \widetilde{D}_x(u_e\omega_e) + O(h^2)(\|\omega_e \partial_y^3 \psi_e\|_{C^1} + \|u_e(\partial_x^4 + \partial_y^4)\psi_e\|_{C^1} \\ &\quad + \|\psi_e\|_{C^6}\|\psi_e\|_{C^4}) \\ &= \widetilde{D}_x(u_e\omega_e) + O(h^2)\|\psi_e\|_{C^4}\|\psi_e\|_{C^6} \,. \end{aligned} \tag{4.9}$$

Moreover, Taylor expansion for $u_e \omega_e$ gives

$$\begin{aligned} \widetilde{D}_x(u_e\omega_e) &= \partial_x(u_e\omega_e) + O(h^2)\|u_e\omega_e\|_{C^3} \\ &= \partial_x(u_e\omega_e) + O(h^2)\|\psi_e\|_{C^5}\|\psi_e\|_{C^3} \,. \end{aligned} \tag{4.10}$$

Then we arrive at

$$\widetilde{D}_x(U\Omega) = \partial_x(u_e\omega_e) + O(h^2)\|\psi_e\|_{C^4}\|\psi_e\|_{C^6} \,. \tag{4.11}$$

The similar result can be obtained for $\widetilde{D}_y(V\Omega)$:

$$\widetilde{D}_y(V\Omega) = \partial_y(v_e\omega_e) + O(h^2)\|\psi_e\|_{C^4}\|\psi_e\|_{C^6} \,. \tag{4.12}$$

Next, we deal with the time marching term $\partial_t \Omega$. The strategy here is to control the difference between $\partial_t \Omega$ and $\partial_t \omega_e$ by $O(h^2)$ of $\|\partial_t \psi_e\|_{C^4}$:

$$\begin{aligned} \partial_t \Omega - \partial_t \omega_e &= \Delta_h \partial_t \psi_e - \Delta \partial_t \psi_e = (\Delta_h - \Delta)\partial_t \psi_e \\ &= O(h^2)\|\partial_t \psi_e\|_{C^4} \,. \end{aligned} \tag{4.13}$$

Yet, to get an estimate of $\|\partial_t\psi_e\|_{C^4}$, we have to apply Schauder estimate to the following Poisson equation

$$\begin{cases} \Delta(\partial_t\psi_e) = \partial_t\omega_e\,, \qquad \text{in} \quad \Omega\,, \\ \partial_t\psi_e = 0\,, \qquad \text{on}\ \Gamma\,, \end{cases} \tag{4.14}$$

which gives us that for $\alpha > 0$,

$$\begin{aligned} \|\partial_t\psi_e\|_{C^{4,\alpha}} &\le C\|\partial_t\omega_e\|_{C^{2,\alpha}} \\ &\le C(\|\psi_e\|_{C^{6,\alpha}} + \|\psi_e\|_{C^{5,\alpha}}\|\psi_e\|_{C^{3,\alpha}})\,. \end{aligned} \tag{4.15}$$

In the second step, we applied the exact vorticity equation that $\partial_t\omega_e + \boldsymbol{u}_e\cdot\nabla\omega_e = \nu\Delta\omega_e$. The combination of (4.13) and (4.15) gives us

$$\partial_t\Omega - \partial_t\omega_e = O(h^2)(\|\psi_e\|_{C^{6,\alpha}} + \|\psi_e\|_{C^{5,\alpha}}\|\psi_e\|_{C^{3,\alpha}})\,. \tag{4.16}$$

Combining (4.7), (4.11), (4.12) and (4.16), and applying the original PDE of the exact solution that $\partial_t\omega_e + \nabla\cdot(\boldsymbol{u}_e\omega_e) = \nu\Delta\omega_e$, we conclude that

$$\begin{aligned} &\partial_t\Omega + \widetilde{D}_x(U\Omega) + \widetilde{D}_y(V\Omega) \\ &\qquad = \Delta_h\Omega + O(h^2)\|\psi_e\|_{C^{6,\alpha}}(1 + \|\psi_e\|_{C^4})\,, \end{aligned} \tag{4.17}$$

which verifies our claim.

Finally, we look at the constructed Ω on the boundary. Our aim is to show that Ω satisfies Wilkes' formula applied to Ψ up to an $O(h^2)$ error. The verification of it is straightforward. We only consider Γ_x, $j = 0$ here. The other three boundaries can be dealt with in the same way. One-sided Taylor expansion for Ψ in the y-th direction near the boundary shows that

$$\begin{aligned} \frac{1}{h^2}\left(4\Psi_{i,1} - \frac{1}{2}\Psi_{i,2}\right) &= \partial_y^2\psi_e(x_i,0) + O(h^2)\|\psi_e\|_{C^4} \\ &= \omega_e(x_i,0) + O(h^2)\|\psi_e\|_{C^4}\,. \end{aligned} \tag{4.18}$$

On the other hand, (4.4) gives us that the difference between $\Omega_{i,0}$ and $\omega_e(x_i,0)$ on Γ_x is also of order $O(h^2)\|\psi_e\|_{C^4}$, then we arrive at

$$\Omega_{i,0} = \frac{1}{h^2}\left(4\Psi_{i,1} - \frac{1}{2}\Psi_{i,2}\right) + O(h^2)\|\psi_e\|_{C^4}\,. \tag{4.19}$$

Thus the consistency analysis is completed.

4.2. Error estimate

For $0 \le i, j \le N$, we define

$$\begin{aligned} \widetilde{\psi}_{i,j} &= \psi_{i,j} - \Psi_{i,j} \,, & \widetilde{\omega}_{i,j} &= \omega_{i,j} - \Omega_{i,j} \,, \\ \widetilde{u}_{i,j} &= u_{i,j} - U_{i,j} \,, & \widetilde{v}_{i,j} &= v_{i,j} - V_{i,j} \,. \end{aligned} \tag{4.20}$$

Then the above consistency analysis gives the following system for the error functions

$$\begin{cases} \partial_t \widetilde{\omega} + \widetilde{D}_x(\widetilde{u}\Omega + u\widetilde{\omega}) + \widetilde{D}_y(\widetilde{v}\Omega + v\widetilde{\omega}) = \nu \Delta_h \widetilde{\omega} + \boldsymbol{f} \,, \\ \Delta_h \widetilde{\psi} = \widetilde{\omega} \,, \qquad \widetilde{\psi} \mid_{\Gamma} = 0 \,, \\ \widetilde{u} = -\widetilde{D}_y \widetilde{\psi} \,, \quad \widetilde{v} = \widetilde{D}_x \widetilde{\psi} \,, \qquad \widetilde{u} \mid_{\Gamma_y} = 0 \,, \qquad \widetilde{v} \mid_{\Gamma_x} = 0 \,, \end{cases} \tag{4.21}$$

where $|\boldsymbol{f}| \le Ch^2 \|\boldsymbol{u}_e\|_{C^{5,\alpha}}(1 + \|\boldsymbol{u}_e\|_{C^3})$. On the boundary, (say at Γ_x, $j = 0$) we have

$$\widetilde{\omega}_{i,0} = \frac{1}{h^2}(4\widetilde{\psi}_{i,1} - \frac{1}{2}\widetilde{\psi}_{i,2}) + h^2 \boldsymbol{e}_i \,, \tag{4.22}$$

where $|\boldsymbol{e}_i| \le C\|\boldsymbol{u}_e\|_{C^3}$. The identity (4.22) comes from Wilkes' formula (2.7) and our estimate (4.19). In other words, the error function of vorticity and the error function of stream function satisfy Wilkes' formula up to an $O(h^2)$ error.

As we can see, the system (4.21), (4.22) is very similar to the second order scheme (2.1) along with the Wilkes' formula (2.7) except for the error terms $\boldsymbol{f}$ and $h^2\boldsymbol{e}$. In other words, as we showed in the consistency part, the constructed solutions satisfy the numerical scheme except for some local truncation errors. We have already shown the stability of the scheme in Sect. 3, so we can apply the same procedure to estimate the error functions. The Cauchy inequality is used to deal with the error terms corresponding to $\boldsymbol{f}$ and $\boldsymbol{e}$.

Multiplying the vorticity dynamic error equation in (4.21) by $-\widetilde{\psi}$, we have

$$\begin{aligned} &-\langle \widetilde{\psi}, \partial_t \widetilde{\omega} \rangle + \langle \widetilde{\psi}, \Delta_h \widetilde{\omega} \rangle \\ &\quad = \langle \widetilde{\psi}, \widetilde{D}_x(\widetilde{u}\Omega + u\widetilde{\omega}) \rangle + \langle \widetilde{\psi}, \widetilde{D}_y(\widetilde{v}\Omega + v\widetilde{\omega}) \rangle - \langle \widetilde{\psi}, \boldsymbol{f} \rangle \,. \end{aligned} \tag{4.23}$$

The first term, which corresponds to time evolution term, can be dealt with in the same way as in (3.5) since $\widetilde{\psi}$ also vanishes on the boundary, i.e.

$$-\langle \widetilde{\psi}, \partial_t \widetilde{\omega} \rangle = -\langle \widetilde{\psi}, \partial_t \Delta_h \widetilde{\psi} \rangle = \frac{1}{2}\frac{d}{dt}\|\nabla_h \widetilde{\psi}\|^2 \,. \tag{4.24}$$

The term $-\langle\widetilde{\psi}, \boldsymbol{f}\rangle$ can be controlled by standard Cauchy inequality. Then the rest of our work will be concentrated on the estimates of the diffusion term and the convection terms. We will rely upon Lemma 4.2 and Lemma 4.3 as below.

