EXISTENCE OF WEAK SOLUTIONS FOR PARTICLE-LADEN FLOW WITH SURFACE TENSION

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Abstract. We prove the existence of solutions for a coupled system modeling the flow of a suspension of fluid and negatively buoyant non-colloidal particles in the thin film limit. The equations take the form of a fourth-order nonlinear degenerate parabolic equation for the film height h coupled to a second-order degenerate parabolic equation for the particle density ψ. We prove the existence of physically relevant solutions, which satisfy the uniform bounds $0 \leq \psi/h \leq 1$ and $h \geq 0$.

1. Introduction

In lubrication theory, the free surface height of a thin liquid film is governed by a degenerate fourth-order parabolic equation which in one dimension typically has the form

$$h_t + (f_0(h))_x = -(f_1(h)h_{xxx})_x + (f_2(h)h_x)_x, \tag{1}$$

where the coefficients f_0, f_1, f_2 depend on the relevant physics (e.g. $f_0 = f_1 = f_2 = h^3$ for the flow of a fluid driven by gravity down an incline) [16]. Equations of this type have been the subject of considerable theoretical study; the tools for analysis can provide insight into important phenomena such as instabilities in spreading films [7, 9, 17, 4], and can be utilized to design efficient numerical schemes [20]. Bernis and Friedman [5] first demonstrated existence and positivity of solutions to the equation $h_t = -(h^n h_{xxx})_x$ through the use of energy and entropy estimates. In later work, Bertozzi and Pugh explicated the theory for the equation (1) with $f_0 = 0$, using different choices of regularization and entropy functions to study regularity, long-time behavior [3] and the growth of singularities [4].

There are a wide variety of problems in multiphase thin-film flows that lead to more complicated systems. Lubrication models of such flows reduce to coupled systems for the film height and a quantity tracking the second phase, whose complex dynamics have been the subject of considerable interest in recent research [11]. Here we consider one such model for gravity-driven suspension flow in one dimension that

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accounts for the non-uniform distribution of particles within the bulk of the fluid, proposed in [15] and recently extended to include surface tension in [18]. The model equations [18] for the film height $h(x,t)$ and depth-integrated particle density $\psi(x,t)$ have the form

$$
\begin{aligned}
h_t + (h^3 f_0(\phi))_x &= -\beta(h^3 f_1(\phi)h_{xxx})_x + (h^3(f_2(\phi)h_x + f_3(\phi)\psi_x))_x, \\
\psi_t + (h^3 g_0(\phi))_x &= -\beta(h^3 g_1(\phi)h_{xxx})_x + (h^3(g_2(\phi)h_x + g_3(\phi)\psi_x))_x,
\end{aligned} \tag{2}
$$

where $\phi = \psi/h$ is the depth-averaged concentration of particles that cannot exceed a maximum packing fraction ϕ_m, normalized here so that $\phi_m = 1$. The fluxes vanish at the maximum packing fraction (i.e. $f_i(1) = g_i(1) = g_i(0) = 0$), where flow of the suspension is completely inhibited by the particles. This adds an additional degeneracy into the equations (along with the standard degeneracy for thin films as $h \to 0$), which has been studied in the related problem of non-linear diffusion equations for sedimenting particles [2].

The flux functions in the model equations (2) have a particular behavior in the dilute limit ($\phi \to 0$) and the high-concentration limit ($\phi \to 1$). In particular, for negatively buoyant particles,

$$
\begin{aligned}
& f_i(\phi) \sim \frac{1}{3}, \quad g_i(\phi) \sim b_i\phi^{3/2} \text{ as } \phi \to 0, \quad i = 0,1,2 \\
& f_3 \sim a_3\phi, \quad g_3 \sim b_3\phi^2 \text{ as } \phi \to 0, \\
& f_i(\phi) \sim c_i(1-\phi)^2, \quad g_i(\phi) \sim d_i(1-\phi)^2 \text{ as } \phi \to 1, \quad i = 0,1,2,3
\end{aligned} \tag{3}
$$

for constants $a_i, b_i, c_i, d_i > 0$ [19]. These fluxes arise from depth-integrating the fluid (f) and particle (g) volume fluxes, which depend on the distribution of particles in the fluid depth. The exponents of ϕ in the dilute limit are a consequence of the particle accumulation towards the substrate of the fluid [18]. The quadratic decay in the high concentration limit is due to the singularity in the suspension viscosity $\mu \sim (1-\phi)^{-2}$, a law that captures the inhibiting of the flow near the maximum packing fraction [6].

The system (2) is closely related to the equations governing transport of insoluble surfactant on the fluid surface [11], for which the concentration Γ satisfies an equation with a non-degenerate diffusion term. Existence and positivity of weak solutions was established in [13, 1] using a finite element approach and studied for more general systems in later work by [8, 14, 10]. The techniques employed there are almost applicable to (2), but must be modified to account for a few key differences in the structure of the equations. First, we do not include the non-degenerate Brownian diffusion term for ψ, leaving only the degenerate diffusion term for the ψ equation which vanishes when $\phi = 0, \phi = 1$ or $h = 0$. Second, the fluxes depend on the ratio ψ/h of the conserved variables h and ψ and vanish when $\psi/h \geq 1$, so it is critical to establish this bound.

Here we are concerned with the existence of physically relevant solutions in the sense that $h \geqslant 0$ and $0 \leqslant \psi/h \leq 1$ when the initial data satisfies the same, with periodic boundary conditions. Under assumptions on the behavior of the flux coefficients f_i and g_i compatible with the properties (3) of the physical model, we prove existence of such solutions and the bound $\phi \leq 1$ when $f_i = g_i = 0$ for $\phi \geq 1$. In Section 2, the governing system and assumptions used in the existence result are introduced. In Section 3 the relevant notion of a weak solution is defined and the main result is stated, which is proven in Section 4.

2. System and assumptions

Next, we assume that $f_2 = g_2 = 0$ for simplicity (due to the fourth order diffusion, this second order term is not important to the existence result). Let us consider the following system of equations:

$$h_t + (|h|^3[\beta f_1(\tfrac{\psi}{h})h_{xxx} + f_0(\tfrac{\psi}{h})])_x = (D_1(h,\psi)\psi_x)_x, \tag{4}$$

$$\psi_t + (|h|^3[\beta g_1(\tfrac{\psi}{h})h_{xxx} + g_0(\tfrac{\psi}{h})])_x = (D_2(h,\psi)\psi_x)_x \tag{5}$$

in $Q_T = (0,T)\times\Omega$ with periodic boundary conditions

$$\tfrac{\partial^i h}{\partial x^i}(-a,t) = \tfrac{\partial^i h}{\partial x^i}(a,t),\ \tfrac{\partial^k \psi}{\partial x^k}(-a,t) = \tfrac{\partial^k \psi}{\partial x^k}(a,t)\ \forall\, t>0, \tag{6}$$

$i = \overline{0,3}$, $k = 0,1$, and initial conditions

$$h(x,0) = h_0(x),\ \psi(x,0) = \psi_0(x), \tag{7}$$

where $\Omega := (-a,a) \subset \mathbb{R}^1$ is bounded domain,

$$h_0(x) \in H^1(\Omega),\quad \psi_0(x) \in L^2(\Omega),\quad 0 \leqslant \psi_0(x) \leqslant h_0(x), \tag{8}$$

and D_i, f_i, g_i are continuous functions such that

$$0 \leqslant f_1(z) \leqslant a_0(1+|z|)^{-m},\ |f_0(z)| \leqslant a_1 f_1^{\frac{1}{2}}(z),\ \text{where } m \geqslant 0, \tag{9}$$

$$|g_1(z)| \leqslant b_0 f_1^{\frac{1}{2}}(z)|z|^{\frac{3}{2}},\quad |g(z)| \leqslant b_1|z|^{\frac{3}{2}}\ \forall\,|z| \leqslant 1, \tag{10}$$

$$|f_2(z)| \leqslant a_2 f_1^{\frac{1}{2}}(z)|z|^{\frac{3}{2}},\quad b_2|z|^3 \leqslant g_2(z)\ \forall\,|z| \leqslant 1, \tag{11}$$

$$D_1(a,b) := |a|^3 f_2(\tfrac{b}{a}),\quad D_2(a,b) := |a|^3 g_2(\tfrac{b}{a}), \tag{12}$$

$$f_i(z) = g_i(z) = 0\ \forall\,|z| > 1,\ i = 0,1,2, \tag{13}$$

where a_2, b_0, b_2 satisfy the following restrictions:

$$\tfrac{a_2^2}{4\beta b_2} < \sqrt{1 - \sqrt{\tfrac{\beta b_0^2}{4b_2}}} \text{ and } \tfrac{\beta b_0^2}{4b_2} < 1, \text{ or } \tfrac{a_2^2}{4\beta b_2} > \max\Big\{\tfrac{\beta b_0^2}{4b_2} - 1, \sqrt{1 + \sqrt{\tfrac{\beta b_0^2}{4b_2}}}\Big\}.$$

While these restrictions are technical, the upper bounds on f_0, f_1, g_0, g_1 and lower bound on g_2 are physically relevant, as is evident in comparing to the behavior of the coefficients (3) in the physical model.

