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# GLOBAL CONVERGENCE OF A STICKY PARTICLE METHOD FOR THE MODIFIED CAMASSA–HOLM EQUATION*

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**Abstract.** In this paper, we prove convergence of a sticky particle method for the modified Camassa–Holm equation (mCH) with cubic nonlinearity in one dimension. As a byproduct, we prove global existence of weak solutions $u$ with regularity: $u$ and $u_x$ are space-time BV functions. The total variation of $m(\cdot,t) = u(\cdot,t) - u_{xx}(\cdot,t)$ is bounded by the total variation of the initial data $m_0$. We also obtain $W^{1,1}(\mathbb{R})$-stability of weak solutions when solutions are in $L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$. (Notice that peakon weak solutions are not in $W^{2,1}(\mathbb{R})$.) Finally, we provide some examples of nonuniqueness of peakon weak solutions to the mCH equation.

**Key words.** $N$-peakon solutions, global existence, nonuniqueness, sticky collisions, space-time BV estimates

**AMS subject classifications.** 35C08, 35D30, 82C22

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## 1. Introduction.

This paper is concerned with the following nonlinear partial differential equation in $\mathbb{R}$:

$$m_t + \left[\left(u^2 - u_x^2\right)m\right]_x = 0, \quad m = u - u_{xx}, \quad x\in\mathbb{R}, \quad t>0, \tag{1}$$

subject to the initial condition

$$m(x,0) = m_0(x), \quad x\in\mathbb{R}. \tag{2}$$

This equation is referred to as the modified Camassa–Holm (mCH) equation with cubic nonlinearity. It was introduced by several different authors [12, 14, 24, 25]. Originally, the mCH equation was proposed by Fokas [12], Fuchssteiner [14], and Olver and Rosenau [24] in the context of integrable systems. In a physical context, Qiao [25] gave a derivation of the mCH equation from a two-dimensional Euler equation, where the functions $u$ and $m$ represent, respectively, the velocity of the fluid and its potential density. From the fundamental solution $G(x) = \frac{1}{2}e^{-|x|}$ for the Helmholtz operator $1-\partial_{xx}$, the velocity function $u$ can be written as a convolution of $m$ with the kernel $G$

$$u(x,t) = G * m = \int_{\mathbb{R}} G(x-y)m(y,t)dy.$$

If we set $m = u - \alpha^2 u_{xx}$ (by scaling $u_\alpha(x,t) = u(\alpha x, \alpha t)$), the fundamental solution for $1-\alpha^2\partial_{xx}$ is given by $G_\alpha(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}$ while the corresponding equation is

$$m_t + \left[\left(u^2 - \alpha^2 u_x^2\right)m\right]_x = 0. \tag{3}$$

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Taking $\alpha \to 0$ in (3), we formally obtain the following scalar conservation law:

$$u_t + (u^3)_x = 0.$$

For smooth solutions to the mCH equation (1), there are two conserved quantities (called Hamiltonian functionals of the mCH equation):

$$H_0 = \int_{\mathbb{R}} mu dx, \quad H_1 = \frac{1}{4}\int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) dx. \tag{4}$$

Equation (1) can be written in the bi-Hamiltonian form [17, 24],

$$m_t = -\left(\left(u^2 - u_x^2\right) m\right)_x = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m},$$

where

$$J = -\partial_x m \partial_x^{-1} m \partial_x, \quad K = \partial_x^3 - \partial_x$$

are compatible Hamiltonian operators. (If the Hamiltonian operators $J$ and $K$ are compatible, then any constant coefficients linear combination $\alpha J + \beta K$ is also a Hamiltonian operator.) The Hamiltonian pair $J$,$K$ is nondegenerate in the sense that one of the associated Poisson structure is symplectic. According to the fundamental theorem of Magri [22], any bi-Hamiltonian system associated with a nondegenerate Hamiltonian pair induces a hierarchy of commuting Hamiltonians and flows and, provided enough of these Hamiltonians are functionally independent, is therefore completely integrable. The mCH equation (1) is a complete integrable system, and it possesses a Lax pair [25, 26].

Equation (1) has $N$-peakon weak solutions (see Definition 2.1) of the form,

$$u^N(x,t) = \sum_{i=1}^N p_i G(x - x_i(t)) \ \text{ and } \ m^N(x,t) = \sum_{i=1}^N p_i \delta(x - x_i(t)),$$

where $p_i$ is the amplitude (momentum) of peakons and when $x_1(t) < x_2(t) < \cdots < x_N(t)$ the traveling speed is given by (see [17])

$$\frac{d}{dt}x_i = \frac{1}{6}p_i^2 + \frac{1}{2}\sum_{j<i} p_i p_j e^{x_j - x_i} + \frac{1}{2}\sum_{j>i} p_i p_j e^{x_i - x_j} + \sum_{1\le m<i<n\le N} p_m p_n e^{x_m - x_n}. \tag{5}$$

Set $N = 1$ in (5), and we can see that the mCH equation has solitary wave solitons (one peakon solutions) of the form

$$u(x,t) = pG(x - x(t)), \quad m(x,t) = p\delta(x - x(t)), \ \text{ and } \ x(t) = \frac{1}{6}p^2 t. \tag{6}$$

Moreover, $pG(x - x(t))$ is a solitary wave soliton if and only if the traveling speed of the soliton is $\frac{1}{6}p^2$ (see Proposition 4.3), which can be viewed as a "jump condition" (or "Rankine–Hugoniot condition"). From the characteristics equation for the mCH (1), the speed of the solitary wave solution is given by

$$\frac{d}{dt}x(t) = u^2(x(t),t) + u_x^2(x(t),t) = \frac{1}{4}p^2 - p^2(G_x^2)(0) = \frac{1}{6}p^2. \tag{7}$$

This is an unusual property because $(G_x^2)(0-) = (G_x^2)(0+) = \frac{1}{4}$ and (7) implies that to obtain the solitary wave solutions the correct definition of $G_x^2$ at 0 is

$$(G_x^2)(0) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}. \tag{8}$$

Notice that Hamiltonians $H_0$ and $H_1$ defined by (4) are not conserved for $N$-peakon solutions when $N \geq 2$, which is different with the Camassa–Holm (CH) equation [6, 15],

$$m_t + (um)_x + mu_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

The CH equation also has $N$-peakon solutions of the form

$$u^N(x,t) = \sum_{i=1}^{N} p_i(t) e^{-|x - x_i(t)|}.$$

The momentum $p_i(t)$ evolves with time which is different with the mCH equation, where $p_i$ is a constant. $p_i(t)$ and $x_i(t)$ satisfy the following Hamiltonian system of ODEs:

$$\begin{cases} \dfrac{d}{dt} x_i(t) = \displaystyle\sum_{j=1}^{N} p_j(t) e^{-|x_i(t) - x_j(t)|}, \quad i = 1, \ldots, N, \\ \dfrac{d}{dt} p_i(t) = \displaystyle\sum_{j=1}^{N} p_i(t) p_j(t) \operatorname{sgn}\big(x_i(t) - x_j(t)\big) e^{-|x_i(t) - x_j(t)|}, \quad i = 1, \ldots, N, \end{cases} \tag{9}$$

and the Hamiltonian function is given by

$$\mathcal{H}_0(t) = \frac{1}{2} \sum_{i,j=1}^{N} p_i(t) p_j(t) e^{-|x_i(t) - x_j(t)|}.$$

In comparison, system (5) is not a Hamiltonian system. It is a Hamiltonian system with an intrinsic speed as in the following description. Set

$$\mathcal{H}(t) := \sum_{1 \leq i < j \leq N} p_i p_j e^{x_i(t) - x_j(t)} \quad \text{and} \quad A = (a_{ij})_{N \times N} \quad \text{and} \quad a_{ij} = \begin{cases} -\frac{1}{2}, & i < j; \\ 0, & i = j; \\ \frac{1}{2}, & i > j. \end{cases} \tag{10}$$

Hence, $A$ is an antisymmetry metric. (Notice that $\mathcal{H}$ is the same as $\mathcal{H}_0$ up to a constant when $x_1 < x_2 < \cdots < x_N$: $\mathcal{H} = \mathcal{H}_0 - \frac{1}{2} \sum_{i=1}^{N} p_i^2$.) Denote

$$X(t) := (x_1(t), \ldots, x_N(t))^T, \quad P := \left( \frac{1}{6} p_1^2, \ldots, \frac{1}{6} p_N^2 \right)^T,$$

and

$$\frac{\delta \mathcal{H}}{\delta X} := \left( \frac{\partial \mathcal{H}}{\partial x_1}, \ldots, \frac{\partial \mathcal{H}}{\partial x_N} \right).$$

Then, (5) can be rewritten as

$$\frac{dX}{dt} = P + A \frac{\delta \mathcal{H}}{\delta X}. \tag{11}$$

This system resembles the structure of Kuramoto model as described below. Consider a population of $N$ coupled nonlinear oscillators where the phase $\theta_i(t)$ of the $i$th oscillator evolves in time according to

$$\frac{d}{dt}\theta_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \tag{12}$$

where $\Omega_i$ is the natural frequency of $i$th oscillator and $K > 0$ is the coupling strength. Each oscillator has its own frequency $\Omega_i$ while it interacts with other oscillators through a gradient system for alignment. In comparison, each peakon (or particle) $x_i(t)$ in system (11) has its own intrinsic speed $\frac{1}{6}p_i^2$ while it interacts with other peakons through a Hamiltonian system.

Although (5) has a unique global solution, only the solution before collision can be used to construct peakon weak solutions. In general, collisions between peakons can happen. For example, consider the case $N = 2$ in (5) and assume $x_1(0) < x_2(0)$. Becasue $\frac{d}{dt}(x_1(t) - x_2(t)) = \frac{1}{6}(p_1^2 - p_2^2)$, we know that $x_1(t)$ and $x_2(t)$ will collide in finite time if $p_1^2 > p_2^2$. In comparison, for the CH equation with positive $p_i$ ($i = 1, \dots, N$), the trajectories of $N$-peakon solutions obtained by (9) never collide [7, 9]. The peakons in the CH equation elastically bounce back after becoming close to each other, and they exchange momentum. Hence, the total energy $p_1^2 + p_2^2$ is conserved [10]. However, two peakons for the mCH equation can collide in finite time and for the sticky collision, the energy becomes $(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1p_2$, which is not conserved. In Proposition 4.5, we also construct a peakon weak solution passing through each other after collision and it conserves energy. This example also shows nonuniqueness of peakon weak solutions.

In this work, we provide a sticky particle model by assuming the particles stick together whenever they collide in system (5). The solutions of this sticky model gives sticky peakon weak solutions to the mCH equation (1). The collisions between the particles are inelastic, and the mean field limit of this model gives a global weak solution to the mCH equation.