Lemma 4.2 *For sufficiently small h, we have*

$$\langle\widetilde{\psi}, \Delta_h\widetilde{\omega}\rangle \geq \frac{1}{2}\|\widetilde{\omega}\|^2 - h^4\,. \tag{4.25}$$

Proof. Our proof of (4.25) follows the procedure of stability analysis in Sect. 3. Summing by parts and using the fact that $\widetilde{\psi}\mid_\Gamma = 0$ gives us

$$\begin{aligned}\langle\widetilde{\psi}, \Delta_h\widetilde{\omega}\rangle &= \langle\widetilde{\psi}, (D_x^2 + D_y^2)\widetilde{\omega}\rangle = \langle D_x^2\widetilde{\psi}, \widetilde{\omega}\rangle + \langle D_y^2\widetilde{\psi}, \widetilde{\omega}\rangle + \mathcal{B}\\ &= \|\widetilde{\omega}\|^2 + \mathcal{B}\,,\end{aligned} \tag{4.26}$$

where the boundary term $\mathcal{B}$ can also be decomposed into $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$ as in (3.7)

$$\begin{aligned}\mathcal{B}_1 &= \sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}\widetilde{\omega}_{i,0}\,, \mathcal{B}_2 = \sum_{i=1}^{N-1}\widetilde{\psi}_{i,N-1}\widetilde{\omega}_{i,N}\,,\\ \mathcal{B}_3 &= \sum_{j=1}^{N-1}\widetilde{\psi}_{1,j}\widetilde{\omega}_{0,j}\,, \mathcal{B}_4 = \sum_{j=1}^{N-1}\widetilde{\psi}_{N-1,j}\widetilde{\omega}_{N,j}\,.\end{aligned} \tag{4.27}$$

The estimate of $\mathcal{B}_1$ is also similar to that in Sect. 3. The only difference here is that $\widetilde{\omega}_{i,0}$, the error of vorticity on the boundary as in (4.22), includes one more error term $h^2\boldsymbol{e}_i$, whose L^2 product with $\widetilde{\psi}$ can be estimated by Cauchy inequality. By (4.22), we can express $\mathcal{B}_1$ as

$$\begin{aligned}\mathcal{B}_1 = \sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}\widetilde{\omega}_{i,0} &= \frac{1}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(4\widetilde{\psi}_{i,1} - \frac{1}{2}\widetilde{\psi}_{i,2})\\ &+h^2\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}\boldsymbol{e}_i \equiv I_1 + I_2\,.\end{aligned} \tag{4.28}$$

As we mentioned just now, I_2 can be controlled by Cauchy inequality directly

$$\begin{aligned}I_2 = \sum_i h^2\widetilde{\psi}_{i,1}\boldsymbol{e}_i &\geq -\frac{1}{2}\sum_{i=1}^{N-1}\frac{\widetilde{\psi}_{i,1}^2}{h^2} - \frac{1}{2}\sum_{i=1}^{N-1}h^6\boldsymbol{e}_i^2\\ &\geq -\frac{1}{2}\sum_{i=1}^{N-1}\frac{\widetilde{\psi}_{i,1}^2}{h^2} - Ch^5\|\boldsymbol{u}_e\|_{C^3}^2\,,\end{aligned} \tag{4.29}$$

where in the last step we applied our estimate that $|\boldsymbol{e}_i| \le C\|\boldsymbol{u}_e\|_{C^3}$ and the fact that $h = \frac{1}{N}$. The estimate of I_1 follows our stability analysis in Sect. 3. First, we rewrite the term appearing in the parentheses as we did in (3.10):

$$4\widetilde{\psi}_{i,1} - \frac{1}{2}\widetilde{\psi}_{i,2} = 3\widetilde{\psi}_{i,1} - \frac{1}{2}h^2(D_y^2\widetilde{\psi})_{i,1}\,, \tag{4.30}$$

which is still valid since $\widetilde{\psi}$ vanishes on the boundary. The purpose of this transformation is still to control local terms by global terms as can be seen later. Next, plugging (4.30) back into I_1

$$\begin{aligned} I_1 &= \frac{1}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}\left(3\widetilde{\psi}_{i,1} - \frac{1}{2}h^2(D_y^2\widetilde{\psi})_{i,1}\right) \\ &= \frac{3}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2 - \frac{1}{2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(D_y^2\widetilde{\psi})_{i,1}\,, \end{aligned} \tag{4.31}$$

and applying Cauchy inequality to the second term $\widetilde{\psi}_{i,1}(D_y^2\widetilde{\psi})_{i,1}$, we arrive at

$$\begin{aligned} I_1 &\ge \frac{3}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2 - \frac{1}{8h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2 - \frac{1}{2}\sum_{i=1}^{N-1}\left|(D_y^2\widetilde{\psi})_{i,1}\right|^2 h^2 \\ &\ge \frac{2}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2 - \frac{1}{2}\sum_{i=1}^{N-1}\left|D_y^2\widetilde{\psi}_{i,1}\right|^2 h^2\,. \end{aligned} \tag{4.32}$$

Finally, the combination of (4.29) and (4.32) gives that for sufficiently small h,

$$\mathcal{B}_1 \ge \frac{1}{h^2}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2 - \frac{1}{2}\sum_{i=1}^{N-1}\left|D_y^2\widetilde{\psi}_{i,1}\right|^2 h^2 - \frac{1}{4}h^4\,. \tag{4.33}$$

The treatment of the other three boundary terms is essentially the same. Now we recover $\mathcal{B}$ by global terms $\|D_x^2\widetilde{\psi}\|^2$ and $\|D_y^2\widetilde{\psi}\|^2$

$$\mathcal{B} \ge -\frac{1}{2}\|D_y^2\widetilde{\psi}\|^2 - \frac{1}{2}\|D_x^2\widetilde{\psi}\|^2 - h^4\,. \tag{4.34}$$

It can be argued that, since $\widetilde{\psi}\,|_{\Gamma} = 0$, Lemma 3.2 is still valid for $\widetilde{\psi}$ and $\widetilde{\omega}$, i.e.

$$\|D_x^2\widetilde{\psi}\|^2 + \|D_y^2\widetilde{\psi}\|^2 \le \|(D_x^2 + D_y^2)\widetilde{\psi}\|^2 = \|\widetilde{\omega}\|^2\,. \tag{4.35}$$

Substituting (4.35) into (4.34), plugging back into (4.26), we obtain (4.25) finally. Lemma 4.2 is proven. □

Lemma 4.3 *Assume a-prior that the error function for the velocity field satisfy*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \le 1\,, \tag{4.36}$$

then we have

$$\begin{aligned} \langle \widetilde{\psi}, \widetilde{D}_x(\widetilde{u}\Omega + u\widetilde{\omega})\rangle &\le \frac{8C_1^2}{\nu}\|\nabla_h\widetilde{\psi}\|^2 + \frac{\nu}{6}\|\widetilde{\omega}\|^2\,, \\ \langle \widetilde{\psi}, \widetilde{D}_y(\widetilde{v}\Omega + v\widetilde{\omega})\rangle &\le \tfrac{8C_1^2}{\nu}\|\nabla_h\widetilde{\psi}\|^2 + \tfrac{\nu}{6}\|\widetilde{\omega}\|^2\,, \end{aligned} \tag{4.37}$$

where $C_1 = 1 + \|\boldsymbol{u}_e\|_{C^1}$.