Integrating (4) on Ω, due to periodic boundary conditions (6), we obtain the mass conservation

$$\int\limits_\Omega h\,dx = \int\limits_\Omega h_0\,dx. \tag{14}$$

3. Main result

Definition 3.1. [weak solution] A generalized weak solution of the problem (4)–(7) with initial data (h_0, ψ_0) satisfying (8) is a pair (h, ψ) has the following regularity properties

$$h \geqslant 0 \text{ in } Q_T, \quad 0 \leqslant \psi \leqslant h \text{ a.e. in } Q_T,$$
$$h \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\bar{Q}_T) \cap L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*),$$
$$\psi \in L^\infty(0,T;L^2(\Omega)) \cap W^1_{\frac{3}{2}}(0,T;(W^1_3(\Omega))^*),$$
$$I := \beta f_1(\phi) h^3 h_{xxx} + f_0(\phi) h^3 - D_1(h,\psi)\psi_x \in L^2(\{h>0\}),$$
$$\beta g_1(\phi) h^3 h_{xxx} - D_2(h,\psi)\psi_x \in L^{\frac{3}{2}}(\{\psi > 0\}), \; g_0(\phi) h^3 \in L^6(\{h>0\}),$$

where $\phi := \frac{\psi}{h}$. Furthermore, (h,ψ) satisfies (4)–(5) in the following sense:

$$\int\limits_0^T \langle h_t(t), \xi(t)\rangle dt - \iint\limits_{\{h>0\}} I \xi_x dxdt = 0,$$

$$\int\limits_0^T \langle \psi_t(t), \zeta(t)\rangle dt - \iint\limits_{\{h>0\}} g_0(\phi) h^3 \zeta_x dxdt - \iint\limits_{\{\psi>0\}} (\beta g_1(\phi) h^3 h_{xxx} - D_2(h,\psi)\psi_x)\zeta_x dxdt = 0$$

for all $\xi \in L^2(0,T;H^1(\Omega))$ and $\zeta \in L^3(0,T;W^1_3(\Omega))$: $\xi(-a,t) = \xi(a,t)$, $\zeta(-a,t) = \zeta(a,t)$ for all $t \in (0,T)$. Moreover, the initial conditions for h and ψ are attained in the sense of traces in the spaces $H^1(0,T;(H^1(\Omega))^*)$ and $W^1_{\frac{3}{2}}(0,T;(W^1_3(\Omega))^*)$, respectively.

Theorem 3.2. *[existence] Let (9)–(13) hold. Assume that the initial data (h_0, ψ_0) satisfy (8) and (14). Then, for any time $T > 0$, there exists a weak solution (h, ψ) of the problem (4)–(7) in the sense of Definition 3.1.*

4. Proof of Theorem 3.2

4.1. **Auxiliary problems.** We regularize the degeneracy which is apparent for $h = 0, \psi = 0$ and $\psi = h$. For this purpose we approximate the system by a family of non-degenerate equations:

$$h_t + (\beta F_{\delta\varepsilon}(h) f_{1,\delta}(\tfrac{\psi}{h}) h_{xxx} + F_\delta(h) f_0(\tfrac{\psi}{h}))_x = (D_{1,\delta}(h,\psi)\psi_x)_x, \tag{15}$$

$$\psi_t + (F_\delta(h)[\beta g_{1,\delta}(\tfrac{\psi}{h}) h_{xxx} + g_0(\tfrac{\psi}{h})])_x = (D_{2,\varepsilon}(h,\psi)\psi_x)_x, \tag{16}$$

in $Q_T = (0,T) \times \Omega$ with periodic boundary conditions

$$\tfrac{\partial^i h}{\partial x^i}(-a,t) = \tfrac{\partial^i h}{\partial x^i}(a,t), \; \tfrac{\partial^k \psi}{\partial x^k}(-a,t) = \tfrac{\partial^k \psi}{\partial x^k}(a,t) \; \forall \, t > 0, \tag{17}$$

$i = \overline{0,3}$, $k = 0,1$, and initial conditions

$$h(x,0) = h_{0,\delta}(x) \geqslant h_0 + \delta^\theta, \; \psi(x,0) = \psi_{0,\varepsilon}(x) \geqslant \psi_0 + \varepsilon^\mu \tag{18}$$

for all $\theta \in (0, \frac{2}{s+4})$ and $\mu \in (0,1)$, where $\varepsilon > 0$, $\delta > 0$, and

$$F_{\delta\varepsilon}(z) := F_\delta(z) + \varepsilon = \tfrac{|z|^{s+3}}{|z|^s + \delta |z|^3} + \varepsilon, \; s \geqslant 8, \; f_{1,\delta}(z) := f_1(z) + \delta;$$
$$|g_{1,\delta}(z)| \leqslant b_0 f^{\frac{1}{2}}_{1,\delta}(z)|z|^{\frac{3}{2}} \text{ if } |z| \leqslant 1, \; g_{1,\delta}(z) = \delta \text{ if } |z| > 1;$$
$$D_{1,\delta}(h,\psi) = F_\delta(h) f_2(\tfrac{\psi}{h}), \quad D_{2,\varepsilon}(h,\psi) = D_2(h,\psi) + \varepsilon.$$

Here $h_{0,\delta}$ and $\psi_{0,\varepsilon}$ are smooth enough approximation functions. Integrating (15) and (16) in Q_T by (17), we get the mass conservation

$$\int\limits_{\Omega} h(x,t)\,dx = \int\limits_{\Omega} h_{0,\delta}(x)\,dx,\ \int\limits_{\Omega} \psi(x,t)\,dx = \int\limits_{\Omega} \psi_{0,\varepsilon}(x)\,dx. \tag{19}$$

Let us denote by $\phi := \frac{\psi}{h}$, and

$$\chi_\phi = 1 \text{ if } |\phi| \leqslant 1,\ \ \chi_\phi = 0 \text{ if } |\phi| > 1.$$

Note that

$$D_{1,\delta}(h,\psi) = 0\ \forall\,|\phi| > 1,$$
$$D_{2,\varepsilon}(h,\psi) \geqslant D_{2,\varepsilon}(\psi) := b_2|\psi|^3 + \varepsilon\ \forall\,|\phi| \leqslant 1,\ \text{ and } D_{2,\varepsilon}(h,\psi) = \varepsilon\ \forall\,|\phi| > 1.$$

4.2. **Galerkin approximation.** Now we use a Galerkin approximation which transforms the system of partial differential equations into a system of ordinary differential equations. As basis functions for the finite dimensional space we select an L^2-orthonormal basis of eigenfunctions which are solutions of the periodic boundary value problem:

$$-v_i'' = \lambda_i v_i \text{ in } \Omega,\ \ v_i(-a) = v_i(a).$$

We make a Galerkin ansatz for $h^N_{\epsilon\delta}(x,t)$ and $\psi^N_{\epsilon\delta}(x,t)$ of the form

$$h^N_{\epsilon\delta} = \sum_{i=0}^{N} a_i(t) v_i(x), \quad \psi^N_{\epsilon\delta} = \sum_{i=0}^{N} b_i(t) v_i(x).$$

According to (15) and (16) the functions $a_i(t)$ and $b_i(t)$ are subject to the following Galerkin equations which have to hold for $j = \overline{0,N}$:

$$\begin{aligned}\dot a_j(t) = {} & -\beta\delta\varepsilon\lambda_j \|v_j'\|_2^2 a_j(t) - \\ & \beta \sum_{i=0}^{N} \lambda_i a_i(t) \int\limits_{\Omega} \big(F_\delta(h^N_{\epsilon\delta}) f_{1,\delta}(\tfrac{\psi^N_{\epsilon\delta}}{h^N_{\epsilon\delta}}) + \varepsilon f_1(\tfrac{\psi^N_{\epsilon\delta}}{h^N_{\epsilon\delta}})\big) v_i' v_j' dx + \\ & \int\limits_{\Omega} F_\delta(h^N_{\epsilon\delta}) f_0(\tfrac{\psi^N_{\epsilon\delta}}{h^N_{\epsilon\delta}}) v_j' dx - \sum_{i=0}^{N} b_i(t) \int\limits_{\Omega} D_{1,\delta}(h^N_{\epsilon\delta}, \psi^N_{\epsilon\delta}) v_i' v_j' dx,\end{aligned}$$

$$\begin{aligned}\dot b_j(t) = {} & -\varepsilon\lambda_j b_j(t) - \sum_{i=0}^{N} b_i(t) \int\limits_{\Omega} D_2(h^N_{\epsilon\delta}, \psi^N_{\epsilon\delta}) v_i' v_j' dx - \\ & \beta \sum_{i=0}^{N} \lambda_i a_i(t) \int\limits_{\Omega} F_\delta(h^N_{\epsilon\delta}) g_{1,\delta}(\tfrac{\psi^N_{\epsilon\delta}}{h^N_{\epsilon\delta}}) v_i' v_j' dx + \int\limits_{\Omega} F_\delta(h^N_{\epsilon\delta}) g_0(\tfrac{\psi^N_{\epsilon\delta}}{h^N_{\epsilon\delta}}) v_j' dx\end{aligned}$$

with

$$a_j(0) = (h_{0,\delta}, v_j)_{L^2(\Omega)}, \quad b_j(0) = (\psi_0, v_j)_{L^2(\Omega)}.$$

Due to (9)–(12), the right-hand side of this system is Lipschitz continuous on a_j and b_j. Thus, by the Picard-Lindelöf theorem a unique local solution of the system exists. Solvability for some $T > 0$ can be proved by using a priori estimates (uniformly in N, ε and δ).

