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THE MODIFIED CAMASSA-HOLM EQUATION IN LAGRANGIAN COORDINATES

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Abstract. In this paper, we study the modified Camassa-Holm (mCH) equation in Lagrangian coordinates. For some initial data m_0, we show that classical solutions to this equation blow up in finite time T_{max}. Before T_{max}, existence and uniqueness of classical solutions are established. Lifespan for classical solutions is obtained: $T_{max} \geq \frac{1}{||m_0||_{L^\infty}||m_0||_{L^1}}$. And there is a unique solution $X(\xi,t)$ to the Lagrange dynamics which is a strictly monotonic function of ξ for any $t \in [0, T_{max})$: $X_\xi(\cdot,t) > 0$. As t approaching T_{max}, we prove that the classical solution $m(\cdot,t)$ in Eulerian coordinates has a unique limit $m(\cdot,T_{max})$ in Radon measure space and there is a point ξ_0 such that $X_\xi(\xi_0,T_{max}) = 0$ which means T_{max} is an onset time of collisions of characteristics. We also show that in some cases peakons are formed at T_{max}. After T_{max}, we regularize the Lagrange dynamics to prove global existence of weak solutions m in Radon measure space.

1. **Introduction.** In this work, we consider the following nonlinear partial differential equation in $\mathbb{R}$:

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

subject to an initial condition

$$m(x,0) = m_0(x). \tag{2}$$

This equation is referred to as the modified Camassa-Holm(mCH) eqaution with cubic nonlinearity, which was introduced as a new integrable system by several

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different researchers [14, 16, 28, 29]. It has a bi-Hamiltonian structure [18, 28] and a Lax-pair [29]. Equation (1) also has solitary wave solutions of the form [18]:

$$u(x,t) = pG(x - x(t)), \quad m(x,t) = p\delta(x - x(t)), \quad \text{and} \quad x(t) = \frac{1}{6}p^2 t,$$

where p is a constant representing the amplitude of the soliton and $G(x) = \frac{1}{2}e^{-|x|}$ is the fundamental solution for the Helmholtz operator $1-\partial_{xx}$. With this fundamental solution G, we have the following relation between functions u and m:

$$u(x,t) = G * m = \int_{\mathbb{R}} G(x-y)m(y,t)dy.$$

Moreover, global existence of N-peakon weak solutions of the following form was obtained in [17]:

$$u^N(x,t) = \sum_{i=1}^{N} p_i G(x - x_i(t)), \; m^N(x,t) = \sum_{i=1}^{N} p_i \delta(x - x_i(t)).$$

In the present paper, we study local well-posedness for classical solutions and global weak solutions to (1) in Lagrangian coordinates. Below we introduce the Lagrange dynamics for the mCH equation. To this end, we first review the Lagrange dynamics for incompressible $2D$ Euler equation:

$$\begin{cases} \omega_t(x,t) + \nabla \cdot \big(u(x,t)\omega(x,t)\big) = 0, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \\ \omega(x,0) = \omega_0(x), \end{cases}$$

where the velocity u is determined from the vorticity ω by the Biot-Savart law

$$u(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy, \quad x \in \mathbb{R}^2,$$

involving the kernel $K_2(x) = (2\pi|x|^2)^{-1}(-x_2, x_1)$. Assume $X(\xi,t)$ is the flow map generated by the velocity field $u(x,t)$:

$$\begin{cases} \dot{X}(\xi,t) = u(X(\xi,t),t), \quad \xi \in \mathbb{R}^2, \; t > 0, \\ X(\xi,0) = \xi. \end{cases}$$

By the incompressible property $\nabla \cdot u = 0$, we know

$$\omega(X(\xi,t),t) = \omega_0(\xi). \tag{3}$$

The $2D$ Euler equation can be rewritten in the Lagrange dynamics

$$\begin{cases} \dot{X}(\xi,t) = u(X(\xi,t),t), \; X(\xi,0) = \xi \in \mathbb{R}^2, \; t > 0, \\ \omega(X(\xi,t),t) = \omega_0(\xi), \\ u(x,t) = (K_2 * \omega)(x,t). \end{cases}$$

Comparing with the incompressible $2D$ Euler equation, assume $X(\xi,t)$ is the flow map for the mCH equation generated by the velocity field $u^2 - u_x^2$:

$$\dot{X}(\xi,t) = (u^2 - u_x^2)(X(\xi,t),t), \; X(\xi,0) = \xi \in \mathbb{R}, \; t > 0.$$

In contrast with (3), we have the following property for the mCH equation:

$$m(X(\xi,t),t)X_\xi(\xi,t) = m_0(\xi).$$

Combining the above two equalities, the mCH equation (1) can be rewritten in the Lagrange dynamics:

$$\begin{cases} \dot{X}(\xi,t) = (u^2-u_x^2)(X(\xi,t),t), \ \ X(\xi,0)=\xi\in\mathbb{R}, \ \ t>0, \\ m(X(\xi,t),t)X_\xi(\xi,t) = m_0(\xi), \\ u(x,t) = (G*m)(x,t). \end{cases} \tag{4}$$

Changing of variable gives

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} G(x-y)m(y,t)dy = \int_{\mathbb{R}} G(x-X(\theta,t))m(X(\theta,t))X_\theta(\theta,t)d\theta \\ &= \int_{\mathbb{R}} G(x-X(\theta,t))m_0(\theta)d\theta. \end{aligned}$$

Set

$$\begin{aligned} U(x,t) : &= u^2(x,t) - u_x^2(x,t) \\ &= \left(\int_{\mathbb{R}} G(x-X(\theta,t))m_0(\theta)d\theta\right)^2 - \left(\int_{\mathbb{R}} G_x(x-X(\theta,t))m_0(\theta)d\theta\right)^2. \end{aligned} \tag{5}$$

Then, Equation (4) can be rewritten as

$$\begin{cases} \dot{X}(\xi,t) = U(X(\xi,t),t), \\ X(\xi,0) = \xi\in\mathbb{R}. \end{cases} \tag{6}$$

When $m_0 \in L^1(\mathbb{R})$, the following useful properties can be easily obtained:

$$|u(x,t)| \le \frac{1}{2}||m_0||_{L^1}, \quad |u_x(x,t)| \le \frac{1}{2}||m_0||_{L^1} \ \text{ and } \ |U(x,t)| \le \frac{1}{2}||m_0||_{L^1}^2. \tag{7}$$

In the rest of this paper, we assume the initial m_0 satisfying $\text{supp}\{m_0\}\subset(-L,L)$ for some constant $L>0$. Next, we summarize our main results in four theorems.

Theorem 1.1. *Suppose $m_0 \in C_c^k(-L,L)$ $(k\in\mathbb{N}, k\ge 1)$. Then, there exists a unique maximum existence time $T_{max}\le+\infty$ such that Lagrange dynamics (6) has a unique solution*

$$X \in C_1^{k+1}([-L,L]\times[0,T_{max})),$$

which satisfies

$$X_\xi(\xi,t) > 0 \ \ \text{for} \ \ (\xi,t)\in[-L,L]\times[0,T_{max}).$$

(The solution space is defined by (13).) The mCH equation (1)-(2) has a unique classical solution

$$u\in C_1^{k+2}(\mathbb{R}\times[0,T_{max})), \quad m\in C_1^k(\mathbb{R}\times[0,T_{max})),$$

which can be represented by $X(\xi,t)$ as

$$u(x,t) = \int_{-L}^{L} G(x-X(\theta,t))m_0(\theta)d\theta \ \ \text{and} \ \ m(x,t) = \int_{-L}^{L}\delta(x-X(\theta,t))m_0(\theta)d\theta. \tag{8}$$

Moreover, m satisfies:

$$\text{supp}\{m(\cdot,t)\}\subset(-L,L) \ \ \text{for} \ \ t\in[0,T_{max}). \tag{9}$$

If $T_{max}<+\infty$, then the following holds:
(i) We have

$$X\in C_1^{k+1}([-L,L]\times[0,T_{max}]).$$

(ii) The following equivalent statements hold:

(a)

$$\limsup_{t\to T_{max}} ||m(\cdot,t)||_{L^\infty} = +\infty,$$

(b)

$$\begin{cases} X_\xi(\xi,t) > 0 \quad \textit{for} \quad (\xi,t) \in [-L,L] \times [0,T_{max}); \\ \min\limits_{\xi\in[-L,L]} X_\xi(\xi,T_{max}) = 0. \end{cases}$$

(c)

$$\liminf_{t\to T_{max}} \left\{ \inf_{\xi\in[-L,L]} \int_0^t (mu_x)(X(\xi,s),s)ds \right\} = -\infty,$$

(d)

$$\liminf_{t\to T_{max}} \left\{ \inf_{x\in\mathbb{R}} (mu_x)(x,t) \right\} = -\infty,$$

(e)

$$\limsup_{t\to T_{max}} ||m(\cdot,t)||_{W^{1,p}} = +\infty, \quad \textit{for } p \geq 1,$$

(f)

$$\int_0^{T_{max}} ||m(\cdot,t)||_{L^\infty} dt = +\infty.$$

(iii) There exists a unique function $u(\cdot,T_{max})$ such that

$$\lim_{t\to T_{max}} u(x,t) = u(x,T_{max}), \qquad \lim_{t\to T_{max}} u_x(x,t) = u_x(x,T_{max}) \textit{ for every } x \in \mathbb{R}.$$

Moreover, for any $t \in [0,T_{max}]$ we have

$$u(\cdot,t),\ u_x(\cdot,t) \in BV(\mathbb{R})$$

and

$$\text{Tot.Var.}\{u(\cdot,t)\} \leq M_1, \quad \text{Tot.Var.}\{u_x(\cdot,t)\} \leq 2M_1.$$

Here, $BV(\mathbb{R})$ is the space of functions with bounded variation (see definition 5.1).
(iv) There exists a unique $m(\cdot,T_{max}) \in \mathcal{M}(\mathbb{R})$ (Radon measure space on $\mathbb{R}$) such that

$$m(\cdot,t) \overset{*}{\rightharpoonup} m(\cdot,T_{max}) \quad \textit{in} \quad \mathcal{M}(\mathbb{R}), \quad \textit{as} \quad t \to T_{max}.$$

(a) and (b) tell us that T_{max} is an onset time of collisions of characteristics. (9) implies that the supports for classical solutions will not change.

Our another main theorem is about finite time blow-up behaviors and the lifespan of classical solutions. Let $T_{max}(m_0)$ be the maximum existence time of the classical solution to the mCH equation subject to an initial condition m_0. Then we have the following theorem about lifespan for classical solutions.

Theorem 1.2. *Assume $m_0 \in C_c^k(\mathbb{R})$ $(k \in \mathbb{N}, k \geq 1)$.*
(i) We have

$$T_{max}(m_0) \geq \frac{1}{||m_0||_{L^\infty}||m_0||_{L^1}}. \tag{10}$$

(ii) If there exists $\bar{\xi} \in [-L,L]$ such that

$$m_0(\bar{\xi})\partial_x u_0(\bar{\xi}) < 0, \quad |m_0(\bar{\xi})|(\partial_x u_0(\bar{\xi}))^2 > \frac{1}{2}||m_0||^3_{L^1}, \tag{11}$$

then the classical solution to the mCH equation will blow up in finite time. Moreover, for any $\epsilon > 0$ we have

$$\frac{1}{||m_0||_{L^\infty}||m_0||_{L^1}} \cdot \frac{1}{\epsilon^2} \leq T_{max}(\epsilon m_0) \leq \frac{1}{||m_0||^2_{L^1}} \cdot \frac{1}{\epsilon^2}. \tag{12}$$

This theorem implies that there are smooth initial data with arbitrary small support and arbitrary small $C^k(\mathbb{R})$-norm, $k \in \mathbb{N}$, for which the classical solution does not exist globally.

Next, we give a theorem to show the formation of peakons at finite blow-up time T_{max}. From Theorem 1.1, we know there is a point $\xi_0 \in [-L, L]$ such that $X_\xi(\xi_0, T_{max}) = 0$. Set

$$F_{T_{max}} := \{X(\xi, T_{max}) : \xi \in [-L, L],\ X_\xi(\xi, T_{max}) = 0\}.$$

For any $x \in F_{T_{max}}$, because $X_\xi(\cdot, T_{max}) \geq 0$, we know that $X^{-1}(x, T_{max})$ is either a single point or a closed interval. Denote

$$\widehat{F}_{T_{max}} := \{x \in F_{T_{max}} : X^{-1}(x, T_{max}) = [\xi_1, \xi_2] \ \text{ for some } \ \xi_1 < \xi_2\}.$$

The figure below describe these singular points.

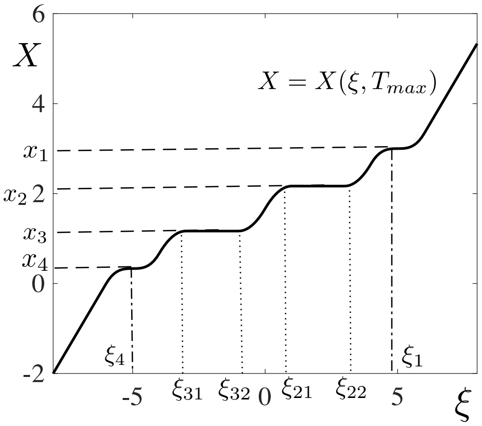

FIGURE 1. At T_{max}, $X_\xi(\cdot, T_{max}) \geq 0$ and $X_\xi(\xi, T_{max}) = 0$ for $\xi \in \{\xi_1, \xi_4\} \cup [\xi_{21}, \xi_{22}] \cup [\xi_{31}, \xi_{32}]$. $F_{T_{max}} = \{x_1, x_2, x_3, x_4\}$ and $\widehat{F}_{T_{max}} = \{x_2, x_3\}$.

For $x \in \widehat{F}_{T_{max}}$ and $X^{-1}(x, T_{max}) = [\xi_1, \xi_2]$, we show that m_0 will not change sign in $[\xi_1, \xi_2]$ (see Proposition 3). Hence, $\int_{\xi_1}^{\xi_2} m_0(\xi)d\xi \neq 0$. We have the following theorem.

Theorem 1.3. *Assume* $F_{T_{max}} = \{x_i\}_{i=1}^{N_1}$ *and* $\widehat{F}_{T_{max}} = \{x_i\}_{i=1}^{N}$ *(*$N \leq N_1$*). Let* $X^{-1}(x_i, T_{max}) = [\xi_{i1}, \xi_{i2}]$ *and* $p_i = \int_{\xi_{i1}}^{\xi_{i2}} m_0(\xi)d\xi$ *for* $1 \leq i \leq N$*. Then*

$$m(x, T_{max}) = m_1(x) + \sum_{i=1}^{N} p_i \delta(x - x_i)$$

where $m_1 \in L^1(\mathbb{R})$ *is given by* (99).

At last, we give a theorem to show global existence of weak solutions (see Definition 6.3). Theorem 1.1 (iv) tells that classical solutions become Radon measures when blow-up happens. After the blow-up time T_{max}, we can extend our solution $m(x, t)$ globally in the Radon measure space. We have:

Theorem 1.4. *Let $m_0 \in \mathcal{M}(\mathbb{R})$ with compact support. Then there exists a global weak solution to the mCH equation satisfying:*

$$u \in C([0,+\infty); H^1(\mathbb{R})) \cap L^\infty(0,+\infty; W^{1,\infty}(\mathbb{R})),$$

and

$$m = u - u_{xx} \in \mathcal{M}(\mathbb{R} \times [0,T)) \ \ \textit{for any} \ \ T > 0.$$

Now, we compare the mCH equation with the Camassa-Holm (CH) equation:

$$\partial_t m + \partial_x(um) + m\partial_x u = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, t > 0.$$

The CH equation was established by Camassa and Holm [6] to model the unidirectional propagation of waves at free surface of a shallow layer of water ($u(x,t)$ representing the height of water's free surface above a flat bottom). It is also a complete integrable system which has a bi-Hamiltonian structure and a Lax pair [6].

There are some different properties between the CH equation and the mCH equation.

- *Classical solutions and blow-up criteria.* For a large class of initial data, classical solutions to the CH equation blow up in finite time (see [2] and references in it). McKean [25] provided a necessary and sufficient condition for wave-breaking of the CH equation. In 2012, Jiang et. al [22] gave a new and direct proof for McKean's theorem in [25]. Moreover, the only way that a classical solution of the CH equation fails to exist globally is that the wave breaks [10] in the sense that the solution u remains bounded while the spatial derivative u_x becomes unbounded. For the mCH equation, blow-up behaviors also happen for a large class of initial data (see [8, 18, 24]). However, u_{xx} (hence m) becomes unbounded when blow-up happens, while u and u_x remain bounded.
- *Lifespan for classical solutions.* Comparing with (12), the lower bound for lifespan of strong solutions to the CH equation with initial data $\epsilon u_0(x)$ is given by [13, 26]:

$$T_{max} \geq \frac{C}{\epsilon} \ \text{ for some } \ C = C(u_0).$$

- *N-peakon weak solutions.* Trajectories for N-peakon weak solutions to the CH equation never collide [7, 9] provided that the initial datum $m_0(x) = \sum_{i=1}^N p_i \delta(x - c_i)$ satisfies $p_i > 0$ and $c_i \neq c_j$ for $i \neq j$. However, the trajectories for N-peakon solutions of the mCH equation may collide in finite time even if $m_0 \geq 0$ [17]. Moreover, for the CH equation, when blow-up happens at finite time T_{max}, we have $\liminf_{t\to T_{max}} u_x(x,t) = -\infty$ (see [10, 26]). Peakon solutions u and its derivative u_x are in BV space, which are bounded functions. Hence, peakon solutions can not be formed when blow-up happens (comparing with Theorem 1.3) for the CH equation.
- *General weak solutions.* In [17], the authors proved nonuniqueness of weak solutions obtained by Theorem 1.4. Comparing with Theorem 1.4, there is a unique global weak solution $u \in C([0,+\infty); H^1(\mathbb{R}))$ and $m \in \mathcal{M}_+(\mathbb{R})$ (see [9, 12]) to the CH equation when $u_0 \in H^1(\mathbb{R})$ and $0 \leq m_0 \in \mathcal{M}(\mathbb{R})$. For general initial data $u_0 \in H^1(\mathbb{R})$, global existence of weak solutions to the CH equation was obtained by several different methods (see [4, 5, 19, 20, 31, 32]).

For more results about local well-posedness and blow up behavior of strong solutions to the Cauchy problem (1)-(2), one can refer to [8, 15, 18, 24]. For weak solutions, one can refer to [17, 33].

The rest of this article is organized as follows. In Section 2, we use contraction mapping theorem to prove local existence and uniqueness of solutions $X(\xi,t)$ to the Lagrange dynamics (6). Then, we use $X(\xi,t)$ to obtain $(u(x,t),m(x,t))$ and prove that it is a unique classical solution to the mCH equation (1)-(2). Besides, when $\sup_{t\in[0,T)}||m(\cdot,t)||_{L^\infty}$ is finite, we can extend this classical solution in time. In Section 3, we show some blow-up criteria for classical solutions. In Section 4, we prove that for some initial data classical solutions blow up in a finite time and the estimates for blow-up rates are given. For small initial data, almost global existence of classical solutions is obtained. In Section 5, we study classical solutions at blow-up time T_{max}. $u(\cdot,T_{max})$ and $u_x(\cdot,T_{max})$ are BV functions while $m(\cdot,t)$ has a unique limit $m(\cdot,T_{max})$ in Radon measure space as $t\to T_{max}$. Moreover, we prove that in some cases peakons are formed at T_{max}. In the last section, we use regularized Lagrange dynamics to prove global existence of weak solutions in Radon measure space.