Proof. We will only prove the first one of (4.37). The proof of the second one is essentially the same. By the a-prior bound (4.36) and our construction of U and Ω, we have

$$\begin{aligned} \|u\|_{L^\infty} &\le \|U\|_{L^\infty} + \|\widetilde{u}\|_{L^\infty} \\ &\le \|\partial_y\psi_e\|_{C^1} + 1 \le \|\boldsymbol{u}_e\|_{C^0} + 1 \le C_1 \\ \|\Omega\|_{L^\infty} &\le \|\partial_x^2\psi_e\|_{C^0} + \|\partial_y^2\psi_e\|_{C^0} \le \|\boldsymbol{u}_e\|_{C^1} \le C_1\,, \end{aligned} \tag{4.38}$$

where $C_1 = \|\boldsymbol{u}_e\|_{C^1} + 1$. Summing by parts and applying (4.36), we obtain

$$\begin{aligned} \langle \widetilde{\psi}, \widetilde{D}_x(\widetilde{u}\Omega + u\widetilde{\omega})\rangle &= -\langle \widetilde{D}_x\widetilde{\psi}\,,\, \widetilde{u}\Omega + u\widetilde{\omega}\rangle \le C_1\|\nabla_h\widetilde{\psi}\|(\|\widetilde{u}\| + \|\widetilde{\omega}\|) \\ &\le \frac{8C_1^2}{\nu}\|\nabla_h\widetilde{\psi}\|^2 + \frac{\nu}{6}\|\widetilde{\omega}\|^2\,. \end{aligned} \tag{4.39}$$

We used the fact that the norms $\|\widetilde{D}_x\widetilde{\psi}\|$, $\|\widetilde{D}_y\widetilde{\psi}\|$ are bounded by $\|\nabla_h\widetilde{\psi}\|$, i.e.

$$\|\widetilde{u}\| = \|\widetilde{D}_y\widetilde{\psi}\| \le \|\nabla_h\widetilde{\psi}\|\,, \qquad \|\widetilde{v}\| = \|\widetilde{D}_x\widetilde{\psi}\| \le \|\nabla_h\widetilde{\psi}\|\,, \tag{4.40}$$

since $\widetilde{\psi}$ vanishes on the boundary. Lemma 4.3 is proven. □

Now we go back to the convergence analysis. First, we assume that (4.36) holds. Plugging (4.37), (4.25) along with (4.24) back into (4.23), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h\widetilde{\psi}\|^2 \le C\|\boldsymbol{f}\|^2 + \frac{16C_1^2}{\nu}\|\nabla_h\widetilde{\psi}\|^2 - \frac{\nu}{6}\|\widetilde{\omega}\|^2 + h^4\,. \tag{4.41}$$

In (4.41), we absorbed the term $C\|\widetilde{\psi}\|^2$ generated by Cauchy inequality: $|\langle\widetilde{\psi}, \boldsymbol{f}\rangle| \le C\|\widetilde{\psi}\|^2 + C\|\boldsymbol{f}\|^2$ into the coefficient of $\|\nabla_h\widetilde{\psi}\|^2$, which is valid since we can apply Poincare inequality for $\widetilde{\psi}$ that

$$\|\widetilde{\psi}\|^2 \le C\|\nabla_h\widetilde{\psi}\|^2\,, \tag{4.42}$$

by the fact that $\widetilde{\psi}$ vanishes on the boundary. Applying Gronwall inequality to (4.42), we have

$$\begin{aligned} &\|\nabla_h \widetilde{\psi}\|^2 + \frac{\nu}{6}\int_0^t \|\widetilde{\omega}\|^2\,dt \\ &\le C\left(\exp\frac{16C_1^2 t}{\nu}\right)\int_0^t (\|\boldsymbol{f}(\cdot,s)\|^2 + h^4)\,ds + CTh^4 \\ &\le Ch^4 \exp\left\{\frac{16C_1^2 t}{\nu}\right\}\left(\|\boldsymbol{u}_e\|_{C^{5,\alpha}}^2 (1+\|\boldsymbol{u}_e\|_{C^3})^2 + T\right) . \end{aligned} \tag{4.43}$$

Thus, we have proven

$$\begin{aligned} &\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}(t)\|_{L^2} + \sqrt{\nu}\left(\int_0^T \|\widetilde{\omega}\|^2 dt\right)^{\frac{1}{2}} \\ &\le Ch^2\left(\|\boldsymbol{u}_e\|_{C^{5,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^3})\exp\left\{\frac{CT}{\nu}(1+\|\boldsymbol{u}_e\|_{C^1})^2\right\} + T\right) , \end{aligned} \tag{4.44}$$

which implies (4.1). Using the inverse inequality, we have

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \le Ch\,. \tag{4.45}$$

Now we can resort to a standard trick which asserts that (4.36) will never be violated if h is small enough. Theorem 4.1 is proven.

5. Analysis of EC4 scheme in a 1-D Stokes model

The methodology carried out in the second order scheme is quite general and can be applied to fourth order scheme in a similar fashion. To explain the idea of the fourth order method more clearly, in this section we consider a simple 1-D model for Stokes equations, where nonlinear terms are neglected. The purpose of the introduction of this model is to catch main features and difficulties both in computation and analysis. This 1-D model reads

$$\begin{cases} \partial_t \omega = \nu(\partial_x^2 - k^2)\omega\,, \\ (\partial_x^2 - k^2)\psi = \omega\,, \\ \psi = \partial_x \psi = 0\,, \qquad \text{at } x = -1, 1\,, \end{cases} \tag{5.1}$$

whose solution is the k-th mode solution of the unsteady Stokes equations in the domain $[-1,1]\times[0,2\pi]$

$$\begin{cases} \partial_t \omega = \nu \Delta \omega\,, \\ \Delta \psi = \omega\,, \end{cases} \tag{5.2}$$

where the no-flow, no-slip boundary condition, $\psi = \partial_x \psi = 0$, is imposed at $x = -1, 1$, and periodic boundary condition is imposed in the y direction. An exact solution of (5.1) is

$$\omega_e(x,t) = \cos(\mu x)\exp\left\{-\nu(k^2+\mu^2)t\right\}, \tag{5.3}$$

where μ satisfies $\mu\tan\mu + k\tanh k = 0$. See [12] and [4] for details. For simplicity we take $k = 1$.

5.1. Description of fourth order scheme

Essentially compact fourth order scheme (EC4) for 2-D Navier-Stokes equations was proposed by E and Liu in [5]. We can use the similar idea to deal with the 1-D model (5.1). As can be seen, the operator $\partial_x^2 - 1$ can be approximated by compact difference operator

$$\partial_x^2 - 1 = \frac{(1-\frac{h^2}{12})D_x^2 - 1}{1+\frac{h^2}{12}D_x^2} + O(h^4). \tag{5.4}$$

Applying (5.4) to both the diffusion term in vorticity equation and the kinematic relation between stream function and vorticity in (5.1), and multiplying both equations by $1+\frac{h^2}{12}D_x^2$, we obtain the following system

$$\begin{cases} \partial_t \overline{\omega} = \nu\left((1-\dfrac{h^2}{12})D_x^2 - 1\right)\omega, \\ \left((1-\dfrac{h^2}{12})D_x^2 - 1\right)\psi = \overline{\omega}, \qquad \psi = 0, \quad \text{on } x = -1, 1, \end{cases} \tag{5.5}$$

where the auxiliary term $\overline{\omega}$ was introduced as

$$\overline{\omega} = (1+\frac{h^2}{12}D_x^2)\omega. \tag{5.6}$$

As pointed out in Sect. 2, there are two boundary conditions for ψ. The situation here is similar. The Dirichlet boundary condition $\psi = 0$ on $x = -1, 1$ can be implemented to solve the stream function via (5.5). Yet the normal boundary condition $\partial_x \psi = 0$, which cannot be enforced directly, will be converted into the boundary condition for vorticity. For example, Briley's formula

$$\omega_0 = \frac{1}{h^2}(6\psi_1 - \frac{3}{2}\psi_2 + \frac{2}{9}\psi_3) - \frac{11}{3h}\left(\frac{\partial\psi}{\partial x}\right)_0, \tag{5.7}$$

was used in the EC4 scheme (see [2], [5]). It should be noted here that Briley's formula is only third order accurate for vorticity on the boundary by formal local Taylor expansion. Later we will show that it still preserves fourth order accuracy. This fact was first proven in [5].

The system (5.5), (5.6), along with Briley's formula (5.7), can be implemented very efficiently via an explicit time stepping procedure introduced by E and Liu in [5].

5.2. Stability analysis of the scheme and vorticity boundary condition

As can be seen, Briley's formula (5.7) is also a long-stencil formula. One of the main concerns is its stability. The technique used in Sect. 3 to deal with Wilkes' formula can be used here in a similar, yet more tricky way.

The first step here is to multiply (5.5) by $-(1+\frac{h^2}{12}D_x^2)\psi$

$$
\begin{aligned}
&-\langle(1+\frac{h^2}{12}D_x^2)\psi,(1+\frac{h^2}{12}D_x^2)\partial_t\omega\rangle \\
&\qquad +\left\langle(1+\frac{h^2}{12}D_x^2)\psi\,,\,\left((1-\frac{h^2}{12})D_x^2-1\right)\omega\right\rangle=0\,.
\end{aligned}
\tag{5.8}
$$