For brevity, we denote by $h := h^N_{\varepsilon\delta}$, $\psi := \psi^N_{\varepsilon\delta}$ and $\phi = \frac{\psi}{h}$. Multiplying (15) by $h - h_{xx}$ and integrating on Ω, we deduce that

$$\begin{gathered}
\tfrac{1}{2}\tfrac{d}{dt}\|h\|^2_{H^1(\Omega)} + \beta \int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx = \\
\beta \int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h_x h_{xxx}dx - \int\limits_{\Omega} f_0(\phi)F_\delta(h)h_{xxx}dx + \int\limits_{\Omega} f_0(\phi)F_\delta(h)h_x dx + \\
\int\limits_{\Omega} D_{1,\delta}(h,\psi)\psi_x h_{xxx}dx - \int\limits_{\Omega} D_{1,\delta}(h,\psi)\psi_x h_x dx \leqslant \\
\beta\Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_x dx\Big)^{\frac{1}{2}} + \\
\Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} F_\delta(h)\tfrac{f_0^2(\phi)}{f_{1,\delta}(\phi)}dx\Big)^{\frac{1}{2}} + \\
\Big(\int\limits_{\Omega} f_0^2(\phi)F_\delta^2(h)dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} h^2_x dx\Big)^{\frac{1}{2}} + \\
\Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)}\psi^2_x dx\Big)^{\frac{1}{2}} + \\
\Big(\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{F_\delta(h)}\psi^2_x dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} F_\delta(h)h^2_x dx\Big)^{\frac{1}{2}} \leqslant \\
C(\beta(a_0+\delta)^{\frac{1}{2}}(\|h\|^{\frac{5}{2}}_{H^1(\Omega)} + \|h\|_{H^1(\Omega)}) + a_1\chi_\phi\|h\|^{\frac{3}{2}}_{H^1(\Omega)})\Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}} + \\
Ca_1a_0^{\frac{1}{2}}\chi_\phi\|h\|^4_{H^1(\Omega)} + \Big(\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}}\Big(\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)}\psi^2_x dx\Big)^{\frac{1}{2}} + \\
C\|h\|^2_{H^1(\Omega)}\Big(\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{F_\delta(h)}\psi^2_x dx\Big)^{\frac{1}{2}},
\end{gathered} \tag{20}$$

whence we find that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|h\|^2_{H^1(\Omega)} + (\beta - \varepsilon_1 - \varepsilon_2)\int\limits_{\Omega} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx \\
&\quad\leqslant \tfrac{C^2}{2\varepsilon_1}(\beta^2(a_0+\delta)(\|h\|^5_{H^1(\Omega)} + \|h\|^2_{H^1(\Omega)}) + a_1^2\chi_\phi\|h\|^3_{H^1(\Omega)}) + Ca_1a_0^{\frac{1}{2}}\chi_\phi\|h\|^4_{H^1(\Omega)} \\
&\quad+ \tfrac{1}{4\varepsilon_2}\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{f_1(\phi)F_\delta(h)}\psi^2_x dx + \varepsilon_3\int\limits_{\Omega} \tfrac{D^2_{1,\delta}(h,\psi)}{F_\delta(h)}\psi^2_x dx + \tfrac{C^2}{4\varepsilon_3}\chi_\phi\|h\|^4_{H^1(\Omega)} \\
&\quad\leqslant C_1\max\{1, \|h\|^5_{H^1(\Omega)}\} + a_2^2\chi_\phi(\tfrac{1}{4\varepsilon_2} + \varepsilon_3)\int\limits_{\Omega} |\psi|^3\psi^2_x dx,
\end{aligned} \tag{21}$$

where $C_1 > 0$ is independent of N, ε and $\delta < \delta_0$.

Next, multiplying (16) by $\Phi'_\varepsilon(\psi)$, we deduce that

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(\psi)dx + \int_\Omega D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2 dx =$$
$$\beta\int_\Omega F_\delta(h)h_{xxx}g_{1,\delta}(\phi)\Phi''_\varepsilon(\psi)\psi_x dx + \int_\Omega F_\delta(h)g_0(\phi)\Phi''_\varepsilon(\psi)\psi_x dx \leqslant$$
$$\beta\Big(\int_\Omega F_\delta(h)\tfrac{g^2_{1,\delta}(\phi)\Phi''_\varepsilon(\psi)}{f_{1,\delta}(\phi)D_{2,\varepsilon}(h,\psi)}D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx\Big)^{\frac{1}{2}} +$$
$$\Big(\int_\Omega D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega F^2_\delta(h)g^2(\phi)\tfrac{\Phi''_\varepsilon(\psi)}{D_{2,\varepsilon}(h,\psi)}dx\Big)^{\frac{1}{2}},$$

where $\Phi''_\varepsilon(z) = |z|^{-3}D_{2,\varepsilon}(z) = b_2\chi_\phi + \varepsilon|z|^{-3} > 0$, i.e. $\Phi_\varepsilon(z) = \frac{b_2}{2}\chi_\phi z^2 + \frac{\varepsilon}{2}|z|^{-1}$. Note that

$$F_\delta(h)\tfrac{g^2_{1,\delta}(\phi)\Phi''_\varepsilon(\psi)}{f_{1,\delta}(\phi)D_{2,\varepsilon}(h,\psi)} \leqslant |h|^3\tfrac{\delta^2\Phi''_\varepsilon(\psi)}{\delta\varepsilon} \leqslant \tfrac{\delta}{\varepsilon}|\psi|^3\Phi''_\varepsilon(\psi) \leqslant \delta \text{ if } |\phi|>1,$$

$$F_\delta(h)\tfrac{g^2_{1,\delta}(\phi)\Phi''_\varepsilon(\psi)}{f_{1,\delta}(\phi)D_{2,\varepsilon}(h,\psi)} \leqslant |h|^3\tfrac{g^2_{1,\delta}(\phi)|\psi|^{-3}(b_2|\psi|^3+\varepsilon)}{f_{1,\delta}(\phi)(b_2|\psi|^3+\varepsilon)} = \tfrac{g^2_{1,\delta}(\phi)|\phi|^{-3}}{f_{1,\delta}(\phi)} \leqslant b_0^2 \text{ if } |\phi|\leqslant 1.$$

Then, due to (9)–(13), we obtain that

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(\psi)dx + (1-\varepsilon_4-\varepsilon_5)\int_\Omega D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dx \leqslant \tag{22}$$
$$\tfrac{\beta^2(b_0^2+\delta)}{4\varepsilon_4}\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx + \tfrac{b_1^2\chi_\phi}{4\varepsilon_5}\int_\Omega |h|^3dx \leqslant$$
$$\tfrac{\beta^2(b_0^2+\delta)}{4\varepsilon_4}\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx + C_2\|h\|^3_{H^1(\Omega)},$$

where $C_2>0$ is independent of N, ε and $\delta<\delta_0$.