2. **Lagrange dynamics and short time classical solutions.**

In this section, we study the existence and uniqueness of solutions to Lagrange dynamics (6). Then, we prove $(u(x,t),m(x,t))$ defined by (8) is a unique classical solution to (1)-(2).

First, let's introduce the spaces for solutions. For nonnegative integers k,n and real number $T>0$, we denote

$$U_T:=[-L,L]\times[0,T]$$

and the function space

$$C_n^k(U_T):=\big\{u:U_T\to\mathbb{R}:\ \partial_x^\beta u\in C(U_T),\ |\beta|\le k;\quad \partial_t^\alpha u\in C(U_T),\ |\alpha|\le n\big\}. \tag{13}$$

Similarly, we can define $C_n^k(\mathbb{R}\times[0,T])$.

We will present the results of this section in three subsections as follows.

1. In Subsection 2.1, when $m_0\in C_c^k(-L,L)$, we prove local existence and uniqueness of a solution $X\in C_1^{k+1}([-L,L]\times[0,t_1])$ to (6) such that
$$\min\{X_\xi(\xi,t):(\xi,t)\in[-L,L]\times[0,t_1]\}>0.$$
2. In Subsection 2.2, we prove u defined by (8) belongs to $C_1^{k+2}(\mathbb{R}\times[0,t_1])$ and (u,m) is a unique classical solution to the mCH equation.
3. In Subsection 2.3, we prove that whenever the classical solution m satisfies
$$\sup_{t\in[0,T)}||m(\cdot,t)||_{L^\infty}<\infty,$$
we can extend the classical solution in time.

2.1. **Local existence and uniqueness of solutions to Lagrange dynamics.**

In this subsection, we use the contraction mapping theorem to prove short time existence and uniqueness of solutions to the Lagrange dynamics (6), which is equivalent to the following integral equation:

$$X(\xi,t)=\xi+\int_0^t U(X(\xi,s),s)ds, \tag{14}$$

where U is defined by (5). Set

$$T_X(\xi,t):=\xi+\int_0^t U(X(\xi,s),s)ds. \tag{15}$$

For constants $C_2 > C_1 > 0$ and $t_1 > 0$, we define

$$\mathcal{Q}_{t_1}(C_1, C_2) := \Big\{X \in C(U_{t_1}) : C_1(\xi-\eta)^2 \le (X(\xi,t)-X(\eta,t))(\xi-\eta) \le C_2(\xi-\eta)^2, \text{for any } \xi,\eta \in [-L,L] \text{ and } t \in [0,t_1]\Big\}. \tag{16}$$

Obviously, $\mathcal{Q}_{t_1}(C_1, C_2)$ is a closed subset of $C(U_{t_1})$. We will look for suitable constants C_1, C_2, t_1 and then use the contraction mapping theorem in the set $\mathcal{Q}_{t_1}(C_1, C_2)$.

Before presenting the existence and uniqueness theorem, we give two useful lemmas.

Lemma 2.1. *Assume $g \in L^\infty(-L, L)$ and $X(\xi,t) \in \mathcal{Q}_{t_1}(C_1, C_2)$ for some constants $C_2 > C_1 > 0$ and $t_1 > 0$. Let*

$$A(x,t) := \int_{-L}^{L} G'(x - X(\theta,t))g(\theta)d\theta.$$

Then, we have $A \in C(\mathbb{R} \times [0,t_1])$.

Proof. According to (16), $X(\xi,t)$ is monotonic about ξ. For given $(x,t) \in U_{t_1}$, we separate the proof into three parts.

Step 1. Continuity at $(x,t) \in \mathbb{R} \times [0,t_1]$ when $x > X(L,t)$.

For (y,s) closed to (x,t) and because $X \in C(U_{t_1})$ is monotonic, we can assume $y > X(\theta,s)$ for $\theta \in (-L,L)$. A direct estimate gives

$$\begin{aligned}|A(y,s) - A(x,t)| &= \left|\int_{-L}^{L} G'(y - X(\theta,s))g(\theta)d\theta - \int_{-L}^{L} G'(x - X(\theta,t))g(\theta)d\theta\right|\\ &\le \int_{-L}^{L} |G(x - X(\theta,t)) - G(y - X(\theta,s))| \cdot |g(\theta)|d\theta\\ &\le \frac{1}{2}||g||_{L^\infty}\int_{-L}^{L} |y-x| + |X(\theta,t) - X(\theta,s)|d\theta.\end{aligned}$$

Therefore, according to the uniform continuity of X, A is continuous at (x,t). The proof of the case $x < X(-L,t)$ is similar.

Step 2. Continuity at $(x,t) \in \mathbb{R} \times [0,t_1]$ when $x = X(\xi,t)$ for some $\xi \in (-L,L)$.

Due to the continuity of X, for (y,s) closed to (x,t), there exists $\eta \in [-L,L]$ such that $X(\eta,s) = y$. Without lose of generality, we assume $\xi > \eta$.

$$\begin{aligned}&|A(y,s) - A(x,t)|\\ &= \left|\int_{-L}^{L} G'(X(\eta,s) - X(\theta,s))g(\theta)d\theta - \int_{-L}^{L} G'(X(\xi,t) - X(\theta,t))g(\theta)d\theta\right|\\ &\le \int_{-L}^{\eta} |G'(X(\eta,s) - X(\theta,s)) - G'(X(\xi,t) - X(\theta,t))||g(\theta)|d\theta\\ &\quad + \int_{\eta}^{\xi} |G'(X(\eta,s) - X(\theta,s)) - G'(X(\xi,t) - X(\theta,t))||g(\theta)|d\theta\\ &\quad + \int_{\xi}^{L} |G'(X(\eta,s) - X(\theta,s)) - G'(X(\xi,t) - X(\theta,t))||g(\theta)|d\theta.\end{aligned}$$

Then, the monotonicity of $X(\theta,t)$ implies that

$$|A(y,s)-A(x,t)| \le ||g||_{L^\infty}|\xi-\eta| + ||g||_{L^\infty}\int_{-L}^{L}|x-y|+|X(\theta,s)-X(\theta,t)|d\theta.$$

From the definition of $\mathcal{Q}_{t_1}(C_1,C_2)$, we have

$$\begin{aligned}|x-y| = |X(\xi,t)-X(\eta,s)| &\ge |X(\xi,s)-X(\eta,s)| - |X(\xi,t)-X(\xi,s)|\\ &\ge C_1|\xi-\eta| - |X(\xi,t)-X(\xi,s)|. \qquad (17)\end{aligned}$$

Therefore, $|\xi-\eta| \le \frac{1}{C_1}(|x-y|+|X(\xi,t)-X(\xi,s)|)$. Hence, $A(x,t)$ is continuous at (x,t).

Step 3. Continuity at $(x,t)\in\mathbb{R}\times[0,t_1]$ when $x=X(L,t)$. The case $x=X(-L,t)$ is similar.

For (y,s) closed to (x,t), we have two cases. When $y>X(L,s)$, we can use Step 1. When there exists $\xi\in(-L,L)$ such that $y=X(\xi,s)$, we can use Step 2.

This is the end of the proof. $\square$

Lemma 2.2. *Assume* $m_0\in L^\infty(-L,L)$ *and* $X\in\mathcal{Q}_{t_1}(C_1,C_2)$ *for some constants* $C_2>C_1>0$ *and* $t_1>0$. *Then, for* $-L\le\eta<\xi\le L$, *we have*

$$\begin{aligned}[1-(M_1M_\infty+C_2M_1^2)t_1](\xi-\eta) &\le T_X(\xi,t)-T_X(\eta,t)\\ &\le [1+(M_1M_\infty+C_2M_1^2)t_1](\xi-\eta), \qquad (18)\end{aligned}$$

where $M_1:=||m_0||_{L^1}$ *and* $M_\infty:=||m_0||_{L^\infty}$.

Proof. Assume $X\in\mathcal{Q}_{t_1}(C_1,C_2)$ for some constants $C_2>C_1>0$ and $t_1>0$. For $-L\le\eta<\xi\le L$, $t\in[0,t_1]$, we have

$$T_X(\xi,t)-T_X(\eta,t) = \xi-\eta+\int_0^t[U(X(\xi,s),s)-U(X(\eta,s),s)]ds. \qquad (19)$$

By (7), we obtain

$$\begin{aligned}&|U(X(\xi,s),s)-U(X(\eta,s),s)|\\ &\le |u^2(X(\xi,s),s)-u^2(X(\eta,s),s)| + |u_x^2(X(\xi,s),s)-u_x^2(X(\eta,s),s)|\\ &\le M_1|u(X(\xi,s),s)-u(X(\eta,s),s)| + M_1|u_x(X(\xi,s),s)-u_x(X(\eta,s),s)|\\ &=: I_1+I_2.\end{aligned}$$

Because $X\in\mathcal{Q}_{t_1}(C_1,C_2)$, we have

$$\begin{aligned}&|u(X(\xi,s),s)-u(X(\eta,s),s)|\\ &=\Big|\int_{-L}^{L}m_0(\theta)\Big(G(X(\xi,s)-X(\theta,s))-G(X(\eta,s)-X(\theta,s))\Big)d\theta\Big|\\ &\le \frac{1}{2}M_1(X(\xi,s)-X(\eta,s)) \le \frac{1}{2}M_1C_2(\xi-\eta).\end{aligned}$$

Thus,

$$I_1 \le \frac{1}{2}M_1^2C_2(\xi-\eta).$$

Next, we estimate I_2.

When $X \in \mathcal{Q}_{t_1}(C_1, C_2)$, we have $(X(\xi, s) - X(\theta, s))(X(\eta, s) - X(\theta, s)) > 0$ for $\theta \in [-L, \eta) \cap (\xi, L]$. On the other hand, we know $|G'(a) - G'(b)| = |G(a) - G(b)| \leq \frac{1}{2}|a - b|$ when $ab > 0$. Therefore,

$$\begin{aligned}
&|u_x(X(\xi, s), s) - u_x(X(\eta, s), s)| \\
\leq &\int_{[-L,\eta)\cap(\xi,L]} m_0(\theta)|G'(X(\xi, s) - X(\theta, s)) - G'(X(\eta, s) - X(\theta, s))|d\theta \\
&+ \int_{\eta}^{\xi} m_0(\theta)|G'(X(\xi, s) - X(\theta, s)) - G'(X(\eta, s) - X(\theta, s))|d\theta \\
\leq &(\frac{1}{2}M_1C_2 + M_\infty)(\xi - \eta).
\end{aligned}$$

Thus

$$I_2 \leq (\frac{1}{2}C_2M_1^2 + M_1M_\infty)(\xi - \eta).$$

Combining I_1 and I_2 gives

$$\begin{aligned}
-(C_2M_1^2 + M_1M_\infty)t_1(\xi - \eta) \leq &\int_0^t [U(X(\xi, s), s) - U(X(\eta, s), s)]ds \\
\leq &(C_2M_1^2 + M_1M_\infty)t_1(\xi - \eta).
\end{aligned}$$

Together with (19), we obtain (18). □

We have the following existence and uniqueness theorem.

Theorem 2.3. *Assume* $m_0 \in C_c^k(-L, L)$ $(k \in \mathbb{N}, k \geq 1)$. *Let* $M_1 := ||m_0||_{L^1}$ *and* $M_\infty := ||m_0||_{L^\infty}$. *Then, for any* t_1 *with*

$$0 < t_1 < \frac{1}{2M_1^2 + M_1M_\infty}, \tag{20}$$

there exist constants $C_2 > C_1 > 0$ *satisfying*

$$\frac{1 + M_1M_\infty t_1}{1 - M_1^2t_1} < C_2 < \frac{1 - M_1M_\infty t_1}{M_1^2t_1}, \tag{21}$$

and

$$0 < C_1 < 1 - (M_1M_\infty + C_2M_1^2)t_1, \tag{22}$$

such that (14) *has a unique solution* $X \in C_1^{k+1}(U_{t_1})$ *satisfying*

$$C_1 \leq X_\xi(\xi, t) \leq C_2 \tag{23}$$

for $(\xi, t) \in [-L, L] \times [0, t_1]$.

Moreover, for any $\ell \in \mathbb{N}$, $0 \leq \ell \leq k + 1$, *there exists a constant* $\widehat{C}_\ell$ *(depending on* $||m_0||_{C^k}$, $||m_0||_{L^1}$ *and* t_1*) such that*

$$|\partial_\xi^\ell X(\xi, t)| \leq \widehat{C}_\ell. \tag{24}$$

Proof. We separate this proof into two parts.

Part I.(Existence and Uniqueness) We use the contraction mapping theorem to prove the existence of a unique solution $X \in C_1^0(U_{t_1})$ to (14).

Step 1. When $0 < t_1 < \frac{1}{2M_1^2+M_1M_\infty}$, we prove there are constants $C_2 > C_1 > 0$ such that when $X \in \mathcal{Q}_{t_1}(C_1, C_2)$, we have $T_X \in \mathcal{Q}_{t_1}(C_1, C_2)$, where T_X is defined by (15).

When t_1 satisfies (20), we have

$$M_1^2 t_1 < \frac{1}{2} \text{ and } M_1 M_\infty t_1 < 1. \tag{25}$$

A simple computation shows that

$$\frac{1 + M_1 M_\infty t_1}{1 - M_1^2 t_1} < \frac{1 - M_1 M_\infty t_1}{M_1^2 t_1}.$$

Hence, there is a constant C_2 satisfying (21). Moreover, inequality (21) implies

$$1 + (M_1 M_\infty + C_2 M_1^2) t_1 \leq C_2, \tag{26}$$

and

$$0 < 1 - (M_1 M_\infty + C_2 M_1^2) t_1.$$

Therefore, we can choose C_1 satisfying (22).

When $X \in \mathcal{Q}_{t_1}(C_1, C_2)$, combining (18), (22) and (26) gives

$$T_X \in \mathcal{Q}_{t_1}(C_1, C_2)$$

and Step 1 is completed.

Step 2. We prove T_X is a contraction map on $\mathcal{Q}_{t_1}(C_1, C_2)$.

For $X, Y \in \mathcal{Q}_{t_1}(C_1, C_2)$, combining (7) we have

$$\begin{aligned}
&|T_X(\xi, t) - T_Y(\xi, t)| \leq \int_0^t |U(X(\xi, s), s) - U(Y(\xi, s), s)| ds \\
&\leq M_1 \int_0^t |u(X(\xi, s), s) - u(Y(\xi, s), s)| ds + M_1 \int_0^t |u_x(X(\xi, s), s) - u_x(Y(\xi, s), s)| ds \\
&=: J_1 + J_2.
\end{aligned} \tag{27}$$

For the first term J_1, we estimate

$$\begin{aligned}
&u(X(\xi, s), s) - u(Y(\xi, s), s) \\
&= \int_{-L}^{L} m_0(\theta) \Big(G(X(\xi, s) - X(\theta, s)) - G(Y(\xi, s) - Y(\theta, s)) \Big) d\theta \\
&\leq \frac{1}{2} \int_{-L}^{L} m_0(\theta) (|X(\xi, s) - Y(\xi, s)| + |X(\theta, s) - Y(\theta, s)|) d\theta \\
&\leq M_1 ||X - Y||_{C(U_{t_1})}.
\end{aligned} \tag{28}$$

For the second term, due to $(X(\xi, s) - X(\theta, s))(Y(\xi, s) - Y(\theta, s)) > 0$, we obtain

$$\begin{aligned}
&u_x(X(\xi, s), s) - u_x(Y(\xi, s), s) \\
&= \int_{-L}^{L} m_0(\theta) \Big(G'(X(\xi, s) - X(\theta, s)) - G'(Y(\xi, s) - Y(\theta, s)) \Big) d\theta \\
&\leq M_1 ||X - Y||_{C(U_{t_1})}.
\end{aligned} \tag{29}$$

Combining (27), (28), and (29), we have

$$|T_X(\xi, t) - T_Y(\xi, t)| \leq J_1 + J_2 \leq 2 M_1^2 t_1 ||X - Y||_{C(U_{t_1})},$$

which implies

$$||T_X - T_Y||_{C(U_{t_1})} \leq 2 M_1^2 t_1 ||X - Y||_{C(U_{t_1})}.$$

Inequality (25) shows that T_X is a contraction map.

At last, by the contraction mapping theorem, the system (14) (or (6)) has a unique solution in $C(U_{t_1})$.

On the other hand, using Lemma 2.1 we can see $U = u^2 - u_x^2 \in C(\mathbb{R} \times [0, t_1])$, which means

$$\partial_t X \in C(U_{t_1}).$$

Hence, $X \in C_1^0(U_{t_1})$ and Part I is finished.

Part II. (Regularity) We show X obtained in the first part belongs to $C_1^{k+1}(U_{t_1})$.

From the first part, we can see solution X belongs to $C_1^0(U_{t_1})$. For this solution we have the following properties

$$X(\xi, t) - X(\theta, t) > 0, \quad -L < \theta < \xi; \quad X(\xi, t) - X(\theta, t) < 0, \quad \xi < \theta < L.$$

On the other hand, $G(x) = \frac{1}{2}e^{-|x|}$ satisfies:

$$G'(x) = G(x), \quad x < 0; \quad G'(x) = -G(x), \quad x > 0.$$

We obtain

$$\int_{-L}^{\xi} G'(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta = -\int_{-L}^{\xi} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta, \tag{30}$$

$$\int_{\xi}^{L} G'(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta = \int_{\xi}^{L} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta, \tag{31}$$

Hence,

$$\begin{aligned} &u_x(X(\xi, t), t) \\ &= \int_{-L}^{\xi} G'(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta + \int_{\xi}^{L} G'(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta \\ &= -\int_{-L}^{\xi} G(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta + \int_{\xi}^{L} G(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta. \end{aligned}$$

We obtain

$$\begin{aligned} &U(X(\xi, t), t) = u^2(X(\xi, t), t) - u_x^2(X(\xi, t), t) \\ &= 4\left(\int_{-L}^{\xi} G(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta\right)\left(\int_{\xi}^{L} G(X(\xi, t) - X(\theta, t))m_0(\theta)d\theta\right). \end{aligned} \tag{32}$$

Thus

$$\begin{aligned} X(\xi, t) =& \xi + 4\int_0^t \left(\int_{-L}^{\xi} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta\right) \\ & \left(\int_{\xi}^{L} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta\right) ds. \end{aligned} \tag{33}$$

Because $X(\xi, t)$ is monotonic about ξ, its derivative exists for a.e. $\xi \in [-L, L]$. Differentiating with respect to ξ shows that for a.e. $\xi \in [-L, L]$,

$$\begin{aligned} X_\xi(\xi, t) = 1 + 4G(0)m_0(\xi)\int_0^t \bigg(&\int_{\xi}^{L} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta \\ &- \int_{-L}^{\xi} G(X(\xi, s) - X(\theta, s))m_0(\theta)d\theta\bigg) ds \end{aligned}$$

$$+4\int_0^t X_\xi(\xi,s)\left(\int_{-L}^{\xi} G'(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)$$
$$\left(\int_{\xi}^{L} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)ds$$
$$+4\int_0^t X_\xi(\xi,s)\left(\int_{-L}^{\xi} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)$$
$$\left(\int_{\xi}^{L} G'(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)ds, \quad (34)$$

Due to (30) and (31), the sum of the last two terms in (34) is zero, which leads to

$$X_\xi(\xi,t)=1+2m_0(\xi)\int_0^t\left(\int_{\xi}^{L} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right.$$
$$\left.-\int_{-L}^{\xi} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)ds. \quad (35)$$

Because $m_0\in C_c^k(-L,L)$, we have $X_\xi\in C(U_{t_1})$ which means $X\in C_1^1(U_{t_1})$.