The first term, which corresponds to the time marching term, can be estimated by the discrete kinematic relation between ψ and ω as in (5.5)

$$
\begin{aligned}
&-\langle(1+\frac{h^2}{12}D_x^2)\psi,(1+\frac{h^2}{12}D_x^2)\partial_t\omega\rangle \\
&=-\left\langle(1+\tfrac{h^2}{12}D_x^2)\psi\,,\,\partial_t\left((1-\tfrac{h^2}{12})D_x^2-1\right)\psi\right\rangle \\
&=-\langle(1+\frac{h^2}{12}D_x^2)\psi\,,\,\partial_t(1-\frac{h^2}{12})D_x^2\psi\rangle-\langle(1+\frac{h^2}{12}D_x^2)\psi\,,\,-\partial_t\psi\rangle \\
&=\frac{1}{2}(1-\frac{h^2}{12})\frac{d}{dt}\|\nabla_h\psi\|^2-\frac{h^2}{24}(1-\frac{h^2}{12})\frac{d}{dt}\|D_x^2\psi\|^2+\frac{1}{2}\frac{d}{dt}\|\psi\|^2 \\
&\quad-\frac{h^2}{24}\frac{d}{dt}\|\nabla_h\psi\|^2 \\
&=\frac{1}{2}\frac{d}{dt}\left((1-\frac{h^2}{6})\|\nabla_h\psi\|^2+\|\psi\|^2-\frac{h^2}{12}(1-\frac{h^2}{12})\|D_x^2\psi\|^2\right)\,,
\end{aligned}
\tag{5.9}
$$

and the second term, which corresponds to the diffusion term, can be estimated via summing by parts

$$
\begin{aligned}
&\left\langle(1+\frac{h^2}{12}D_x^2)\psi\,,\,\left((1-\frac{h^2}{12})D_x^2-1\right)\omega\right\rangle \\
&\quad=\langle\widetilde{\psi}\,,\,(1-\frac{h^2}{12})D_x^2\widetilde{\omega}\rangle+\langle\psi\,,\,-\omega\rangle+\langle\frac{h^2}{12}D_x^2\psi\,,\,(1-\frac{h^2}{12})D_x^2\omega\rangle \\
&\qquad+\langle\frac{h^2}{12}D_x^2\psi\,,\,-\omega\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle (1-\frac{h^2}{12})D_x^2\psi\,,\,\omega\rangle + \frac{1}{h}(1-\frac{h^2}{12})(\psi_1\omega_0+\psi_{N-1}\omega_N) \\
&\quad + \langle -\psi\,,\,\omega\rangle + \langle (1-\frac{h^2}{12})D_x^2\psi\,,\,\frac{h^2}{12}D_x^2\omega\rangle + \langle -\psi\,,\,\frac{h^2}{12}D_x^2\omega\rangle \\
&\quad + \frac{1}{h}\cdot\frac{h^2}{12}(\psi_1\omega_0+\psi_{N-1}\omega_N) \\
&= \left\langle \left((1-\frac{h^2}{12})D_x^2 - 1\right)\psi\,,\,(1+\frac{h^2}{12}D_x^2)\omega \right\rangle \\
&\quad + \frac{1}{h}(\psi_1\omega_0+\psi_{N-1}\omega_N) \\
&= \|(1+\frac{h^2}{12}D_x^2)\omega\|^2 + \frac{1}{h}(\psi_1\omega_0+\psi_{N-1}\omega_N)\,. \qquad (5.10)
\end{aligned}
$$

As can be seen in (5.10), the boundary term has to be controlled to ensure stability. Here we decompose the boundary term into two parts $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where $\mathcal{B}_1 = \frac{1}{h}\psi_1\omega_0$ and $\mathcal{B}_2 = \frac{1}{h}\psi_{N-1}\omega_N$. For simplicity of presentation, we only consider $\mathcal{B}_1$ here. $\mathcal{B}_2$ can be treated in the same way.

Briley's formula (5.7) can be used to update $\mathcal{B}_1$

$$
\mathcal{B}_1 = \frac{\psi_1}{h^3}\left(6\psi_1 - \frac{3}{2}\psi_2 + \frac{2}{9}\psi_3\right). \qquad (5.11)
$$

Still, a straightforward calculation cannot guarantee a bound for $\mathcal{B}_1$. The difficulty comes from the fact that three interior points of stream function are involved in Briley's formula. A similar technique used in Sect. 3, where we dealt with second order scheme with Wilkes' formula, can be applied: the term $(6\psi_1 - \frac{3}{2}\psi_2 + \frac{2}{9}\psi_3)$ can be rewritten as

$$
6\psi_1 - \frac{3}{2}\psi_2 + \frac{2}{9}\psi_3 = \frac{11}{3}\psi_1 - \frac{19}{18}h^2D_x^2\psi_1 + \frac{2}{9}h^2D_x^2\psi_2\,, \qquad (5.12)
$$

which is valid since $\psi_0 = \psi_N = 0$. The purpose of this transformation is similar to that of (3.1): to control the local terms in (5.11) by global quantities. Now (5.12), along with Cauchy inequalities applied to $\psi_1\cdot D_x^2\psi_1$ and $\psi_2\cdot D_x^2\psi_2$, gives the estimate of $\mathcal{B}_1$

$$
\begin{aligned}
\mathcal{B}_1 &= \frac{\psi_1}{h^3}\left(\frac{11}{3}\psi_1 - \frac{19}{18}h^2D_x^2\psi_1 + \frac{2}{9}h^2D_x^2\psi_2\right) \\
&\geq \frac{11\psi_1^2}{3h^3} - \frac{1}{2h^3}\frac{19^2}{18^2}\psi_1^2 - \frac{1}{2}|D_x^2\psi_1|^2h - \frac{1}{2h^3}\frac{2^2}{9^2}\psi_1^2 - \frac{1}{2}|D_x^2\psi_2|^2h \\
&\geq \frac{2\psi_1^2}{h^3} - \frac{1}{2}|D_x^2\psi_1|^2h - \frac{1}{2}|D_x^2\psi_2|^2h\,, \qquad (5.13)
\end{aligned}
$$

where in the last step we used the fact that $\frac{11}{3} - \frac{1}{2}(\frac{19^2}{18^2} + \frac{2^2}{9^2}) \geq 2$. The term $\mathcal{B}_2$ can be estimated in a similar fashion. Now we arrive at

$$\mathcal{B} \geq -\frac{1}{2} \sum_{i=1,2N-1,N-2} h|D_x^2\psi_i|^2 \geq -\frac{1}{2}\|D_x^2\psi\|^2 \,. \tag{5.14}$$

As mentioned earlier, the transformation (5.12) and the application of Cauchy inequalities give us a bound of the boundary term, which is a local term, by a global term $\|D_x^2\psi\|^2$. Next, we need to control $\|D_x^2\psi\|^2$ by the diffusion term $\|\overline{\omega}\|^2$, using the following lemma, which is analogous to Lemma 3.2 in Sect. 3.

Lemma 5.1 *For $\psi_0 = \psi_N = 0$, we have*

$$\|(1-\frac{h^2}{12})D_x^2\psi\| \leq \|\left((1-\frac{h^2}{12})D_x^2 - 1\right)\psi\| = \|(1+\frac{h^2}{12}D_x^2)\omega\| = \|\overline{\omega}\| \,. \tag{5.15}$$

Proof. The boundary condition $\psi_0 = \psi_N = 0$ indicates that we can Sine transform ψ,

$$\psi_i = \sum_k \widehat{\psi}_k \sin(k\pi x_i) \,. \tag{5.16}$$

The Parserval equality gives that

$$\sum_i (\psi_i)^2 = \sum_k (\widehat{\psi}_k)^2 \,. \tag{5.17}$$

We let $f_k = -\frac{4}{h^2}\sin^2(\frac{k\pi h}{2})$, then we have $D_x^2\psi_i = \sum_k f_k\widehat{\psi}_k \sin(k\pi x_i)$, which in turn shows the Parserval equality for $(1-\frac{h^2}{12})D_x^2\psi$ and $((1-\frac{h^2}{12})D_x^2 - 1)\psi$

$$\begin{aligned} \left\|\left(1-\frac{h^2}{12}\right)D_x^2\psi\right\|^2 &= h\sum_k \left(1-\frac{h^2}{12}\right)^2 f_k^2\widehat{\psi}_k^2 \,, \\ \left\|\left((1-\frac{h^2}{12})D_x^2 - 1\right)\psi\right\|^2 &= h\sum_k \left((1-\frac{h^2}{12})f_k - 1\right)^2 \widehat{\psi}_k^2 \,. \end{aligned} \tag{5.18}$$

On the other hand, $f_k \leq 0$ implies that $(1-\frac{h^2}{12})^2 f_k^2 \leq ((1-\frac{h^2}{12})f_k - 1)^2$. Then we obtain (5.15). Lemma 5.1 is proven. □

Plugging (5.15) back into (5.14), we obtain

$$\mathcal{B} \geq -\frac{1}{2(1-\frac{h^2}{12})}\|(1+\frac{h^2}{12}D_x^2)\omega\|^2 \geq -\frac{2}{3}\|\overline{\omega}\|^2 \,. \tag{5.19}$$

Substituting (5.19) into (5.10), along with (5.9), and denoting "energy" E as

$$E = \left(1 - \frac{h^2}{6}\right) \|\nabla_h \psi\|^2 + \|\psi\|^2 - \frac{h^2}{12}\left(1 - \frac{h^2}{12}\right) \|D_x^2 \psi\|^2 , \tag{5.20}$$

we finally arrive at

$$\frac{1}{2}\frac{dE}{dt} + \frac{1}{2}\nu \|\overline{\omega}\|^2 \leq 0 . \tag{5.21}$$

This completes the stability analysis of the fourth order scheme (5.5), (5.6) with Briley's formula (5.7).