The sticky or adhesion model appears in many subjects in science. In the early 1970s Zeldovich [28] described a simple PDE model designed to explain the pancake-like clumping structure of matter in the universe, which has gathered considerable interest from cosmologists over time [18, 27]. Comparing with dynamics governed by (5) in one dimension, this adhesion model describes the behavior of a finite collection of particles, freely moving in the absence of force and sticking, combining their mass and momentum upon collision. They can be mathematically represented by a time-dependent discrete measure $\rho_t^N := \sum_{i=1}^N m_i\delta(x - x_i(t))$ concentrated in a finite set of $N$ particles $P_i(t) := (m_i, x_i(t), v_i(t))$, $i = 1, \dots, N$ with positive mass $m_i$, ordered positions $x_1(t) \le x_2(t) \le \cdots \le x_N(t)$, and velocities $v_i(t)$. This particle model addresses the nature of singular solutions of the pressureless Euler system, consisting of conservation laws for mass and momentum. Global existence of singular solutions was first studied by E, Rykov, and Sinai [11] and Brenier and Grenier [3]. Recent work by Natile and Savaré [23] and Brenier et al. [2] brings recent progress in optimal transportation theory to bear on the problem.

In this paper, we use the system (5) to construct global sticky trajectories $\{x_i(t)\}_{i=1}^N$ subject to initial data $\{x_i(0) = c_i\}_{i=1}^N$ with $c_1 < c_2 < \cdots < c_N$ and for a fixed set of momentum $\{p_i\}_{i=1}^N$. Our strategy is as follows. Assume the peakons stick and combine their momentum ($p_i$) whenever they collide. Then, view the collision time as a new starting point and use system (5) again to extend the trajectories.

Repeat this process whenever collision happens. Because the trajectories can collide at most $N-1$ times, we can construct global sticky trajectories $\{x_i(t)\}_{i=1}^N$ subject to initial data $\{x_i(0)=c_i\}_{i=1}^N$.

Then, we use the global sticky trajectories $\{x_i(t)\}_{i=1}^N$ to construct a global sticky peakon weak solution as

$$u^N(x,t) := \sum_{i=1}^N p_i G(x-x_i(t)) \quad \text{and} \quad m^N(x,t) := \sum_{i=1}^N p_i \delta(x-x_i(t)). \tag{13}$$

$(u^N, m^N)$ is a global weak solution to the mCH equation subject to $m_0^N = \sum_{i=1}^N p_i\delta(x-c_i)$ (see Proposition 2.3).

For general initial data $m_0 \in \mathcal{M}(\mathbb{R})$ (Radon measure space), a sticky particle method is used to show global existence of weak solutions to the mCH equation. We construct an initial sequence $\{m_0^N\}_{N=1}^\infty$ which approximates $m_0$ in measure sense. Each $m_0^N$ is a summation of $N$ delta measures with weights $\{p_i\}_{i=1}^N$, and we obtain a sticky peakon weak solution defined by (13) for initial data $m_0^N$. Then, some uniform (in $N$) space and time BV estimates for $u^N$ and $u_x^N$ (see Proposition 3.3) are obtained. These estimates shows that there is a subsequence of $u^N$ converging to a function $u$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ as $N\to+\infty$. And the limiting function $u$ is a weak solution to the mCH equation with regularity that $u$ and $u_x$ are space-time BV functions (see Theorem 3.4). Moreover, there is a Radon measure $m$ such that $m^N \overset{*}{\rightharpoonup} m$ in $\mathcal{M}(\mathbb{R}\times[0,T))$ (as $N\to+\infty$) for any $T>0$. We also prove the total variation stability of $m(\cdot,t)$ in Theorem 3.5. That is $|m(\cdot,t)|(\mathbb{R}) \le |m_0|(\mathbb{R})$ for a.e. $t\ge 0$.

In Theorem 4.1, we obtain $W^{1,1}(\mathbb{R})$-stability of weak solutions when $u\in L^\infty(0,\infty; W^{2,1}(\mathbb{R}))$ and this implies uniqueness of weak solutions in the solution class $W^{2,1}(\mathbb{R})$. Notice that peakon solutions are not in the solution class $W^{2,1}(\mathbb{R})$. We provide two examples to show the nonuniqueness of peakon weak solutions. When the initial data has a single atom $m_0 = p\delta(x-c)$, one peakon weak solution is obtained (Proposition 4.3). However, if we split the initial data into two atoms $p_1\delta(x-c)+p_2\delta(x-c)$ ($p_1+p_2=p$ and $p_1\neq p_2$), we can also obtain a 2-peakon weak solution (Proposition 4.4). Both the one peakon and 2-peakon solutions are weak solutions to the mCH equation with the same initial data $m_0=p\delta(x-c)$. Hence, this example implies that peakon weak solutions are not unique. On the other hand, when the initial data has two atoms $m_0 = p_1\delta(x-c_1)+p_2\delta(x-c_2)$ with $0<p_2<p_1,\ c_1<c_2$, we show the two peakons collide in finite time. After the collision, two peakons can stick together or cross with each other (see Proposition 4.5). This provides another example for nonuniqueness of peakon weak solutions to the mCH equation.

For more results about local well-posedness and blow-up behaviors of strong solutions to the mCH euqation, one can refer to [8, 13, 17, 19, 21].

When initial data $u_0$ is in $W^{2,1}(\mathbb{R})$, which implies $m_0 = u_0 - \partial_{xx}u_0 \in L^1(\mathbb{R})$, Zhang [29] used the dissipative regularization to prove global existence of entropy weak solutions $u$ to (1) in space $H^1(\mathbb{R})$ with its derivative in BV space. Assuming the entropy weak solution $u(\cdot,t)\in W^{2,1}(\mathbb{R})$ ($t\ge 0$), Zhang also proved uniqueness of the solutions by doubling of variables. The main differences between the results in [29] and this work are listed as follows. (i) Comparing with the method of dissipative regularization in [29], we provide a sticky particle model. The solutions for this model give the sticky peakon weak solutions to the mCH equation (1), and the mean field limit for this model provides a global weak solution to the mCH equation. (ii) Comparing with the existence result in [29], we obtain global existence of weak solutions when the initial data $m_0\in\mathcal{M}(\mathbb{R})$. Moreover, we also obtain some explicit solutions, which we call sticky peakon weak solutions. (iii) Comparing with the uniqueness result in [29], we use a more direct method (comparing with doubling of variables) to show that the uniqueness result can be obtained for any weak solutions in the solution class $W^{2,1}(\mathbb{R})$ (there is no need for the entropy). Besides, peakon weak solutions are not in this solution class, and we give some examples for the nonuniqueness of peakon weak solutions.

Because the sticky particle method is a dispersive approximation to the mCH equation (comparing with dissipative approximation in [29]), we do not expect the weak solutions obtained in this paper are entropy weak solutions.

The rest of this paper is organized as follows. In section 2, we use the system (5) to construct global sticky trajectories $\{x_i(t)\}_{i=1}^N$ and prove $(u^N, m^N)$ defined by (13) is a global sticky peakon weak solution to the mCH equation. In section 3, we use the sticky particle method to study weak solutions to the mCH equation for general Radon measure initial data. Space-time BV estimates are established. Then by using a compactness argument, we obtain global existence of a weak solution with regularity that $u$ and $u_x$ are space-time BV functions. Moreover, we prove the total variation of $m(\cdot,t)(= (1-\partial_{xx})u(\cdot,t))$ is bounded by the total variation of the initial data $m_0$. In section 4, we show that weak solutions are unique if $u$ are in the solution class $W^{2,1}(\mathbb{R})$. And then, we construct some peakon weak solutions to show that weak solutions to the mCH equation are not unique.

## 2. Sticky peakon weak solutions.

In this section, we use (5) to construct global sticky peakon weak solutions to the mCH equation. Notice that in some cases, solutions (peakons) to (5) can collide in finite time. We assume that peakons stick together when they collide. First, let's give a definition of weak solutions.

Rewrite (1) as an equation of $u$,

$$\begin{aligned}(1-\partial_{xx})u_t &+ \left[\left(u^2-u_x^2\right)(u-u_{xx})\right]_x \\ &= (1-\partial_{xx})u_t + \left(u^3+uu_x^2\right)_x - \frac{1}{3}\left(u^3\right)_{xxx} + \frac{1}{3}\left(u_x^3\right)_{xx} = 0.\end{aligned}$$

We introduce the definition of weak solutions in terms of $u$. To this end, for test function $\phi \in C_c^\infty(\mathbb{R}\times[0,T))$ $(T>0)$ we denote the functional

$$\begin{aligned}\mathcal{L}(u,\phi) := &\int_0^T\int_{\mathbb{R}} u(x,t)[\phi_t(x,t)-\phi_{txx}(x,t)]dxdt \\ &- \frac{1}{3}\int_0^T\int_{\mathbb{R}} u_x^3(x,t)\phi_{xx}(x,t)dxdt - \frac{1}{3}\int_0^T\int_{\mathbb{R}} u^3(x,t)\phi_{xxx}(x,t)dxdt \\ &+ \int_0^T\int_{\mathbb{R}}\left(u^3+uu_x^2\right)\phi_x(x,t)dxdt.\end{aligned} \tag{14}$$

Definition 2.1. *For $m_0 \in \mathcal{M}(\mathbb{R})$, a function*

$$u \in C\left([0,T);H^1(\mathbb{R})\right) \cap L^\infty\left(0,T;W^{1,\infty}(\mathbb{R})\right)$$

*is said to be a weak solution of the mCH equation* (1)–(2) *if*

$$\mathcal{L}(u,\phi) = -\int_{\mathbb{R}}\phi(x,0)dm_0$$

*holds for all $\phi \in C_c^\infty(\mathbb{R}\times[0,T))$. If $T=+\infty$, we call $u$ a global weak solution of the mCH equation.*

Given an initial datum

$$m_0^N(x) = \sum_{i=1}^N p_i \delta(x - c_i) \quad \text{with} \quad \sum_{i=1}^N |p_i| \le M_0 < +\infty \text{ and } c_1 < c_2 < \cdots < c_N, \tag{15}$$

we have the following lemma.

LEMMA 2.2. *Let $m_0^N(x)$ given by (15). Then, the following statements hold:*

(i) *There is a unique global solution $\{x_i(t)\}_{i=1}^N$ to (5) subject to $\{x_i(0) = c_i\}_{i=1}^N$.*

(ii) *Assume that the first collision time of the solutions is $t_1 \in \mathbb{R}_+ \cup \{+\infty\}$, which means*

$$t_1 := \sup\{t_0 : x_1(t) < x_2(t) < \cdots < x_N(t) \;\; \text{for} \;\; t \in [0, t_0)\}.$$

*Then, we have*

$$\left|\frac{d}{dt} x_i(t)\right| \le \frac{1}{2} M_0^2, \quad t \in [0, t_1), \quad i = 1, \ldots, N. \tag{16}$$

(iii) *There is a weak solution to the mCH equation subject to initial data $m_0^N$ in $[0, t_1)$, which is given by*

$$u^N(x,t) := \sum_{i=1}^N p_i G(x - x_i(t)), \quad t \in [0, t_1). \tag{17}$$

*Proof.* The statement (i) is obvious, and we only prove (ii) and (iii).

*Step* 1. We prove (16).

By the definition of $t_1$, we know $x_1(t) < x_2(t) < \cdots < x_N(t)$ when $t \in [0, t_1)$. Hence, by (5) we obtain

$$\begin{aligned}\left|\frac{d}{dt} x_i(t)\right| &\le \frac{1}{6}|p_i|^2 + \frac{1}{2}\sum_{j<i}|p_i||p_j| + \frac{1}{2}\sum_{j>i}|p_i||p_j| + \sum_{1\le m<i<n\le N}|p_m||p_n| \\ &\le \frac{1}{2}\Big(\sum_{i=1}^N |p_i|\Big)^2 \le \frac{1}{2}M_0^2.\end{aligned}$$

*Step* 2. We prove that $u^N$ defined by (17) is a weak solution.