From (35), we have

$$|X_\xi(\xi,t)|\le 1+M_1M_\infty t_1=1+||m_0||_C||m_0||_{L^1}t_1 \quad \text{for} \quad t\in[0,t_1]. \quad (36)$$

Differentiating (35) with respect to ξ shows that

$$X_{\xi\xi}(\xi,t)=1+2m_0'(\xi)\int_0^t\left(\int_{\xi}^{L} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right.$$
$$\left.-\int_{-L}^{\xi} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta\right)ds-2m_0^2(\xi)t$$
$$+2m_0(\xi)\int_0^t X_\xi(\xi,s)\int_{-L}^{L} G(X(\xi,s)-X(\theta,s))m_0(\theta)d\theta ds. \quad (37)$$

Hence, we obtain $X_{\xi\xi}\in C(U_{t_1})$ and

$$|X_{\xi\xi}(\xi,t)|\le 1+2||m_0||_{C^1}||m_0||_{L^1}t_1+2||m_0||_C^2t_1+2||m_0||_C^2||m_0||_{L^1}^2t_1^2.$$

We have $X\in C_1^2(U_{t_1})$.

Similarly, taking derivative about ξ for k times on both sides of (35) gives that

$$X\in C_1^{k+1}(U_{t_1})$$

and (24) holds. □

Remark 1. Monotonicity of $X(\cdot,t)$ plays an important role in our proof. Without monotonicity, the vector field for the Lagrange dynamics may be not Lipschitz. From (35), we know $\mathrm{supp}\{X_\xi(\cdot,t)-1\}\subset(-L,L)$. Hence, we can continuously extend $X_\xi(\cdot,t)$ globally as

$$X_\xi(\xi,t)=1 \quad \text{for} \quad \xi\in\mathbb{R}\setminus[-L,L].$$

2.2. **Classical solutions to the mCH equation.** Next, we prove the short time existence and uniqueness of the classical solutions to (1)-(2).

The following lemma shows that we can construct classical solutions to the mCH equation (1)-(2) from the solutions to the Lagrange dynamics (6). Moreover, we show that the support of $m(\cdot,t)$ will not change.

Lemma 2.4. *Let* $m_0 \in C_c^k(-L,L)$ *for some integer* $k \geq 1$*. Assume that* $X \in C_1^{k+1}(U_\delta)$ *(for some* $\delta > 0$*) is the solution of* (6) *and strictly monotonic about* ξ *for any fixed time* $t \in [0,\delta]$*.* $u(x,t)$*,* $m(x,t)$ *are defined by* (8)*. And assume* $u \in C_1^{k+2}(\mathbb{R}\times[0,\delta])$*. Then,* $(u(x,t), m(x,t))$ *is a classical solution of* (1)-(2)*.*

Moreover, we have

$$\operatorname{supp}\{m(\cdot,t)\} \subset (-L,L), \quad t \in [0,\delta]. \tag{38}$$

Proof. We denote $(\phi,\psi) := \int_{\mathbb{R}} \phi(x)\psi(x)dx$. For any test function $\phi \in C_c^\infty(\mathbb{R})$, we have

$$(\phi, m) = \int_{\mathbb{R}} \phi(x) \int_{-L}^{L} m_0(\theta)\delta(x - X(\theta,t))d\theta dx = \int_{-L}^{L} m_0(\theta)\phi(X(\theta,t))d\theta.$$

$$\begin{aligned}(\phi, m_t) &= \frac{d}{dt}(\phi,m) = \int_{-L}^{L} m_0(\theta)\phi'(X(\theta,t))\dot{X}(\theta,t)d\theta \\ &= \int_{-L}^{L} m_0(\theta)\phi'(X(\theta,t))U(X(\theta,t),t)d\theta = \int_{\mathbb{R}} \phi'(x)U(x,t)m(x,t)dx \\ &= -\int_{\mathbb{R}} \phi(x)(U(x,t)m(x,t))_x dx.\end{aligned}$$

Since that ϕ is arbitrary, we have

$$m_t + (Um)_x = m_t + [(u^2 - u_x^2)m]_x = 0.$$

Next, we prove (38). Because $X(\xi,t)$ is monotonic and $G'(x) = -G(x)$ for $x > 0$, we obtain

$$u_x(X(L,t),t) = \int_{-L}^{L} G'(X(L,t) - X(\theta,t))m_0(\theta)d\theta = -u(X(L,t),t).$$

Hence, we have

$$\dot{X}(L,t) = u^2(X(L,t),t) - u_x^2(X(L,t),t) = 0 \ \text{ for } \ t \in [0,\delta],$$

which implies

$$X(L,t) \equiv X(L,0) = L.$$

Similarly, we have $X(-L,t) \equiv X(-L,0) = -L$.

For any $\phi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\{\phi\} \subset \mathbb{R} \setminus (-L,L)$ gives

$$(\phi, m) = \int_{-L}^{L} m_0(\theta)\phi(X(\theta,t))d\theta = 0.$$

Hence, (38) holds. ☐

Remark 2. Consider the following general equation with $\alpha > 0$,

$$m_t + [m(u^2 - \alpha^2 u_x^2)]_x = 0. \tag{39}$$

When $\text{supp}\{m_0\} \subset (-L, L)$, the support of the classical solution $m(x,t)$ to (39) is also contained in $(-L, L)$. Indeed, by scaling $\tilde{u}(x,t) = u(\alpha x, \alpha t)$ and $\tilde{m}(x,t) = m(\alpha x, \alpha t) = \tilde{u}(x,t) - \tilde{u}_{xx}(x,t)$, $\tilde{u}$ and $\tilde{m}$ satisfy

$$\tilde{m}_t + [(\tilde{u}^2 - \tilde{u}_x^2)\tilde{m}]_x = 0.$$

Due to $\text{supp}\{\tilde{m}_0\} \subset (-\alpha L, \alpha L)$, by (38) we know $\text{supp}\{\tilde{m}(\cdot,t)\} \subset (-\alpha L, \alpha L)$. Hence, we have $\text{supp}\{m(\cdot,t)\} \subset (-L, L)$.

Next, we present a useful lemma which is similar to Lemma 2.1.

Lemma 2.5. *Assume $g \in C(U_{t_1})$ and $g(\cdot,t) \in C_c(-L,L)$ for any fixed time $t \in [0,t_1]$. Let $X \in C_1^1(U_{t_1})$ satisfy (23) for some constants $C_2 > C_1 > 0$. Set*

$$A(x,t) := \int_{-L}^{L} \delta(x - X(\theta,t)) g(\theta,t) d\theta.$$

Then, we have $A \in C(\mathbb{R} \times [0,t_1])$ and

$$\int_{-L}^{L} G''(x - X(\theta,t)) g(\theta,t) d\theta \in C(\mathbb{R} \times [0,t_1]).$$

Proof. From the proof of Lemma 2.4, we know $[X(-L,t), X(L,t)] = [-L, L]$. However, in order to make no confusion, we still use $[X(-L,t), X(L,t)]$ in this proof.

By using the inverse function theorem, for any $t \in [0,t_1]$, there is a continuously differentiable function $Z(\cdot,t) \in C^1[X(-L,t), X(L,t)]$ such that

$$Z(X(\theta,t),t) = \theta \text{ for } \theta \in [-L,L]$$

and

$$X(Z(x,t),t) = x \text{ for } x \in [X(-L,t), X(L,t)].$$

Moreover, we have

$$\frac{1}{C_2} \le Z_x(x,t) \le \frac{1}{C_1}.$$

Changing variable and using the property of Dirac measure, we have

$$\begin{aligned} A(x,t) &= \int_{-L}^{L} \delta(x - X(\theta,t)) g(\theta,t) d\theta = \int_{X(-L,t)}^{X(L,t)} \delta(x-y) g(Z(y,t),t) Z_x(y,t) dy \\ &= \begin{cases} 0, & \text{for } x > X(L,t) \text{ or } x < X(-L,t); \\ g(Z(x,t),t) Z_x(x,t), & \text{for } x \in [X(-L,t), X(L,t)]. \end{cases} \end{aligned} \tag{40}$$

Next, we separate the proof into three parts, which is similar to the proof of Lemma 2.1.

Step 1. Continuity at $(x,t) \in \mathbb{R} \times [0,t_1]$ when $x > X(L,t)$. Then case for $x < X(-L,t)$ is similar.

In this case, we have $A(x,t) = 0$. For any (y,s) closed to (x,t) and because $X \in C(U_{t_1})$, we can assume $y \ge X(L,s)$. Because $g(\cdot,s) \in C_c(-L,L)$, we have $A(y,s) = 0$. Hence, A is continuous at (x,t).

Step 2. Continuity at $(x,t) \in \mathbb{R} \times [0,t_1]$ when $x = X(\xi,t)$ for some $\xi \in (-L,L)$. This means $x \in (X(-L,t), X(L,t))$.

Due to the continuity of X, for (y,s) closed enough to (x,t), we can assume $y \in [X(-L,s), X(L,s)]$. In other words, there exists $\eta \in [-L,L]$ such that $X(\eta,s) = y$. Because

$$|A(y,s) - A(x,t)| = |g(Z(x,t),t) Z_x(x,t) - g(Z(y,s),s) Z_x(y,s)|,$$

we only have to prove Z and Z_x are continuous at (x,t). (17) shows that

$$|Z(x,t)-Z(y,s)|=|\xi-\eta|\leq\frac{1}{C_1}(|x-y|+|X(\xi,t)-X(\xi,s)|),\tag{41}$$

which means Z is continuous at (x,t).

Because $Z_x(x,t)=\frac{1}{X_\xi(\xi,t)}$ and $Z_x(y,s)=\frac{1}{X_\xi(\eta,s)}$, we have

$$|Z_x(x,t)-Z_x(y,s)|=\left|\frac{1}{X_\xi(\xi,t)}-\frac{1}{X_\xi(\eta,s)}\right|\leq\frac{1}{C_1^2}|X_\xi(\xi,t)-X_\xi(\eta,s)|.$$

From (41) we can see $(\eta,s)\to(\xi,t)$ as $(y,s)\to(x,t)$. Together with $X\in C_1^1(U_{t_1})$ implies the continuity of $Z_x(x,t)$ at (x,t).

Hence, $A(x,t)$ is continuous at (x,t).

Step 3. $x=X(L,t)$. The case $x=X(-L,t)$ is similar.

For (y,s) closed to (x,t), we have two cases. When $y>X(L,s)$, we can use Step 1. When there exists $\xi\in(-L,L)$ such that $y=X(\xi,s)$, we can use Step 2.

Put Step 1,2,3 together and we can see $A\in C(\mathbb{R}\times[0,t_1])$.

At last, because $G(x)$ is fundamental solution for Helmholtz operator $1-\partial_{xx}$, we have

$$\begin{aligned}&\int_{-L}^{L}G''(x-X(\theta,t))g(\theta,t)d\theta\\&=\int_{-L}^{L}G(x-X(\theta,t))g(\theta,t)d\theta-\int_{-L}^{L}\delta(x-X(\theta,t))g(\theta,t)d\theta.\end{aligned}$$

Hence, $\int_{-L}^{L}G''(x-X(\theta,t))g(\theta,t)d\theta\in C(\mathbb{R}\times[0,t_1])$. □

Now we prove that $u(x,t),m(x,t)$ defined by (8) is a unique classical solution of (1)-(2).

Theorem 2.6. *Assuming* $m_0\in C_c^k(-L,L)$ $(k\in\mathbb{N},k\geq1)$. *Then, for*

$$t_1<\frac{1}{2||m_0||_{L^1}^2+||m_0||_{L^1}||m_0||_{L^\infty}},$$

u given by (8) belongs to $C_1^{k+2}(\mathbb{R}\times[0,t_1])$ and m belongs to $C_1^k(\mathbb{R}\times[0,t_1])$. $(u(x,t)$, $m(x,t))$ is a unique classical solution to (1)-(2).

Proof. Let $M_1:=||m_0||_{L^1}$ and $M_\infty:=||m_0||_{L^\infty}$. For $t_1<\frac{1}{2M_1^2+M_1M_\infty}$, by Theorem 2.3, we know there exist a solution $X\in C_1^{k+1}(U_{t_1})$ to (6) satisfying (23) for C_1,C_2 given by (21) and (22).

Part I. Regularity.

Step 1. When $k=1$, we have $X\in C_1^2(U_{t_1})$ and we prove $u\in C_1^3(\mathbb{R}\times[0,t_1])$.

Taking derivative about t for $u(x,t)$ in (8) gives that

$$\partial_t u(x,t)=-\int_{-L}^{L}U(X(\theta,t),t)G'(x-X(\theta,t))m_0(\theta)d\theta.$$

Because $m_0(\theta)U(X(\theta,t),t)\in C(U_{t_1})$ and $m_0(\cdot)U(X(\cdot,t),t)\in C_c(-L,L)$ for any fix time $t\in[0,t_1]$, Lemma 2.5 shows that $\partial_t u\in C(\mathbb{R}\times[0,t_1])$.

For the spatial variable x, integration by parts leads to

$$u_x(x,t)=\int_{-L}^{L}G'(x-X(\theta,t))m_0(\theta)d\theta=\int_{-L}^{L}G(x-X(\theta,t))\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)d\theta,$$

$$u_{xx}(x,t)=\int_{-L}^{L}G'(x-X(\theta,t))\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)d\theta,$$

and

$$u_{xxx}(x,t)=\int_{-L}^{L}G''(x-X(\theta,t))\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)d\theta. \tag{42}$$

Set $g(\theta,t):=\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)$. Then, $g(\theta,t)$ satisfies the assumption of Lemma 2.5. Hence

$$u_{xxx}\in C(\mathbb{R}\times[0,t_1]) \text{ and } u\in C_1^3(\mathbb{R}\times[0,t_1]).$$

Step 2. When $k=2$, we have $X\in C_1^3(U_{t_1})$. Integration by parts changes (42) into

$$u_{xxx}(x,t)=\int_{-L}^{L}G'(x-X(\theta,t))\partial_\theta\left(\frac{1}{X_\theta}\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)\right)d\theta.$$

Hence

$$\partial_x^4u(x,t)=\int_{-L}^{L}G''(x-X(\theta,t))\partial_\theta\left(\frac{1}{X_\theta}\partial_\theta\left(\frac{m_0(\theta)}{X_\theta(\theta,t)}\right)\right)d\theta.$$

And Lemma 2.5 shows that $u\in C_1^4(\mathbb{R}\times[0,t_1])$.

Step 3. If $k>2$, we can keep using integration by parts and Lemma 2.5 and obtain

$$u\in C_1^{k+2}(\mathbb{R}\times[0,t_1]).$$

Step 4. Because $m=u-u_{xx}$, from the above steps, we already know $\partial_x^k m\in C(\mathbb{R}\times[0,t_1])$. In this step, we show $\partial_t m\in C(\mathbb{R}\times[0,t_1])$. Due to (38), we only have to show $\partial_t m\in C([-L,L]\times[0,t_1])$. From (40), for $x\in(-L,L)$ and $X(\xi,t)=x$, we have

$$m(X(\xi,t),t)=m_0(Z(x,t))Z_x(x,t)=\frac{m_0(\xi)}{X_\xi(\xi,t)}. \tag{43}$$

Taking derivative of both sides of (43), we have

$$\begin{aligned}\frac{d}{dt}m(X(\xi,t),t)&=m_x(X(\xi,t),t)X_t(\xi,t)+\partial_t m(X(\xi,t),t)\\&=[m_x(u^2-u_x^2)](x,t)+m_t(x,t),\end{aligned} \tag{44}$$

and

$$\frac{d}{dt}\frac{m_0(\xi)}{X_\xi(\xi,t)}=-\frac{2m_0(\xi)mu_x(X(\xi,t),t)}{X_\xi(\xi,t)}=-2m^2u_x(x,t). \tag{45}$$

Combining (43), (44) and (45), we obtain

$$m_t=-[m(u^2-u_x^2)]_x\in C([-L,L]\times[0,t_1]).$$

From the above proof (or Lemma 2.4), we can see that $u(x,t),m(x,t)$ is a classical solution to (1)-(2).

Part II. Uniqueness of the classical solution to (1)-(2).

Assume there is another classical solution $m_1\in C_1^k(\mathbb{R}\times[0,t_1])$ to (1)-(2). $u_1=G*m_1\in C_1^{k+2}(\mathbb{R}\times[0,t_1])$. We prove that $u_1(x,t)$ can also be defined by the solution $X(\xi,t)$ to (6), which means

$$u_1(x,t)=\int_{-L}^{L}G(x-X(\theta,t))m_0(\theta)d\theta=u(x,t). \tag{46}$$

To this end, define another characteristics $Y(\xi,t)$ by

$$\dot{Y}(\xi,t)=(u_1^2-\partial_x u_1^2)(Y(\xi,t),t),$$

subject to

$$Y(\xi,0)=\xi\in\mathbb{R}.$$

By standard ODE theory, we can obtain a solution $Y\in C_1^{k+1}(\mathbb{R}\times[0,t_1])$.

Step 1. We prove

$$u_1(x,t)=\int_{-L}^{L}G(x-Y(\theta,t))m_0(\theta)d\theta.$$

Taking derivative with respect to ξ shows that

$$\dot{Y}_\xi(\xi,t)=2(m_1\partial_x u_1)(Y(\xi,t),t)Y_\xi(\xi,t). \tag{47}$$

Taking time derivative of $m_1(Y(\xi,t),t)Y_\xi(\xi,t)$ gives that

$$\begin{aligned}\frac{d}{dt}[m_1(Y(\xi,t),t)Y_\xi(\xi,t)]&=[\partial_t m_1(Y,t)+\partial_x m_1(Y,t)Y_t]Y_\xi+m_1(Y,t)Y_{\xi t}\\&=[\partial_t m_1+(u_1^2-\partial_x u_1^2)\partial_x m_1]Y_\xi+2\partial_x u_1 m_1^2 Y_\xi\\&=[\partial_t m_1+[(u_1^2-\partial_x u_1^2)m_1]_x]Y_\xi=0.\end{aligned}$$

This implies

$$m_1(Y(\theta,t),t)Y_\xi(\theta,t)=m_0(\theta),\quad\text{for}\quad\theta\in[-L,L]. \tag{48}$$

Hence, we can see

$$\begin{aligned}u_1(x,t)=\int_{\mathbb{R}}G(x-y)m_1(y,t)dy&=\int_{\mathbb{R}}G(x-Y(\theta,t))m_1(Y(\theta,t),t)Y_\xi(\theta,t)d\theta\\&=\int_{-L}^{L}G(x-Y(\theta,t))m_0(\theta)d\theta.\end{aligned} \tag{49}$$

Step 2. We prove $Y(\xi,t)=X(\xi,t)$.

From (49), we obtain

$$\begin{aligned}&\dot{Y}(\xi,t)=(u_1^2-\partial_x u_1^2)(Y(\xi,t),t)\\&=\left(\int_{-L}^{L}G(Y(\xi,t)-Y(\theta,t))m_0(\theta)d\theta\right)^2-\left(\int_{-L}^{L}G'(Y(\xi,t)-Y(\theta,t))m_0(\theta)d\theta\right)^2,\end{aligned}$$

which means that $Y(\xi,t)$ is also a solution to (6).

From Theorem 2.3 we know that the strictly monotonic solution to (6) is unique. Therefore, to prove $Y(\xi,t)=X(\xi,t)$, we only have to prove $Y(\cdot,t)$ is strictly monotonic for $t\in[0,t_1]$.