5.3. Convergence analysis of the fourth order scheme

In this section, we will give a convergence analysis of the fourth order method. The stability of it has been been established in **5.2**. The consistency analysis of it turns out to be quite technical, by which we hope to explain the methodology of Strang-type expansion. Direct truncation error analysis gives us fourth order accuracy for the momentum equation, but only third order accuracy for the vorticity on the boundary, if Briley's formula is used. Below, a more careful truncation error analysis will be carried out by including a higher order term to construct approximate stream function. In addition, the construction of the approximate vorticity needs some technique: an approximate intermediate vorticity variable is constructed via finite difference of the approximate stream function and the approximate vorticity field is constructed by solving a linear system through the approximate intermediate vorticity variable. The eigenvalues corresponding to the linear system are controlled. Moreover, an $O(h^4)$ correction term to the exact vorticity on the boundary is added when its boundary condition is set. The purpose of that correction term is to maintain higher order consistency for the approximate vorticity. This makes the computation of its finite differences convenient.

5.3.1. Consistency analysis. Denote ψ_e, ω_e as the exact solutions, extend ψ_e smoothly to $[-1-\delta, 1+\delta]$, and construct the approximate stream function $\Psi = \psi_e + h^4 \widehat{\psi}$ with

$$\widehat{\psi}(x,t) = \frac{1}{4}\alpha(t)(x+1)(1-x)^2 - \frac{1}{4}\beta(t)(x+1)^2(1-x) , \tag{5.22}$$

where $\alpha(t) = -\frac{30}{11}\partial_x^5 \psi_e(-1)$, $\beta(t) = -\frac{30}{11}\partial_x^5 \psi_e(1)$. The choices of $\alpha(t)$ and $\beta(t)$ will guarantee Ψ to satisfy higher order truncation errors in Briley's formula, which can be seen later. It is obvious that (say at the left boundary $x_0 = -1$)

$$\widehat{\psi}(-1) = 0\,, \qquad \partial_x \widehat{\psi}(-1) = -\frac{30}{11}\partial_x^5 \psi_e(-1)\,. \tag{5.23}$$

To estimate $\widehat{\psi}$, we can see that $\partial_x^4 \widehat{\psi} = 0$, which implies that

$$\|\widehat{\psi}\|_{C^m} = \|\widehat{\psi}\|_{C^3} \le C\|\psi_e\|_{C^5}\,, \qquad \text{if } m \ge 3\,. \tag{5.24}$$

Moreover, our definition of $\alpha(t)$ and $\beta(t)$ implies that $|\partial_t \alpha(t)|, |\partial_t \beta(t)| \le C\|\partial_t \partial_x^5 \psi_e\|_{C^0}$. To have a good estimate of $\|\partial_t \partial_x^5 \psi_e\|_{C^0} = \|\partial_x^5 \partial_t \psi_e\|_{C^0}$, we see that $\partial_t \psi_e$ satisfies

$$\begin{cases} (\partial_x^2 - 1)\partial_t \psi_e = \partial_t \omega_e\,, & \text{in } [-1,1]\,, \\ \partial_t \psi_e = 0 & \text{at } x = -1, 1\,, \end{cases} \tag{5.25}$$

which implies that $\|\partial_t \psi_e\|_{C^5} \le C\|\partial_t \omega_e\|_{C^3}$. On the other hand, $\|\partial_t \omega_e\|_{C^3}$ can be controlled by the order of $\|\omega_e\|_{C^5}$ from the original vorticity equation that $\partial_t \omega_e = (\partial_x^2 - 1)\omega_e$. The combination of the above arguments indicates that

$$\begin{aligned} |\partial_t \alpha(t)|, |\partial_t \beta(t)| &\le C\|\partial_t \partial_x^5 \psi_e\|_{C^0} \le C\|\partial_t \omega_e\|_{C^3} \\ &\le C\|\omega_e\|_{C^5} \le C\|\psi_e\|_{C^7}\,. \end{aligned} \tag{5.26}$$

By the fact that $\partial_t \widehat{\psi} = \frac{1}{4}(x+1)(1-x)^2 \partial_t \alpha(t) - \frac{1}{4}(x+1)^2(1-x)\partial_t \beta(t)$, we have

$$\|\partial_t \widehat{\psi}\|_{C^m} = \|\partial_t \widehat{\psi}\|_{C^3} \le C\|\psi_e\|_{C^7}\,, \qquad \text{if } m \ge 3\,. \tag{5.27}$$

The construction of the approximate vorticity is quite tricky. First we define

$$\overline{\Omega}_i = \left((1 - \frac{h^2}{12})D_x^2 - 1 \right) \Psi_i\,, \qquad \text{for } 1 \le i \le N-1\,, \tag{5.28}$$

and then recover Ω by solving the following system

$$\left(1 + \frac{h^2}{12} D_x^2 \right) \Omega_i = \overline{\Omega}_i\,. \tag{5.29}$$

We should mention that (5.29) always has a solution since the eigenvalues of the matrix corresponding to $1 + \frac{h^2}{12}D_x^2$ are all non-zero. On the other hand, the implementation of (5.29) requires the boundary value for Ω. To maintain the higher order consistency needed in the truncation error estimate below for the discrete derivatives of the constructed vorticity, we introduce

$$\widehat{\omega} = \widehat{\omega}_1 + \widehat{\omega}_2\,, \qquad \text{where } \widehat{\omega}_1 = -\frac{1}{240}\partial_x^6 \psi_e\,, \quad \widehat{\omega}_2 = (\partial_x^2 - 1)\widehat{\psi}\,, \tag{5.30}$$

where $h^4\widehat{\omega}_1$ is the $O(h^4)$ truncation error of $\left((1-\frac{h^2}{12})D_x^2-1\right)\psi_e-(1+\frac{h^2}{12}D_x^2)\omega_e$, $h^4\widehat{\omega}_2$ is the $O(h^4)$ part of $h^4\left((1-\frac{h^2}{12})D_x^2-1\right)\widehat{\psi}$ The boundary condition for Ω (say at $x_0=-1$) is imposed as

$$\Omega_0=\omega_e(x_0)+h^4\widehat{\omega}_0\,, \tag{5.31}$$

and Ω_N can be determined similarly. The purpose of this choice can be seen in the following lemma.

Lemma 5.2 *We have on the grid points x_i, $0\le i\le N$,*

$$\Omega=\omega_e+h^4\widehat{\omega}+O(h^6)\|\psi_e\|_{C^8}\,. \tag{5.32}$$

Proof. First we note that

$$\begin{aligned}\left(1+\frac{h^2}{12}D_x^2\right)\Omega&=\left((1-\frac{h^2}{12})D_x^2-1\right)\Psi\\ &=\left((1-\frac{h^2}{12})D_x^2-1\right)\psi_e+h^4\left((1-\frac{h^2}{12})D_x^2-1\right)\widehat{\psi}\,.\end{aligned} \tag{5.33}$$

The first term can be estimated via local Taylor expansion

$$\begin{aligned}\left((1-\frac{h^2}{12})D_x^2-1\right)\psi_e&=(1+\frac{h^2}{12}D_x^2)\omega_e-\frac{1}{240}h^4\partial_x^6\psi_e+O(h^6)\|\psi_e\|_{C^8}\\ &=(1+\frac{h^2}{12}D_x^2)\omega_e+h^4\widehat{\omega}_1+O(h^6)\|\psi_e\|_{C^8}\,,\end{aligned} \tag{5.34}$$

where $\widehat{\omega}_1$ was introduced in (5.30). The second term appearing on the right hand side of (5.33) can be treated as

$$\begin{aligned}h^4\left((1-\frac{h^2}{12})D_x^2-1\right)\widehat{\psi}&=h^4(\partial_x^2-1)\widehat{\psi}+O(h^6)\|\widehat{\psi}\|_{C^4}\\ &=h^4\widehat{\omega}_2+O(h^6)\|\psi_e\|_{C^5}\,,\end{aligned} \tag{5.35}$$

where we applied (5.24) and $\widehat{\omega}_2$ was also introduced in (5.30). The combination of (5.33), (5.34) and (5.35) gives

$$\begin{aligned}\left(1+\frac{h^2}{12}D_x^2\right)\Omega&=(1+\frac{h^2}{12}D_x^2)\omega_e+h^4\widehat{\omega}_1+h^4\widehat{\omega}_2+O(h^6)\|\psi_e\|_{C^8}\\ &=(1+\frac{h^2}{12}D_x^2)\omega_e+h^4\widehat{\omega}+O(h^6)\|\psi_e\|_{C^8}\,.\end{aligned} \tag{5.36}$$

On the other hand, the fact that $\|D_x^2\widehat{\omega}\|_{C^0}$ is bounded by the order of $\|\psi_e\|_{C^8}$ from our construction of $\widehat{\omega}$ indicates

$$\begin{aligned}&\left(1+\frac{h^2}{12}D_x^2\right)(\omega_e+h^4\widehat{\omega})\\ &\quad=\left(1+\frac{h^2}{12}D_x^2\right)\omega_e+h^4\widehat{\omega}+O(h^6)\|\psi_e\|_{C^8}\,.\end{aligned} \tag{5.37}$$

The combination of (5.36) and (5.37) shows that at interior grid points x_i, $1 \le i \le N-1$,

$$\left| \left(1 + \frac{h^2}{12} D_x^2\right) (\Omega - \omega_e - h^4 \widehat{\omega}) \right| \le C h^6 \|\psi_e\|_{C^8} \,. \tag{5.38}$$