Summing (21) and (22), we have

$$\tfrac{1}{2}\tfrac{d}{dt}\|h\|^2_{H^1(\Omega)} + \tfrac{d}{dt}\int_\Omega \Phi_\varepsilon(\psi)dx + \big(\beta-\varepsilon_1-\varepsilon_2-\tfrac{\beta^2(b_0^2+\delta)}{4\varepsilon_4}\big)\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx + \tag{23}$$
$$\big(1-\varepsilon_4-\varepsilon_5-\chi_\phi\tfrac{a_2^2}{b_2^2}(\tfrac{1}{4\varepsilon_2}+\varepsilon_3)\big)\int_\Omega D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dx \leqslant$$
$$C_3\max\{1,\|h\|^5_{H^1(\Omega)}\},$$

where $C_3=\max\{C_1,C_2\}>0$. Choosing ε_i such that

$$\beta-\varepsilon_1-\varepsilon_2-\tfrac{\beta^2(b_0^2+\delta)}{4\varepsilon_4}>0,\ \ 1-\varepsilon_4-\varepsilon_5-\chi_\phi\tfrac{a_2^2}{b_2^2}(\tfrac{1}{4\varepsilon_2}+\varepsilon_3)>0,$$

namely,

$$0<\varepsilon_1,\varepsilon_3,\varepsilon_5\ll 1,\ \ \chi_\phi\tfrac{a_2^2}{4b_2^2(1-\varepsilon_4)}<\varepsilon_2<\beta-\tfrac{\beta^2(b_0^2+\delta)}{4\varepsilon_4},$$

$$\max\{\tfrac{\beta(b_0^2+\delta)}{4}, \tfrac{1}{8\beta}(4\beta+\beta^2(b_0^2+\delta)-\chi_\phi \tfrac{a_2^2}{b_2^2} - \sqrt{(4\beta+\beta^2(b_0^2+\delta)-\chi_\phi \tfrac{a_2^2}{b_2^2})^2-16\beta^3(b_0^2+\delta)}\} < \varepsilon_4 <$$

$$\min\{1, \tfrac{1}{8\beta}(4\beta+\beta^2(b_0^2+\delta)-\chi_\phi \tfrac{a_2^2}{b_2^2} + \sqrt{(4\beta+\beta^2(b_0^2+\delta)-\chi_\phi \tfrac{a_2^2}{b_2^2})^2-16\beta^3(b_0^2+\delta)}\}$$

provided

$$|4\beta+\beta^2(b_0^2+\delta)-\chi_\phi \tfrac{a_2^2}{b_2^2}| > 4(b_0^2+\delta)^{\frac{1}{2}}\beta^{\frac{3}{2}},$$

we get

$$\tfrac{1}{2}\tfrac{d}{dt}\|h\|^2_{H^1(\Omega)} + \tfrac{d}{dt}\int\limits_\Omega \Phi_\varepsilon(\psi)dx + C\int\limits_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dx + C\int\limits_\Omega D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dx \leqslant C_3\max\{1,\|h\|^5_{H^1(\Omega)}\}. \tag{24}$$

Applying Grönwall's lemma to (24) with $y(t) = \max\{1, \|h\|^2_{H^1(\Omega)}\} + 2\|\Phi_\varepsilon(\psi)\|_1$, we obtain that

$$\|h\|^2_{H^1(\Omega)} \leqslant \frac{\max\{1,\|h^N_{0\varepsilon}\|^2_{H^1(\Omega)}\}+2\|\Phi_\varepsilon(\psi^N_{0\varepsilon})\|_1}{(1-3C_3(\max\{1,\|h^N_{0\varepsilon}\|^2_{H^1(\Omega)}\}+2\|\Phi_\varepsilon(\psi^N_{0\varepsilon})\|_1)^{\frac{3}{2}}t)^{\frac{2}{3}}} \tag{25}$$

for all $t < T_N := [3C_3(\max\{1, \|h^N_{0\varepsilon}\|^2_{H^1(\Omega)}\} + 2\|\Phi_\varepsilon(\psi^N_{0\varepsilon})\|_1)^{\frac{3}{2}}]^{-1}$. Because $h^N_{0\varepsilon} \to h_0$ strongly in $H^1(\Omega)$ and $\Phi_\varepsilon(\psi^N_{0\varepsilon}) \to \Phi_0(\psi_0)$ strongly in $L^1(\Omega)$ as $N \to +\infty$ and $\varepsilon \to 0$ then we can select a time $T_0 := [6C_3(\max\{1, \|h_0\|^2_{H^1(\Omega)}\} + 2\|\Phi_0(\psi_0)\|_1)^{\frac{3}{2}}]^{-1} < T_N$ which is independent of N, ε and δ. As a result, we have the following a priori estimate

$$\|h\|^2_{H^1(\Omega)} + \int\limits_\Omega \Phi_\varepsilon(\psi)dx + C\iint\limits_{Q_T} f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h^2_{xxx}dxdt + C\iint\limits_{Q_T} D_{2,\varepsilon}(h,\psi)\Phi''_\varepsilon(\psi)\psi_x^2dxdt \leqslant C_4 \tag{26}$$

for all $T \leqslant T_0$, where C_4 is independent of N, ε and δ.

Hence, from (26) we obtain that the solution $(a_i(t), b_i(t))$ can be extended up to T_0. As a conclusion, we have shown that the Galerkin equations have solutions

$$h^N_{\varepsilon\delta},\ \psi^N_{\varepsilon\delta} \in C^1(0,T;C^\infty(\Omega)) \text{ for all } T \leqslant T_0.$$

4.3. **Limit processes.**

4.3.1. *Limits of $N \to +\infty$ and $\varepsilon \to 0$.* Let $\phi^N_{\varepsilon\delta} = \frac{\psi^N_{\varepsilon\delta}}{h^N_{\varepsilon\delta}}$. Next, we have to show that in the following weak formulation we can pass to the limit for $N \to +\infty$:

$$\int\limits_0^T \langle h^N_{\varepsilon\delta,t}(t), \xi^N(t)\rangle dt - \beta\iint\limits_{Q_T} f_{1,\delta}(\phi^N_{\varepsilon\delta})F_{\delta\varepsilon}(h^N_{\varepsilon\delta})h^N_{\varepsilon\delta,xxx}\xi^N_x dxdt - \iint\limits_{Q_T} f_0(\phi^N_{\varepsilon\delta})F_\delta(h^N_{\varepsilon\delta})\xi^N_x dxdt = -\iint\limits_{Q_T} D_{1,\delta}(h^N_{\varepsilon\delta},\psi^N_{\varepsilon\delta})\psi^N_{\varepsilon\delta,x}\xi^N_x dxdt, \tag{27}$$

$$(28)\quad \int_0^T \langle \psi^N_{\varepsilon\delta,t}(t), \zeta^N(t)\rangle dt - \beta \iint_{Q_T} g_{1,\delta}(\phi^N_{\varepsilon\delta}) F_\delta(h^N_{\varepsilon\delta}) h^N_{\varepsilon\delta,xxx} \zeta^N_x \,dxdt - \iint_{Q_T} g_0(\phi^N_{\varepsilon\delta}) F_\delta(h^N_{\varepsilon\delta}) \zeta^N_x \,dxdt = -\iint_{Q_T} D_{2,\varepsilon}(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta}) \psi^N_{\varepsilon\delta,x} \zeta^N_x \,dxdt$$

for all $\xi^N \in L^2(0,T;H^1(\Omega))$ and $\zeta^N \in L^3(0,T;W^1_3(\Omega))$ such that $\xi^N \to \xi$ in $L^2(0,T;H^1(\Omega))$ and $\zeta^N \to \zeta$ in $L^3(0,T;W^1_3(\Omega))$ with $\xi(-a,t) = \xi(a,t)$, $\zeta(-a,t) = \zeta(a,t)$ for all $t \in (0,T)$.

To ensure convergence we have to establish appropriate convergence properties. By (26) we have the following (uniformly in N, ε and δ) boundedness for all $T \leqslant T_0$

$$(29)\quad \{h^N_{\varepsilon\delta}\} \text{ in } L^\infty(0,T;H^1(\Omega)),$$

$$(30)\quad \{\Phi_\varepsilon(\psi^N_{\varepsilon\delta})\} \text{ in } L^\infty(0,T;L^1(\Omega)),$$

$$(31)\quad \{(f_{1,\delta}(\phi^N_{\varepsilon\delta}) F_{\delta\varepsilon}(h^N_{\varepsilon\delta}))^{\frac{1}{2}} h^N_{\varepsilon\delta,xxx}\} \text{ in } L^2(Q_T),$$

$$(32)\quad \{D_{1,\delta}(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta}) \psi^N_{\varepsilon\delta,x}\} \text{ in } L^2(Q_T),$$

$$(33)\quad \{(D_{2,\varepsilon}(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta}) \Phi''_\varepsilon(\psi^N_{\varepsilon\delta}))^{\frac{1}{2}} \psi^N_{\varepsilon\delta,x}\} \text{ in } L^2(Q_T),$$

$$(34)\quad \{(\delta\varepsilon)^{\frac{1}{2}} h^N_{\varepsilon\delta,xxx}\} \text{ in } L^2(Q_T).$$

By (29) and the embedding theorem, we have

$$(35)\quad \{h^N_{\varepsilon\delta}\} \text{ is uniformly bounded in } L^\infty(Q_T).$$

Note that from (30) it follows

$$(36)\quad \{\chi_\phi \psi^N_{\varepsilon\delta}\} \text{ in } L^\infty(0,T;L^2(\Omega)),$$

and from (33), (36) we have

$$(37)\quad \{\chi_\phi |\psi^N_{\varepsilon\delta}|^{\frac{5}{2}}\} \text{ in } L^2(0,T;H^1(\Omega)),$$