Combining (47) and (48) gives that

$$Y_\xi(\xi,t)=\exp\left(2\int_0^t(m_1\partial_x u_1)(Y(\xi,s),s)ds\right),\quad(\xi,t)\in[-L,L]\times[0,t_1].$$

Because $||Y||_{L^\infty([-L,L]\times[0,t_1])}<+\infty$, $u_1\in C_1^{k+2}(\mathbb{R}\times[0,t_1])$ and $m_1\in C_1^k(\mathbb{R}\times[0,t_1])$, the minimum and maximum of $(m_1\partial_x u_1)(Y(\xi,s),s)$ can be obtained on $[-L,L]\times[0,t_1]$. Hence

$$e^{2K_1t_1}\le Y_\xi(\xi,t)\le e^{2K_2t_1},\quad\text{for}\quad t\in[0,t_1],$$

where

$$K_1 = \min_{(\xi,s)\in[-L,L]\times[0,t_1]} (m_1\partial_x u_1)(Y(\xi,s),s)$$

and

$$K_2 = \max_{(\xi,s)\in[-L,L]\times[0,t_1]} (m_1\partial_1 u_1)(Y(\xi,s),s).$$

Hence, $Y(\cdot,t)$ is strictly monotonic for $t\in[0,t_1]$.

Combining Step 1 and Step 2, we obtain (46). ☐

Remark 3. (48) also can be easily obtained by [30, Theorem 5.34]

The strictly monotonic property of X plays a crucial role in the proof of the above Theorem. Whenever X is strictly monotonic, we can use integration by parts to obtain the regularity of $u(x,t)$. Conversely, if $m(x,t)$ is a classical solution, then the characteristics for the mCH equation is strictly monotonic.

For the convenience of the rest proof, we summarize the results in the proof of Part II of Theorem 2.6 and give a corollary.

Corollary 1. *Let $m_0\in C_c^k(-L,L)$ $(k\in\mathbb{N},k\geq 1)$ and $X\in C_1^{k+1}([-L,L]\times[0,T])$ be the solution to (6). $u\in C_1^{k+2}(\mathbb{R}\times[0,T])$, $m\in C_1^k(\mathbb{R}\times[0,T])$ is a classical solution to (1)-(2). Then, we have*

$$X_\xi(\xi,t) = \exp\left(2\int_0^t (mu_x)(X(\xi,s),s)ds\right) \quad \textit{for} \quad (\xi,t)\in[-L,L]\times[0,T] \tag{50}$$

and

$$e^{2K_1T} \leq X_\xi(\xi,t) \leq e^{2K_2T} \quad \textit{for} \quad (\xi,t)\in[-L,L]\times[0,T], \tag{51}$$

where

$$K_1 = \min_{(\xi,s)\in[-L,L]\times[0,T]} (mu_x)(X(\xi,s),s)$$

and

$$K_2 = \max_{(\xi,s)\in[-L,L]\times[0,T]} (mu_x)(X(\xi,s),s).$$

Moreover, we have

$$m(X(\xi,t),t)X_\xi(\xi,t) = m_0(\xi) \quad \textit{for} \quad (\xi,t)\in(-L,L)\times[0,T]. \tag{52}$$

Proof. The proof for (51) and (52) is the same as the proof for uniqueness in Theorem 2.6. ☐

Remark 4. From (52), we know that $m(X(\theta,t),t)$ does not change sign for any $t\in[0,T]$. We present a precise argument here.

Set

$$A^+ := \{\xi\in(-L,L): m_0(\xi)>0\},\ A^- := \{\xi\in(-L,L): m_0(\xi)<0\},$$

and

$$A^0 := \{\xi\in(-L,L): m_0(\xi)=0\}.$$

Hence,

$$A^+\cup A^-\cup A^0 = (-L,L).$$

For $t\in[0,T]$, denote

$$A_t^+ := \{X(\xi,t)\in\mathbb{R}: \xi\in A^+\},\ A_t^- := \{X(\xi,t)\in\mathbb{R}: \xi\in A^-\},$$

and

$$A_t^0 := \{X(\xi,t)\in\mathbb{R}: \xi\in A^0\}.$$

Then, we have $A_0^+ = A^+$, $A_0^- = A^-$ and $A_0^0 = A^0$. Due to the monotonicity of $X(\cdot,t)$, one can easily show that A_t^+ and A_t^- are open sets while A^0 is a closed set for $t \in [0,T]$. Also we have

$$A_t^+ \cup A_t^- \cup A_t^0 = (X(-L,t), X(L,t))$$

and (by (52))

$$m(x,t) \begin{cases} > 0, & \text{for } \ x \in A_t^+ \\ = 0, & \text{for } \ x \in A_t^0 \\ < 0, & \text{for } \ x \in A_t^-. \end{cases}$$

Due to

$$\dot{X}_\xi(\xi,t) = 2(mu_x)(X(\xi,t),t) \equiv 0 \ \text{ for } \ \xi \in A^0,$$

we obtain

$$X_\xi(\xi,t) \equiv X_\xi(\xi,0) = 1 \ \text{ for } \ \xi \in A^0, \ t \in [0,T].$$

This can also be obtained by (35).

2.3. **Solution extension.** In this subsection, we will show that as long as classical solutions to (1)-(2) satisfying $||m(\cdot,t)||_{L^\infty} < \infty$ we can extend the solutions X and m in time.

Proposition 1. *Assume* $m_0 \in C_c^k(-L,L)$ *and* $X \in C_1^{k+1}([-L,L] \times [0,T_0))$ *is the solution to* (6). *Let* $m \in C_1^k(\mathbb{R} \times [0,T_0))$ *be the corresponding solution to* (1)-(2). *If*

$$\sup_{t\in[0,T_0)} ||m(\cdot,t)||_{L^\infty} < +\infty,$$

then there exists $\widetilde{T}_0 > T_0$ *such that*

$$X \in C_1^{k+1}([-L,L] \times [0,\widetilde{T}_0])$$

is a solution to (6), *and*

$$u \in C_1^{k+2}(\mathbb{R} \times [0,\widetilde{T}_0]), \quad m \in C_1^k(\mathbb{R} \times [0,\widetilde{T}_0])$$

is a solution to (1)-(2).

Proof. There exists a constant $\widetilde{M_\infty}$ satisfies

$$\sup_{t\in[0,T_0)} ||m(\cdot,t)||_{L^\infty} \le \widetilde{M_\infty}.$$

From Lemma 2.4, we know $m(\cdot,t)$ has a uniform (in t) support. Hence, there exists a constant $\widetilde{M_1}$ such that

$$\sup_{t\in[0,T_0)} ||m(\cdot,t)||_{L^1} \le \widetilde{M_1}.$$

Consider time $T_1 = T_0 - \frac{1}{3(2\widetilde{M_1}^2+\widetilde{M_1}\widetilde{M_\infty})}$. Our target is to prove that the classical solution can be extend to $\widetilde{T}_0 := T_1 + \frac{1}{2(2\widetilde{M_1}^2+\widetilde{M_1}\widetilde{M_\infty})} > T_0$. We will show this in two steps.

Step 1. In this step we consider a dynamic system from time T_1.

From (38) we know $m(\cdot,T_1) \in C_c^k(-L,L)$. Set

$$\widetilde{m}_0(\widetilde{\theta}) := m(\widetilde{\theta},T_1) \ \text{ for } \ \widetilde{\theta} \in [-L,L].$$

Consider dynamics for $\widetilde{X}(\widetilde{\xi},t)$:

$$\begin{cases} \dfrac{d}{dt}\widetilde{X}(\widetilde{\xi},t)=\left(\displaystyle\int_{-L}^{L}G(\widetilde{X}(\widetilde{\xi},t))-\widetilde{X}(\widetilde{\theta},t))\widetilde{m}_0(\widetilde{\theta})d\widetilde{\theta}\right)^2 \\ \qquad\qquad\qquad\qquad -\left(\displaystyle\int_{-L}^{L}G'(\widetilde{X}(\widetilde{\xi},t))-\widetilde{X}(\widetilde{\theta},t))\widetilde{m}_0(\widetilde{\theta})d\widetilde{\theta}\right)^2, \\ \widetilde{X}(\widetilde{\xi},0)=\widetilde{\xi}\in[-L,L]. \end{cases} \tag{53}$$

Because $\widetilde{m}_0(\cdot)=m(\cdot,T_1)\in C_c^k(-L,L)$, by Theorem 2.6, we know that for any

$$0<t_1<\frac{1}{2\widetilde{M}_1^2+\widetilde{M}_1\widetilde{M}_\infty},$$

there exists a solution $\widetilde{X}(\widetilde{\xi},t)$ to (53) and a classical solution $(\widetilde{u}(x,t),\widetilde{m}(x,t))$ to (1) subject to initial condition

$$\widetilde{m}(x,0)=\widetilde{m}_0(x)=m(x,T_1).$$

Moreover,

$$\widetilde{X}\in C_1^{k+1}([-L,L]\times[0,t_1]),$$
$$\widetilde{u}\in C_1^{k+2}(\mathbb{R}\times[0,t_1]) \text{ and } \widetilde{m}\in C_1^{k}(\mathbb{R}\times[0,t_1]).$$

Choose $t_1=\frac{1}{2(2\widetilde{M}_1^2+\widetilde{M}_1\widetilde{M}_\infty)}$ and set $\widetilde{T}_0=T_1+t_1$. Thus $T_0<\widetilde{T}_0$.

Step 2. In this step we extend the solutions to $[0,\widetilde{T}_0]$.

Changing variable by $\widetilde{\xi}=X(\xi,T_1)$, initial value $\widetilde{X}\big(X(\xi,T_1),0\big)=X(\xi,T_1)$ allows us to define

$$X(\xi,T_1+t):=\widetilde{X}\big(X(\xi,T_1),t\big) \text{ for } \xi\in[-L,L],t\in[0,t_1] \tag{54}$$

and we have

$$X\in C_1^{k+1}([-L,L]\times[0,\widetilde{T}_0]).$$

Similarly, because $\widetilde{m}(x,0)=m(x,T_1)$, we can use $\widetilde{u}(x,t),\widetilde{m}(x,t)$ to define

$$u(x,T_1+t):=\widetilde{u}(x,t),\quad m(x,T_1+t):=\widetilde{m}(x,t)\quad \text{for } (x,t)\in\mathbb{R}\times[0,t_1]$$

and we have

$$u\in C_1^{k+2}(\mathbb{R}\times[0,\widetilde{T}_0]),\quad m\in C_1^{k}(\mathbb{R}\times[0,\widetilde{T}_0]).$$

Moreover, we can see $(u(x,t),m(x,t))$ we defined is a classical solution to (1)-(2) in $[0,\widetilde{T}_0]$.

Next, we show $X(\xi,t)$ satisfies (6) in $[0,\widetilde{T}_0]$.

Actually, changing variable by $\widetilde{\theta}=X(\theta,T_1)$ and combining (54) and (52) lead to

$$\begin{aligned} u(x,T_1+t)=\widetilde{u}(x,t)&=\int_{-L}^{L}G(x-\widetilde{X}(\widetilde{\theta},t))\widetilde{m}_0(\widetilde{\theta})d\widetilde{\theta} \\ &=\int_{-L}^{L}G(x-X(\theta,T_1+t))m(X(\theta,T_1),T_1+t)X_\theta(\theta,T_1+t)d\theta \\ &=\int_{-L}^{L}G(x-X(\theta,T_1+t))m_0(\theta)d\theta. \end{aligned}$$

Similarly,

$$\int_{-L}^{L}G'(x-\widetilde{X}(\widetilde{\theta},t))\widetilde{m}_0(\widetilde{\theta})d\widetilde{\theta}=u_x(x,T_1+t).$$

Therefore, (53) turns into

$$\begin{cases} \dot{X}(\xi, T_1+t) = u^2(X(\xi,T_1+t),T_1+t) - u_x^2(X(\xi,T_1+t),T_1+t), \\ X(\xi,T_1+0) = \widetilde{X}(X(\xi,T_1),0) = X(\xi,T_1), \end{cases}$$

for $\xi \in [-L,L]$ and $t \in [0,t_1]$.

Hence, $X \in C_1^{k+1}([-L,L]\times[0,\widetilde{T}_0])$ is a solution to (6). Corollary 1 ensures the strictly monotonicity of $X(\cdot,t)$ for $t\in[0,\widetilde{T}_0]$. Therefore, $X(\xi,t)$ is the unique solution which extends the solution to $\widetilde{T}_0$. □

3. Blow-up criteria.

In this section, we give some criteria on finite time blow-up of classical solutions to the mCH equation.

Let $T_{max} > 0$ be the maximal existence time of classical solution to the mCH equation. In other words, T_{max} satisfies

$$\begin{cases} ||m(\cdot,t)||_{L^\infty} < +\infty, \quad 0 \le t < T_{max}, \\ \limsup\limits_{t\to T_{max}} ||m(\cdot,t)||_{L^\infty} = +\infty. \end{cases}$$

Next lemma shows that the solution to Lagrange dynamics (6) can be extended to the blow-up time T_{max}.

Lemma 3.1. *Let $m_0 \in C_c^k(-L,L)$. Let T_{max} be the maximal existence time for the classical solution $m(x,t)$ to (1)-(2) and $X \in C_1^{k+1}([-L,L]\times[0,T_{max}))$ be the solution to (6). Then we have*

$$X \in C_1^{k+1}([-L,L]\times[0,T_{max}]). \tag{55}$$

Proof. Let t go to T_{max} in (33) and we obtain $X(\xi,T_{max})$. Using (36) and Lipschitz property of $G(x)=\frac{1}{2}e^{-|x|}$, we can obtain that

$$X \in C([-L,L]\times[0,T_{max}]).$$

Let t go to T_{max} in (35) and (37). Similarly, combining (24) gives

$$X \in C_0^2([-L,L]\times[0,T_{max}]).$$

Keep doing like this and we can see

$$X \in C_0^{k+1}([-L,L]\times[0,T_{max}]).$$

At last, let t go to T_{max} in (32) and combining (6), we have $\partial_t X \in C([-L,L]\times[0,T_{max}])$. □

We have the following blow up criteria.

Theorem 3.2. *Let $m_0 \in C_c^k(-L,L)$ $(k\in\mathbb{N}, k\ge 1)$. $X(\xi,t)$ is the solution to Lagrange dynamics (6). Assume $T_{max} < +\infty$ is the maximum existence time for the classical solution to (1)-(2). Then, the following equivalent statements hold.*

(i)

$$\limsup_{t\to T_{max}} ||m(\cdot,t)||_{L^\infty} = +\infty, \tag{56}$$

(ii)

$$\begin{cases} X_\xi(\xi,t) > 0 \ \ \text{for} \ \ (\xi,t)\in[-L,L]\times[0,T_{max}); \\ \min\limits_{\xi\in[-L,L]} X_\xi(\xi,T_{max}) = 0. \end{cases} \tag{57}$$

(iii)

$$\liminf_{t\to T_{max}} \left\{ \inf_{\xi\in[-L,L]} \int_0^t (mu_x)(X(\xi,s),s)ds \right\} = -\infty, \tag{58}$$

(iv)

$$\liminf_{t\to T_{max}} \left\{ \inf_{x\in\mathbb{R}} (mu_x)(x,t) \right\} = -\infty, \tag{59}$$

(v)

$$\limsup_{t\to T_{max}} ||m(\cdot,t)||_{W^{1,p}} = +\infty \quad \textit{for } p \geq 1, \tag{60}$$

(vi)

$$\int_0^{T_{max}} ||m(\cdot,t)||_{L^\infty} dt = +\infty. \tag{61}$$

Proof. We follow the following lines to prove this theorem,

$$(56) \Rightarrow (57) \Rightarrow (58) \Rightarrow (59) \Rightarrow (60) \Rightarrow (56)$$

and

$$(58) \Rightarrow (61) \Rightarrow (56).$$

Step 1. We prove (56) $\Rightarrow$ (57).

Assume $m(x,t)$ blows up in finite time T_{max}. We prove (57) by contradiction. From Lemma 3.1, we know $X \in C_1^2([-L,L]\times[0,T_{max}])$. If (57) does not hold, then we have

$$\min\left\{X_\xi(\xi,t) : (\xi,t)\in[-L,L]\times[0,T_{max}]\right\} > C_1 > 0.$$

Combining (52) and (38), we have

$$\begin{aligned}\sup_{t\in[0,T_{max})} ||m(\cdot,t)||_{L^\infty(\mathbb{R})} &= \sup_{t\in[0,T_{max})} ||m(X(\cdot,t),t)||_{L^\infty(-L,L)} \\ &= \sup_{t\in[0,T_{max})} \left|\left|\frac{m_0(\cdot)}{X_\theta(\cdot,t)}\right|\right|_{L^\infty(-L,L)} \leq \frac{||m_0||_{L^\infty}}{C_1}.\end{aligned}$$

This is a contradiction to (56).

Step 2. We prove (57) $\Rightarrow$ (58).

From (57), we have

$$\liminf_{t\to T_{max}} \left\{ \inf_{\xi\in[-L,L]} X_\xi(\xi,t) \right\} = 0.$$

Together with (50), we can see (57) $\Rightarrow$ (58).

Step 3. We prove (58) $\Rightarrow$ (59).

(58) implies that

$$\liminf_{t\to T_{max}} \left\{ \inf_{\xi\in[-L,L]} (mu_x)(X(\xi,t),t) \right\} = -\infty. \tag{62}$$

Because of (38), for any $t \in [0,T_{max})$ we have

$$\inf_{\xi\in[-L,L]} (mu_x)(X(\xi,t),t) = \inf_{x\in[-L,L]} mu_x(x,t) = \inf_{x\in\mathbb{R}} mu_x(x,t).$$

Hence, we can see that (62) and (59) are equivalent.

Step 4. We prove (59) $\Rightarrow$ (60).

Assume (59) holds. We prove (60) by contradiction. For any $1 \leq p \leq +\infty$, if

$$\limsup_{t \to T_{max}} ||m(\cdot, t)||_{W^{1,p}} < +\infty,$$

then

$$\sup_{t \in [0, T_{max})} ||m(\cdot, t)||_{W^{1,p}} < +\infty.$$

$W^{1,p}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ with continuous injection for all $1 \leq p \leq +\infty$ implies that

$$\sup_{t \in [0, T_{max})} ||m(\cdot, t)||_{L^\infty} < +\infty.$$

On the other hand, we have

$$\sup_{t \in [0, T_{max})} ||u_x(\cdot, t)||_{L^\infty} \leq \sup_{t \in [0, T_{max})} \left|\left| \int_{-L}^{L} G'(\cdot - X(\theta, t)) m_0(\theta) d\theta \right|\right|_{L^\infty} \leq \frac{1}{2} ||m_0||_{L^1}. \tag{63}$$

Hence we obtain $\sup_{t \in [0, T_{max})} ||m u_x(\cdot, t)||_{L^\infty} < +\infty$, which is a contradiction with (59). Therefore, (60) holds.

Step 5. We prove (60) ⇒ (56).

Assume (60) holds. If $\sup_{t \in [0, T_{max})} ||m(\cdot, t)||_{L^\infty} < +\infty$, by Proposition 1, there exists $T > T_{max}$ such that $m \in C_1^1(\mathbb{R} \times [0, T])$. Because $m(\cdot, t)$ has uniform compact support for $t \in [0, T]$, we have

$$\sup_{t \in [0, T_{max})} ||m(\cdot, t)||_{W^{1,p}} \leq \sup_{t \in [0, T]} ||m(\cdot, t)||_{W^{1,p}} < +\infty,$$

which is a contradiction.