On the boundary (say at $i = 0$), (5.32) indicates

$$\Omega_0 - (\omega_e + h^4 \widehat{\omega})_0 = 0 \,. \tag{5.39}$$

Since the matrix $I + \frac{h^2}{12} D_x^2$ is uniformly diagonally dominant, we obtain (5.32) from (5.38) and (5.39). Lemma 5.2 is proven. □

Next we look at the truncation error of the diffusion term. By Lemma 5.2 and the fact that $\widehat{\omega}$ and its divided differences up to second order are bounded by the order of $\|\psi_e\|_{C^8}$, we have

$$|D_x^2 (\Omega - \omega_e)| \le C h^4 \|\psi_e\|_{C^8} \,, \tag{5.40}$$

which along with (5.32) gives us

$$\begin{aligned} \left((1 - \frac{h^2}{12}) D_x^2 - 1\right) \Omega = &\left(\left(1 - \frac{h^2}{12}\right) D_x^2 - 1\right) \omega_e \\ &+ O(h^4) \|\psi_e\|_{C^8} \,. \end{aligned} \tag{5.41}$$

Meanwhile, local Taylor expansion of ω_e shows that

$$\begin{aligned} \left((1 - \frac{h^2}{12}) D_x^2 - 1\right) \omega_e = &\left(1 + \frac{h^2}{12} \partial_x^2\right) (\partial_x^2 - 1) \omega_e \\ &+ O(h^4) \|\omega_e\|_{C^6} \,. \end{aligned} \tag{5.42}$$

The combination of (5.41) and (5.42) implies that

$$\begin{aligned} \left(\left(1 - \frac{h^2}{12}\right) D_x^2 - 1\right) \Omega = &\left(1 + \frac{h^2}{12} \partial_x^2\right) (\partial_x^2 - 1) \omega_e \\ &+ O(h^4) \|\psi_e\|_{C^8} \,. \end{aligned} \tag{5.43}$$

Now we estimate the time marching term. As we can see, at the interior grid points x_i, $1 \le i \le N-1$,

$$\begin{aligned} \partial_t \overline{\Omega} &= \partial_t \left((1 - \tfrac{h^2}{12}) D_x^2 - 1\right) \Psi \\ &= \partial_t \left((1 - \tfrac{h^2}{12}) D_x^2 - 1\right) \psi_e + h^4 \partial_t \left((1 - \tfrac{h^2}{12}) D_x^2 - 1\right) \widehat{\psi} \,, \end{aligned} \tag{5.44}$$

where the first term can be treated via local Taylor expansion and the kinematic relation between ψ_e and ω_e

$$\begin{aligned}\partial_t \left((1 - \frac{h^2}{12}) D_x^2 - 1 \right) \psi_e = \\ \partial_t (1 + \frac{h^2}{12} \partial_x^2) \omega_e + h^4 \partial_t (\frac{1}{360} \partial_x^6 \psi_e - \frac{1}{144} \partial_x^4 \psi_e) \\ + O(h^6) \| \partial_t \psi_e \|_{C^6} \,,\end{aligned} \tag{5.45}$$

and the second term can also be controlled by

$$h^4 \partial_t \left(\left(1 - \frac{h^2}{12} \right) D_x^2 - 1 \right) \widehat{\psi} = h^4 \partial_t (\partial_x^2 - 1) \widehat{\psi} + O(h^6) \| \partial_t \widehat{\psi} \|_{C^4} \,. \tag{5.46}$$

Again, by (5.25), we have the following estimate

$$\| \partial_t \psi_e \|_{C^6} \le C \| \partial_t \omega_e \|_{C^4} \le C \| \omega_e \|_{C^6} \le C \| \psi_e \|_{C^8} \,, \tag{5.47}$$

where the original PDE that $\partial_t \omega_e = \nu (\partial_x^2 - 1) \omega_e$ was applied in the second step. The term $\| \partial_t \widehat{\psi} \|_{C^4}$ appearing in (5.46) can be controlled by (5.27). The combination of (5.44)-(5.47) shows that

$$\partial_t \left(1 + \frac{h^2}{12} D_x^2 \right) \Omega - \partial_t \left(1 + \frac{h^2}{12} \partial_x^2 \right) \omega_e = O(h^4) \| \psi_e \|_{C^8} \,. \tag{5.48}$$

Combining (5.48) and (5.43), the estimates for time marching term and diffusion term respectively, and applying the original vorticity equation, which implies that $(1 + \frac{h^2}{12} \partial_x^2) \left(\partial_t \omega_e - \nu (\partial_x^2 - 1) \omega_e \right) = 0$, we arrive at

$$\partial_t \left(1 + \frac{h^2}{12} D_x^2 \right) \Omega - \nu \left((1 - \frac{h^2}{12}) D_x^2 - 1 \right) \Omega = O(h^4) \| \psi_e \|_{C^8} \,, \tag{5.49}$$

at grid points x_i, $1 \le i \le N - 1$.

Finally we look at the boundary condition for Ω. We will show that Ω satisfies Briley's formula applied to Ψ up to $O(h^4)$ error. To verify it, first we have a look at the expression appearing in Briley's formula (say near the left boundary $x_0 = -1$)

$$\begin{aligned}6 \Psi_1 - \frac{3}{2} \Psi_2 + \frac{2}{9} \Psi_3 = \\ \left(6 \psi_e(x_1) - \frac{3}{2} \psi_e(x_2) + \frac{2}{9} \psi_e(x_3) \right) + h^4 \left(6 \widehat{\psi}_1 - \frac{3}{2} \widehat{\psi}_2 + \frac{2}{9} \widehat{\psi}_3 \right) \,.\end{aligned} \tag{5.50}$$

The first term can be estimated via Taylor expansion of ψ_e, keeping in mind that $\psi_e(x_0) = \partial_x \psi_e(x_0) = 0$

$$6\psi_e(x_1) - \frac{3}{2}\psi_e(x_2) + \frac{2}{9}\psi_e(x_3) = h^2 \partial_x^2 \psi_e(x_0) + \frac{1}{10} h^5 \partial_x^5 \psi_e(x_0) + O(h^6)\|\psi_e\|_{C^6} \,. \tag{5.51}$$

The estimate of the second term appearing in (5.50) can also be carried out via Taylor expansion and our construction of $\widehat{\psi}$

$$\begin{aligned} h^4 \left(6\widehat{\psi}_1 - \frac{3}{2}\widehat{\psi}_2 + \frac{2}{9}\widehat{\psi}_3 \right) &= \tfrac{11}{3} h^5 \partial_x \widehat{\psi}(x_0) + O(h^6)\|\widehat{\psi}\|_{C^2} \\ &= -\tfrac{1}{10} h^5 \partial_x^5 \psi_e(x_0) + O(h^6)\|\psi_e\|_{C^5} \,, \end{aligned} \tag{5.52}$$

where we used (5.23) and (5.24). As we can see, the $O(h^5)$ terms appearing in (5.51) and (5.52) cancel each other if we put them into a combined term $6\Psi_1 - \frac{3}{2}\Psi_2 + \frac{2}{9}\Psi_3$ because of our special choice of $\alpha(t)$ and $\beta(t)$. The reason of the choice can be seen more clearly here. The combination of (5.50), (5.51) and (5.52), along with the fact that $\omega_e(x_0) = (\partial_x^2 - 1)\psi_e(x_0) = \partial_x^2 \psi_e(x_0)$ since ψ_e vanishes on the boundary, gives us

$$6\Psi_1 - \frac{3}{2}\Psi_2 + \frac{2}{9}\Psi_3 = h^2 \omega_e(x_0) + O(h^6)\|\psi_e\|_{C^6} \,. \tag{5.53}$$

In other words,

$$\omega_e(x_0) = \frac{1}{h^2} \left(6\Psi_1 - \frac{3}{2}\Psi_2 + \frac{2}{9}\Psi_3 \right) + O(h^4)\|\psi_e\|_{C^6} \,. \tag{5.54}$$

Meanwhile, our definition of Ω_0 in (5.32), combined with the fact that $|\widehat{\omega}_0| \le C\|\psi_e\|_{C^6}$, implies that the difference between Ω_0 and $\omega_e(x_0)$ is of order $O(h^4)\|\psi_e\|_{C^6}$. Then we obtain the boundary condition for Ω:

$$\Omega_0 = \frac{1}{h^2} \left(6\Psi_1 - \frac{3}{2}\Psi_2 + \frac{2}{9}\Psi_3 \right) + h^4 \boldsymbol{e}_0 \,, \tag{5.55}$$

where $|\boldsymbol{e}_0| \le C\|\psi_e\|_{C^6}$.