Obviously, we have

$$u^N \in C\left([0, t_1); H^1(\mathbb{R})\right) \cap L^\infty\left(0, t_1; W^{1,\infty}(\mathbb{R})\right).$$

In the following proof we denote $u := u^N$. For any test function $\phi(x,t) \in C_c^\infty(\mathbb{R} \times [0, t_1))$, let

$$\begin{aligned}\mathcal{L}(u, \phi) &= \int_0^{t_1}\int_{\mathbb{R}} u(\phi_t - \phi_{txx})dxdt \\ &\quad - \int_0^{t_1}\int_{\mathbb{R}}\left[\frac{1}{3}\left(u_x^3\phi_{xx} + u^3\phi_{xxx}\right) - \left(u^3 + uu_x^2\right)\phi_x\right]dxdt \\ &=: I_1 + I_2.\end{aligned} \tag{18}$$

Denote $x_0 := -\infty$, $x_{N+1} := +\infty$ and $p_0 = p_{N+1} = 0$. By integration by parts for space variable $x$, we calculate $I_1$ as

$$
\begin{aligned}
I_1 &= \int_0^{t_1}\int_{\mathbb{R}} u(\phi_t - \phi_{txx})dxdt = \sum_{i=0}^{N}\int_0^{t_1}\int_{x_i}^{x_{i+1}} u(\phi_t-\phi_{txx})dxdt \\
&= \sum_{i=0}^{N}\int_0^{t_1}\int_{x_i}^{x_{i+1}} \left(\frac{1}{2}\sum_{j\le i} p_j e^{x_j - x} + \frac{1}{2}\sum_{j>i} p_j e^{x-x_j}\right)(\phi_t-\phi_{txx})dxdt \\
&= \int_0^{t_1}\sum_{i=1}^{N} p_i\phi_t(x_i(t),t)dt. \qquad (19)
\end{aligned}
$$

Similarly, by (5), we have

$$
\begin{aligned}
I_2 &= -\int_0^{t_1}\int_{\mathbb{R}}\left[\frac{1}{3}\left(u_x^3\phi_{xx} + u^3\phi_{xxx}\right) - \left(u^3 + uu_x^2\right)\phi_x\right]dxdt \\
&= \int_0^{t_1}\sum_{i=1}^{N} p_i\phi_x(x_i(t))\Bigg(\frac{1}{6}p_i^2 + \frac{1}{2}\sum_{j<i} p_ip_je^{x_j-x_i} + \frac{1}{2}\sum_{j>i}p_ip_je^{x_i-x_j} \\
&\qquad\qquad + \sum_{1\le m<i<n\le N} p_mp_ne^{x_m-x_n}\Bigg)dt \\
&= \int_0^{t_1}\sum_{i=1}^{N} p_i\phi_x(x_i(t))\frac{d}{dt}x_i(t)dt. \qquad (20)
\end{aligned}
$$

Combining (18), (19), and (20) gives

$$
\begin{aligned}
\mathcal{L}(u,\phi) &= \sum_{i=1}^{N} p_i\int_0^{t_1}\frac{d}{dt}\phi(x_i(t),t)dt \\
&= \sum_{i=1}^{N} p_i\phi(x(t_1),t_1) - \sum_{i=1}^{N} p_i\phi(c_i,0) = -\int_{\mathbb{R}}\phi(x,0)dm_0^N. \qquad (21)
\end{aligned}
$$

By Definition 2.1 we know $u^N$ defined by (17) is a weak solution. □

In Lemma 2.2, $u^N$ defined by (17) is a global weak solution when $t_1 = +\infty$. However, collisions of $x_i(t)$ might happen and $t_1 < +\infty$. Whenever trajectories collide, we assume they stick together.

Next, we use the following four steps to extend the solution $x_i$ and $u^N$ when $t_1 < +\infty$. Denote $I := \{1,\dots,N\}$.

1. Sticky momentum $q_k$ and index $i_k$ after collision:
   For $1 \le i \le N$, denote the collection of the indices of $x_j$ coinciding with $x_i$ at time $t_1$

   $$J_i := \{j : x_j(t_1) = x_i(t_1)\}. \qquad (22)$$

   We pick up the collection of minimal indices from $J_i$

   $$\overline{I} := \{\min J_i : i = 1,\dots,N\} = \{i_1 < \cdots < i_{N_1}\} \subset I. \qquad (23)$$

Define

$$q_k := \sum_{j \in J_{i_k}} p_j, \quad 1 \le k \le N_1. \tag{24}$$

Each set $J_{i_k}$ corresponds to a single peakon $q_k \delta(x - x_{i_k}(t_1))$. $\{J_{i_k}\}_{k=1}^{N_1}$ is a partition of $I$, and we have

$$\sum_{k=1}^{N_1} q_k = \sum_{i=1}^{N} p_i \quad \text{and} \quad \sum_{k=1}^{N_1} |q_k| \le \sum_{i=1}^{N} |p_i| \le M_0. \tag{25}$$

2. Initial data $m_1$ and $y_k^0$:
Set
$$y_k^0 := x_{i_k}(t_1) \quad \text{for} \quad 1 \le k \le N_1$$
and
$$m_1(x) := \sum_{k=1}^{N_1} q_k \delta\left(x - y_k^0\right).$$
By the definition of $q_k$ we know
$$m_1(x) = \sum_{k=1}^{N_1} q_k \delta\left(x - y_k^0\right) = \sum_{i=1}^{N} p_i \delta(x - x_i(t_1)). \tag{26}$$

3. $N_1$ peakon solution between two collision time $t_1$ and $t_2$:
Consider the system (5) with $N_1$ particles ($1 \le k \le N_1$) (view $t_1$ as initial time)
(27)
$$\begin{cases} \dfrac{d}{dt} y_k = \dfrac{1}{6} q_k^2 + \dfrac{1}{2} \displaystyle\sum_{j<k} q_k q_j e^{y_j - y_k} + \dfrac{1}{2} \sum_{j>k} q_k q_j e^{y_k - y_j} + \sum_{m<k<n} q_m q_n e^{y_m - y_n}, \\ y_k(t_1) = y_k^0. \end{cases}$$
There exists a unique solution $\{y_k(t)\}_{k=1}^{N_1}$ to (27). Because $x_{i_1}(t_1) < x_{i_2}(t_1) < \cdots < x_{i_{N_1}}(t_1)$, we know $y_1^0 < y_2^0 < \cdots < y_{N_1}^0$. Set
$$t_2 := \sup\{t_0 : y_1(t) < y_2(t) < \cdots < y_{N_1}(t) \quad \text{for} \quad t \in [t_1, t_0)\}$$
and $t_2 > t_1$. Then, by (25), similarly to (16) we can obtain that
$$\left| \frac{d}{dt} y_k(t) \right| \le \frac{1}{2} M_0^2, \quad t \in [t_1, t_2), \quad i = 1, \ldots, N, \tag{28}$$
and
$$v^{N_1}(x,t) := \sum_{k=1}^{N_1} q_k G(x - y_k(t)), \quad t \in [t_1, t_2), \tag{29}$$
is a weak solution to the mCH equation subject to initial data
$$(1 - \partial_{xx}) v^{N_1}(x, t_1) = m_1(x) = \sum_{i=1}^{N} p_i \delta(x - x_i(t_1)).$$

4. Extend in time for $x_i$ and $u^N$:
Extend $x_i$ in time by

$$x_i(t) = y_k(t) \quad \text{for} \quad i \in J_{i_k} \quad \text{and} \quad t \in [t_1, t_2). \tag{30}$$

Because $\{J_{i_k}\}_{k=1}^{N_1}$ is a partition of $I$, all the trajectories $x_i$ $(1 \le i \le N)$ have been extended and $x_i(t)$ stick together as one trajectory $y_k(t)$ when $i \in J_{i_k}$.
Extend $u^N$ in time by

$$u^N(x,t) := \sum_{i=1}^{N} p_i G(x - x_i(t)) \quad \text{for} \quad t \in [t_1, t_2). \tag{31}$$

Combining (25), (29), (30), and (31), we know

$$u^N(x,t) = v^{N_1}(x,t) \quad \text{for} \quad t \in [t_1, t_2).$$

Hence, $u^N(x,t)$ is a weak solution to the mCH equation when $t \in [t_1, t_2)$.

Next, we prove that $u^N$ defined by (17) and (31) is a weak solution in $[0, t_2)$. For any test function $\phi \in C_c^\infty(\mathbb{R} \times [0, t_2))$, by using (21) and (26) we have

$$\begin{aligned}
\mathcal{L}(u^N, \phi) &= \sum_{i=1}^{N} p_i \int_0^{t_1} \frac{d}{dt}\phi(x_i(t), t)dt + \sum_{k=1}^{N_1} q_k \int_{t_1}^{t_2} \frac{d}{dt}\phi(y_k(t), t)dt \\
&= \sum_{i=1}^{N} p_i \phi(x_i(t_1), t_1) - \sum_{i=1}^{N} p_i \phi(c_i, 0) - \sum_{k=1}^{N_1} q_k \phi(x_{i_k}(t_1), t_1) \\
&= -\sum_{i=1}^{N} p_i \phi(c_i, 0) = -\int_{\mathbb{R}} \phi(x, 0) dm_0^N,
\end{aligned}$$

which means $u^N(x,t)$ is a weak solution to the mCH equation subject to initial data $m_0^N$ (given by (15)) when $t \in [0, t_2)$. Moreover, by (16) and (28) we know

$$\left| \frac{d}{dt} x_i(t) \right| \le \frac{1}{2} M_0^2, \quad t \in [0, t_2), \quad i = 1, \dots, N.$$

If $t_2 = +\infty$, then we obtain a global weak solution to the mCH equation. If $t_2 < +\infty$, then we can repeat the above process to extend trajectories $x_i$ and weak solution $u^N$ in time. By the sticky assumption, collisions between $x_i$ can only happen $N-1$ times (at most), and we can extend $x_i$ and $u^N$ globally. Moreover, at each time interval $x_i$ is unique. Hence, the sticky weak solution constructed by the above method is unique.

We have the following proposition.

Proposition 2.3. *Let $m_0^N$ given by* (15). *Then, there are unique global sticky trajectories $\{x_i(t)\}_{i=1}^N$ (defined similarly by* (27) *and* (30) *at each collision time interval) with initial data $\{x_i(0) = c_i\}_{i=1}^N$ and the following estimate holds:*

$$\left| \frac{d}{dt} x_i(t) \right| \le \frac{1}{2} M_0^2, \quad t \ge 0, \quad i = 1, \dots, N. \tag{32}$$

*Moreover, there is a global sticky peakon weak solution to the mCH equation with initial data $m_0^N$ and it is given by*

$$u^N(x,t) := \sum_{i=1}^{N} p_i G(x - x_i(t)) \quad \textit{and} \quad m^N(x,t) := \sum_{i=1}^{N} p_i \delta(x - x_i(t)). \tag{33}$$

**3. A sticky particle method and convergence theorem.** Assume that the initial data $m_0$ satisfies

$$m_0 \in \mathcal{M}(\mathbb{R}), \quad \operatorname{supp}\{m_0\} \subset (-L, L), \quad M_0 := \int_{\mathbb{R}} d|m_0| < +\infty. \tag{34}$$

In this section, we use a sticky particle method to obtain a global weak solution to the mCH equation for initial data $m_0$.