Step 6. At last, we prove

$$(58) \Rightarrow (61) \Rightarrow (56).$$

When (61) holds, one can easily obtain (56). So, we only have to prove (58) ⇒ (61). (58) implies

$$\limsup_{t \to T_{max}} \left\{ \sup_{x \in \mathbb{R}} \int_0^t |m u_x(x, s)| ds \right\} = +\infty.$$

Due to (63), we obtain

$$\sup_{x \in \mathbb{R}} \int_0^t |m u_x(x, s)| ds \leq C \int_0^t ||m(\cdot, s)||_{L^\infty} ds \leq C \int_0^{T_{max}} ||m(\cdot, t)||_{L^\infty} dt$$

and this gives (61). □

Remark 5. (57) shows that there is a ξ_0 such that $X_\xi(\xi_0, T_{max}) = 0$. This means T_{max} is an onset time of collision of characteristics. Now, we can conclude that if $m(x, t)$ blows up in finite time T_{max}, then we have

$$X \in C_1^{k+1}([-L, L] \times [0, T_{max}]) \quad \text{and} \quad m \in C_1^k(\mathbb{R} \times [0, T_{max})).$$

The blow-up criterion (59) can also be found in [18]. Besides, (61) is similar to the well known blow-up criterion for smooth solutions to $3D$ Euler equation [1].

Remark 6 (Other equivalent criteria)**.** Because $m(x, t)$ has compact support for $t \in [0, T_{max})$, by Poincaré inequality, (60) is equivalent to (for any $1 \leq p \leq +\infty$)

$$\limsup_{t \to T_{max}} ||m_x(\cdot, t)||_{L^p} = +\infty. \tag{64}$$

Because $m = u - u_{xx}$ and $|u(x,t)| = \left|\int_{-L}^{L} G(x - X(\theta,t))m_0(\theta)d\theta\right| \leq \frac{1}{2}||m_0||_{L^1}$, we know that (56) is equivalent to

$$\limsup_{t\to T_{max}} ||u_{xx}(\cdot,t)||_{L^\infty} = +\infty.$$

(63) tells us u_x is bounded. Hence the blow up behavior is different with the Camassa-Holm equation, where u_x becomes unbounded [10, 11].

When $m_0(x) \geq 0$, equality (52) implies $m(x,t) \geq 0$ for any $t \in [0, T_{max})$. Then, all the above blow-up criteria are equivalent to

$$\limsup_{t\to T_{max}} \left\{ \sup_{x\in\mathbb{R}} m(x,t) \right\} = +\infty.$$

4. Finite time blow up and almost global existence of classical solutions.

In the rest of this paper, we assume $m_0 \in C_c^1(-L, L)$.

In this section, we show that for some initial data solutions to the mCH equation blow up in finite time. Some blow-up rates are obtained. Moreover, for any $\epsilon > 0$ and initial data $\epsilon m_0(x) \in C_c^1(\mathbb{R})$, we prove that the lifespan of the classical solutions satisfies

$$T_{max}(\epsilon m_0) \sim \frac{C}{\epsilon^2},$$

where C is a constant depends on $m_0(x)$.

Our finite time blow-up results are similar to the blow-up results in [8, 18, 24] but with some subtle differences. All these three papers apply the idea from transport equation and focus on the derivative of $u^2 - u_x^2$ which is $2mu_x$. Comparing with [18, Theorem 5.2,5.3], we show finite time blow-up for m_0 which can change its sign. Besides, our starting point do not have to be the maximum point of m_0 in contrast with [24, Theorem 1.3]. The main idea of our proof is similar to [8, Theorem 1.5] which shows blow-up for a sign-changing m_0 with the effect of the linear dispersion term γu_x ($\gamma \geq 0$).

We have the following proposition.

Proposition 2. *Suppose $m_0 \in C_c^1(-L, L)$. Let T_{max} be the maximal time of the existence of the corresponding classical solution $m(x,t)$ to (1)-(2). $X \in C_1^2([-L, L] \times [0, T_{max}))$ is the solution to (6).*
(i) If $\xi_0 \in [-L, L]$ satisfies $m_0(\xi_0) \neq 0$, then we have

$$X_\xi(\xi_0, t) = 1 + 2m_0(\xi_0) \int_0^t u_x(X(\xi_0, s), s)ds \quad \text{for} \quad t \in [0, T_{max}). \tag{65}$$

(ii) We have the following lower bound for blow-up time

$$T_{max} \geq \frac{1}{||m_0||_{L^\infty}||m_0||_{L^1}}. \tag{66}$$

Proof. (i) The mCH equation (1) can be rewritten as

$$m_t + (u^2 - u_x^2)m_x = -2m^2 u_x. \tag{67}$$

Therefore, we have

$$\frac{d}{dt} m(X(\xi,t),t) = -2(m^2 u_x)(X(\xi,t),t).$$

By (52), when $m_0(\xi_0) \neq 0$ we know $m(X(\xi,t),t) \neq 0$ and it will keep sign (positive or negative) for $t \in [0, T_{max})$. Hence

$$\frac{1}{m^2(X(\xi_0,t),t)}\frac{d}{dt}m(X(\xi_0,t),t) = -2u_x(X(\xi_0,t),t). \tag{68}$$

This implies

$$\frac{d}{dt}\left(\frac{1}{m(X(\xi_0,t),t)}\right) = 2u_x(X(\xi_0,t),t).$$

Integrating from 0 to t leads to

$$\frac{1}{m(X(\xi_0,t),t)} = \int_0^t 2u_x(X(\xi_0,s),s)ds + \frac{1}{m_0(\xi_0)}, \tag{69}$$

and combining (52) gives (65).

(ii) If $T_{max} < \frac{1}{||m_0||_{L^\infty}||m_0||_{L^1}}$, then (65) and (7) give that

$$X_\xi(\xi_0, T_{max}) \geq 1 - ||m_0||_{L^\infty}||m_0||_{L^1}T_{max} > 0,$$

which is a contradiction with the assumption of blow-up at T_{max}. □

In view of equation (67), the most natural way to study blow-up behavior is following the characteristics. This method was used for the Burgers equation and the CH equation. Equality (69) reminds us the proof for finite time blow-up of Burgers equation:

$$u_t + uu_x = 0, \quad \text{for} \quad x \in \mathbb{R}, \ t > 0. \tag{70}$$

Consider its characteries $\dot{X}(x,t) = u(X(x,t),t)$ and we have

$$\frac{d}{dt}u(X(x,t),t) = 0.$$

Taking derivative of (70) gives

$$u_{xt} + u_x^2 + uu_{xx} = 0.$$

Then we have

$$\frac{d}{dt}u_x(X(x,t),t) = (uu_{xx})(X(x,t),t) + u_{xt}(X(x,t),t) = -u_x^2(X(x,t),t),$$

which implies

$$\frac{1}{u_x(X(x,t),t)} = t + \frac{1}{u_{0x}(x)}. \tag{71}$$

Hence, if there exists $x_0 \in \mathbb{R}$ such that $u_{0x}(x_0) < 0$, then u_x goes to $-\infty$ in finite time.

(69) is similar to (71). But we can not have direct estimate on the blow-up time like the Burgers equation. Hence we need to give some estimate about u_x. We have the following lemma.

Lemma 4.1. *Suppose $m_0 \in C_c^1(-L,L)$ and $M_1 := ||m_0||_{L^1}$. Let T_{max} be the maximal time of existence of the corresponding classical solution $m(x,t)$ to (1)-(2). $X \in C_1^2([-L,L] \times [0,T_{max}))$ is the solution to (6). Then we have*

$$\left|\frac{d}{dt}u_x(X(\xi,t),t)\right| \leq \frac{M_1^3}{2}. \tag{72}$$

Proof. From (1), we obtain

$$u_t+(u^2-u_x^2)u_x=-(1-\partial_{xx})^{-1}[2u_xm^2+6u_xu_{xx}m+2u_x^2m_x]. \tag{73}$$

Taking derivative to (73) with respect to x and after some calculation we obtain

$$u_{xt}+(u^2-u_x^2)u_{xx}=-uu_x^2-G*(uu_x^2)+\frac{2}{3}u^3-\frac{2}{3}G*(u^3)-G'*\left(\frac{1}{3}u_x^3\right)$$

Combining Young's inequality and (7) gives

$$\begin{aligned}&|u_{xt}+(u^2-u_x^2)u_{xx}|\\ &\leq ||uu_x^2||_{L^\infty}+\left|\left|uu_x^2-\frac{2}{3}u^3\right|\right|_{L^\infty}||G||_{L^1}+\frac{2}{3}||u||_{L^\infty}^3+\left|\left|\frac{1}{3}u_x^3\right|\right|_{L^\infty}||G'||_{L^1}\\ &\leq\frac{1}{2}M_1^3,\end{aligned}$$

which implies (72). □

Next, we state and prove our main results in this section.

Theorem 4.2. *Suppose $m_0\in C_c^1(-L,L)$ and $M_1:=||m_0||_{L^1}$. Let T_{max} be the maximal time of existence of the classical solution $m(x,t)$ to (1)-(2). $X\in C_1^2([-L,L]\times[0,T_{max}))$ is the solution to (6). If there is a $\xi_0\in[-L,L]$ such that $m_0(\xi_0)>0$ and*

$$-\partial_xu_0(\xi_0)>\sqrt{\frac{M_1^3}{2m_0(\xi_0)}}, \tag{74}$$

then $m(x,t)$ defined by (8) blows up at a time

$$T_{max}\leq t^*:=\frac{2}{M_1^3}\left(-\partial_xu_0(\xi_0)-\sqrt{\Big[\partial_xu_0(\xi_0)\Big]^2-\frac{M_1^3}{2m_0(\xi_0)}}\right). \tag{75}$$

Moreover, when $T_{max}=t^*$, we have the following estimate of the blow-up rate for m:

$$||m(\cdot,t)||_{L^\infty}\geq\frac{1}{C(T_{max}-t)}\ \textit{for}\ t\in[0,T_{max}), \tag{76}$$

and for X_ξ we have

$$\inf_{\xi\in(-L,L)}X_\xi(\xi,t)\leq Cm_0(\xi_0)(T_{max}-t)\ \textit{for}\ t\in[0,T_{max}), \tag{77}$$

Where

$$C=-\partial_xu_0(\xi_0)+\sqrt{\Big[\partial_xu_0(\xi_0)\Big]^2-\frac{M_1^3}{2m_0(\xi_0)}}.$$

Proof. **Step 1.**

Assume $m_0(\xi_0)>0$. Combining (68) and (72) shows that

$$\frac{d}{dt}\left(\frac{1}{m^2(X(\xi_0,t),t)}\frac{d}{dt}m(X(\xi_0,t),t)\right)=-2\frac{d}{dt}u_x(X(\xi_0,t),t)\geq-M_1^3. \tag{78}$$

Integrating (78) shows that

$$\frac{1}{m^2(X(\xi_0,t),t)}\frac{d}{dt}m(X(\xi_0,t),t)\geq-M_1^3t-2\partial_xu_0(\xi_0) \tag{79}$$

where we used

$$\frac{1}{m^2(X(\xi_0,t),t)}\frac{d}{dt}m(X(\xi_0,t),t)\Big|_{t=0}=-2\partial_x u_0(\xi_0).$$

Integrating (79) gives

$$\frac{1}{m(X(\xi_0,t),t)}\leq\frac{1}{2}M_1^3t^2+2\partial_x u_0(\xi_0)t+\frac{1}{m_0(\xi_0)}.$$

If ξ_0 satisfies (74), then we have

$$\frac{1}{2}M_1^3t^2+2\partial_x u_0(\xi_0)t+\frac{1}{m_0(\xi_0)}=\frac{1}{2}M_1^3(t-t^*)(t-t_*),$$

where

$$t^*=\frac{2}{M_1^3}\left(-\partial_x u_0(\xi_0)-\sqrt{\Big[\partial_x u_0(\xi_0)\Big]^2-\frac{M_1^3}{2m_0(\xi_0)}}\right)$$

and

$$t_*=\frac{2}{M_1^3}\left(-\partial_x u_0(\xi_0)+\sqrt{\Big[\partial_x u_0(\xi_0)\Big]^2-\frac{M_1^3}{2m_0(\xi_0)}}\right).$$

Hence

$$0<\frac{1}{m(X(\xi_0,t),t)}\leq\frac{1}{2}M_1^3(t-t^*)(t-t_*). \tag{80}$$

This implies that there is a time $0<T_{max}\leq t^*$ such that

$$m(X(\xi_0,t),t)\to+\infty,\ \text{ as } t\to T_{max}$$

which means $m(x,t)$ blows up at the time T_{max}.

Step 2.

Assume $T_{max}=t^*$. From (80), we have

$$\begin{aligned}||m(\cdot,t)||_{L^\infty}\geq m(X(\xi_0,t),t)&\geq\frac{2}{M_1^3(t-t^*)(t-t_*)}\\&\geq\frac{2}{M_1^3t_*(T_{max}-t)}=\frac{1}{C(T_{max}-t)}.\end{aligned}$$

Hence, we have (76).

From (52) and (80), we have

$$\begin{aligned}\inf_{\xi\in(-L,L)}X_\xi(\xi,t)\leq X_\xi(\xi_0,t)=\frac{m_0(\xi_0)}{m(X(\xi_0,t),t)}&\leq\frac{1}{2}m_0(\xi_0)M_1^3(t-t^*)(t-t_*)\\&\leq\frac{1}{2}m_0(\xi_0)M_1^3t_*(T_{max}-t)\leq Cm_0(\xi_0)(T_{max}-t).\end{aligned}$$

Hence, (77) follows and this ends the proof. □

Similarly, we have the following theorem.

Theorem 4.3. *Suppose $m_0\in C_c^1(-L,L)$ and $M_1:=||m_0||_{L^1}$. Let T_{max} be the maximal time of existence of the classical solution $m(x,t)$ to (1)-(2). $X\in C_1^2([-L,L]\times[0,T_{max}))$ is the solution to (6). If there is a $\xi_1\in[-L,L]$ such that $m_0(\xi_1)<0$ and*

$$\partial_x u_0(\xi_1)>\sqrt{\frac{M_1^3}{-2m_0(\xi_1)}}, \tag{81}$$

then $m(x,t)$ *defined by* (8) *blows up at a time*

$$T_{max} \le t^* := \frac{2}{M_1^3}\left(\partial_x u_0(\xi_1) - \sqrt{\left[\partial_x u_0(\xi_1)\right]^2 + \frac{M_1^3}{2m_0(\xi_1)}}\right).$$

Moreover, when $T_{max} = t^*$*, we have the following estimate of the blow-up rate for* $m(x,t)$*:*

$$||m(\cdot,t)||_{L^\infty} \ge \frac{1}{C(T_{max}-t)} \text{ for } t \in [0, T_{max}),$$

and for X_ξ *we have*

$$\inf_{\xi\in(-L,L)} X_\xi(\xi,t) \le Cm_0(\xi_1)(t-T_{max}) \text{ for } t \in [0, T_{max}),$$

Where

$$C = \partial_x u_0(\xi_1) + \sqrt{\left[\partial_x u_0(\xi_1)\right]^2 + \frac{M_1^3}{2m_0(\xi_1)}}.$$

From conditions (74) and (81), if there exists $\bar{\xi} \in [-L, L]$ such that (11) holds, then the classical solution will blow up in finite time.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. (i) (10) follows from (66).

(ii) Let m_0 satisfies the assumptions in Theorem 4.2. Then, for any $\epsilon > 0$ we know ϵm_0 also satisfies the assumptions. Hence, from (75) we have

$$T_{max}(\epsilon m_0) \le \frac{2||\epsilon u_x||_{L^\infty}}{||\epsilon m_0||_{L^1}^3} \le \frac{1}{||m_0||_{L^1}^2}\cdot\frac{1}{\epsilon^2},$$

where (7) was used. Together with (10) we can obtain (12). □

5. Solutions at blow-up time and formation of peakons.

In this section, we study the behavior of classical solutions at blow-up time T_{max}.

First, we show that u and u_x are uniformly BV function for $t \in [0, T_{max}]$ (including the blow-up time T_{max}) and $m(\cdot,t)$ has a unique limit in Radon measure space as t approaching T_{max}.

Let us recall the concept of the space $BV(\mathbb{R})$.

Definition 5.1. (i) For dimension $d \ge 1$ and an open set $\Omega \subset \mathbb{R}^d$, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if

$$Tot.Var.\{f\} := \sup\left\{\int_\Omega f(x)\nabla\cdot\phi(x)dx : \phi \in C_c^1(\Omega;\mathbb{R}^d), ||\phi||_{L^\infty} \le 1\right\} < \infty.$$

(ii) (Equivalent definition for one dimension case) A function f belongs to $BV(\mathbb{R})$ if for any $\{x_i\} \subset \mathbb{R}$, $x_i < x_{i+1}$, the following statement holds:

$$Tot.Var.\{f\} := \sup_{\{x_i\}}\left\{\sum_i |f(x_i) - f(x_{i-1})|\right\} < \infty.$$

Remark 7. Let $\Omega \subset \mathbb{R}^d$ for $d \ge 1$ and $f \in BV(\Omega)$. $Df := (D_{x_1}f, \ldots, D_{x_d}f)$ is the distributional gradient of f. Then, Df is a vector Radon measure and the total variation of f is equal to the total variation of $|Df|$: $Tot.Var.\{f\} = |Df|(\Omega)$. Here, $|Df|$ is the total variation measure of the vector measure Df ([23, Definition (13.2)]).

If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Definition 5.1 (ii), then f satisfies Definition (i). On the contrary, if f satisfies Definition 5.1 (i), then there exists a right continuous representative which satisfies Definition (ii). See [23, Theorem 7.2] for the proof.

We have the following theorem about u and u_x at T_{max}.

Theorem 5.2. *Let* $m_0 \in C_c^1(-L, L)$ *and* $M_1 := ||m_0||_{L^1}$. *Let* T_{max} *be the maximal existence time for the classical solution* $m(x, t)$ *to* (1)-(2) *and* $X \in C_1^2([-L, L] \times [0, T_{max}])$ *be the solution to* (6). *Then, the following assertions hold:*
(i) There exists a function $u(x, T_{max})$ *such that*

$$\lim_{t\to T_{max}} u(x,t) = u(x, T_{max}), \quad \lim_{t\to T_{max}} u_x(x,t) = u_x(x, T_{max}) \text{ for every } x \in \mathbb{R}. \tag{82}$$

(ii) For any $t \in [0, T_{max}]$ *we have*

$$u(\cdot, t),\ u_x(\cdot, t) \in BV(\mathbb{R})$$

and

$$\text{Tot.Var.}\{u(\cdot, t)\} \le M_1, \quad \text{Tot.Var.}\{u_x(\cdot, t)\} \le 2M_1. \tag{83}$$

Proof. We use three steps to prove (i) and (ii) together.

Step 1. We prove $u \in C(\mathbb{R} \times [0, T_{max}])$.

Due to (55) and $u(x,t) = \int_{-L}^{L} G(x - X(\theta, t))m_0(\theta)d\theta$ for $t \in [0, T_{max})$, let t go to T_{max} and we obtain

$$u(x, T_{max}) = \int_{-L}^{L} G(x - X(\theta, T_{max}))m_0(\theta)d\theta.$$

Moreover, we have $u \in C(\mathbb{R} \times [0, T_{max}])$.

Step 2. For $0 \le t < T_{max}$, we prove (83).