By (37) and (36), due to the embedding theorem for parabolic function spaces from [12, Proposition 3.2, p. 8] applied to $w = \chi_\phi |\psi^N_{\varepsilon\delta}|^{\frac{5}{2}} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^{\frac{4}{5}}(\Omega))$, we can derive the following estimate for $\psi^N_{\varepsilon\delta}$:

$$(38)\quad \{\chi_\phi \psi^N_{\varepsilon\delta}\} \text{ in } L^9(Q_T).$$

The previous statements allow us to prove that

$$(39)\quad I^N_{\varepsilon\delta} := \beta(F_\delta(h^N_{\varepsilon\delta}) + \varepsilon) f_{1,\delta}(\phi^N_{\varepsilon\delta}) h^N_{\varepsilon\delta,xxx} + F_\delta(h^N_{\varepsilon\delta}) f_0(\phi^N_{\varepsilon\delta}) - D_{1,\delta}(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta}) \psi^N_{\varepsilon\delta,x} \text{ is u. b. in } L^2(Q_T),$$

$$(40)\quad J^N_{\varepsilon\delta} := F_\delta(h^N_{\varepsilon\delta})[\beta g_{1,\delta}(\phi^N_{\varepsilon\delta}) h^N_{\varepsilon\delta,xxx} + g_0(\phi^N_{\varepsilon\delta})] - D_{2,\varepsilon}(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta}) \psi^N_{\varepsilon\delta,x} \text{ is u. b. in } L^{\frac{3}{2}}(Q_T),$$

and therefore by (29), (31) and (34), we find that

$$\{h^N_{\varepsilon\delta,t}\} \text{ is uniformly bounded in } L^2(0,T;(H^1(\Omega))^*), \tag{41}$$

and by (36) and (33), we deduce that

$$\{\psi^N_{\varepsilon\delta,t}\} \text{ is uniformly bounded in } L^{\frac{3}{2}}(0,T;(W^1_3(\Omega))^*). \tag{42}$$

By (29) and (41) we find (see [5]) that

$$\{h^N_{\varepsilon\delta}\} \text{ is uniformly bounded in } C^{\frac{1}{8},\frac{1}{2}}_{t,x}(\bar{Q}_T). \tag{43}$$

Therefore, we conclude that there exists a subsequence $N = N_k$, $\varepsilon = \varepsilon_l$ such that

$$h^N_{\varepsilon\delta} \to h_\delta \text{ uniformly as } N \to +\infty,\ \varepsilon \to 0. \tag{44}$$

Moreover, by (34) we have

$$h^N_{\varepsilon\delta} \to h_{\varepsilon\delta} \text{ weakly in } L^2(0,T;W^3_2(\Omega)) \text{ as } N \to +\infty. \tag{45}$$

From (36), (19) and (37) it follows that there exists a subsequence such that

$$\psi^N_{\varepsilon\delta} \to \psi_\delta \text{ *-weakly in } L^\infty(0,T;L^1(\Omega)) \text{ and a.e. in } Q_T, \tag{46}$$

$$\chi_\phi\psi^N_{\varepsilon\delta} \to \chi_\phi\psi_\delta \text{ *-weakly in } L^\infty(0,T;L^2(\Omega)) \text{ and a.e. in } Q_T, \tag{47}$$

$$\chi_\phi(\psi^N_{\varepsilon\delta})^{\frac{5}{2}} \to \chi_\phi(\psi_\delta)^{\frac{5}{2}} \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{48}$$

as $N \to +\infty$, $\varepsilon \to 0$. Thus, by (41) and (42), we have for correspondent subsequences

$$h^N_{\varepsilon\delta,t} \to h_{\delta,t} \text{ *-weakly in } L^2(0,T;(H^1(\Omega))^*), \tag{49}$$

$$\psi^N_{\varepsilon\delta,t} \to \psi_{\delta,t} \text{ *-weakly in } L^{\frac{3}{2}}(0,T;(W^1_3(\Omega))^*). \tag{50}$$

In particular, by (44) and (46) we get

$$\phi^N_{\varepsilon\delta} := \tfrac{\psi^N_{\varepsilon\delta}}{h^N_{\varepsilon\delta}} \to \phi_\delta := \tfrac{\psi_\delta}{h_\delta} \text{ a.e. on } \{|h_\delta| > \mu\} \tag{51}$$

for all $\mu > 0$ as $N \to +\infty$ and $\varepsilon \to 0$. Due to (51), we can take limit in all terms of (27)–(28), connected with $\phi^N_{\varepsilon\delta}$, as $N \to +\infty$ and $\varepsilon \to 0$ on the set $\{|h_\delta| > \mu\}$. On the next subsections, we will prove that $h_\delta > 0$ and $\psi_\delta \geqslant 0$. For this reason, instead of convergence (51) on the set $\{|h_\delta| > \mu\}$, we obtain this convergence a.e. in Q_T.

Applying these convergence results to (27)–(28) we get that the Galerkin solutions $(h^N_{\varepsilon\delta}, \psi^N_{\varepsilon\delta})$ converge for any fixed $\delta > 0$ to a weak solution (h_δ, ψ_δ) of the degenerate problem

$$\int\limits_0^T \langle h_{\delta,t}(t), \xi(t)\rangle dt - \beta \iint\limits_{Q_T} f_{1,\delta}(\phi_\delta)F_\delta(h_\delta)h_{\delta,xxx}\xi_x dxdt - \iint\limits_{Q_T} f_0(\phi_\delta)F_\delta(h_\delta)\xi_x dxdt = -\iint\limits_{Q_T} D_{1,\delta}(h_\delta,\psi_\delta)\psi_{\delta,x}\xi_x dxdt, \tag{52}$$

$$\int_0^T \langle \psi_{\delta,t}(t), \zeta(t)\rangle dt - \beta \iint_{Q_T} g_{1,\delta}(\phi_\delta) F_\delta(h_\delta) h_{\delta,xxx}\zeta_x dxdt - \iint_{Q_T} g_0(\phi_\delta)F_\delta(h_\delta)\zeta_x dxdt = -\iint_{Q_T} D_2(h_\delta,\psi_\delta)\psi_{\delta,x}\zeta_x dxdt \tag{53}$$

for all $T \leqslant T_0$ and $\xi \in L^2(0,T;H^1(\Omega))$, and $\zeta \in L^3(0,T;W_3^1(\Omega))$ with $\xi(-a,t) = \xi(a,t)$, $\zeta(-a,t) = \zeta(a,t)$ for all $t \in (0,T_0)$.

4.3.2. *Positivity of h_δ.* Next, we show $h_\delta > 0$ for all $\delta < \delta_0$. This allows us to extend the corresponding integrals in (52), (53) on all Q_T. Multiplying (15) by $G'_{\delta\varepsilon}(h)$, we get

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega G_{\delta\varepsilon}(h)dx &= \beta \int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)G''_{\delta\varepsilon}(h)h_x h_{xxx}dx + \\ &\int_\Omega f_0(\phi)F_\delta(h)G''_{\delta\varepsilon}(h)h_x dx - \int_\Omega D_{1,\delta}(h,\psi)\psi_x G''_{\delta\varepsilon}(h)h_x dx = \\ &\beta\int_\Omega f_{1,\delta}(\phi)|h|^\alpha h_x h_{xxx}dx + \int_\Omega \tfrac{f_0(\phi)F_\delta(h)}{F_{\delta\varepsilon}(h)}|h|^\alpha h_x dx - \int_\Omega \tfrac{f_2(\phi)F_\delta(h)}{F_{\delta\varepsilon}(h)}|h|^\alpha h_x\psi_x dx,\end{aligned}$$

where $G''_{\delta\varepsilon}(z) = \frac{|z|^\alpha}{F_{\delta\varepsilon}(z)}$. Using (9), we have

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega G_{\delta\varepsilon}(h)dx &\leqslant a_1\Big(\int_\Omega h_x^2 dx\Big)^{\frac12}\Big(\int_\Omega f_{1,\delta}(\phi)|h|^{2\alpha}dx\Big)^{\frac12} + \\ &\beta\Big(\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h_{xxx}^2 dx\Big)^{\frac12}\Big(\int_\Omega f_{1,\delta}(\phi)\tfrac{|h|^{2\alpha}}{F_{\delta\varepsilon}(h)}h_x^2 dx\Big)^{\frac12} + \\ &a_2\chi_\phi\Big(\int_\Omega |h|^{2\alpha-3}h_x^2 dx\Big)^{\frac12}\Big(\int_\Omega |\psi|^3\psi_x^2 dx\Big)^{\frac12}.\end{aligned}$$