5.3.2. Error estimate. For $0 \le i \le N$, we define

$$\widetilde{\psi}_i = \psi_i - \Psi_i \,, \qquad \widetilde{\omega}_i = \omega_i - \Omega_i \,, \tag{5.56}$$

and the error function for $\overline{\omega}$ is defined at interior grid points x_i, $1 \le i \le N-1$,

$$\widetilde{\overline{\omega}}_i = \overline{\omega}_i - \overline{\Omega}_i = \left(1 + \frac{h^2}{12} D_x^2 \right) \widetilde{\omega}_i \,. \tag{5.57}$$

Our consistency analysis above gives a closed system for error functions

$$
\left\{
\begin{aligned}
&\left(1+\frac{h^2}{12}D_x^2\right)\partial_t\widetilde{\omega} = \nu\left((1-\frac{h^2}{12})D_x^2-1\right)\widetilde{\omega}+\boldsymbol{f}\,,\\
&\left((1-\frac{h^2}{12})D_x^2-1\right)\widetilde{\psi} = \left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\omega}\,,\ \widetilde{\psi}_0=\widetilde{\psi}_N=0\,,
\end{aligned}
\right. \tag{5.58}
$$

where the local truncation error $\boldsymbol{f}$ satisfies $|\boldsymbol{f}_i| \le Ch^4\|\psi_e\|_{C^8}$. On the boundary, (say at the left boundary point $x_0=-1$)

$$
\widetilde{\omega}_0 = \frac{1}{h^2}\left(6\widetilde{\psi}_1-\frac{3}{2}\widetilde{\psi}_2+\frac{2}{9}\widetilde{\psi}_3\right)+h^4\boldsymbol{e}_0\,, \tag{5.59}
$$

where $|\boldsymbol{e}_0| \le C\|\psi_e\|_{C^6}$. The identity (5.59) comes from Briley's formula (5.7) and the estimate for Ω_0 in (5.55). In other words, the error function of vorticity and the error function of stream function satisfy Briley's formula up to $O(h^4)$ error.

The system of the error functions (5.58), along with (5.59), is very similar to the fourth order scheme (5.5) with Briley's formula (5.7), except for local truncation error terms $\boldsymbol{f}$, $\boldsymbol{h}^4\boldsymbol{e}_0$. The procedure of stability analysis carried out in **5.2** can be implemented here similarly.

Multiplying (5.58) by $-(1+\frac{h^2}{12}D_x^2)\widetilde{\psi}$ gives

$$
\begin{aligned}
&\left\langle -\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi},(1+\frac{h^2}{12}D_x^2)\partial_t\widetilde{\omega}\right\rangle\\
&\qquad+\left\langle\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi},\left(\left(1-\frac{h^2}{12}\right)D_x^2-1\right)\widetilde{\omega}\right\rangle\\
&= \langle-\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi},\boldsymbol{f}\rangle\,.
\end{aligned} \tag{5.60}
$$

The term corresponding to local truncation error can be controlled by Cauchy inequality

$$
\langle-\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi},\boldsymbol{f}\rangle \le C\|\widetilde{\psi}\|^2+C\|\boldsymbol{f}\|^2\,, \tag{5.61}
$$

and the results corresponding to the time marching term and diffusion term are analogous to (5.9) and (5.10)

$$
\begin{aligned}
&-\left\langle\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi}\,,\left(1+\frac{h^2}{12}D_x^2\right)\partial_t\widetilde{\omega}\right\rangle\\
&=\frac{1}{2}\frac{d}{dt}\left(\left(1-\frac{h^2}{6}\right)\|\nabla_h\widetilde{\psi}\|^2\right.\\
&\left.+\|\widetilde{\psi}\|^2-\tfrac{h^2}{12}\left(1-\tfrac{h^2}{12}\right)\|D_x^2\widetilde{\psi}\|^2\right)\,,
\end{aligned} \tag{5.62}
$$

$$\left\langle \left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi}\,,\ \left(\left(1-\frac{h^2}{12}\right)D_x^2-1\right)\widetilde{\omega}\right\rangle = \left\|\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\omega}\right\|^2+\frac{1}{h}(\widetilde{\psi}_1\widetilde{\omega}_0+\widetilde{\psi}_{N-1}\widetilde{\omega}_N)\,. \tag{5.63}$$

The estimate of the boundary term $\mathcal{B}=\mathcal{B}_1+\mathcal{B}_2$, where $\mathcal{B}_1=\frac{1}{h}(\widetilde{\psi}_1\widetilde{\omega}_0)$ and $\mathcal{B}_2=\frac{1}{h}(\widetilde{\psi}_{N-1}\widetilde{\omega}_N)$ is similar to that in **5.2**. The boundary condition for $\widetilde{\omega}_0$ in (5.59) gives

$$\mathcal{B}_1=\widetilde{\psi}_1\left(\frac{1}{h^3}\left(6\widetilde{\psi}_1-\frac{3}{2}\widetilde{\psi}_2+\frac{2}{9}\widetilde{\psi}_3\right)+h^3\boldsymbol{e}_0\right)\,. \tag{5.64}$$

Analogous to (5.12), the term $6\widetilde{\psi}_1-\frac{3}{2}\widetilde{\psi}_2+\frac{2}{9}\widetilde{\psi}_3$ can be rewritten as $\frac{11}{3}\widetilde{\psi}_1-\frac{19}{18}h^2D_x^2\widetilde{\psi}_1+\frac{2}{9}h^2D_x^2\widetilde{\psi}_2$. Then the procedure in (5.13), combined with Cauchy inequality, can be repeated to estimate $\mathcal{B}_1$

$$\begin{aligned}\mathcal{B}_1&=\frac{\widetilde{\psi}_1}{h^3}\left(\frac{11}{3}\widetilde{\psi}_1-\frac{19}{18}h^2D_x^2\widetilde{\psi}_1+\frac{2}{9}h^2D_x^2\widetilde{\psi}_2\right)+h^3\widetilde{\psi}_1\boldsymbol{e}_0\\&\geq\frac{11\widetilde{\psi}_1^2}{3h^3}-\frac{1}{2h^3}\frac{19^2}{18^2}\widetilde{\psi}_1^2-\frac{1}{2}|D_x^2\widetilde{\psi}_1|^2h-\frac{1}{2h^3}\frac{2^2}{9^2}\widetilde{\psi}_1^2\\&\quad-\frac{1}{2}|D_x^2\widetilde{\psi}_2|^2h-\frac{1}{4}\frac{\widetilde{\psi}_1^2}{h^3}-h^9\boldsymbol{e}_0^2\\&\geq\frac{\widetilde{\psi}_1^2}{h^3}-\frac{1}{2}|D_x^2\widetilde{\psi}_1|^2h-\frac{1}{2}|D_x^2\widetilde{\psi}_2|^2h-h^9\boldsymbol{e}_0^2\,,\end{aligned} \tag{5.65}$$

Similar to (5.14), we arrive at

$$\begin{aligned}\mathcal{B}&\geq-\frac{1}{2}\sum_{i=1,2N-1,N-2}h|D_x^2\widetilde{\psi}_i|^2-h^9(\boldsymbol{e}_0^2+\boldsymbol{e}_N^2)\\&\geq-\frac{1}{2}\|D_x^2\widetilde{\psi}\|^2-h^8\,,\end{aligned} \tag{5.66}$$

if h is small enough since $|\boldsymbol{e}_0|$, $|\boldsymbol{e}_N|\leq C\|\psi_e\|_{C^8}$.

Since $\widetilde{\psi}_0=\widetilde{\psi}_N=0$, we can use the same argument as in Lemma 5.1 to conclude that

$$\begin{aligned}\left\|\left(1-\frac{h^2}{12}\right)D_x^2\widetilde{\psi}\right\|&\leq\left\|\left((1-\frac{h^2}{12})D_x^2-1\right)\widetilde{\psi}\right\|\\&=\left\|\left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\omega}\right\|\,.\end{aligned} \tag{5.67}$$

Substituting (5.67) into (5.66) and (5.63), we obtain

$$\left\langle \left(1+\frac{h^2}{12}D_x^2\right)\widetilde{\psi}\,,\ \left(\left(1-\frac{h^2}{12}\right)D_x^2-1\right)\widetilde{\omega}\right\rangle \ge -h^8\,. \tag{5.68}$$

Combining (5.68), (5.63) and (5.62), denoting $\widetilde{E}$ as

$$\widetilde{E}=\left(1-\frac{h^2}{6}\right)\|\nabla_h\widetilde{\psi}\|^2+\|\widetilde{\psi}\|^2-\frac{h^2}{12}\left(1-\frac{h^2}{12}\right)\|D_x^2\widetilde{\psi}\|^2\,, \tag{5.69}$$

and using the Poincare inequality for $\widetilde{\psi}$, which states that $\|\widetilde{\psi}\|\le C\|\nabla_h\widetilde{\psi}\|$, we arrive at

$$\frac{1}{2}\frac{d\widetilde{E}}{dt}\le C\|\boldsymbol{f}\|^2+C\|\widetilde{\psi}\|^2+h^8\le C\|\boldsymbol{f}\|^2+C\|\nabla_h\widetilde{\psi}\|^2+h^8\,. \tag{5.70}$$