Let us choose the initial data $\{x_i(0)\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ to approximate $m_0(x)$. Divide the interval $[-L, L]$ into $N$ nonoverlapping subintervals $I_j$ by using a uniform grid with size $h = \frac{2L}{N}$. We choose $x_i(0)$ and $p_i$ as

$$c_i := -L + \left(i - \frac{1}{2}\right) h; \quad p_i := \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} dm_0, \quad i = 1, 2, \ldots, N. \tag{35}$$

Hence, we have

$$\sum_{i=1}^N |p_i| \le \int_{[-L,L]} d|m_0| = M_0. \tag{36}$$

Using (35), one can easily prove that $m_0$ is approximated by

$$m_0^N(x) := \sum_{i=1}^N p_i \delta(x - c_i) \tag{37}$$

in the sense of measures. Actually, for any test function $\phi \in C_b(\mathbb{R})$, we know $\phi$ is uniformly continuous on $[-L, L]$. Hence, for any $\eta > 0$, there exists a $\delta > 0$ such that when $x, y \in [-L, L]$ and $|x - y| < \delta$, we have $|\phi(x) - \phi(y)| < \eta$. Hence, choose $\frac{h}{2} < \delta$ (or $N > \frac{L}{\delta}$), and we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} \phi(x) dm_0 - \int_{\mathbb{R}} \phi(x) dm_0^N \right| &= \left| \int_{[-L,L]} \phi(x) dm_0 - \int_{[-L,L]} \phi(x) dm_0^N \right| \\
&= \left| \sum_{i=1}^N \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} \big(\phi(x) - \phi(c_i)\big) dm_0 \right| \\
&\le \eta \sum_{i=1}^N \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} d|m_0| \le M_0 \eta.
\end{aligned} \tag{38}$$

Let $N \to +\infty$, and we obtain narrow convergence of $m_0^N$ to $m_0$.

**3.1. Space and time BV estimates.** By Proposition 2.3, we know there exist trajectories $\{x_i(t)\}_{i=1}^N$ subject to $\{x_i(0) = c_i\}_{i=1}^N$ ($c_i$ defined by (35)) and a sticky peakon weak solution $u^N$ (defined by (33)) to the mCH equation with initial data $m_0^N$ defined by (37). Moreover, (32) holds for $x_i(t)$, $i = 1, \ldots, N$.

Next, we show some space-time BV estimates for $u^N$ and $u_x^N$. To this end, let us recall the definition of BV functions.

DEFINITION 3.1. (i) *For dimension $d \ge 1$ and an open set $\Omega \subset \mathbb{R}^d$, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if*

$$Tot.Var.\{f\} := \sup \left\{ \int_\Omega f(x) \nabla \cdot \phi(x) dx : \phi \in C_c^1\left(\Omega; \mathbb{R}^d\right), \|\phi\|_{L^\infty} \le 1 \right\} < \infty.$$

(ii) (Equivalent definition for one dimension case) *A function $f$ belongs to $BV(\mathbb{R})$ if for any $\{x_i\} \subset \mathbb{R}$, $x_i < x_{i+1}$, the following statement holds:*

$$Tot.Var.\{f\} := \sup_{\{x_i\}} \left\{ \sum_i |f(x_i) - f(x_{i-1})| \right\} < \infty.$$

*Remark* 3.2. Let $\Omega \subset \mathbb{R}^d$ for $d \geq 1$ and $f \in BV(\Omega)$. $Df := (D_{x_1}f, \dots, D_{x_d}f)$ is the distributional gradient of $f$. Then, $Df$ is a vector Radon measure and the total variation of $f$ is equal to the total variation of $|Df|$: $Tot.Var.\{f\} = |Df|(\Omega)$. Here, $|Df|$ is the total variation measure of the vector measure $Df$ [20, Definition (13.2)].

If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Definition 3.1 (ii), then $f$ satisfies Definition (i). On the contrary, if $f$ satisfies Definition 3.1 (i), then there exists a right continuous representative which satisfies Definition (ii). See [20, Theorem 7.2] for the proof.

We have the following estimates for $u^N$ and $u_x^N$.

PROPOSITION 3.3. *Assume the initial value $m_0$ satisfies* (34). *$\{p_i\}_{i=1}^N$ and $\{c_i\}_{i=1}^N$ are given by* (35). *$u^N$ is the sticky peakon weak solution defined by* (33) *in Proposition* 2.3 *with initial data $m_0^N$ given by* (37). *Then, the following statements hold.*

(i) *For any $t \in [0, \infty)$, we have*

$$Tot.Var.\{u^N(\cdot, t)\} \leq M_0, \quad Tot.Var.\{u_x^N(\cdot, t)\} \leq 2M_0 \text{ uniformly in } N. \tag{39}$$

(ii)

$$\|u^N\|_{L^\infty} \leq \frac{1}{2}M_0, \quad \|u_x^N\|_{L^\infty} \leq \frac{1}{2}M_0 \text{ uniformly in } N. \tag{40}$$

(iii) *For $t, s \in [0, \infty)$, we have*

$$\begin{aligned} \int_{\mathbb{R}} \left| u^N(x,t) - u^N(x,s) \right| dx \leq \frac{1}{2} M_0^3 |t - s|, \\ \int_{\mathbb{R}} \left| u_x^N(x,t) - u_x^N(x,s) \right| dx \leq M_0^3 |t - s|. \end{aligned} \tag{41}$$

(iv) *For any $T > 0$, there exist subsequences of $u^N$, $u_x^N$ (also denoted as $u^N$, $u_x^N$) and two functions $u$, $u_x \in BV(\mathbb{R} \times [0, T))$ such that*

$$u^N \to u, \quad u_x^N \to u_x \text{ in } L^1_{loc}(\mathbb{R} \times [0, +\infty)) \text{ as } N \to +\infty \tag{42}$$

*and $u$, $u_x$ also satisfy all the above properties in* (i), (ii), *and* (iii).

*Proof.* By (35), it is clear that $\sum_{i=1}^N |p_i| \leq M_0$.
*Step* 1. We first prove (39) and (40) ((i) and (ii)).
$G(x) = \frac{1}{2}e^{-|x|}$ satisfies

$$\begin{aligned} \|G\|_{L^\infty} \leq \frac{1}{2}, \quad \|G_x\|_{L^\infty} \leq \frac{1}{2}; \quad \|G\|_{L^1} = 1, \quad \|G_x\|_{L^1} = 1; \\ \text{and } \ Tot.Var.\{G\} \leq 1, \quad Tot.Var.\{G_x\} \leq 2. \end{aligned}$$

For any $t \geq 0$, $x \in \mathbb{R}$, using the above inequalities we have

$$Tot.Var.\left\{u^N(\cdot,t)\right\} \leq \sum_{i=1}^N |p_i| Tot.Var.\{G\} \leq M_0;$$

$$Tot.Var.\left\{u_x^N(\cdot,t)\right\} \leq \sum_{i=1}^N |p_i| Tot.Var.\{G_x\} \leq 2M_0.$$

$$\left\|u^N(\cdot,t)\right\|_{L^1} \leq M_0\|G\|_{L^1} = M_0, \quad \left\|u_x^N(\cdot,t)\right\|_{L^1} \leq M_0\|G_x\|_{L^1} = M_0;$$

$$\left|u^N(x,t)\right| \leq \|G\|_{L^\infty} \sum_{i=1}^N |p_i| \leq \frac{1}{2}M_0;$$

$$\left|u_x^N(x,t)\right| \leq \|G_x\|_{L^\infty} \sum_{i=1}^N |p_i| \leq \frac{1}{2}M_0.$$

Therefore, the assertions (39) and (40) hold.

*Step* 2. We now prove (41) (Proposition 3.3 (iii)).

Because $x_i$ satisfies (32), by [4, Lemma 2.3] we have

$$\begin{aligned}\int_{\mathbb{R}} \left|u^N(x,t) - u^N(x,s)\right| dx &\leq \int_{\mathbb{R}} \sum_{j=1}^N |p_j||G(x-x_j(t)) - G(x-x_j(s))|dx \\ &\leq Tot.Var.\{G\} \sum_{j=1}^N |p_j||x_j(t)-x_j(s)| \leq \frac{1}{2}M_0^3|t-s|\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} \left|u_x^N(x,t) - u_x^N(x,s)\right| dx &\leq \int_{\mathbb{R}} \sum_{j=1}^N |p_j||G_x(x-x_j(t)) - G_x(x-x_j(s))|dx \\ &\leq Tot.Var.\{G_x\} \sum_{j=1}^N |p_j||x_j(t)-x_j(s)| \leq M_0^3|t-s|.\end{aligned}$$

Hence, (41) holds.

*Step* 3. Combining (i), (ii), and (iii), the statement (iv) can be obtained by [4, Theorem 2.4, 2.6].

This ends the proof. ∎

**3.2. Global weak solutions and convergence theorem.** Now we state and prove our main theorem.

THEOREM 3.4. *Assume that the initial data $m_0$ satisfies* (34). *$p_j, c_j$ are given by* (35). *$(u^N(x,t), m^N(x,t))$ is the sticky peakon weak solution given by* (33) *with initial data $m_0^N$ defined by* (37). *Then, the following statements hold.*

(i) *The limiting function $u$ obtained in Proposition* 3.3 (iv) *satisfies*

$$u \in C([0,+\infty); H^1(\mathbb{R})) \cap L^\infty(0,+\infty; W^{1,\infty}(\mathbb{R})), \tag{43}$$

*and it is a global weak solution of the mCH equation* (1)–(2).

(ii) *For any $T > 0$, we have*

$$m = (1-\partial_{xx})u \in \mathcal{M}(\mathbb{R} \times [0,T))$$

*and there exists a subsequence of* $m^N$ *(also labeled as* $m^N$*) such that*

$$m^N \overset{*}{\rightharpoonup} m \text{ in } \mathcal{M}(\mathbb{R}\times[0,T)) \quad (\text{as } N\to+\infty). \tag{44}$$

(iii) *For a.e.* $t\geq 0$ *we have (in subsequence sense)*

$$m^N(\cdot,t) \overset{*}{\rightharpoonup} m(\cdot,t) \text{ in } \mathcal{M}(\mathbb{R}) \text{ as } N\to+\infty \tag{45}$$

*and*

$$\operatorname{supp}\{m(\cdot,t)\}\subset\left(-L-\frac{1}{2}M_0^2t, L+\frac{1}{2}M_0^2t\right). \tag{46}$$

*Proof.*

*Step* 1. Proof of (i).

We first prove $u$ obtained in Proposition 3.3 satisfies (43).