Choose $\alpha \geqslant \frac{s}{2}$ and $s \geqslant 3$. Then

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega G_{\delta\varepsilon}(h)dx &\leqslant Ca_1 a_0^{\frac12}\|h\|^{\alpha+1}_{H^1(\Omega)} + \\ &C\beta(a_0+\delta)^{\frac12}\big(\|h_x\|_2^{\frac{2\alpha+1}{2}} + \delta^{\frac12}\|h_x\|_2^{\frac{2\alpha-s+2}{2}}\big)\Big(\int_\Omega f_{1,\delta}(\phi)F_{\delta\varepsilon}(h)h_{xxx}^2 dx\Big)^{\frac12} + \\ &Ca_2\chi_\phi\|h_x\|_2^{\frac{2\alpha-1}{2}}\Big(\int_\Omega |\psi|^3\psi_x^2 dx\Big)^{\frac12}.\end{aligned}$$

Integrating this inequality in time, taking into account (26), we deduce that

$$\int_\Omega G_{\delta\varepsilon}(h)dx \leqslant \int_\Omega G_{\delta\varepsilon}(h_{0,\delta})dx + C_4(T) \tag{54}$$

for all $T \leqslant T_0$, where $C_4(T)$ is independent of N, ε and $\delta < \delta_0$. By Fatou's lemma, (44) and from the uniform (in N, ε, δ) bound of $\int\limits_\Omega G_{\delta\varepsilon}(h_{0,\delta})dx$ we deduce that $\int\limits_\Omega G_\delta(h)dx$ dx is uniformly bounded in N, ε and δ.

First of all, we show that $h_\delta \geqslant 0$ in Q_{T_0} when $s \geqslant 4$. If this is not true, then there is a point $(x_0, t_0) \in Q_{T_0}$ such that $h_\delta(x_0, t_0) < 0$. Since convergence $h_{\delta\varepsilon}$ to h_δ is uniform as $\varepsilon \to 0$ then there exist $\gamma > 0$ and $\varepsilon_0 > 0$ such that $h_{\delta\varepsilon}(x, t_0) < -\gamma$ if $|x - x_0| < \gamma$ and $\varepsilon < \varepsilon_0$. But for such x, by the monotone convergence theorem

$$G_{\delta\varepsilon}(h_{\delta\varepsilon}(x,t_0)) = \int\limits_A^{h_{\delta\varepsilon}(x,t_0)} \int\limits_A^v \tfrac{|z|^\alpha}{F_{\delta\varepsilon}(z)} dzdv \geqslant \int\limits_{-\gamma}^0 \int\limits_A^v \tfrac{|z|^\alpha}{F_{\delta\varepsilon}(z)} dzdv \underset{\varepsilon\to 0}{\to}$$

$$\int\limits_{-\gamma}^0 \int\limits_A^v \tfrac{|z|^\alpha}{F_{\delta}(z)} dzdv = +\infty \text{ for } s \geqslant 4,\ \alpha \in [\tfrac{s}{2}, s-2],$$

where $A > \max|h_{\delta\varepsilon}|$ for all small δ, ε. Hence, $\lim\limits_{\varepsilon\to 0} \int\limits_\Omega G_{\delta\varepsilon}(h_{\delta\varepsilon})dx = \infty$ and this is in contradiction with (54).

Next, we show that $h_\delta > 0$ on $\bar\Omega$ when $s \geqslant 8$. Indeed, if h_δ is not positive everywhere in Q_{T_0}, then there exists a point (x_0, t_0) in Q_{T_0} such that $h_\delta(x_0, t_0) = 0$. Then by the Hölder continuity of $h_\delta \in C_x^{1/2}$, we have $|h_\delta(x,t)| = |h_\delta(x,t) - h_\delta(x_0,t)| \leqslant C|x - x_0|^{1/2}$. Hence, taking into account $G_\delta(z) \sim \frac{\delta |z|^{\alpha-s+2}}{(\alpha-s+1)(\alpha-s+2)}$ for $|z| \ll 1$, we come to a contradiction

$$\infty > \int\limits_\Omega G_\delta(h_\delta)dx \geqslant C \int\limits_\Omega |x - x_0|^{\frac{\alpha-s+2}{2}} dx = \infty \text{ if } s \geqslant 8,\ \alpha \in [\tfrac{s}{2}, s-4].$$

As $G_\delta(z) - G_0(z) = \frac{\delta z^{\alpha-s+2}}{(\alpha-s+1)(\alpha-s+2)}$ for all $z \geqslant 0$ then, due to (18), $G_\delta(h_{0,\delta}) - G_0(h_{0,\delta}) \leqslant \frac{\delta^{1+\theta(\alpha-s+2)}}{(\alpha-s+1)(\alpha-s+2)}$ as $\delta \to 0$. Hence, $\int\limits_\Omega G_0(h)dx$ is bounded provided $\int\limits_\Omega h_0^{\alpha-1}dx < \infty$, hence it follows that $h \geqslant 0$ if $h_0 \geqslant 0$ in $\bar\Omega$.

4.3.3. *Nonnegativity of* ψ_δ. Now, we can use the bound for $\int\limits_\Omega \Phi_\varepsilon(\psi_{\varepsilon\delta})dx$ (see (26)) to derive the lower bound $\psi_\delta \geqslant 0$. If $z < 0$ and $0 < \varepsilon < \varepsilon_0$, then, due to $\Phi''_\varepsilon(z) = \chi_\phi b_2 + \varepsilon|z|^{-3}$ and $\Phi_\varepsilon(z) = \frac{b_2\chi_\phi}{2}z^2 + \frac{\varepsilon}{2}|z|^{-1}$, we have

$$\Phi_\varepsilon(z) \geqslant \Phi_\varepsilon(\varepsilon) + \Phi'_\varepsilon(\varepsilon)(z-\varepsilon) + \tfrac{1}{2}\Phi''_\varepsilon(\varepsilon)(z-\varepsilon)^2 \geqslant \Phi'_\varepsilon(\varepsilon)(z-\varepsilon) + \tfrac{\chi_\phi b_2 \varepsilon^2 + 1}{2\varepsilon^2} z^2.$$

It follows that

$$z^2 \leqslant \tfrac{2\varepsilon^2}{\chi_\phi b_2 \varepsilon^2 + 1}(\Phi_\varepsilon(z) - \Phi'_\varepsilon(\varepsilon)(z-\varepsilon)).$$

This implies

$$\int\limits_\Omega (-\psi_{\varepsilon\delta})_+^2 dx \leqslant \tfrac{2\varepsilon^2}{\chi_\phi b_2\varepsilon^2+1} \int\limits_\Omega \Phi_\varepsilon(\psi_{\varepsilon\delta})dx - \tfrac{2\varepsilon^2 \Phi'_\varepsilon(\varepsilon)}{\chi_\phi b_2\varepsilon^2+1} \int\limits_\Omega (\psi_{\varepsilon\delta} - \varepsilon)dx \leqslant$$

$$2\varepsilon^2 \int\limits_\Omega \Phi_\varepsilon(\psi_{\varepsilon\delta})dx + \varepsilon \int\limits_\Omega \psi_{0,\varepsilon}\, dx + 2|\Omega|\varepsilon^2.$$

Then, taking into account (30) and (19), passing to the limit in this inequality as $\varepsilon \to 0$ yields $\psi_\delta \geqslant 0$ a.e. in Q_{T_0}.

4.3.4. *Estimate $\psi_\delta \leqslant h_\delta$.* Let us denote by

$$v := h_\delta - \psi_\delta.$$

We want to show that $v \geqslant 0$. Subtracting (16) from (15) with $\varepsilon = 0$, we arrive at

$$v_t + (L_\delta)_x = (\tilde{D}_\delta(h,\psi) v_x)_x \text{ in } Q_{T_0}, \tag{55}$$

$$v(x,0) = v_{0,\delta}(x) := h_{0,\delta} - \psi_0 \geqslant 0, \tag{56}$$

$$v(-a,t) = v(a,t),\ v_x(-a,t) = v_x(a,t)\ \forall\, t \in (0,T_0), \tag{57}$$

where

$$L_\delta := \beta F_\delta(h)(f_{1,\delta}(\phi) - g_{1,\delta}(\phi)) h_{xxx} + F_\delta(h)(f_0(\phi) - g_0(\phi)) + \tilde{D}_\delta(h,\psi) h_x,$$

$$\tilde{D}_\delta(h,\psi) := D_2(h,\psi) - D_{1,\delta}(h,\psi) = h^3 g_2(\phi) - F_\delta(h) f_2(\phi).$$

By (13) we find that

$$L_\delta = \tilde{D}_\delta = 0 \text{ if } v < 0,\ \text{i.e. } \phi_\delta = \tfrac{\psi_\delta}{h_\delta} > 1. \tag{58}$$