For $G = \frac{1}{2}e^{-|x|}$, we know $G,\ G_x \in BV(\mathbb{R})$ and the following holds

$$\text{Tot.Var.}\{G\} = 1, \quad \text{Tot.Var.}\{G_x\} = 2.$$

When $t \in [0, T_{max})$, for any $\{x_i\} \subset \mathbb{R}$, $x_i < x_{i+1}$, we have

$$\sum_i |u(x_i, t) - u(x_{i-1}, t)| \le \int_{-L}^{L} \sum_i |G(x_i - X(\theta, t)) - G(x_{i-1} - X(\theta, t))||m_0(\theta)|d\theta$$
$$\le \text{Tot.Var.}\{G\}||m_0||_{L^1} = M_1,$$

which means $\text{Tot.Var.}\{u(\cdot, t)\} \le M_1$. Similarly, we can obtain $\text{Tot.Var.}\{u_x(\cdot, t)\} \le 2M_1$ for $t \in [0, T_{max})$.

Step 3. We prove (82) and show that $u(x, T_{max})$ satisfies (83).

The first part of (82) is deduced by $u \in C(\mathbb{R} \times [0, T_{max}])$. To prove the second part, we have to do a little more job.

Combining (7), step 2, and [3, Theorem 2.3], we know that there exists a consequence $\{t_k\}(\to T_{max})$ and two BV functions $\widetilde{u}(x), \widetilde{v}(x)$ such that

$$\lim_{k\to\infty} u(x, t_k) = \widetilde{u}(x), \quad \lim_{k\to\infty} u_x(x, t_k) = \widetilde{v}(x) \text{ for every } x \in \mathbb{R},$$

and

$$\text{Tot.Var.}\{\widetilde{u}\} \le M_1, \quad |\widetilde{u}| \le \frac{1}{2}M_1 \text{ and Tot.Var.}\{\widetilde{v}\} \le 2M_1, \quad |\widetilde{v}| \le \frac{1}{2}M_1.$$

Because

$$\lim_{t\to T_{max}} u(x,t) = u(x,T_{max}) \text{ for every } x\in\mathbb{R},$$

we know $\widetilde{u}(x) = u(x,T_{max})$.

For any test function $\phi\in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}-\int_{\mathbb{R}} u(x,T_{max})\phi_x(x)dx &= -\int_{\mathbb{R}}\widetilde{u}(x)\phi_x(x)dx = -\lim_{k\to\infty}\int_{\mathbb{R}} u(x,t_k)\phi_x(x)dx\\ &= \lim_{k\to\infty}\int_{\mathbb{R}} u_x(x,t_k)\phi(x)dx = \int_{\mathbb{R}}\widetilde{v}(x)\phi(x)dx,\end{aligned}$$

which means $\widetilde{v}(x)$ is the derivative of $u(x,T_{max})$ in distribution sense. Define $u_x(x,T_{max}) = \widetilde{v}(x)$ for every $x\in\mathbb{R}$ and we obtain

$$\lim_{k\to\infty} u_x(x,t_k) = \widetilde{u}_x(x) = u_x(x,T_{max}) \text{ for every } x\in\mathbb{R}.$$

Because $u_x(x,t)$ is continuous in $[0,T_{max})$, we know

$$\lim_{t\to T_{max}} u_x(x,t) = \widetilde{u}_x(x) = u_x(x,T_{max}) \text{ for every } x\in\mathbb{R}.$$

This is the end of the proof. □

Next we give a theorem to prove that $m(\cdot,t)$ has a unique limit in Radon measure space $\mathcal{M}(\mathbb{R})$ as t approaching T_{max}. Before this, let's recall the definition A_t^+ and A_t^- in Remark 4 and denote

$$A_{T_{max}}^+ := \{X(\xi,T_{max})\in\mathbb{R} : \xi\in A^+\},\quad A_{T_{max}}^- := \{X(\xi,T_{max})\in\mathbb{R} : \xi\in A^-\}.$$

Because $X(\xi,T_{max})$ may not be strictly monotonic, it is not obvious to see that $A_{T_{max}}^+$ and $A_{T_{max}}^-$ are open sets. We give a lemma to show this.

Lemma 5.3. *$A_{T_{max}}^+$ and $A_{T_{max}}^-$ are open sets.*

Proof. We only deals with $A_{T_{max}}^+$ and the proof for $A_{T_{max}}^-$ is similar.

For $x_0\in A_{T_{max}}^+$, there exist $\xi\in(-L,L)$ such that $m_0(\xi)>0$ and $x_0 = X(\xi,T_{max})$. Set

$$\xi_1 := \min\{\xi\in[-L,L] : m_0(\xi)\geq 0 \ \text{ and } \ X(\xi,T_{max}) = x_0\}$$

and

$$\xi_2 := \max\{\xi\in[-L,L] : m_0(\xi)\geq 0 \ \text{ and } \ X(\xi,T_{max}) = x_0\}.$$

By continuity of m_0 and $X(\xi,T_{max})$, ξ_1 and ξ_2 can be obtained.

1. If $\xi_1=\xi_2$, then there is only one point $\xi_0=\xi_1$ such that $m_0(\xi_0)>0$ and $x_0 = X(\xi_0,T_{max})$. In this case, set

$$\eta_1 := \max\{\xi : m_0(\xi) = 0 \ \text{ and } \ \xi<\xi_0\}$$

and

$$\eta_2 := \min\{\xi : m_0(\xi) = 0 \ \text{ and } \ \xi>\xi_0\}.$$

Because $m_0(\xi_0)>0$, we know $\eta_1<\xi_0<\eta_2$ and $m_0(\xi)>0$ for $\xi\in(\eta_1,\eta_2)$. Hence

$$X(\xi,T_{max})\in A_{T_{max}}^+,\quad \text{for } \ \xi\in(\eta_1,\eta_2).$$

Because $X(\xi,T_{max})$ is nondecreasing, we obtain

$$x_0 = X(\xi_0,T_{max})\in(X(\eta_1,T_{max}),X(\eta_2,T_{max}))\subset A_{T_{max}}^+.$$

2. If $\xi_1<\xi_2$, we have

$$X(\xi,T_{max})\equiv x_0,\quad \text{for } \ \xi\in[\xi_1,\xi_2]. \tag{84}$$

By definition we know $m_0(\xi_i) \geq 0$ for $i = 1, 2$. When $m_0(\xi_i) = 0$ for $i = 1$ or $i = 2$, from Remark 4 we know $X_\xi(\xi_i, T_{max}) = 1$. This implies that $X(\xi, T_{max})$ is strictly monotonic in a neighborhood of ξ_i which is a contradiction with (84). Hence, we have

$$m_0(\xi_i) > 0, \quad \text{for} \quad i = 1, 2.$$

Hence, there exist $\xi_3 < \xi_1$ and $\xi_4 > \xi_2$ such that

$$m_0(\xi) > 0 \;\; \text{for} \;\; \xi \in (\xi_3, \xi_4).$$

Therefore, we have

$$X(\xi_3, T_{max}) < X(\xi_1, T_{max}) = x_0 = X(\xi_2, T_{max}) < X(\xi_4, T_{max}),$$

and

$$X(\xi, T_{max}) \in A^+_{T_{max}} \quad \text{for} \;\; \xi \in (\xi_3, \xi_4),$$

which imply

$$x_0 = X(\xi_1, T_{max}) \in (X(\xi_3, T_{max}), X(\xi_4, T_{max})) \subset A^+_{T_{max}}.$$

□

For any Radon measure μ and measurable set A, we use $\mu|_A$ to stand for the restriction of μ on the set A. We have the following Theorem.

Theorem 5.4. *Let the assumptions in Theorem 5.2 holds. Then there exists a unique Radon measure $m(\cdot, T_{max})$ such that*

$$m(\cdot, t) \overset{*}{\rightharpoonup} m(\cdot, T_{max}) \quad in \;\; \mathcal{M}(\mathbb{R}), \quad as \;\; t \to T_{max}. \tag{85}$$

Moreover, $m(\cdot, T_{max})$ has the following properties:
(i) Compact support:

$$\text{supp}\{m(\cdot, T_{max})\} \subset (-L, L). \tag{86}$$

(ii) Denote

$$m^+_{T_{max}} := m(\cdot, T_{max})\Big|_{A^+_{T_{max}}} \quad and \quad m^-_{T_{max}} := m(\cdot, T_{max})\Big|_{A^-_{T_{max}}}.$$

Then $m^+_{T_{max}}$ is a positive Radon measure and $m^-_{T_{max}}$ is a negative Radon measure. Besides, we have

$$m(\cdot, T_{max}) = m^+_{T_{max}} + m^-_{T_{max}}. \tag{87}$$

(iii) The following equality holds:

$$\int_{\mathbb{R}} |m|(dx, T_{max}) = \int_{\mathbb{R}} |m(x, t)| dx = \int_{-L}^{L} |m_0(x)| dx, \;\; t \in [0, T_{max}). \tag{88}$$

Proof. **Step 1.** Proof of (85).

Because $u_x(\cdot, T_{max})$ is a BV function, its derivative $u_{xx}(\cdot, T_{max})$ is a Radon measure. We know

$$m(\cdot, T_{max}) = u(\cdot, T_{max}) - u_{xx}(\cdot, T_{max})$$

is a Radon measure and for any test function $\phi \in C_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \phi(x) m(dx, T_{max}) = \int_{\mathbb{R}} u(x, T_{max})\phi(x) + u_x(x, T_{max})\phi_x(x) dx. \tag{89}$$

Then, we have

$$\begin{aligned}\lim_{t\to T_{max}}\int_{\mathbb{R}} m(x,t)\phi(x)dx &= \lim_{t\to T_{max}}\int_{\mathbb{R}} u(x,t)\phi(x)+u_x(x,t)\phi_x(x)dx\\ &= \int_{\mathbb{R}} u(x,T_{max})\phi(x)+u_x(x,T_{max})\phi_x(x)dx\\ &= \int_{\mathbb{R}} \phi(x)m(dx,T_{max}).\end{aligned}$$

This proves (85).

Step 2. Proof of (i).

For any test function $\phi\in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)m(dx,T_{max}) &= \lim_{t\to T_{max}}\int_{\mathbb{R}} m(x,t)\phi(x)dx\\ &= \lim_{t\to T_{max}}\int_{\mathbb{R}} m(X(\xi,t),t)\phi(X(\xi,t))X_\xi(\xi,t)d\xi\\ &= \lim_{t\to T_{max}}\int_{-L}^{L} m_0(\xi)\phi(X(\xi,t))d\xi\\ &= \int_{-L}^{L} m_0(\xi)\phi(X(\xi,T_{max}))d\xi, \end{aligned}\tag{90}$$

where (52) was used. Because $X(L,t)=L$ and $X(-L,t)=-L$ for $t\in[0,T_{max}]$, we have $X(\cdot,T_{max})\in(-L,L)$. Let test function ϕ satisfy $\text{supp}\{\phi\}\subset\mathbb{R}\setminus(-L,L)$. Then we obtain

$$\int_{\mathbb{R}}\phi(x)m(dx,T_{max})=\int_{-L}^{L}m_0(\xi)\phi(X(\xi,T_{max}))d\xi=0,$$

which implies (86).

Step 3. Proof of (ii).

Due to (90), we know

$$m(\cdot,T_{max})=X(\cdot,T_{max})\#m_0.$$

For $\phi\in C_c^\infty(\mathbb{R})$ and $\phi\geq 0$, by the definition of A^+ we have

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)dm^+_{T_{max}} &= \int_{A^+_{T_{max}}}\phi(x)m(dx,T_{max})\\ &= \int_{A^+} m_0(\xi)\phi(X(\xi,T_{max}))d\xi\geq 0.\end{aligned}$$

Hence, $m^+_{T_{max}}$ is a positive Radon measure. With the same argument, we can see that $m^-_{T_{max}}$ is a negative Radon measure.

On the other hand, by using (90), we have

$$\begin{aligned}&\int_{\mathbb{R}}\phi(x)d(m^+_{T_{max}}+m^-_{T_{max}}) = \int_{A^+_{T_{max}}\cup A^-_{T_{max}}}\phi(x)m(dx,T_{max})\\ &= \int_{A^+\cup A^-} m_0(\xi)\phi(X(\xi,T_{max}))d\xi = \int_{A^+\cup A^-\cup A^0} m_0(\xi)\phi(X(\xi,T_{max}))d\xi\\ &= \int_{-L}^{L} m_0(\xi)\phi(X(\xi,T_{max}))d\xi = \int_{\mathbb{R}}\phi(x)m(dx,T_{max}),\end{aligned}$$

which implies (87).

Step 4. Proof of (iii).

From (52), we have

$$|m(X(\xi,t),t)|X_\xi(\xi,t)=|m_0(\xi)|,$$

which implies

$$\int_{\mathbb{R}}|m(x,t)|dx=\int_{\mathbb{R}}|m(X(\xi,t),t)|X_\xi(\xi,t)d\xi=\int_{-L}^{L}|m_0(\xi)|d\xi \quad \text{for} \quad t\in[0,T_{max}).$$

For any test function $\phi\in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)|m|(dx,T_{max})&=\int_{A^+_{T_{max}}}\phi(x)m(dx,T_{max})-\int_{A^-_{T_{max}}}\phi(x)m(dx,T_{max})\\&=\int_{A^+}m_0(\xi)\phi(X(\xi,T_{max}))d\xi-\int_{A^-}m_0(\xi)\phi(X(\xi,T_{max}))d\xi.\end{aligned}$$

Choose $\phi\in C_c^\infty(\mathbb{R})$ satisfying

$$\phi(x)\equiv 1,\quad x\in(X(-L,T_{max}),X(L,T_{max})).$$

Hence, we have

$$\int_{\mathbb{R}}|m|(dx,T_{max})=\int_{A^+}m_0(\xi)d\xi-\int_{A^-}m_0(\xi)d\xi=\int_{-L}^{L}|m_0(\xi)|d\xi.$$

This ends the proof. ☐

Remark 8. In Section 6, we will prove the global existence of weak solutions to the mCH equation when initial datum m_0 belongs to $\mathcal{M}(\mathbb{R})$. Hence, we can extend m globally in time after blow up time. Similar results can be found in [17], where a sticky particle method was used.

Next, we introduce another two sets to study solutions at T_{max}. Assume $m_0\in C_c^1(\mathbb{R})$ and $X\in C_1^2([-L,L]\times[0,T_{max}])$ is the solution to the Lagrange dynamics (6). Set

$$F:=\{\xi\in(-L,L):X_\xi(\xi,T_{max})=0\}$$

and

$$O:=\{\xi\in(-L,L):X_\xi(\xi,T_{max})>0\}.$$

Then, F is a closed set and O is an open set. Moreover, we have

$$F\cup O=(-L,L).$$

Because the classical solution blows up in finite time T_{max}, we know F is not empty. On the other hand, due to $m_0(\pm L)=0$, Remark 4 tells that $X_\xi(\pm L,T_{max})=1$ which implies O is not empty.

Set

$$O_{T_{max}}:=\{X(\xi,T_{max}):\xi\in O\}\ \text{ and }\ F_{T_{max}}:=\{X(\xi,T_{max}):\xi\in F\}. \tag{91}$$

Then, we have

$$O_{T_{max}}\cup F_{T_{max}}=(X(-L,T_{max}),X(L,T_{max})).$$

$X(\cdot,T_{max})$ is strictly monotonic in O. Hence, $O_{T_{max}}$ is also an open set and $F_{T_{max}}$ is a closed set. Moreover, we claim that

$$\overline{O_{T_{max}}}=[X(-L,T_{max}),X(L,T_{max})]. \tag{92}$$

To show (92), we only have to prove $F_{T_{max}} \subset \overline{O_{T_{max}}}$. For any $x \in F_{T_{max}}$, there exists ξ_0 such that $x = X(\xi_0, T_{max})$ and $X_\xi(\xi_0, T_{max}) = 0$. Let $\xi_1 = \max\{\xi : X(\xi, T_{max}) = x\}$. $\forall \epsilon > 0$, there is $\xi_\epsilon \in (\xi_1, \xi_1 + \epsilon)$ such that $X_\xi(\xi_\epsilon, T_{max}) > 0$ and $X(\xi_\epsilon, T_{max}) \in \overline{O_{T_{max}}}$. Hence, $\lim_{\epsilon\to 0} X(\xi_\epsilon, T_{max}) = X(\xi_1, T_{max})$.

We have the following theorem.

Theorem 5.5. *Let assumptions in Theorem 5.2 hold. Then we have*

$$u(\cdot, T_{max}) \in C^3(\mathbb{R} \setminus F_{T_{max}})$$

and

$$m(\cdot, T_{max}) \in C^1(O_{T_{max}}) \cap L^1(O_{T_{max}}).$$

Moreover, the following holds

$$m(X(\xi, T_{max}), T_{max}) X_\xi(\xi, T_{max}) = m_0(\xi) \quad \text{for} \quad \xi \in O.$$

Proof. **Step 1.** We first consider the cases when $x \notin (X(-L, T_{max}), X(L, T_{max}))$.

Because $m_0(L) = 0$, from Remark 4 we know $X_\xi(L, T_{max}) = 1$, which means $X(\xi, T_{max})$ is strictly monotonic in a small neighborhood of L. Hence,

$$X(L, T_{max}) > X(\xi, T_{max}) \quad \text{for} \quad \xi \in [-L, L).$$

From this we know, if $x \geq X(L, T_{max})$, we have $x - X(\xi, T_{max}) > 0$ for $\xi \in (-L, L)$. From Theorem 5.2, we know

$$u(x, T_{max}) = \int_{-L}^{L} G(x - X(\theta, T_{max})) m_0(\theta) d\theta.$$

Thus

$$\begin{aligned} u_x(x, T_{max}) &= \int_{-L}^{L} G'(x - X(\theta, T_{max})) m_0(\theta) d\theta \\ &= -\int_{-L}^{L} G(x - X(\theta, T_{max})) m_0(\theta) d\theta = -u(x, T_{max}). \end{aligned}$$

This shows

$$u(x, T_{max}) = u(X(L, T_{max}), T_{max}) e^{-x + X(L, T_{max})}.$$

Hence, $u(\cdot, T_{max}) \in C^\infty[X(L, T_{max}), +\infty)$.

Similarly, we can show $u(\cdot, T_{max}) \in C^\infty(-\infty, X(-L, T_{max})]$.

Step 2. We only left the case for $x \in O_{T_{max}}$.