As shown in Proposition 3.3, there exists $u, u_x$ such that (42) holds. Moreover, for any $T>0$, the limiting functions $u, u_x$ satisfy the following properties:

$$u\in BV(\mathbb{R}\times[0,T)),\quad u_x\in BV(\mathbb{R}\times[0,T)),$$
$$\|u\|_{L^\infty}\leq\frac{1}{2}M_0,\quad \|u_x\|_{L^\infty}\leq\frac{1}{2}M_0$$

and

$$\int_{\mathbb{R}}|u(x,t)-u(x,s)|dx\leq\frac{1}{2}M_0^3|t-s|,\quad \int_{\mathbb{R}}|u_x(x,t)-u_x(x,s)|dx\leq M_0^3|t-s| \tag{47}$$

for $t,s\in[0,+\infty)$. Using the above properties we obtain

$$\begin{aligned}\|u(\cdot,t)-u(\cdot,s)\|_{L^2}^2&=\int_{\mathbb{R}}|u(x,t)-u(x,s)|^2dx\\&\leq M_0\int_{\mathbb{R}}|u(x,t)-u(x,s)|dx\leq\frac{1}{2}M_0^4|t-s|\end{aligned}$$

and

$$\|u_x(\cdot,t)-u_x(\cdot,s)\|_{L^2}^2\leq M_0^4|t-s|.$$

Hence,

$$\begin{aligned}\|u(\cdot,t)-u(\cdot,s)\|_{H^1}^2&\leq 2\left(\|u(\cdot,t)-u(\cdot,s)\|_{L^2}^2+\|u_x(\cdot,t)-u_x(\cdot,s)\|_{L^2}^2\right)\\&\leq 3M_0^4|t-s|.\end{aligned}$$

Therefore, (43) holds.

Next, we prove $u$ is a global weak solution.

For each $\phi\in C_c^\infty(\mathbb{R}\times[0,+\infty))$, there exists $T=T(\phi)$ such that $\phi\in C_c^\infty(\mathbb{R}\times[0,T))$. Because $u^N$ is a sticky peakon weak solution with initial data $m_0^N$, we have

$$\mathcal{L}\left(u^N,\phi\right)=-\int_{\mathbb{R}}\phi(x,0)dm_0^N. \tag{48}$$

We now consider convergence for each term of $\mathcal{L}(u^N,\phi)$, where

$$\begin{aligned}\mathcal{L}\left(u^N,\phi\right)=&\int_0^T\int_{\mathbb{R}}u^N(\phi_t-\phi_{txx})dxdt-\frac{1}{3}\int_0^T\int_{\mathbb{R}}\left(u_x^N\right)^3\phi_{xx}dxdt\\&-\frac{1}{3}\int_0^T\int_{\mathbb{R}}\left(u^N\right)^3\phi_{xxx}dxdt+\int_0^T\int_{\mathbb{R}}\left(\left(u^N\right)^3+u^N\left(u_x^N\right)^2\right)\phi_xdxdt.\end{aligned} \tag{49}$$

For the first term on the right-hand side of (49), using (42) and the fact that supp$\{\phi\}$ is compact we obtain

$$\int_0^T \int_{\mathbb{R}} u^N(\phi_t - \phi_{txx})dxdt \to \int_0^T \int_{\mathbb{R}} u(\phi_t - \phi_{txx})dxdt \quad (N \to +\infty).$$

The second term of (49) is estimated as follows

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}} \left[ \left(u_x^N\right)^3 - u_x^3 \right] \phi_{xx} dxdt \right| \\
&\quad = \left| \int_0^T \int_{\mathbb{R}} \left(u_x^N - u_x\right) \left[ \left(u_x^N\right)^2 + u_x^2 + u_x^N u_x \right] \phi_{xx} dxdt \right| \\
&\quad \leq \frac{3}{4} M_0^2 \|\phi_{xx}\|_{L^\infty} \int\int_{\text{supp}\{\phi\}} \left|u_x^N - u_x\right| dxdt \to 0 \quad (N \to +\infty).
\end{aligned}$$

Similarly, we have the following estimates for the rest terms on the right-hand side of (49):

$$\int_0^T \int_{\mathbb{R}} \left[ \left(u^N\right)^3 - u^3 \right] \phi_{xxx} dxdt \to 0 \quad (N \to +\infty),$$

$$\int_0^T \int_{\mathbb{R}} \left[ \left(u^N\right)^3 - u^3 \right] \phi_x dxdt \to 0 \quad (N \to +\infty),$$

and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left[ u^N \left(u_x^N\right)^2 - u u_x^2 \right] \phi_x dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}} \left[ \left(u^N - u\right) \left(u_x^N\right)^2 + u \left( \left(u_x^N\right)^2 - u_x^2 \right) \right] \phi_x dxdt \\
&\quad = \int_0^T \int_{\mathbb{R}} \left[ \left(u^N - u\right) \left(u_x^N\right)^2 + u \left(u_x^N + u_x\right) \left(u_x^N - u_x\right) \right] \phi_x dxdt \\
&\quad \to 0 \quad (N \to +\infty).
\end{aligned}$$

Letting $N \to +\infty$ in (48) and combining (38) gives

$$\mathcal{L}(u, \phi) = -\int_{\mathbb{R}} \phi(x, 0) dm_0.$$

This proves that $u$ is a global weak solution to the mCH equation.

*Step* 2. Proof of (ii).

Due to $u, u_x \in BV(\mathbb{R} \times [0, T))$ for any $T > 0$, we know

$$m = (1 - \partial_{xx})u \in \mathcal{M}(\mathbb{R} \times [0, T)).$$

Now we prove (44), which means that for any test function $\phi \in C_c(\mathbb{R} \times [0, T))$ the following holds:

$$\lim_{N \to +\infty} \int_0^T \int_{\mathbb{R}} \phi(x, t) m^N(dx, dt) = \int_0^T \int_{\mathbb{R}} \phi(x, t) m(dxdt). \tag{50}$$

First, we prove that (50) holds for any test function $\phi \in C_c^1(\mathbb{R} \times [0,T))$. Using the relationship $m^N = (1-\partial_{xx})u^N$ and integrating by parts give that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \phi(x,t) m^N(dx,dt) &= \int_0^T \int_{\mathbb{R}} \phi(x,t)(1-\partial_{xx})u^N(x,t)dxdt \\ &= \int_0^T \int_{\mathbb{R}} \phi(x,t)u^N(x,t) + \phi_x(x,t)u_x^N(x,t)dxdt.\end{aligned}$$

Taking $N \to +\infty$ and combining (42), the right-hand side of the above equality converges to

$$\int_0^T \int_{\mathbb{R}} \phi(x,t)u(x,t) + \phi_x(x,t)u_x(x,t)dxdt = \int_0^T \int_{\mathbb{R}} \phi(x,t)m(dxdt).$$

Hence, (50) holds for any test function $\phi \in C_c^1(\mathbb{R} \times [0,T))$.

Next, we prove that (50) holds for any test function $\phi \in C_c(\mathbb{R} \times [0,T))$. In this case, there exists a sequence $\{\phi_n\}_{n\geq 1} \subset C_c^1(\mathbb{R} \times [0,T))$ and two constants $R > 0, 0 < T_1 < T$ such that

$$\operatorname{supp}\{\phi_n\} \subset [-R,R] \times [0,T_1] \quad \text{and} \quad \operatorname{supp}\{\phi\} \subset [-R,R] \times [0,T_1]$$

and

$$\phi_n \to \phi \quad (\text{as} \quad n \to \infty) \quad \text{uniformly in} \quad \mathbb{R} \times [0,T). \tag{51}$$

This implies

$$\phi_n \to \phi \quad \text{in} \quad C_c(\mathbb{R} \times [0,T)).$$

Because $m$ is a Radon measure (which defines a bounded linear functional of $C_c(\mathbb{R} \times [0,T))$ by integration with respect to $m$), for any $\eta > 0$, there exists $n_0 > 0$ such that

$$\left| \int_{\mathbb{R}} (\phi_{n_0} - \phi) m(dxdt) \right| \leq \frac{\eta}{2}. \tag{52}$$

Due to (51), we choose $n_0$ big enough such that the following holds:

$$\sup_{(x,t)\in\mathbb{R}\times[0,T)} |\phi_{n_0}(x,t) - \phi(x,t)| < \frac{\eta}{2M_0T}.$$

At this time, we have

$$\left| \int_0^T \int_{\mathbb{R}} (\phi_{n_0}(x,t) - \phi(x,t)) m^N(dx,dt) \right| \leq \frac{\eta}{2M_0T} \int_0^T \sum_{i=1}^N |p_i| dt \leq \frac{\eta}{2}. \tag{53}$$

Because $\phi_{n_0} \in C_c^1(\mathbb{R} \times [0,T))$, we also obtain

$$\lim_{N\to+\infty} \left| \int_0^T \int_{\mathbb{R}} \phi_{n_0} m^N(dxdt) - \int_0^T \int_{\mathbb{R}} \phi_{n_0} m(dxdt) \right| = 0. \tag{54}$$

Combining (52), (53), and (54), we have ($n > n_0$)

$$\lim_{N\to+\infty}\left|\int_0^T\int_{\mathbb{R}}\phi(x,t)m^N(dx,dt)-\int_0^T\int_{\mathbb{R}}\phi(x,t)m(dxdt)\right|$$
$$\leq \lim_{N\to+\infty}\left|\int_0^T\int_{\mathbb{R}}(\phi-\phi_{n_0})m^N(dxdt)\right|+\lim_{N\to+\infty}\left|\int_0^T\int_{\mathbb{R}}(\phi_{n_0}-\phi)m(dxdt)\right|$$
$$+\lim_{N\to+\infty}\left|\int_0^T\int_{\mathbb{R}}\phi_{n_0}m^N(dxdt)-\int_0^T\int_{\mathbb{R}}\phi_{n_0}m(dxdt)\right|\leq\frac{\eta}{2}+\frac{\eta}{2}=\eta.$$

Because $\eta > 0$ is arbitrary, we know (50) holds.

*Step* 3. Proof of (iii).

We first prove that for $N > 0$ we have

$$\operatorname{supp}\{m^N(\cdot,t)\}\subset\left(-L-\frac{1}{2}M_0^2t,L+\frac{1}{2}M_0^2t\right). \tag{55}$$

Because $c_i$ defined by (35) satisfies $-L < c_1 < c_2 < \cdots < c_N < L$, due to (32) we obtain

$$-L-\frac{1}{2}M_0^2t<|x_i(t)|<L+\frac{1}{2}M_0^2t,\quad i=1,\ldots,N,$$

where $x_i(t)$ is obtained by Proposition 2.3 subject to $x_i(0) = c_i$. Therefore, $m^N$ defined by (33) satisfies (55).

Next, we prove (45).

Let $u^N(x,t)$ be defined by (33). Assume that $u^N$ and $u^N_x$ are the convergence sequences in (42). We first prove that there exists a subsequence of $u^N$ (still denote as $u^N$) such that for a.e. $t \geq 0$,

$$u^N(\cdot,t)\to u(\cdot,t)\ \text{ in }\ L^1_{loc}(\mathbb{R})\ \ (N\to\infty). \tag{56}$$

Let $K$ be a positive integer. Due to (42), we have

$$u^N\to u\ \text{ in }\ L^1((-K,K)\times(0,K))\ \ (N\to\infty).$$

Hence, by [5, Theorem 4.9] we know there exists a subsequence $\{u^N_{K_i}\}_{i=1}^N$ of $\{u^N\}$ such that for a.e. $t \in [0,K]$

$$u^N_{K_i}(\cdot,t)\to u(\cdot,t)\ \text{ in }\ L^1(-K,K)\ \ (i\to\infty).$$

By using a diagonalization argument with respect to $K = 1, 2, \ldots$, we obtain a subsequence such that (56) holds.