Choose

$$r \in C^1(\mathbb{R}) : r(z) > 0,\ r'(z) \leqslant 0 \text{ if } z < 0, \quad r(z) = 0 \text{ if } z \geqslant 0.$$

Then

$$R(z) := \int_0^z r(s)\, ds = 0 \text{ if } z \geqslant 0, \quad R(z) > 0 \text{ if } z < 0,$$

and in particular,

$$\int_\Omega R(v_{0,\delta}(x))\, dx = 0.$$

Multiplying (55) by $R'(v) = r(v)$, we deduce that

$$\tfrac{d}{dt} \int_\Omega R(v) dx + \int_\Omega \tilde{D}_\delta(h,\psi) r'(v) v_x^2 dx = -\int_\Omega r(v) (L_\delta)_x dx,$$

whence by (58) we have

$$0 \leqslant \int_\Omega R(v) dx \leqslant \int_\Omega R(v_{0,\delta}) dx = 0.$$

This implies that $R(v) = 0$, thus

$$v \geqslant 0,\ \text{i.e. } \psi_\delta \leqslant h_\delta \Leftrightarrow \phi_\delta \leqslant 1,\ \text{a.e. in } \Omega \text{ for any } t \in [0,T_0]. \tag{59}$$

Passing to the limit in (59) as $\delta \to 0$ yields

$$0 \leqslant \psi \leqslant h \text{ a.e. in } Q_{T_0}. \tag{60}$$

4.3.5. *Global existence.* Using the mass conservation (19), we can extend our local solution for all times. We consider the approximation solutions (h_δ, ψ_δ), where $h_\delta > 0$. Next, instead of (26), taking into account (59), we obtain more exact a priori estimates for (h_δ, ψ_δ). For brevity, we denote by $h := h_\delta$, $\psi := \psi_\delta$ and $\phi := \phi_\delta$.

Multiplying (15) with $\varepsilon = 0$ by $-h_{xx}$, we deduce

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\int\limits_\Omega h_x^2 dx + \beta \int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx = \\
&\quad - \int\limits_\Omega f_0(\phi) F_\delta(h) h_{xxx} dx + \int\limits_\Omega D_{1,\delta}(h,\psi)\psi_x h_{xxx} dx \leqslant \\
&\quad \Big(\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx\Big)^{\frac{1}{2}} \Big(\int\limits_\Omega F_\delta(h) \tfrac{f_0^2(\phi)}{f_{1,\delta}(\phi)} dx\Big)^{\frac{1}{2}} + \\
&\quad \Big(\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx\Big)^{\frac{1}{2}} \Big(\int\limits_\Omega \tfrac{D_{1,\delta}^2(h,\psi)}{f_{1,\delta}(\phi)F_\delta(h)} \psi_x^2 dx\Big)^{\frac{1}{2}} \leqslant \\
&\quad \Big(\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx\Big)^{\frac{1}{2}} \Big(a_1^2 \int\limits_\Omega h^3 dx\Big)^{\frac{1}{2}} + \\
&\quad \Big(\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx\Big)^{\frac{1}{2}} \Big(a_2^2 \int\limits_\Omega \psi^3 \psi_x^2 dx\Big)^{\frac{1}{2}},
\end{aligned}$$

whence we find that

$$\tfrac{1}{2}\tfrac{d}{dt}\int\limits_\Omega h_x^2 dx + (\beta - \varepsilon_1 - \varepsilon_2)\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \leqslant \tfrac{a_1^2}{4\varepsilon_1}\int\limits_\Omega h^3 dx + \tfrac{a_2^2}{4\varepsilon_2}\int\limits_\Omega \psi^3\psi_x^2 dx. \tag{61}$$

Using the Nirenberg-Gagliardo interpolation inequality

$$\|v\|_p \leqslant c_0 \|v_x\|_2^{\frac{2(p-1)}{3p}} \|v\|_1^{\frac{p+2}{3p}} + c_1\|v\|_1 \tag{62}$$

for all $v \in H^1(\Omega)$ with $v = h \geq 0, p = 3$ and the mass conservation $\int h\, dx = \|h_{0,\delta}\|_1$, we arrive at

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\int\limits_\Omega h_x^2 dx + (\beta - \varepsilon_1 - \varepsilon_2)\int\limits_\Omega f_{1,\delta}(\phi) F_\delta(h) h_{xxx}^2 dx \leqslant \\
&\qquad \tfrac{Ca_1^2}{4\varepsilon_1}\Big(\int\limits_\Omega h_x^2 dx\Big)^{\frac{2}{3}} + \tfrac{a_2^2}{4\varepsilon_2 b_2}\int\limits_\Omega D_2(h,\psi)\psi_x^2 dx,
\end{aligned} \tag{63}$$

where the $C's$ is independent of $\delta < \delta_0$.

Next, multiplying (16) with $\varepsilon = 0$ by ψ, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi^2 dx + \int_\Omega D_2(h,\psi)\psi_x^2 dx =$$
$$\beta \int_\Omega F_\delta(h) h_{xxx} g_{1,\delta}(\phi)\psi_x dx + \int_\Omega F_\delta(h) g_0(\phi)\psi_x dx \leqslant$$
$$\beta\Big(\int_\Omega F_\delta(h)\tfrac{g_{1,\delta}^2(\phi)}{f_{1,\delta}(\phi)}\psi_x^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega f_{1,\delta}(\phi)F_\delta(h)h_{xxx}^2 dx\Big)^{\frac{1}{2}} +$$
$$\Big(\int_\Omega D_2(h,\psi)\psi_x^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \tfrac{F_\delta^2(h)g_0^2(\phi)}{D_2(h,\psi)}dx\Big)^{\frac{1}{2}} \leqslant$$
$$\beta\Big(\tfrac{b_0^2}{b_2}\int_\Omega D_2(h,\psi)\psi_x^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega f_{1,\delta}(\phi)F_\delta(h)h_{xxx}^2 dx\Big)^{\frac{1}{2}} +$$
$$\Big(\int_\Omega D_2(h,\psi)\psi_x^2 dx\Big)^{\frac{1}{2}}\Big(\tfrac{b_1^2}{b_2}\int_\Omega h^3 dx\Big)^{\frac{1}{2}}.$$

Then we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi^2 dx + (1-\varepsilon_3-\varepsilon_4)\int_\Omega D_2(h,\psi)\psi_x^2 dx \leqslant \tag{64}$$
$$\tfrac{\beta^2 b_0^2}{4\varepsilon_3 b_2}\int_\Omega f_{1,\delta}(\phi)F_\delta(h)h_{xxx}^2 dx + \tfrac{b_1^2}{4\varepsilon_4 b_2}\int_\Omega h^3 dx \leqslant$$
$$\tfrac{\beta^2 b_0^2}{4\varepsilon_3 b_2}\int_\Omega f_{1,\delta}(\phi)F_\delta(h)h_{xxx}^2 dx + \tfrac{Cb_1^2}{4\varepsilon_4 b_2}\Big(\int_\Omega h_x^2 dx\Big)^{\frac{2}{3}} + \tfrac{Cb_1^2}{4\varepsilon_4 b_2}.$$

Summing (63) and (64), we have

$$\frac{1}{2}\frac{d}{dt}(\|h_x\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}) + (\beta - \varepsilon_1 - \varepsilon_2 - \tfrac{\beta^2 b_0^2}{4\varepsilon_3 b_2})\int_\Omega f_{1,\delta}(\phi)F_\delta(h)h_{xxx}^2 dx + \tag{65}$$
$$(1-\varepsilon_3-\varepsilon_4-\tfrac{a_2^2}{4\varepsilon_2 b_2})\int_\Omega D_2(h,\psi)\psi_x^2 dx \leqslant C_4\|h_x\|^{\frac{4}{3}}_{L^2(\Omega)} + C_5.$$

Choosing ε_i such that

$$\beta - \varepsilon_1 - \varepsilon_2 - \tfrac{\beta^2 b_0^2}{4\varepsilon_3 b_2} > 0,\ \ 1-\varepsilon_3-\varepsilon_4-\tfrac{a_2^2}{4\varepsilon_2 b_2} > 0,$$

namely,

$$0 < \varepsilon_1, \varepsilon_4 \ll 1,\ \ \tfrac{\beta^2 b_0^2}{4b_2(\beta-\varepsilon_2)} < \varepsilon_3 < 1 - \tfrac{a_2^2}{4b_2\varepsilon_2},$$

$$\tfrac{\beta}{2}\Big[\tfrac{a_2^2}{4\beta b_2} - \tfrac{\beta b_0^2}{4b_2} + 1 - \sqrt{(\tfrac{a_2^2}{4\beta b_2} - \tfrac{\beta b_0^2}{4b_2} + 1)^2 - \tfrac{a_2^2}{\beta b_2}}\Big] < \varepsilon_2 <$$
$$\tfrac{\beta}{2}\Big[\tfrac{a_2^2}{4\beta b_2} - \tfrac{\beta b_0^2}{4b_2} + 1 + \sqrt{(\tfrac{a_2^2}{4\beta b_2} - \tfrac{\beta b_0^2}{4b_2} + 1)^2 - \tfrac{a_2^2}{\beta b_2}}\Big]$$

provided

$$\tfrac{a_2^2}{4\beta b_2} < \sqrt{1 - \sqrt{\tfrac{\beta b_0^2}{4b_2}}} \text{ and } \tfrac{\beta b_0^2}{4b_2} < 1 \quad \text{ or } \quad \tfrac{a_2^2}{4\beta b_2} > \max\Big\{ \tfrac{\beta b_0^2}{4b_2} - 1, \sqrt{1 + \sqrt{\tfrac{\beta b_0^2}{4b_2}}} \Big\},$$

we have

$$\tfrac{d}{dt}(\|h_x\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}) + C\int\limits_{\Omega} f_{1,\delta}(\phi)F_\delta(h)h^2_{xxx}dx + C\int\limits_{\Omega} D_2(h,\psi)\psi_x^2 dx \leqslant C_6 \max\{1, \|h_x\|^{\frac{4}{3}}_{L^2(\Omega)}\}, \tag{66}$$

where $C_6 > 0$ is independent of $\delta > 0$.