When $x \in O_{T_{max}}$, there exists a $\eta \in O$ such that $X(\eta, T_{max}) = x$. Because $X_\xi(\eta, T_{max}) > 0$, we know η is the unique point satisfying $X(\eta, T_{max}) = x$. Rewrite $u(x, T_{max})$ as

$$\begin{aligned} u(x, T_{max}) = &\int_{\eta}^{L} G(X(\eta, T_{max}) - X(\theta, T_{max})) m_0(\theta) d\theta \\ &+ \int_{-L}^{\eta} G(X(\eta, T_{max}) - X(\theta, T_{max})) m_0(\theta) d\theta. \end{aligned}$$

Using $X_\eta(\eta, T_{max}) > 0$, we can obtain

$$\begin{aligned} u_x(x, T_{max}) &= \frac{1}{X_\eta(\eta, T_{max})} u_\eta(X(\eta, T_{max}), T_{max}) \\ &= \frac{1}{X_\eta(\eta, T_{max})} \Bigg(\int_{\eta}^{L} G'(X(\eta, T_{max}) - X(\theta, T_{max})) X_\eta(\eta, T_{max}) m_0(\theta) d\theta \end{aligned}$$

$$-\frac{1}{2}m_0(\eta)+\frac{1}{2}m_0(\eta)+\int_{-L}^{\eta}G'(X(\eta,T_{max})-X(\theta,T_{max}))X_\eta(\eta,T_{max})m_0(\theta)d\theta\Big).$$

Hence,

$$\begin{aligned} u_x(x,T_{max}) = &\int_{\eta}^{L}G(X(\eta,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta \\ &-\int_{-L}^{\eta}G(X(\eta,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta. \end{aligned} \tag{93}$$

Taking derivative again shows that

$$\begin{aligned} u_{xx}(x,T_{max}) &= -\frac{m_0(\eta)}{2X_\eta(\eta,T_{max})}+\int_{\eta}^{L}G(X(\eta,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta \\ &\quad -\frac{m_0(\eta)}{2X_\eta(\eta,T_{max})}+\int_{-L}^{\eta}G(X(\eta,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta \\ &= -\frac{m_0(\eta)}{X_\eta(\eta,T_{max})}+u(x,T_{max}). \end{aligned} \tag{94}$$

Because $m_0\in C_c^1(\mathbb{R})$ and $X_\xi(\cdot,T_{max})\in C^1(-L,L)$, which implies

$$u(\cdot,T_{max})\in C^3(O_{T_{max}}).$$

Together with Step 1 and Step 2, we obtain

$$u(\cdot,T_{max})\in C^3(\mathbb{R}\setminus F_{T_{max}}).$$

Step 3. Because $\mathbb{R}\setminus F_{T_{max}}$ is an open set, for any $\phi\in C_c^\infty(\mathbb{R}\setminus F_{T_{max}})$ we have

$$\begin{aligned} \int_{\mathbb{R}}\phi(x)m(dx,T_{max}) &= \int_{\mathbb{R}}u(x,T_{max})\phi(x)+u_x(x,T_{max})\phi_x(x)dx \\ &= \int_{\mathbb{R}\setminus F_{T_{max}}}u(x,T_{max})\phi(x)+u_x(x,T_{max})\phi_x(x)dx \\ &= \int_{\mathbb{R}\setminus F_{T_{max}}}(u(x,T_{max})-u_{xx}(x,T_{max}))\phi(x)dx, \end{aligned}$$

where (89) was used. Because ϕ is arbitrary and $u(\cdot,T_{max})\in C^3(\mathbb{R}\setminus F_{T_{max}})$, we obtain

$$m(\cdot,T_{max})=u(\cdot,T_{max})-u_{xx}(\cdot,T_{max})\in C^1(\mathbb{R}\setminus F_{T_{max}}). \tag{95}$$

From Theorem 5.4, we know $m(\cdot,T_{max})$ has compact support in $(-L,L)$. Hence,

$$m(\cdot,T_{max})\in C^1(O_{T_{max}}).$$

Because the Radon measure $m(\cdot,T_{max})$ has finite total variation, we obtain

$$m(\cdot,T_{max})\in L^1(O_{T_{max}}).$$

From (94), we know

$$m(x,T_{max})=\frac{m_0(\eta)}{X_\eta(\eta,T_{max})}$$

where $x\in O_{T_{max}}$ and $X(\eta,T_{max})=x$. This means (52) holds in the set O:

$$m(X(\xi,T_{max}),T_{max})X_\xi(\xi,T_{max})=m_0(\xi)\quad\text{for}\quad\xi\in O.$$

This finishes our proof. □

Because $u(\cdot,T_{max})$ and $u_x(\cdot,T_{max})$ are BV functions, their discontinuous points are countable. We give a proposition to show discontinuous points of $u_x(\cdot,T_{max})$. First, let us introduce two subsets of $F_{T_{max}}$.

$$\widetilde{F}_{T_{max}}=\{x\in F_{T_{max}}:X^{-1}(x,T_{max})=\{\xi\}\ \text{ for some }\ \xi\in[-L,L]\},$$

and

$$\widehat{F}_{T_{max}}=\{x\in F_{T_{max}}:X^{-1}(x,T_{max})=[\xi_1,\xi_2]\ \text{ for some }\ \xi_1<\xi_2\}. \tag{96}$$

Proposition 3. *Let the assumptions in Theorem 5.2 hold. Then, $u_x(\cdot,T_{max})\in C(\mathbb{R}\setminus\widehat{F}_{T_{max}})$ and $u_x(\cdot,T_{max})$ is not continuous at $y\in\widehat{F}_{T_{max}}$.*

Proof. **Step 1.** Assume $y\in\widetilde{F}_{T_{max}}$ and we prove $u_x(\cdot,T_{max})$ is continuous at y.

By definition of $F_{T_{max}}$, we know there is only one point $\xi_0\in F$, such that $X(\xi_0,T_{max})=y$. Due to (92), there exist two sequence $\{\overline{y}_n\}$ and $\{\widehat{y}_n\}$ such that the following hold:

$$\{\overline{y}_n\}\subset O_{T_{max}},\quad \lim_{n\to+\infty}\overline{y}_n=y,\quad \overline{y}_n \text{ is increasing}$$

and

$$\{\widehat{y}_n\}\subset O_{T_{max}},\quad \lim_{n\to+\infty}\widehat{y}_n=y,\quad \widehat{y}_n \text{ is decreasing.}$$

Because $\overline{y}_n\in O_{T_{max}}$, there is a unique $\overline{\xi}_n\in O$ such that $X(\overline{\xi}_n,T_{max})=\overline{y}_n$. Similarly, we have a unique $\widehat{\xi}_n\in O$ such that $X(\widehat{\xi}_n,T_{max})=\widehat{y}_n$. (Uniqueness is because $X(\xi,T_{max})$ is strictly monotonic in O.) Moreover, we have

$$\overline{\xi}_n<\xi_0<\widehat{\xi}_n,$$

and

$$\lim_{n\to+\infty}\overline{\xi}_n=\xi_0=\lim_{n\to+\infty}\widehat{\xi}_n.$$

Because formula (93) holds for $x\in O_{T_{max}}$, we know

$$\begin{aligned}u_x(\overline{y}_n,T_{max})=&\int_{\overline{\xi}_n}^{L}G(X(\overline{\xi}_n,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta\\&-\int_{-L}^{\overline{\xi}_n}G(X(\overline{\xi}_n,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta.\end{aligned}$$

Let n goes to infinity and we obtain

$$\begin{aligned}u_x(y-,T_{max})=&\int_{\xi_0}^{L}G(X(\xi_0,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta\\&-\int_{-L}^{\xi_0}G(X(\xi_0,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta.\end{aligned}$$

Similarly, we have

$$\begin{aligned}u_x(\widehat{y}_n,T_{max})=&\int_{\widehat{\xi}_n}^{L}G(X(\widehat{\xi}_n,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta\\&-\int_{-L}^{\widehat{\xi}_n}G(X(\widehat{\xi}_n,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta.\end{aligned}$$

and

$$u_x(y+,T_{max})=\int_{\xi_0}^{L}G(X(\xi_0,T_{max})-X(\theta,T_{max}))m_0(\theta)d\theta$$

$$- \int_{-L}^{\xi_0} G(X(\xi_0, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta.$$

This implies $u_x(y-, T_{max}) = u_x(y+, T_{max})$. For any $y \in \widetilde{F}_{T_{max}}$, define

$$\begin{aligned} u_x(y, T_{max}) = &\int_{\xi_0}^{L} G(X(\xi_0, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta \\ &- \int_{-L}^{\xi_0} G(X(\xi_0, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta. \end{aligned}$$

Then using similar argument for any sequence $\mathbb{R} \setminus \widehat{F}_{T_{max}} \ni y_n \to y$, we know

$$u_x(\cdot, T_{max}) \in C(\mathbb{R} \setminus \widehat{F}_{T_{max}}).$$

Step 2. Assume $y \in \widehat{F}_{T_{max}}$ and we prove $u_x(\cdot, T_{max})$ is discontinuous at y.

Set

$$\xi_1 = \min\{\xi \in F : X(\xi, T_{max}) = y\} \text{ and } \xi_2 = \max\{\xi \in F : X(\xi, T_{max}) = y\}.$$

By definition of $\widehat{F}_{T_{max}}$ we know $\xi < \xi_2$. Moreover, we know

$$X(\xi, T_{max}) = y, \quad X_\xi(\xi, T_{max}) = 0 \text{ for } \xi \in [\xi_1, \xi_2].$$

Claim. m_0 will not change sign in $[\xi_1, \xi_2]$.

If this is not true, then we have $\eta \in [\xi_1, \xi_2]$ such that $m_0(\eta) = 0$. Remark 4 tells us that $X_\xi(\eta, T_{max}) = 1$ and we obtain a contradiction.

Similar to Step 1, we have four sequences $\overline{\overline{y}}_n$, $\overline{\overline{\xi}}_n$, $\widehat{\widehat{y}}_n$ and $\widehat{\widehat{\xi}}_n$ which satisfy

$$\lim_{n\to+\infty} \overline{\overline{y}}_n = y = \lim_{n\to+\infty} \widehat{\widehat{y}}_n,$$

$$\overline{\overline{y}}_n \in O_{T_{max}} \text{ increasing}, \quad \widehat{\widehat{y}}_n \in O_{T_{max}} \text{ decreasing},$$

and

$$\lim_{n\to+\infty} \overline{\overline{\xi}}_n = \xi_1, \quad \lim_{n\to+\infty} \widehat{\widehat{\xi}}_n = \xi_2.$$

From (93), we know

$$\begin{aligned} u_x(\overline{\overline{y}}_n, T_{max}) = &\int_{\overline{\overline{\xi}}_n}^{L} G(X(\overline{\overline{\xi}}_n, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta \\ &- \int_{-L}^{\overline{\overline{\xi}}_n} G(X(\overline{\overline{\xi}}_n, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta. \end{aligned}$$

Let n go to $+\infty$ and we obtain

$$\begin{aligned} u_x(y-, T_{max}) = &\int_{\xi_1}^{L} G(X(\xi_1, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta \\ &- \int_{-L}^{\xi_1} G(X(\xi_1, T_{max}) - X(\theta, T_{max}))m_0(\theta)d\theta \\ = &\int_{\xi_1}^{L} G(y - X(\theta, T_{max}))m_0(\theta)d\theta - \int_{-L}^{\xi_1} G(y - X(\theta, T_{max}))m_0(\theta)d\theta. \end{aligned}$$

Similarly, we also have

$$u_x(y+, T_{max}) = \int_{\xi_2}^{L} G(y - X(\theta, T_{max}))m_0(\theta)d\theta - \int_{-L}^{\xi_2} G(y - X(\theta, T_{max}))m_0(\theta)d\theta.$$

Hence, using the above claim, we have

$$\begin{aligned} u_x(y-,T_{max})-u_x(y+,T_{max}) &= 2\int_{\xi_1}^{\xi_2} G(y-X(\theta,T_{max}))m_0(\theta)d\theta \\ &= \int_{\xi_1}^{\xi_2} m_0(\theta)d\theta \neq 0 \end{aligned} \tag{97}$$

which shows that $u_x(\cdot,T_{max})$ is not continuous at y. □

Next, we prove Theorem 1.3. Let's give some notations first.

Assume $F_{T_{max}} = \{x_i\}_{i=1}^{N_1}$ and $x_1 < x_2 \ldots < x_{N_1}$. Let $\widehat{F}_{T_{max}} = \{x_i\}_{i=1}^{N}$ $(N \le N_1)$. From the proof (97), we know that for each $1 \le i \le N$ there exist $\xi_{i1} < \xi_{i2}$ such that

$$u_x(x_i-,T_{max})-u_x(x_i+,T_{max}) = p_i$$

where

$$p_i = \int_{\xi_{i1}}^{\xi_{i2}} m_0(\theta)d\theta. \tag{98}$$

Set

$$m_1(x) = \begin{cases} m(x,T_{max}), & x \in O_{T_{max}}; \\ 0, & x \in \mathbb{R}\setminus O_{T_{max}}. \end{cases} \tag{99}$$

Proof of Theorem 1.3. For any text function $\phi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} \int_{\mathbb{R}} \phi(x)m(dx,T_{max}) &= \int_{\mathbb{R}} u(x,T_{max})\phi(x)+u_x(x,T_{max})\phi_x(x)dx \\ = \Big(\int_{-\infty}^{x_1} + \sum_{i=1}^{N_1-1}\int_{x_i}^{x_{i+1}} + \int_{x_{N_1}}^{+\infty}\Big)&\Big[u(x,T_{max})\phi(x)+u_x(x,T_{max})\phi_x(x)\Big]dx. \end{aligned}$$

Because $u_x(\cdot,T_{max}) \in C^{k+2}(\mathbb{R}\setminus F_{T_{max}})$, integration by parts leads to

$$\begin{aligned} &\int_{\mathbb{R}} \phi(x)m(dx,T_{max}) \\ &= \Big(\int_{-\infty}^{x_1} + \sum_{i=1}^{N_1-1}\int_{x_i}^{x_{i+1}} + \int_{x_{N_1}}^{+\infty}\Big)\Big[\big(u(x,T_{max})-u_{xx}(x,T_{max})\big)\phi(x)\Big]dx \\ &\quad + \sum_{i=1}^{N_1}\big(u_x(x_i-,T_{max})-u_x(x_i+,T_{max})\big)\phi(x_i) \\ &= \int_{O_{T_{max}}} m(x,T_{max})\phi(x)dx + \sum_{i=1}^{N_1}\big(u_x(x_i-,T_{max})-u_x(x_i+,T_{max})\big)\phi(x_i). \end{aligned}$$

Because $u_x(\cdot,T_{max})$ is continuous at x_i for $i \ge N+1$, combining (95) and (98) gives that

$$\begin{aligned} \int_{\mathbb{R}} \phi(x)m(dx,T_{max}) &= \int_{O_{T_{max}}} m(x,T_{max})\phi(x)dx + \sum_{i=1}^{N}\int_{\xi_{i1}}^{\xi_{i2}} m_0(\theta)d\theta\phi(x_i) \\ &= \int_{O_{T_{max}}} m(x,T_{max})\phi(x)dx + \sum_{i=1}^{N} p_i\phi(x_i) \end{aligned}$$

$$
\begin{aligned}
&= \int_{O_{T_{max}}} m(x, T_{max})\phi(x)dx + \sum_{i=1}^{N} \int_{\mathbb{R}} p_i \delta(x - x_i)\phi(x)dx \\
&= \int_{\mathbb{R}} \Big(m_1(x) + \sum_{i=1}^{N} p_i \delta(x - x_i) \Big) \phi(x)dx.
\end{aligned}
$$

□

This theorem tells us that peakons are exactly the points in the set $\widehat{F}_{T_{max}}$. Hence, a peakon is formulated when some Lagrangian labels in a interval $[\xi_1, \xi_2]$ aggregate into one point at T_{max} and the weight of the peakon is the integration of $m_0(x)$ on $[\xi_1, \xi_2]$.

6. **Solutions after blow-up.**

At the blow up time, the solution to the mCH equation m becomes a Radon measure. In this section, we assume initial datum m_0 belongs to the Radon measure space $\mathcal{M}(\mathbb{R})$ and use the Lagrange dynamics to prove that weak solution to (1)-(2) exists globally in Radon measure space.

6.1. **Regularized Lagrange dynamics and BV estimate.**

Let $m_0 \in \mathcal{M}(\mathbb{R})$ satisfies

$$
\mathrm{supp}\{m_0\} \subset (-L, L) \quad \text{and} \quad M_1 := |m_0|(\mathbb{R}) < +\infty. \tag{100}
$$

G' is not continuous and may not be integrable with respect to Radon measure m_0. (6) can not be used directly. Hence, a regularization is needed.

Let's give the definition of mollifier.

Definition 6.1. (i) Define the mollifier $0 \le \rho \in C^k(\mathbb{R})$, $k \ge 2$ satisfying

$$
\int_{\mathbb{R}} \rho(x)dx = 1, \quad \rho(x) = \rho(|x|) \ \text{ and } \ \mathrm{supp}\{\rho\} \subset \{x \in \mathbb{R} : |x| < 1\}.
$$

(ii) For each $\epsilon > 0$, set

$$
\rho_\epsilon(x) := \frac{1}{\epsilon}\rho(\frac{x}{\epsilon}).
$$

With this definition, we define

$$
G^\epsilon(x) := (\rho_\epsilon * G)(x).
$$

Hence, $G^\epsilon \in C^k(\mathbb{R})$ for $k \ge 2$. By Young's inequality we have

$$
||G^\epsilon||_{L^\infty} \le ||G||_{L^\infty} = \frac{1}{2}, \quad ||G^\epsilon_x||_{L^\infty} \le ||G_x||_{L^\infty} = \frac{1}{2} \tag{101}
$$

and

$$
||G^\epsilon||_{L^1} \le ||G||_{L^1} = 1, \quad ||G^\epsilon_x||_{L^1} \le ||G_x||_{L^\infty} = 1.
$$

Because $G_{xx}(x) = G(x)$ when $x \neq 0$, we have

$$
|G^\epsilon_{xx}(x)| = \left| \int_{\mathbb{R}} \rho_\epsilon(y) G_{xx}(x-y)dy \right| = \left| \int_{\mathbb{R}} \rho_\epsilon(y) G(x-y)dy \right| \le \frac{1}{2}, \quad \text{for} \ \ |x| > \epsilon.
$$

On the other hand, because $G^\epsilon_{xx} \in C[-\epsilon, \epsilon]$, there is a constant $\ell^\epsilon > 0$ such that

$$
G^\epsilon_{xx}(x) \le \ell^\epsilon \ \text{ for } \ x \in [-\epsilon, \epsilon].
$$

Hence, $G^\epsilon_x(x)$ is a global Lipschitz function. For any measurable function $X(\xi, t)$, we define

$$
U_\epsilon(x; X) := \left(\int_{\mathbb{R}} G^\epsilon(x - X(\theta, t))dm_0(\theta) \right)^2 - \left(\int_{\mathbb{R}} G^\epsilon_x(x - X(\theta, t))dm_0(\theta) \right)^2
$$

and

$$U^\epsilon(x;X) := [\rho_\epsilon * U_\epsilon](x;X).$$

The regularized Lagrange dynamics is given by

$$\begin{cases} \dot{X}(\xi,t) = U^\epsilon(X(\xi,t);X), \\ X(\xi,0) = \xi \in [-L,L]. \end{cases}$$

Consider this equation in the Banach space $C[-L,L]$ with sup norm. One can easily show that the vector field is globally Lipschitz. Hence, by the Picard theorem for ODEs in a Banach space, we obtain a unique global solution

$$X^\epsilon(\xi,t) \in C([-L,L]\times[0,+\infty)) \text{ for any } \epsilon > 0.$$

Define

$$u^\epsilon(x,t) := \int_{\mathbb{R}} G^\epsilon(x - X^\epsilon(\theta,t))dm_0(\theta),\ m^\epsilon(x,t) := u^\epsilon(x,t) - u^\epsilon_{xx}(x,t) \tag{102}$$

and

$$m_\epsilon(\cdot,t) := X^\epsilon(\cdot,t)\#m_0(\cdot). \tag{103}$$

By the definition, we have

$$u^\epsilon(x,t) = \int_{\mathbb{R}} G^\epsilon(x - X^\epsilon(\theta,t))dm_0(\theta) = \int_{\mathbb{R}} G^\epsilon(x-y)m_\epsilon(dy,t).$$

Hence, we have the following relation between m^ϵ and m_ϵ

$$m^\epsilon(x,t) = (1-\partial_{xx})\int_{\mathbb{R}} G^\epsilon(x-y)m_\epsilon(dy,t) = \int_{\mathbb{R}} \rho_\epsilon(x-y)m_\epsilon(dy,t). \tag{104}$$

In the following of this paper, we denote

$$U_\epsilon(x,t) := (u^\epsilon)^2(x,t) - (u^\epsilon_x)^2(x,t) \text{ and } U^\epsilon(x,t) := [\rho_\epsilon * U_\epsilon](x,t).$$

Hence, we have

$$\dot{X}^\epsilon(\xi,t) = U^\epsilon(X^\epsilon(\xi,t),t). \tag{105}$$

From Definition 5.1 we can easily obtain

$$\text{Tot.Var.}\{G\} = 1, \quad \text{Tot.Var.}\{G_x\} = 2$$

and

$$\text{Tot.Var.}\{G^\epsilon\} = 1, \quad \text{Tot.Var.}\{G^\epsilon_x\} = 2. \tag{106}$$

We have the following Lemma about u^ϵ.