Similarly, for a.e. $t \geq 0$ we also obtain (in subsequence sense)

$$u^N_x(\cdot,t)\to u_x(\cdot,t)\ \text{ in }\ L^1_{loc}(\mathbb{R})\ \ (N\to\infty).$$

Hence, for any test function $\phi \in C^1_c(\mathbb{R})$ and a.e. $t \geq 0$, we have

$$\int_{\mathbb{R}}\phi(x)m^N(dx,t)=\int_{\mathbb{R}}\Big[\phi(x)u^N(x,t)-\phi_x(x)u^N_x(x,t)\Big]dx$$
$$\to\int_{\mathbb{R}}\Big[\phi(x)u(x,t)-\phi_x(x)u_x(x,t)\Big]dx$$
$$=\int_{\mathbb{R}}\phi(x)m(dx,t),\quad(N\to\infty).$$

Similarly to the proof of (44), for any test function $\phi(x) \in C_c(\mathbb{R})$ and a.e. $t \geq 0$ we obtain

$$\int_{\mathbb{R}} \phi(x) m^N(x,t)dx \to \int_{\mathbb{R}} \phi(x) m(dx,t) \quad \text{as} \quad N \to \infty.$$

Hence, (45) holds.

Finally, we prove (46).

From (45) and (55), for any test function $\phi(x) \in C_c(\mathbb{R})$ that satisfies

$$\mathrm{supp}\{\phi\} \subset \mathbb{R} \setminus \left(-L - \frac{1}{2}M_0^2 t, L + \frac{1}{2}M_0^2 t\right),$$

we obtain

$$\int_{\mathbb{R}} \phi(x) m(dx,t) = 0,$$

which implies (46).

This is the end of the proof. ∎

**3.3. Total variation stability of $m(\cdot,t)$.** For initial data $m_0 \in \mathcal{M}(\mathbb{R})$ satisfying (34), assume $u$ is a weak solution obtained in Theorem 3.4 and $m(x,t) = u(x,t) - u_{xx}(x,t)$. Our main target in this subsection is to prove that the total variation of $m(\cdot,t)$ satisfies

$$|m(\cdot,t)|(\mathbb{R}) \leq |m_0|(\mathbb{R}) \quad \text{for a.e.} \quad t \geq 0. \tag{57}$$

Next, let's give some preparations.

From Jordan measure decomposition Theorem, there exist two positive measures $m_0^+$ and $m_0^-$ such that

$$m_0 = m_0^+ - m_0^- \quad \text{and} \quad |m_0| = m_0^+ + m_0^-. \tag{58}$$

Let $(c_i, p_i)$ be defined by (35). Set

$$p_i^+ := \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} dm_0^+ \quad \text{and} \quad p_i^- := \int_{[c_i - \frac{h}{2}, c_i + \frac{h}{2})} dm_0^-, \quad i = 1, 2, \ldots, N.$$

Because $m_0^+$ and $m_0^-$ are positive measures, we know

$$p_i^+ \geq 0, \quad p_i^- \geq 0 \quad \text{and} \quad p_i = p_i^+ - p_i^-, \quad i = 1, 2, \ldots, N.$$

Moreover, by (58) we obtain

$$m_0(\mathbb{R}) = \int_{[-L,L]} d(m_0^+ - m_0^-) = \sum_{i=1}^N p_i^+ - \sum_{i=1}^N p_i^- \tag{59}$$

and

$$M_0 = \int_{[-L,L]} d|m_0| = \int_{[-L,L]} d(m_0^+ + m_0^-) = \sum_{i=1}^N p_i^+ + \sum_{i=1}^N p_i^-. \tag{60}$$

Set

$$u_+^N(x,t) := \sum_{i=1}^N p_i^+ G(x - x_i(t)) \quad \text{and} \quad u_-^N(x,t) := \sum_{i=1}^N p_i^- G(x - x_i(t)), \tag{61}$$

where $x_i(t)$ is given by Proposition 2.3 with initial data $c_i$ defined by (35). Then, we have

$$u_+^N(x,t) \geq 0, \quad u_-^N(x,t) \geq 0 \quad \text{and} \quad u^N(x,t) = u_+^N(x,t) - u_-^N(x,t), \tag{62}$$

where $u^N$ is defined by (33).

Set

$$m_+^N(x,t) := \sum_{i=1}^N p_i^+ \delta(x - x_i(t)) \quad \text{and} \quad m_-^N(x,t) := \sum_{i=1}^N p_i^- \delta(x - x_i(t)). \tag{63}$$

We have

$$m_+^N(x,t) \geq 0, \quad m_-^N(x,t) \geq 0 \quad \text{and} \quad m^N(x,t) = m_+^N(x,t) - m_-^N(x,t), \tag{64}$$

where $m^N$ is defined by (33).

Now, we state and prove our main theorem in this subsection.

THEOREM 3.5. *Let $u$ be a weak solution obtained in Theorem 3.4 and $m(x,t) = (1 - \partial_{xx})u(x,t)$. Then, we have*

$$m(\cdot,t)(\mathbb{R}) = m_0(\mathbb{R}) \quad \textit{for a.e.} \quad t \geq 0, \tag{65}$$

*and the total variation of $m(\cdot,t)$ satisfies* (57).

*Proof.* Use the process as in Proposition 3.3 for $u_+^N$ and $u_-^N$ (defined by (61)). There exist two functions

$$u^+, u^- \in BV(\mathbb{R} \times [0,T)) \quad \text{and} \quad u_x^+, u_x^- \in BV(\mathbb{R} \times [0,T)) \tag{66}$$

such that ($N \to \infty$ in subsequence sense)

$$u_+^N \to u^+, \quad \partial_x u_+^N \to u_x^+ \quad \text{in} \quad L^1_{loc}(\mathbb{R} \times [0,\infty))$$

and

$$u_-^N \to u^-, \quad \partial_x u_-^N \to u_x^- \quad \text{in} \quad L^1_{loc}(\mathbb{R} \times [0,\infty)).$$

Due to (62), we have

$$u^+(x,t) \geq 0, \quad u^-(x,t) \geq 0.$$

Combining (42) and (62), we know

$$u(x,t) = u^+(x,t) - u^-(x,t). \tag{67}$$

Set

$$m^+(x,t) := (1 - \partial_{xx})u^+(x,t), \quad m^-(x,t) := (1 - \partial_{xx})u^-(x,t).$$

Due to (66) and (67), we have

$$m^+, \quad m^- \in \mathcal{M}_+(\mathbb{R} \times [0,T))$$

and

$$m(x,t) = (1 - \partial_{xx})u(x,t) = m^+(x,t) - m^-(x,t). \tag{68}$$

Similarly to part (iii) in Theorem 3.4, for a.e. $t \geq 0$ we have ($N \to \infty$ in subsequence sense)

$$m_+^N(\cdot,t) \overset{*}{\rightharpoonup} m^+(\cdot,t) \text{ in } \mathcal{M}_+(\mathbb{R}), \quad m_-^N(\cdot,t) \overset{*}{\rightharpoonup} m^-(\cdot,t) \text{ in } \mathcal{M}_+(\mathbb{R}) \tag{69}$$

and

$$\text{supp}\{m^+(\cdot,t)\} \subset \left(-L - \frac{1}{2}M_0^2 T, L + \frac{1}{2}M_0^2 T\right),$$

$$\text{supp}\{m^-(\cdot,t)\} \subset \left(-L - \frac{1}{2}M_0^2 T, L + \frac{1}{2}M_0^2 T\right),$$

where $m_+^N$ and $m_-^N$ are defined by (63).

Take test function $\phi \in C_c(\mathbb{R})$ satisfying

$$\phi(x) = 1 \quad \text{for} \quad x \in \left(-L - \frac{1}{2}M_0^2 t, L + \frac{1}{2}M_0^2 t\right).$$

Then, we have

$$\int_{\mathbb{R}} \phi(x) m_+^N(dx,t) = \int_{\mathbb{R}} m_+^N(dx,t) = \sum_{i=1}^N p_i^+.$$

Hence, for a.e. $t \geq 0$, using (69) we have

$$\int_{\mathbb{R}} m^+(dx,t) = \int_{\mathbb{R}} \phi(x) m^+(dx,t) = \lim_{N\to\infty} \int_{\mathbb{R}} \phi(x) m_+^N(x,t)dx = \sum_{i=1}^N p_i^+. \tag{70}$$

Similarly, we have

$$\int_{\mathbb{R}} m^-(dx,t) = \sum_{i=1}^N p_i^- \quad \text{for a.e.} \quad t \geq 0. \tag{71}$$

Therefore, using (59), (60), (68), (70), and (71) we obtain

$$\begin{aligned} m(\cdot,t)(\mathbb{R}) &= \int_{\mathbb{R}} m(dx,t) = \int_{\mathbb{R}} m^+(dx,t) - \int_{\mathbb{R}} m^-(dx,t) \\ &= \sum_{i=1}^N p_i^+ - \sum_{i=1}^N p_i^- = m_0(\mathbb{R}) \quad \text{for a.e.} \quad t \geq 0 \end{aligned}$$

and

$$\begin{aligned} |m(\cdot,t)|(\mathbb{R}) &= \int_{\mathbb{R}} |m|(dx,t) \leq \int_{\mathbb{R}} m^+(dx,t) + \int_{\mathbb{R}} m^-(dx,t) \\ &= \sum_{i=1}^N p_i^+ + \sum_{i=1}^N p_i^- = M_0 \quad \text{for a.e.} \quad t \geq 0. \end{aligned}$$

This completes the proof of (57) and (65). ∎

**4. Uniqueness and nonuniqueness of weak soutions.** In this section, we prove that weak solutions $u$ are unique when $u \in L^\infty(0,\infty; W^{2,1}(\mathbb{R}))$. Notice that peakon weak solutions are not in $W^{2,1}(\mathbb{R})$. We provide some examples of the nonuniqueness of peakon weak solutions to the mCH equation.

**4.1. Stability and uniqueness in the class $W^{2,1}(\mathbb{R})$.** In [29], Zhang used the dissipative regularization and proved the global existence of entropy weak solutions $u$ to the mCH equation. Entropy weak solutions are unique when $u(\cdot,t) \in \dot{W}^{2,1}(\mathbb{R})$ for $t \geq 0$.

When the initial data $u_0 \in W^{2,1}(\mathbb{R})$, the dissipative regularization solution $u^\epsilon$ satisfies $||u^\epsilon_{xx}(\cdot,t)||_{L^1} \leq C(T,u_0)$ for any $T > 0$ and $t \in [0,T)$. Since $L^1$ is not reflexive, the limit function of $u^\epsilon_{xx}$ belongs to $\mathcal{M}(\mathbb{R})$. In general, weak solutions are not unique when $m \in \mathcal{M}(\mathbb{R})$ as shown in subsection 4.2. However, assuming $u \in \dot{W}^{2,1}(\mathbb{R})$, Zhang proved entropy weak solutions to the mCH equation are unique.

In this subsection, under the same assumption, we show $W^{1,1}(\mathbb{R})$-stability for general weak solutions, and this implies the uniqueness of weak solutions in the solution class $W^{2,1}(\mathbb{R})$.