Integrating in time, we obtain

$$\widetilde{E}\le C\int_0^T\|\boldsymbol{f}\|^2\,dt\,+C\int_0^T\|\nabla_h\widetilde{\psi}\|^2\,dt+2Th^8+O(h^8)\,, \tag{5.71}$$

where $O(h^8)$ is the initial term of $\widetilde{E}(\cdot,0)$. By our construction of Ψ, we have $\widetilde{\psi}(\cdot,0)=h^4\widehat{\psi}(\cdot,0)$. Moreover, we get $O(h^8)\le Ch^8\|\psi_e\|^2_{C^8}$. The application of the inequality $\frac{1}{3}\|\nabla_h\widetilde{\psi}\|^2\le\widetilde{E}$, implied by the fact that $\|D_x^2\widetilde{\psi}\|^2\le\frac{4}{h^2}\|\nabla_h\widetilde{\psi}\|^2$, into (5.71) results in

$$\|\nabla_h\widetilde{\psi}\|^2\le C\int_0^T\|\boldsymbol{f}\|^2\,dt+C\int_0^T\|\nabla_h\widetilde{\psi}\|^2\,dt+CTh^8+O(h^8)\,. \tag{5.72}$$

By Gronwall inequality, we have

$$\begin{aligned}\|\nabla_h\widetilde{\psi}\|^2&\le C\int_0^T\|\boldsymbol{f}(\cdot,s)\|^2\,ds+CTh^8+O(h^8)\\&\le Ch^8(\|\psi_e\|^2_{C^8}+T)\,.\end{aligned} \tag{5.73}$$

Thus, we have proven the following theorem for EC4 scheme:

Theorem 5.3 *Let $\psi_e\in L^\infty([0,T];C^8(\overline{\Omega}))$, ω_e be the exact solution of the 1-D Stokes equations (5.1) and ψ_h, ω_h be the approximate solution of the EC4 scheme (5.5) with Briley's formula (5.7), then we have*

$$\|\nabla_h(\psi_e-\psi_h)\|_{L^\infty([0,T],L^2)}\le Ch^4(\|\psi_e\|_{C^8}+T)\,. \tag{5.74}$$

Table 1. Errors and order of accuracy for stream function and vorticity at $t = 1$ when the **second order** schemes with **Thom's formula** for the vorticity at the boundary are used. CFL=1, where CFL= $\frac{2\nu\triangle t}{\triangle x^2}$. We take $\triangle t = \frac{1}{2}\triangle x$ when $N = 16, 32$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
	16	1.86e-02		1.54e-02		1.71e-02	
	32	4.71e-03	1.98	3.88e-03	1.99	4.30e-03	1.99
ψ	64	1.18e-03	2.00	9.72e-04	2.00	1.07e-03	2.00
	128	2.95e-04	2.00	2.43e-04	2.00	2.69e-04	2.00
	256	7.37e-05	2.00	6.08e-05	2.00	6.72e-05	2.00
	16	2.24e-02		2.77e-02		5.25e-02	
	32	7.41e-03	1.60	9.47e-03	1.55	2.02e-02	1.38
ω	64	2.03e-03	1.87	2.55e-03	1.89	5.30e-03	1.93
	128	5.29e-04	1.94	6.62e-04	1.95	1.34e-03	1.98
	256	1.35e-04	1.97	1.67e-04	1.99	3.32e-04	2.01

Table 2. Errors and order of accuracy for stream function and vorticity at $t = 1$ when the **second order** schemes with **Wilkes formula** for the vorticity at the boundary are used. CFL=1, where CFL= $\frac{2\nu\triangle t}{\triangle x^2}$. We take $\triangle t = \frac{1}{2}\triangle x$ when $N = 16, 32$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
	16	1.47e-02		1.31e-02		1.52e-02	
	32	3.85e-03	1.94	3.36e-03	1.96	3.88e-03	1.97
ψ	64	9.85e-04	1.97	8.54e-04	1.98	9.81e-04	1.98
	128	2.49e-04	1.98	2.15e-04	1.99	2.47e-04	1.99
	256	6.27e-05	1.99	5.40e-05	2.00	6.18e-05	2.00
	16	2.27e-02		2.61e-02		4.77e-02	
	32	4.65e-03	2.28	4.85e-03	2.43	9.25e-03	2.36
ω	64	9.46e-04	2.30	8.93e-04	2.44	1.58e-03	2.55
	128	2.02e-04	2.22	1.79e-04	2.32	2.88e-04	2.46
	256	4.64e-05	2.12	3.93e-05	2.19	5.83e-05	2.30

Table 3. Errors and order of accuracy for stream function and vorticity at $t = 1$ when the fourth order schemes with **Briley's formula** at the boundary are used. CFL=$\frac{1}{2}$, where CFL= $\frac{2\nu\triangle t}{\triangle x^2}$. We take $\triangle t = \frac{1}{2}\triangle x$ when $N = 32$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
	32	4.01e-06		3.22e-06		3.75e-06	
ψ	64	2.60e-07	3.95	2.17e-07	3.90	2.67e-07	3.81
	128	1.67e-08	3.96	1.43e-08	3.92	1.77e-08	3.91
	256	1.06e-09	3.98	9.18e-10	3.96	1.14e-09	3.96
	32	6.15e-05		8.75e-05		1.81e-04	
ω	64	3.41e-06	4.17	4.71e-06	4.21	9.60e-06	4.23
	128	1.96e-07	4.12	2.66e-07	4.14	5.30e-07	4.18
	256	1.16e-08	4.07	1.56e-08	4.09	3.06e-08	4.11

6. Numerical test and accuracy check

Finally we present accuracy check for the schemes, including the second and fourth order method mentioned above. The exact solution (5.3) of the system (5.1) with $k = 1$, the viscosity $\nu = 0.01$, and $\mu = 2.88335565358979$ will be used for comparison in our numerical experiments. The final time is taken to be $T = 1.0$. Explicit treatment of the diffusion term and the fourth order Runge-Kutta time stepping were used (see E and Liu [4] for detail). The errors in the tables are all absolute errors between the numerical and exact solutions. As can be seen in the tables, the second order method with Thom's boundary condition achieves second order accuracy for the stream function, and is only slightly less accurate for the vorticity on coarse grids. As the grid is refined, the accuracy order is closer and closer to two. The second order scheme with Wilkes formula on the boundary also achieves second order accuracy for the stream function, and gets more than second order accuracy for the vorticity. In other words, the orders of accuracy of these two formulas for the stream function are almost the same, yet Wilkes formula performs better than Thom's formula in the accuracy of vorticity. Table 3 lists the numerical results of the fourth order scheme. The EC4 method with Briley's boundary condition achieves fourth order accuracy for the stream function, and gets more than fourth order accuracy for the vorticity. That shows both profiles have full fourth order convergence.

References

1. Anderson, C.R., Reider, M.B. (1996): A High order explicit method for the computation of flow about a circular cylinder. J. Comput. Phys. **125**, 207–224
2. Briley, W.R. (1971): A Numerical study of laminar separation bubbles using the Navier-Stokes equations. J. Fluid Mech. **47**, 713–736
3. Dennis, S.C.R., Hudson, J.D. (1989): Compact h^4 finite-difference approximations to operators of Navier-Stokes type. J. Comput. Phys. **85**, 390–416
4. E, Weinan, Liu, J.-G. (1996): Vorticity boundary condition for finite difference Schemes. J. Comput. Phys. **124**, 368–382
5. E, Weinan, Liu, J.-G. (1996): Essentially compact schemes for unsteady viscous incompressible flows. J. Comput. Phys. **126**, 122–138
6. Glowinski, R., Pironneau, O. (1979): Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem. SIAM Rev. **21**, 167–212
7. Henshaw, W.D., Kreiss, H.O., Reyna, L.G.M. (1994): A Fourth-order accurate difference approximation for the incompressible Navier-Stokes equations. Comput. and Fluids **23**, 575–593
8. Hou, T.Y., Wetton, B. (1992): Convergence of a finite difference scheme for the Navier-Stokes equations using vorticity boundary conditions. SIAM J. Numer. Anal. **29**, 615–639
9. Liu, J.-G., Weinan, E. (2001): Simple finite element method in vorticity formulation for incompressible fows. Math. Comp. **70**, 579–593

10. Meth, K.Z. (1988): A Vortex and finite difference hybrid method to compute the flow of an incompressible, inviscid fluid past a semi-infinite plate. Ph.D. thesis, NYU (unpublished)
11. Orszag, S.A., Israeli, M. (1974): Numerical simulation of viscous incompressible flows. Ann. Rev. Fluid Mech. **6**, 281–318
12. Orszag, S.A., Israeli. M., Deville, M.O. (1986): Boundary conditions for incompressible flows. J. Sci. Comput. **1**, 75–111
13. Peyret, R., Taylor, T.: Computational methods for fluid flow. New York, Berlin: Springer 1983
14. Quartapelle, L. (1981): Vorticity conditioning in the computation of two-dimensional Viscous Flows. J. Comput. Phys. **40**, 453–477
15. Quartapelle, L.: Numerical solution of the incompressible Navier-Stokes equations. Berlin: Birkhäuser 1983
16. Strang, G. (1964): Accurate partial differential methods II. non-Linear problems. Numer. Math. **6**, 37–46
17. Thom, A. (1933): The Flow past circular cylinders at low speeds. Proceedings Royal Society, London Sect. A. **141**, 651–669
18. Wang. C., Liu, J.-G.: Fourth order convergence of compact finite difference solver for 2D incompressible flow. Commun. Appl. Anal. (to appear)
19. Wang. C., Liu, J.-G., High order finite difference method for unsteady incompressible flow on multi-connected domain in vorticity-stream function formulation. J. Comput. Phys. (submitted)