Lemma 6.2. *Let $m_0 \in \mathcal{M}(\mathbb{R})$ satisfy (100). For $\epsilon > 0$, $u^\epsilon(x,t)$ is defined by (102). Then, the following statements hold:*

(i)

$$||u^\epsilon||_{L^\infty} \le \frac{1}{2}M_1 \ \textit{ and } \ ||u^\epsilon_x||_{L^\infty} \le \frac{1}{2}M_1 \ \textit{ uniformly in } \ \epsilon.$$

(ii)

$$\text{Tot.Var.}\{u^\epsilon(\cdot,t)\} \le M_1 \ \textit{ and } \ \text{Tot.Var.}\{u^\epsilon(\cdot,t)\} \le 2M_1 \ \textit{ uniformly in } \ \epsilon.$$

(iii) For any $t,\ s \in [0,\infty)$, we have

$$\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)|dx \le \frac{1}{2}M_1^3|t-s| \ \textit{ and } \ \int_{\mathbb{R}} |u^\epsilon_x(x,t) - u^\epsilon_x(x,s)|dx \le M_1^3|t-s|.$$

Moreover, for any $T>0$, there exist subsequences of u^ϵ, u^ϵ_x (also denoted as u^ϵ, u^ϵ_x) and two functions u, $u_x \in BV(\mathbb{R}\times[0,T))$ such that

$$u^\epsilon \to u,\ u^\epsilon_x \to u_x \quad in \quad L^1_{loc}(\mathbb{R}\times[0,+\infty)) \quad as \quad \epsilon\to 0$$

and u, u_x satisfy all the properties in (i), (ii) *and* (iii).

Proof. (i) From (101) and the definition of u^ϵ, we can easily obtain (i).

(ii) For any $\{x_i\}\subset\mathbb{R}$, $x_i<x_{i+1}$, (106) yields

$$\begin{aligned}&\sum_i |u^\epsilon(x_i,t)-u^\epsilon(x_{i-1},t)|\\ \le&\int_{\mathbb{R}}\sum_i |G^\epsilon(x_i-X^\epsilon(\theta,t))-G^\epsilon(x_{i-1}-X^\epsilon(\theta,t))|dm_0(\theta)\\ \le&\text{Tot.Var.}\{G^\epsilon\}M_1=M_1.\end{aligned}$$

Hence, Tot.Var.$\{u^\epsilon(\cdot,t)\}\le M_1$. Similarly, we can obtain Tot.Var.$\{u^\epsilon_x(\cdot,t)\}\le 2M_1$.

(iii)

$$\int_{\mathbb{R}}|u^\epsilon(x,t)-u^\epsilon(x,s)|dx\le\int_{\mathbb{R}}\int_{\mathbb{R}}|G^\epsilon(x-X^\epsilon(\theta,t))-G^\epsilon(x-X^\epsilon(\theta,s))|dm_0(\theta)dx.$$

By the definition of U^ϵ and (105), we know

$$|\dot{X}^\epsilon(\xi,t)|\le\frac{1}{2}M_1^2.$$

Hence,

$$|X^\epsilon(\theta,t)-X^\epsilon(\theta,s)|\le\frac{1}{2}M_1^2|t-s|.$$

[3, Lemma 2.3] gives

$$\begin{aligned}&\int_{\mathbb{R}}|G^\epsilon(x-X^\epsilon(\theta,t))-G^\epsilon(x-X^\epsilon(\theta,s))|dx\\ &\le\text{Tot.Var.}\{G^\epsilon\}|X^\epsilon(\theta,t)-X^\epsilon(\theta,s)|\le\frac{1}{2}M_1^2|t-s|.\end{aligned}$$

Hence

$$\int_{\mathbb{R}}|u^\epsilon(x,t)-u^\epsilon(x,s)|dx\le\frac{1}{2}M_1^3|t-s|.$$

Similarly, we can obtain

$$\int_{\mathbb{R}}|u^\epsilon_x(x,t)-u^\epsilon_x(x,s)|dx\le M_1^3|t-s|.$$

The rest results can be obtained by using [3, Theorem 2.4,2.6]. □

6.2. **Weak consistency and convergence theorem.**

In this subsection, we show that u^ϵ defined by (102) is weak consistent with the mCH equation (1)-(2).

We rewrite (1) as equation of u,

$$\begin{aligned}&(1-\partial_{xx})u_t+[(u^2-u_x^2)(u-u_{xx})]_x\\ =&(1-\partial_{xx})u_t+(u^3+uu_x^2)_x-\frac{1}{3}(u^3)_{xxx}+\frac{1}{3}(u_x^3)_{xx}=0.\end{aligned}$$

Now, we introduce the definition of weak solution in terms of u. To this end, for $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$, we denote the functional

$$\begin{aligned}\mathcal{L}(u,\phi) :&= \int_0^T \int_{\mathbb{R}} u(x,t)[\phi_t(x,t) - \phi_{txx}(x,t)]dxdt \\ &\quad - \frac{1}{3}\int_0^T \int_{\mathbb{R}} u_x^3(x,t)\phi_{xx}(x,t)dxdt - \frac{1}{3}\int_0^T \int_{\mathbb{R}} u^3(x,t)\phi_{xxx}(x,t)dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}} (u^3 + uu_x^2)\phi_x(x,t)dxdt. \qquad (107)\end{aligned}$$

Then, the definition of the weak solution to (1) in terms of $u(x,t)$ is given as follows.

Definition 6.3. For $m_0 \in \mathcal{M}(\mathbb{R})$, a function

$$u \in C([0,T); H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$$

is said to be a weak solution of (1)-(2) if

$$\mathcal{L}(u,\phi) = -\int_{\mathbb{R}} \phi(x,0)dm_0(x)$$

holds for all $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$. If $T = +\infty$, we call $u(x,t)$ as a global weak solution of the mCH equation.

For simplicity in notations, we denote

$$\langle f(x,t), g(x,t)\rangle := \int_0^T \int_{\mathbb{R}} f(x,t)g(x,t)dxdt.$$

For any test function $\phi \in C_c^\infty(\mathbb{R} \times [0,T))$, we have

$$\begin{aligned}&\langle m_\epsilon(x,t), \phi_t\rangle + \langle U^\epsilon m_\epsilon, \phi_x\rangle \\ &= \int_0^T \int_{\mathbb{R}} \phi_t(x,t)m_\epsilon(dx,t)dt + \int_0^T \int_{\mathbb{R}} U^\epsilon(x,t)\phi_x(x,t)m_\epsilon(dx,t)dt \\ &= \int_0^T \int_{\mathbb{R}} \Big[\phi_t(X^\epsilon(\theta,t),t) + U^\epsilon(X^\epsilon(\theta,t),t)\phi_x(X^\epsilon(\theta,t),t)\Big]dm_0(\theta)dt \\ &= \int_0^T \frac{d}{dt}\int_{\mathbb{R}} \phi(X^\epsilon(\theta,t),t)dm_0(\theta)dt = -\int_{\mathbb{R}} \phi(\theta,0)dm_0(\theta). \qquad (108)\end{aligned}$$

On the other hand, combining the definition (102) and (107) gives

$$\begin{aligned}\mathcal{L}(u^\epsilon,\phi) &= \int_0^T \int_{\mathbb{R}} u^\epsilon[\phi_t - \phi_{txx}]dxdt - \frac{1}{3}\int_0^T \int_{\mathbb{R}} (\partial_x u^\epsilon)^3\phi_{xx}dxdt \\ &\qquad - \frac{1}{3}\int_0^T \int_{\mathbb{R}} (u^\epsilon)^3\phi_{xxx}dxdt + \int_0^T \int_{\mathbb{R}} ((u^\epsilon)^3 + u^\epsilon(u_x^\epsilon)^2)\phi_x dxdt \\ &= \langle \phi_t, (1-\partial_{xx})u^\epsilon\rangle + \langle [(u^\epsilon)^2 - (\partial_x u^\epsilon)^2](1-\partial_{xx})u^\epsilon, \phi_x\rangle \\ &= \langle m^\epsilon, \phi_t\rangle + \langle U_\epsilon m^\epsilon, \phi_x\rangle.\end{aligned}$$

Combining the last two equalities, we define

$$E_\epsilon := \langle m^\epsilon - m_\epsilon, \phi_t\rangle + \langle U_\epsilon m^\epsilon - U^\epsilon m_\epsilon, \phi_x\rangle = \mathcal{L}(u^\epsilon,\phi) + \int_{\mathbb{R}} \phi(x,0)dm_0(x). \qquad (109)$$

We now state the main result of this section.

Proposition 4. *We have the following estimate*

$$|E_\epsilon| \le C\epsilon.$$

The constant C is independent of ϵ.

Proof. By the definition of m^ϵ and m_ϵ, the first term in (109) can be estimated as

$$\begin{aligned}\langle m^\epsilon - m_\epsilon, \phi_t\rangle &= \int_0^T \left(\int_{\mathbb{R}} \phi_t(x,t)m^\epsilon(x,t)dx - \int_{\mathbb{R}} \phi_t(x,t)m_\epsilon(dx,t)\right)dt\\ &= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_t(x,t)\rho_\epsilon(x-y)m_\epsilon(dy,t)dx - \int_{\mathbb{R}} \phi_t(y,t)m_\epsilon(dy,t)\right)dt\\ &= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \big[\phi_t(x,t)-\phi_t(y,t)\big]\rho_\epsilon(x-y)m_\epsilon(dy,t)dx\right)dt\\ &= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \big[\phi_t(x,t)-\phi_t(X^\epsilon(\theta,t),t)\big]\rho_\epsilon(x-X^\epsilon(\theta,t))dm_0(\theta)dx\right)dt\\ &\le M_1||\phi_{tx}||_{L^\infty}T\epsilon.\end{aligned}$$

For the second term of (109), because ρ_ϵ is an even function, by the definition of U^ϵ we can obtain

$$\begin{aligned}&\langle U_\epsilon m^\epsilon - U^\epsilon m_\epsilon, \phi_x\rangle\\ &= \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\phi_x(x,t)\rho_\epsilon(x-X^\epsilon(\theta,t))dm_0(\theta)dxdt\\ &\qquad - \int_0^T\int_{\mathbb{R}} U^\epsilon(X^\epsilon(\theta,t),t)\phi_x(X^\epsilon(\theta,t),t)dm_0(\theta)dt\\ &= \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\phi_x(x,t)\rho_\epsilon(x-X^\epsilon(\theta,t))dm_0(\theta)dxdt\\ &\qquad - \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} U_\epsilon(x,t)\rho_\epsilon(x-X^\epsilon(\theta,t))\phi_x(X^\epsilon(\theta,t),t)dm_0(\theta)dxdt\\ &\le M_1||U_\epsilon||_{L^\infty}||\phi_{xx}||_{L^\infty}T\epsilon \le \frac{1}{2}M_1^3||\phi_{xx}||_{L^\infty}T\epsilon.\end{aligned}$$

This ends the proof. ☐

Next, we state our main theorem in this section, which contains Theorem 1.4.

Theorem 6.4. *Assume that initial datum $m_0 \in \mathcal{M}(\mathbb{R})$ satisfies (100). $u^\epsilon(x,t)$ and $m^\epsilon(x,t)$ are defined by (102). Then, the limit function u given by Lemma 6.2 is a global weak solution of the mCH equation (1)-(2) and*

$$u \in C([0,+\infty);H^1(\mathbb{R})) \cap L^\infty(0,+\infty;W^{1,\infty}(\mathbb{R})).$$

Furthermore, for any $T>0$, we have

$$u \in BV(\mathbb{R}\times[0,T)); \quad u_x \in BV(\mathbb{R}\times[0,T)),$$

$$m := (1-\partial_{xx})u \in \mathcal{M}(\mathbb{R}\times[0,T)),$$

and there exists subsequence of m^ϵ (also labeled as m^ϵ) such that

$$m^\epsilon \overset{*}{\rightharpoonup} m \text{ in } \mathcal{M}(\mathbb{R}\times[0,T)) \quad \text{as } \epsilon \to 0.$$

Proof. **Step 1**. Global weak solution.

As it is shown in Lemma 6.2, we have $u,\ u_x \in BV(\mathbb{R}\times[0,T))$ such that

$$u^\epsilon \to u, \quad \partial_x u^\epsilon \to u_x \text{ in } L^1_{loc}(\mathbb{R}\times[0,+\infty)).$$

Moreover, for any $T>0$, the limit functions u, u_x satisfy

$$u \in BV(\mathbb{R}\times[0,T)), \quad u_x \in BV(\mathbb{R}\times[0,T)),$$

$$|u(x,t)| \le \frac{1}{2}M_1, \quad |u_x(x,t)| \le \frac{1}{2}M_1$$

and

$$\int_{\mathbb{R}} |u(x,t)-u(x,s)|dx \le \frac{1}{2}M_1^3|t-s|, \ \int_{\mathbb{R}} |u_x(x,t)-u_x(x,s)|dx \le M_1^3|t-s|$$

for $t,s\in[0,+\infty)$. Hence,

$$\begin{aligned}||u(\cdot,t)-u(\cdot,s)||_{L^2}^2 &= \int_{\mathbb{R}} |u(x,t)-u(x,s)|^2dx \\ &\le M_1\int_{\mathbb{R}} |u(x,t)-u(x,s)|dx \le \frac{1}{2}M_1^4|t-s|.\end{aligned}$$

Similarly, we have

$$||u_x(\cdot,t)-u_x(\cdot,s)||_{L^2}^2 \le M_1^4|t-s|.$$

These two inequalities imply

$$\begin{aligned}||u(\cdot,t)-u(\cdot,s)||_{H^1}^2 &\le 2\Big(||u(\cdot,t)-u(\cdot,s)||_{L^2}^2 + ||u_x(\cdot,t)-u_x(\cdot,s)||_{L^2}^2\Big) \\ &\le 3M_1^4|t-s|.\end{aligned}$$

Therefore

$$u \in C([0,+\infty);H^1(\mathbb{R}))\cap L^\infty(0,+\infty;W^{1,\infty}(\mathbb{R})).$$

For each $\phi\in C_c^\infty(\mathbb{R}\times[0,+\infty))$, there exists $T=T(\phi)$ such that $\phi\in C_c^\infty(\mathbb{R}\times[0,T))$ We now consider convergence for each term of $\mathcal{L}(u^\epsilon,\phi)$,

$$\begin{aligned}\mathcal{L}(u^\epsilon,\phi) = &\int_0^T\int_{\mathbb{R}} u^\epsilon[\phi_t-\phi_{txx}]dxdt - \frac{1}{3}\int_0^T\int_{\mathbb{R}}(\partial_x u^\epsilon)^3\phi_{xx}dxdt \\ &- \frac{1}{3}\int_0^T\int_{\mathbb{R}}(u^\epsilon)^3\phi_{xxx}dxdt + \int_0^T\int_{\mathbb{R}}((u^\epsilon)^3+u^\epsilon(\partial_x u^\epsilon)^2)\phi_x dxdt.\end{aligned}$$

For the first term, because $\text{supp}\{\phi\}$ is compact, we can see

$$\int_0^T\int_{\mathbb{R}} u^\epsilon[\phi_t-\phi_{txx}]dxdt \to \int_0^T\int_{\mathbb{R}} u[\phi_t-\phi_{txx}]dxdt \quad (\epsilon\to 0).$$

The second term can be estimated as follows

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}[(\partial_x u^\epsilon)^3 - u_x^3]\phi_{xx}dxdt \\ =&\int_0^T\int_{\mathbb{R}}(\partial_x u^\epsilon - u_x)[(\partial_x u^\epsilon)^2+u_x^2+\partial_x u^\epsilon u_x]\phi_{xx}dxdt \\ \le& \frac{3}{4}M_1^2||\phi_{xx}||_{L^\infty}\int_{\text{supp}\{\phi\}}|\partial_x u^\epsilon - u_x|dxdt \to 0 \quad (\epsilon\to 0).\end{aligned}$$

Similarly, we obtain

$$\int_0^T \int_{\mathbb{R}} [(u^\epsilon)^3 - u^3]\phi_{xxx} dxdt \to 0 \quad (\epsilon \to 0),$$
$$\int_0^T \int_{\mathbb{R}} [(u^\epsilon)^3 - u^3]\phi_x dxdt \to 0 \quad (\epsilon \to 0),$$

and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} [u^\epsilon(\partial_x u^\epsilon)^2 - uu_x^2]\phi_x dxdt \\
&= \int_0^T \int_{\mathbb{R}} [(u^\epsilon - u)(\partial_x u^\epsilon)^2 + u((\partial_x u^\epsilon)^2 - u_x^2)]\phi_x dxdt \\
&= \int_0^T \int_{\mathbb{R}} [(u^\epsilon - u)(\partial_x u^\epsilon)^2 + u(\partial_x u^\epsilon + u_x)(\partial_x u^\epsilon - u_x)]\phi_x dxdt \\
&\to 0 \quad (\epsilon \to 0).
\end{aligned}$$

Combining the above estimates and Proposition 4 gives

$$\mathcal{L}(u, \phi) = -\int_{\mathbb{R}} \phi(x, 0) dm_0(x).$$

This proves that u is a global weak solution to the mCH equation.

Step 2. Now we prove that

$$m^\epsilon \overset{*}{\rightharpoonup} m \text{ in } \mathcal{M}(\mathbb{R} \times [0, T)) \quad (\epsilon \to 0).$$

For any test function $\phi \in C_c^1(\mathbb{R} \times [0, T))$, integrating by parts and using the relationship $m^\epsilon = (1 - \partial_{xx})u^\epsilon$ imply that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \phi(x, t) dm^\epsilon(x, t) &= \int_0^T \int_{\mathbb{R}} \phi(x, t)(1 - \partial_{xx})u^\epsilon(x, t) dxdt \\
&= \int_0^T \int_{\mathbb{R}} \phi(x, t)u^\epsilon(x, t) + \phi_x(x, t)\partial_x u^\epsilon(x, t) dxdt.
\end{aligned}$$

Taking $\epsilon \to 0$, the right hand side of the above equality converges to

$$\int_0^T \int_{\mathbb{R}} \phi(x, t)u(x, t) + \phi_x(x, t)u_x(x, t) dxdt = \int_0^T \int_{\mathbb{R}} \phi(x, t)m(dx, dt).$$

Hence, $m^\epsilon \overset{*}{\rightharpoonup} m$ in $\mathcal{M}(\mathbb{R} \times [0, T))$. This ends the proof. □

Remark 9. In [17], the authors also prove the total variation stability of $m(\cdot, t)$. That is

$$|m(\cdot, t)|(\mathbb{R}) \leq |m_0|(\mathbb{R}).$$

The weak solution is unique when $u \in L^\infty(0, \infty; W^{2,1}(\mathbb{R}))$. Moreover, examples about nonuniqueness of peakon weak solutions can also be found in [17]. Notice that peakon solutions are not in the solution class $W^{2,1}(\mathbb{R})$.

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