We remark that $u(\cdot,t) \in \dot{W}^{2,1}(\mathbb{R})$ is equivalent to $u \in L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$ for weak solutions $u$. Indeed, due to (57), we obtain (a.e. $t \geq 0$)

$$\|u(\cdot,t)\|_{L^1} = \|(G*m)(\cdot,t)\|_{L^1} = |m(\cdot,t)|(\mathbb{R}) \leq M_0.$$

Moreover, for any $t \geq 0$ we have

$$\|u_x\|_{L^1} = Tot.Var\{u(\cdot,t)\} \leq M_0 \quad \text{and} \quad \|u_{xx}\|_{L^1} = Tot.Var\{u_x(\cdot,t)\} \leq 2M_0.$$

Therefore, we have $u \in L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$ when $u \in \dot{W}^{2,1}(\mathbb{R})$.

The mCH equation can be rewritten as a first-order equation of $u$

$$u_t + u^2u_x - \frac{1}{3}u_x^3 + \frac{1}{3}G*(u_x^3) + G_x * \left(\frac{2}{3}u^3 + uu_x^2\right) = 0. \tag{72}$$

This implies that weak solutions defined by Definition 2.1 satisfy (72) in the sense of distributions.

We have the following theorem.

THEOREM 4.1. *Assume* $(1-\partial_{xx})u_0 = m_0 \in L^1(-L,L)$ *and* $M_0 := \|m_0\|_{L^1}$. *Let* $u \in L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$ *be a weak solution to the mCH equation* (1)–(2). *Then, we have*

$$\|u(\cdot,t)\|_{H^1} = \|u_0\|_{H^1}. \tag{73}$$

*Moreover, if* $v \in L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$ *is another weak solution to the mCH equation subject to initial data* $v_0(x)$, *then we have*

$$\|u(\cdot,t) - v(\cdot,t)\|_{W^{1,1}} \leq e^{CM_0^2 t}\|u_0 - v_0\|_{W^{1,1}}, \tag{74}$$

*where* $C$ *is a constant. This implies the uniqueness of weak solutions in solution class* $W^{2,1}(\mathbb{R})$.

*Proof.*

*Step* 1. We prove (73).

Multiplying (72) by $u$ and taking integration, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2dx + \int_{\mathbb{R}} u^3u_xdx - \frac{1}{3}\int_{\mathbb{R}} uu_x^3dx + \frac{1}{3}\int_{\mathbb{R}} uG*(u_x^3)dx \\ + \int_{\mathbb{R}} uG_x * \left(\frac{2}{3}u^3 + uu_x^2\right)dx = 0.\end{aligned} \tag{75}$$

Because

$$\int_{\mathbb{R}} u^3 u_x dx = 0,$$

(75) turns into

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 dx - \frac{1}{3}\int_{\mathbb{R}} uu_x^3 dx + \frac{1}{3}\int_{\mathbb{R}} uG*(u_x^3)dx + \int_{\mathbb{R}} uG_x*\left(\frac{2}{3}u^3+uu_x^2\right)dx = 0. \tag{76}$$

Due to $u\in W^{2,1}(\mathbb{R})$, taking derivative of (72) gives

$$u_{xt} + uu_x^2 - \frac{2}{3}u^3 + u^2u_{xx} - u_x^2u_{xx} + \frac{1}{3}G_x*(u_x^3) + G*\left(\frac{2}{3}u^3+uu_x^2\right) = 0. \tag{77}$$

Multiplying (77) by $u_x$ and taking integration yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2 dx + \int_{\mathbb{R}} uu_x^3 dx - \frac{2}{3}\int_{\mathbb{R}} u^3u_x dx + \int_{\mathbb{R}} u^2u_xu_{xx}dx - \int_{\mathbb{R}} u_x^3u_{xx}dx\\ + \frac{1}{3}\int_{\mathbb{R}} u_xG_x*(u_x^3)dx + \int_{\mathbb{R}} u_xG*\left(\frac{2}{3}u^3+uu_x^2\right)dx = 0.\end{aligned} \tag{78}$$

Because

$$\int_{\mathbb{R}} u^3u_x dx = \frac{1}{4}\int_{\mathbb{R}}\left(u^4\right)_x dx = 0, \quad \int_{\mathbb{R}} u_x^3u_{xx}dx = \frac{1}{4}\int_{\mathbb{R}}\left(u_x^4\right)_x dx = 0$$

and

$$\int_{\mathbb{R}} uu_x^3 dx + \int_{\mathbb{R}} u^2u_xu_{xx}dx = \frac{1}{2}\int_{\mathbb{R}}\left(u^2u_x^2\right)_x dx = 0,$$

using (78) we obtain

$$\begin{aligned}0 &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2 dx + \frac{1}{3}\int_{\mathbb{R}} u_xG_x*(u_x^3)dx + \int_{\mathbb{R}} u_xG*\left(\frac{2}{3}u^3+uu_x^2\right)dx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2 dx - \frac{1}{3}\int_{\mathbb{R}} uG*(u_x^3)dx + \frac{1}{3}\int_{\mathbb{R}} uu_x^3 dx - \int_{\mathbb{R}} uG_x*\left(\frac{2}{3}u^3+uu_x^2\right)dx.\end{aligned} \tag{79}$$

Combining (76) and (79) gives

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2+u_x^2\right)dx = 0.$$

Hence, (73) holds.

*Step* 2. We prove (74).

First, we estimate $\frac{d}{dt}\int_{\mathbb{R}}|u-v|dx$.

Because $|u|, |u_x| \le \frac{1}{2}M_0$, from (72) we see that $u(x,\cdot)\in W^1(0,T)$ for any $T>0$ and $x\in\mathbb{R}$. Hence, [16, Lemma 7.6] shows that

$$|u-v|_t = (u_t-v_t)\mathrm{sgn}(u-v).$$

Taking the difference of the PDEs for $u$ and $v$ yields

$$
\begin{aligned}
|u-v|_t &= (u_t - v_t)\mathrm{sgn}(u-v)\\
&= [-u^2(u_x - v_x) - (u+v)(u-v)v_x]\mathrm{sgn}(u-v)\\
&\quad + \Big(\frac{1}{3}\left(u_x^2 + u_x v_x + v_x^2\right)(u_x - v_x)\\
&\qquad - \frac{1}{3}G * \Big[\left(u_x^2 + u_x v_x + v_x^2\right)(u_x - v_x)\Big]\Big)\mathrm{sgn}(u-v)\\
&\quad - \Big(G_x * \Big[\frac{2}{3}\left(u^2 + uv + v^2\right)(u-v)\Big]\\
&\qquad + G_x * \Big[u(u_x + v_x)(u_x - v_x)\Big]\Big)\mathrm{sgn}(u-v)\\
&\quad - \Big(G_x * \Big[(u-v)v_x^2\Big]\Big)\mathrm{sgn}(u-v).
\end{aligned}
$$

Taking integration and by Young's inequality, we obtain

$$
\frac{d}{dt}\int_{\mathbb{R}} |u-v|dx \le C_1 M_0^2 ||u-v||_{W^{1,1}}, \tag{80}
$$

where $C_1$ is a positive constant.

Next, we estimate $\frac{d}{dt}\int_{\mathbb{R}} |u_x - v_x|dx$.

Due to $u \in L^\infty(0,\infty; W^{2,1}(\mathbb{R}))$ and $Tot.Var\{u_x(\cdot,t)\} \le 2M_0$ for any $t \ge 0$, we know $||u_{xx}(\cdot,t)||_{L^1} \le 2M_0$. Hence, from (77) we have $u_{xt} \in L^\infty(0,T; L^1(\mathbb{R})) \subset L^1(\mathbb{R}\times(0,T))$ for any $T > 0$. By Fubini Theorem [5, Theorem 4.5], we have $u_{xt}(x,\cdot) \in L^1(0,T)$ for a.e. $x \in \mathbb{R}$. Hence, $u_x(x,\cdot) \in W^1(0,T)$ for a.e. $x \in \mathbb{R}$. Similarly, $v_x(x,\cdot) \in W^1(0,T)$ for a.e. $x \in \mathbb{R}$. From [16, Lemma 7.6], we obtain

$$
|u_x - v_x|_t = (u_{xt} - v_{xt})\mathrm{sgn}(u_x - v_x).
$$

Taking the difference of the equations for $u_x$ (77) and $v_x$ gives

$$
\begin{aligned}
|u_x - v_x|_t &= (u_{xt} - v_{xt})\mathrm{sgn}(u_x - v_x)\\
&= \Big[u(u_x+v_x)(u_x - v_x) + (u-v)v_x^2\\
&\qquad - \frac{2}{3}\left(u^2 + uv + v^2\right)(u-v)\Big]\mathrm{sgn}(u_x - v_x)\\
&\quad + \left(u^2 u_{xx} - v^2 v_{xx}\right)\mathrm{sgn}(u_x - v_x) - \left(u_x^2 u_{xx} - v_x^2 v_{xx}\right)\mathrm{sgn}(u_x - v_x)\\
&\quad + \Big(\frac{1}{3}G_x * \Big[\left(u_x^2 + u_x v_x + v_x^2\right)(u_x - v_x)\Big]\Big)\mathrm{sgn}(u_x - v_x)\\
&\quad + \Big(G * \Big[\frac{2}{3}\left(u^2 + uv + v^2\right)(u-v)\Big]\\
&\qquad + G * \Big[u(u_x + v_x)(u_x - v_x)\Big]\Big)\mathrm{sgn}(u_x - v_x)\\
&\quad + \Big(G * \Big[(u-v)v_x^2\Big]\Big)\mathrm{sgn}(u_x - v_x).
\end{aligned}
$$

Take integration and with some calculations we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|u_x-v_x|dx &\le C_2M_0^2\|u-v\|_{W^{1,1}}+\int_{\mathbb{R}}\left(u^2u_{xx}-v^2v_{xx}\right)\operatorname{sgn}(u_x-v_x)dx\\ &\quad-\int_{\mathbb{R}}\left(u_x^2u_{xx}-v_x^2v_{xx}\right)\operatorname{sgn}(u_x-v_x)dx\\ &=C_2M_0^2\|u-v\|_{W^{1,1}}+I_1-I_2, \qquad (81)\end{aligned}$$

where $C_2$ is a constant and

$$I_1:=\int_{\mathbb{R}}\left(u^2u_{xx}-v^2v_{xx}\right)\operatorname{sgn}(u_x-v_x)dx,\quad I_2:=\int_{\mathbb{R}}\left(u_x^2u_{xx}-v_x^2v_{xx}\right)\operatorname{sgn}(u_x-v_x)dx.$$

Because $Tot.Var\{u_x\}\le 2M_0$, we know $\|u_{xx}\|_{L^1}\le 2M_0$. Due to $|u|,|u_x|,|v|,|v_x|\le \frac{1}{2}M_0$, by using the Sobolev inequality we know there is a constant $C_3$ such that

$$\begin{aligned}I_1&=\int_{\mathbb{R}}(u+v)(u-v)u_{xx}\operatorname{sgn}(u_x-v_x)dx+\int_{\mathbb{R}}v^2|u_x-v_x|_xdx\\ &\le 2M_0^2\|u-v\|_{L^\infty}-2\int_{\mathbb{R}}vv_x|u_x-v_x|dx\le C_3M_0^2\|u-v\|_{W^{1,1}}.\end{aligned}$$

Due to $\operatorname{sgn}(u_x^3-v_x^3)=\operatorname{sgn}(u_x-v_x)$, we have the following estimate for $I_2$:

$$I_2=\frac{1}{3}\int_{\mathbb{R}}\left(u_x^3-v_x^3\right)_x\operatorname{sgn}\left(u_x^3-v_x^3\right)dx=\frac{1}{3}\int_{\mathbb{R}}\left|u_x^3-v_x^3\right|_x dx=0.$$

Put the above estimates for $I_1$ and $I_2$ into (81) and combining (80), we have

$$\frac{d}{dt}\|u-v\|_{W^{1,1}}\le CM_0^2\|u-v\|_{W^{1,1}}$$

for some constant $C$. Therefore, Grönwall's inequality implies

$$\|u(\cdot,t)-v(\cdot,t)\|_{W^{1,1}}\le e^{CM_0^2t}\|u_0-v_0\|_{W^{1,1}}.$$ □

*Remark* 4.2. We remark that the uniqueness results in [1] cannot be used to study the modified Camassa–Holm equation (1). Consider the transport equation with BV vector field $U$

$$n_t+(Un)_x=0 \qquad (82)$$

subject to initial data $n(x,0)=m_0(x)\in\mathcal{M}(\mathbb{R})$. In [1], Ambrosio studied the transport equation with BV vector fields which is similar to (82). When the space derivative of $U$ is absolutely continuous with respect to the Lebesgue measure, he obtained uniqueness and comparison results for bounded and compactly supported (in space) solutions of the transport equation. Although for the weak solution $u$ obtained in Theorem 3.4, we know $U(\cdot,t)=(u^2-u_x^2)(\cdot,t)$ is a BV function for $t\ge 0$. However, the distribution derivative of $U(\cdot,t)$ is a Radon measure which may not be absolutely continuous with respect to the Lebesgue measure. Moreover, $m$ may not be bounded even if $u\in L^\infty(0,\infty;W^{2,1}(\mathbb{R}))$. Hence, we cannot apply the results in [1] to the mCH equation.

**4.2. Examples of nonuniqueness of peakon weak solutions.** In this subsection, we construct peakon weak solutions (for $N = 1, 2$) to (1)–(2) and show that peakon weak solutions are not unique.

First, let's see one peakon situation. Assume $u(x,t) = pG(x - x(t))$ is a one peakon weak solution to (1)–(2). Then, from (5), we know that $x(t)$ satisfies

$$\frac{d}{dt}x(t) = \frac{1}{6}p^2. \tag{83}$$

We have the following result which can be viewed as the "jump condition" (or "Rankine–Hugoniot condition") for solitary wave solutions.

PROPOSITION 4.3. *For some constant $p \neq 0$, assume $u(x,t) = pG(x - x(t))$ and $m(x,t) = u - u_{xx} = p\delta(x - x(t))$. Then, $(u,\ m)$ is a solitary wave solution (or one peakon weak solution) to the mCH equation* (1) *if and only if the traveling speed of the soliton satisfies*

$$\frac{d}{dt}x(t) = \frac{1}{6}p^2. \tag{84}$$

*Proof.* For any test function $\phi \in C_c^\infty(\mathbb{R} \times [0,\infty))$, $\mathcal{L}(u,\phi)$ is defined by (14). By Step 2 in the proof of Lemma 2.2, we obtain

$$\begin{aligned}
\mathcal{L}(u,\phi) &= p\int_0^{+\infty} \phi_t(x(t),t) + \frac{1}{6}p^2\phi_x(x(t),t)dt \\
&= p\int_0^{+\infty} \left[\frac{d}{dt}\phi(x(t),t)dt + \left(\frac{1}{6}p^2 - \frac{d}{dt}x(t)\right)\phi_x(x(t),t)\right]dt \\
&= -p\phi(x(0),0) + p\int_0^{+\infty}\left(\frac{1}{6}p^2 - \frac{d}{dt}x(t)\right)\phi_x(x(t),t)dt. \\
&= -\int_{\mathbb{R}} \phi(x,0)m(dx,0) + p\int_0^{+\infty}\left(\frac{1}{6}p^2 - \frac{d}{dt}x(t)\right)\phi_x(x(t),t)dt.
\end{aligned}$$

Because $\phi$ is arbitrary, combining Definition 2.1 we know that $u$ is a weak solution to (1) if and only if (84) holds. ∎

From Proposition 4.3, we can see that $pG(x - \frac{1}{6}p^2 - c)$ is a weak solution to the mCH equation subject to initial data $m_0 = p\delta(x - c)$. This solution is also the sticky peakon weak solution given by Proposition 2.3.

Next, we show that one peakon can split into two peakons by presenting a two peakon weak solution for initial data $m_0(x) = p\delta(x - c)$. We have the following proposition.

PROPOSITION 4.4. *Assume initial data $m_0(x) = p\delta(x-c) = p_1\delta(x-c)+p_2\delta(x-c)$, where $p_1 + p_2 = p$ and $p_1 \neq p_2$. Then,*

$$u(x,t) = p_1G(x - x_1(t)) + p_2G(x - x_2(t)) \tag{85}$$

*is a global weak solution to* (1)–(2), *where*

$$x_1(t) = \frac{1}{6}p_1^2 t + \frac{3p_1p_2}{p_1^2 - p_2^2}\exp\left(\frac{1}{6}\left(p_1^2 - p_2^2\right)t\right) + c - \frac{3p_1p_2}{p_1^2 - p_2^2} \tag{86}$$

*and*

$$x_2(t) = \frac{1}{6}p_2^2 t + \frac{3p_1p_2}{p_1^2 - p_2^2}\exp\left(\frac{1}{6}\left(p_1^2 - p_2^2\right)t\right) + c - \frac{3p_1p_2}{p_1^2 - p_2^2}. \tag{87}$$

*$x_1(t)$ and $x_2(t)$ are obtained by* (5) *when $N = 2$ subject to initial value $x_1(0) = x_2(0) = c$.*

*Proof.* Without lose of generality, we assume $p_1^2 < p_2^2$. When $N = 2$, (5) can be rewritten as

$$\begin{cases} \frac{d}{dt}x_1(t) = \frac{1}{6}p_1^2 + \frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)}, \\ \frac{d}{dt}x_2(t) = \frac{1}{6}p_2^2 + \frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)}. \end{cases} \tag{88}$$

Taking the difference of the above two equations, we have

$$\frac{d}{dt}(x_1(t) - x_2(t)) = \frac{1}{6}\left(p_1^2 - p_2^2\right) < 0.$$

Combining $x_1(0) = x_2(0) = c$, we have

$$x_1(t) - x_2(t) = \frac{1}{6}\left(p_1^2 - p_2^2\right)t \quad \text{for} \quad t \geq 0. \tag{89}$$

Hence, the two peakons will never collide when $t > 0$. Put (89) into (88), and with some calculations we obtain (86) and (87).

Follow Step 2 in the proof of Lemma 2.2, and we see that (85) is a global weak solution to the mCH equation subject to initial data $m_0 = p\delta(x - c)$. ∎

Combining Propositions 4.3 and 4.4 gives an example that peakon weak solutions are not unique.

Next, we give a proposition to show that after collision the peakons can either stick together or cross with each other.

PROPOSITION 4.5. *Assume initial data $m(x, 0) = m_0(x) = p_1\delta(x - c_1) + p_2\delta(x - c_2)$. $p_1 > p_2 > 0$ and $c_1 < c_2$. Set $t^* = \frac{6(c_2-c_1)}{p_1^2-p_2^2}$. Then, there exist a sticky peakon weak solution to the mCH equation* (1)–(2) *which is given by*

$$\overline{u}(x,t) = \begin{cases} p_1G(x - x_1(t)) + p_2G(x - x_2(t)), & t < t^*; \\ (p_1 + p_2)G(x - x_3(t)), & t \geq t^*, \end{cases} \tag{90}$$

*where*

$$x_1(t) = \frac{1}{6}p_1^2t + \frac{3p_1p_2}{p_1^2 - p_2^2}\exp\left(\frac{1}{6}\left(p_1^2 - p_2^2\right)t + c_1 - c_2\right) + c_1 - \frac{3p_1p_2e^{c_1-c_2}}{p_1^2 - p_2^2},$$

$$x_2(t) = \frac{1}{6}p_2^2t + \frac{3p_1p_2}{p_1^2 - p_2^2}\exp\left(\frac{1}{6}\left(p_1^2 - p_2^2\right)t + c_1 - c_2\right) + c_2 - \frac{3p_1p_2e^{c_1-c_2}}{p_1^2 - p_2^2}$$

*and*

$$x_3(t) = \frac{1}{6}(p_1 + p_2)^2(t - t^*) + c^*, \quad c^* = x_1(t^*) = x_2(t^*), \quad t \geq t^*.$$

*And the mCH equation* (1)–(2) *has another weak solution $\widetilde{u}(x,t)$*

$$\widetilde{u}(x,t) = p_1G(x - \widetilde{x}_1(t)) + p_2G(x - \widetilde{x}_2(t)), \tag{91}$$

*where*

$$\widetilde{x}_1(t) = \begin{cases} x_1(t), \quad t \le t^*, \\ \dfrac{1}{6}p_1^2 t + \dfrac{3p_1p_2}{p_2^2 - p_1^2} \exp\left(\dfrac{1}{6}\left(p_2^2 - p_1^2\right)t + c_2 - c_1\right) \\ \quad + c_1 + \dfrac{3p_1p_2}{p_1^2 - p_2^2}(2 - e^{c_1 - c_2}), \ t > t^*; \end{cases}$$

*and*

$$\widetilde{x}_2(t) = \begin{cases} x_2(t), \quad t \le t^*, \\ \dfrac{1}{6}p_2^2 t + \dfrac{3p_1p_2}{p_2^2 - p_1^2} \exp\left(\dfrac{1}{6}\left(p_2^2 - p_1^2\right)t + c_2 - c_1\right) \\ \quad + c_2 + \dfrac{3p_1p_2}{p_1^2 - p_2^2}(2 - e^{c_1 - c_2}), \ t > t^*. \end{cases}$$

*Proof.* Similarly as proof of Proposition 4.4, taking the difference of (88) yields

$$\frac{d}{dt}(x_1(t) - x_2(t)) = \frac{1}{6}\left(p_1^2 - p_2^2\right).$$

Hence,

$$x_1(t) - x_2(t) = \frac{1}{6}\left(p_1^2 - p_2^2\right)t + c_1 - c_2.$$

Because $p_1 > p_2 > 0$ and $c_1 < c_2$, we have $x_1(t) < x_2(t)$ for $t < t^* = \frac{6(c_2 - c_1)}{p_1^2 - p_2^2}$ and $x_1(t^*) = x_2(t^*) = c^*$. At time $t^*$, the two peakons solution $p_1 G(x - x_1(t^*)) + p_2 G(x - x_2(t^*))$ becomes one peakon $(p_1 + p_2)G(x - c^*)$. From Proposition 2.3, we see that $\overline{u}(x,t)$ defined by (90) is a sticky peakon weak solution to the mCH equation (1)–(2).

Take $t^*$ as a new starting point. The initial data is $m(x, t^*) = (p_1 + p_2)\delta(x - c^*)$. By Proposition 4.4 we know there is another weak solution after $t^*$ with momentum $p_1$ and $p_2$, which is exactly our $\widetilde{u}(x,t)$. ◻

*Remark* 4.6. For the sticky solution $\overline{u}(x,t)$ given by (90), the energy $p_1^2 + p_2^2$ is not conserved. However, it is conserved for the crossing solution $\widetilde{u}$ given by (91).

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