By Grönwall's lemma applied to $y(t) := \max\{1, \|h_x\|^2_{L^2(\Omega)}\} + \|\psi\|^2_{L^2(\Omega)}$, we obtain that

$$\|h_x\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} \leqslant \Big[(\max\{1, \|h_{0\delta,x}\|^2_{L^2(\Omega)}\} + \|\psi_0\|^2_{L^2(\Omega)})^{\frac{1}{3}} + \tfrac{1}{3}C_6\, t\Big]^3 \tag{67}$$

for all $t \geqslant 0$. As a result, we have

$$\|h_x\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} + C\iint\limits_{Q_T} f_{1,\delta}(\phi)F_\delta(h)h^2_{xxx}dxdt + C\iint\limits_{Q_T} D_2(h,\psi)\psi_x^2 dxdt \leqslant C_7(T)\ \forall\, T > 0, \tag{68}$$

where $C_7(T)$ is independent of $\delta < \delta_0$. The a priori estimate (68) allows us to construct the limit solution (h, ψ) as $\delta \to 0$ for all $T > 0$.

4.3.6. *Limit process for $\delta \to 0$.* Similar to (51), in view of (59) and (60), we can take limit in all terms of (52)–(53), connected with $\phi_\delta = \frac{\psi_\delta}{h_\delta}$, as $\delta \to 0$ on the set $\{h > \mu\}$. On the sets $\{\psi \leqslant h \leqslant \mu\}$ and $\{\psi \leqslant \mu < h\}$, we show smallness of the corresponding integrals on this set. Really, if δ is sufficiently small, depending on μ, then

$$\Big|\iint\limits_{\{h\leqslant\mu\}} f_{1,\delta}(\phi_\delta)F_\delta(h_\delta)h_{\delta,xxx}\xi_x dxdt\Big| \leqslant \Big(\iint\limits_{Q_T} f_{1,\delta}(\phi_\delta)F_\delta(h_\delta)h^2_{\delta,xxx}dxdt\Big)^{\frac{1}{2}}\Big(\iint\limits_{\{h\leqslant\mu\}} f_{1,\delta}(\phi_\delta)F_\delta(h_\delta)\xi_x^2 dxdt\Big)^{\frac{1}{2}} \leqslant C\,\mu^{\frac{3}{2}},$$

$$\Big|\iint\limits_{\{h\leqslant\mu\}} f_0(\phi_\delta)F_\delta(h_\delta)\xi_x dxdt\Big| \leqslant \Big(\iint\limits_{Q_T} \xi_x^2 dxdt\Big)^{\frac{1}{2}}\Big(\iint\limits_{\{h\leqslant\mu\}} (f_0(\phi_\delta)F_\delta(h_\delta))^2 dxdt\Big)^{\frac{1}{2}} \leqslant C\,\mu^3,$$

$$\Big|\iint\limits_{\{h\leqslant\mu\}} D_{1,\delta}(h_\delta,\psi_\delta)\psi_{\delta,x}\xi_x dxdt\Big| \leqslant \Big(\iint\limits_{Q_T} D_2(h_\delta,\psi_\delta)\psi^2_{\delta,x}dxdt\Big)^{\frac{1}{2}} \times \Big(\iint\limits_{\{h\leqslant\mu\}} \tfrac{D^2_{1,\delta}(h_\delta,\psi_\delta)}{D_2(h_\delta,\psi_\delta)}\xi_x^2 dxdt\Big)^{\frac{1}{2}} \leqslant C\,\mu^{\frac{3}{2}},$$

$$\Big|\iint\limits_{\{\psi\leqslant\mu\}} g_{1,\delta}(\phi_\delta)F_\delta(h_\delta)h_{\delta,xxx}\zeta_x dxdt\Big| \leqslant$$
$$\Big(\iint\limits_{Q_T} f_{1,\delta}(\phi_\delta)F_\delta(h_\delta)(h_{\delta,xxx})^2 dxdt\Big)^{\frac{1}{2}}\Big(\iint\limits_{Q_T} |\zeta_x|^3 dxdt\Big)^{\frac{1}{3}}\times$$
$$\Big(\iint\limits_{\{\psi\leqslant\mu\}} \big(F_\delta(h_\delta)\tfrac{g^2_{1,\delta}(\phi_\delta)}{f_{1,\delta}(\phi_\delta)}\big)^3 dxdt\Big)^{\frac{1}{6}} \leqslant C\Big(\iint\limits_{\{\psi\leqslant\mu\}} \psi^9_\delta dxdt\Big)^{\frac{1}{6}} \leqslant C\mu^{\frac{3}{2}},$$

$$\Big|\iint\limits_{\{h\leqslant\mu\}} g_0(\phi_\delta)F_\delta(h_\delta)\zeta_x dxdt\Big| \leqslant \Big(\iint\limits_{Q_T}|\zeta_x|^3 dxdt\Big)^{\frac{1}{3}}\Big(\iint\limits_{\{h\leqslant\mu\}} |g_0(\phi_\delta)F_\delta(h_\delta)|^{\frac{3}{2}} dxdt\Big)^{\frac{2}{3}} \leqslant$$
$$C\Big(\iint\limits_{\{h\leqslant\mu\}} (\psi_\delta h_\delta)^{\frac{9}{4}} dxdt\Big)^{\frac{2}{3}} \leqslant C\mu^{\frac{3}{2}}\Big(\iint\limits_{Q_T} \psi_\delta^{\frac{9}{4}} dxdt\Big)^{\frac{2}{3}} \leqslant C\mu^{\frac{3}{2}},$$

$$\Big|\iint\limits_{\{\psi\leqslant\mu\}} D_2(h_\delta,\psi_\delta)\psi_{\delta,x}\zeta_x dxdt\Big| \leqslant \Big(\iint\limits_{Q_T} D_2(h_\delta,\psi_\delta)\psi^2_{\delta,x} dxdt\Big)^{\frac{1}{2}}\times$$
$$\Big(\iint\limits_{Q_T}|\zeta_x|^3 dxdt\Big)^{\frac{1}{3}}\Big(\iint\limits_{\{\psi\leqslant\mu\}} (D_2(h_\delta,\psi_\delta))^3 dxdt\Big)^{\frac{1}{6}} \leqslant C\Big(\iint\limits_{\{\psi\leqslant\mu\}} \psi^9_\delta dxdt\Big)^{\frac{1}{6}} \leqslant C\,\mu^{\frac{3}{2}}.$$

Applying these convergence results to (52)–(53), we get that the solutions (h_δ,ψ_δ) converge to a weak nonnegative solution (h,ψ) of the degenerate problem

$$\int\limits_0^T \langle h_t(t),\xi(t)\rangle dt - \iint\limits_{\{h>0\}} (\beta f_1(\phi)h^3 h_{xxx}\xi_x + f_0(\phi)h^3 - D_1(h,\psi)\psi_x) dxdt = 0,$$
$$\int\limits_0^T \langle \psi_t(t),\zeta(t)\rangle dt - \iint\limits_{\{h>0\}} g_0(\phi)h^3\zeta_x dxdt - \iint\limits_{\{\psi>0\}} (\beta g_1(\phi)h^3 h_{xxx} - D_2(h,\psi)\psi_x)\zeta_x dxdt = 0$$

for all $T>0$.

5. Conclusions

We have obtained an existence result for a coupled system of degenerate parabolic equations governing the height h and particle concentration ψ of a viscous suspension under the effect of surface tension. The solution satisfies the physical bounds $h \geq 0$ and $0 \leq \psi/h \leq 1$ corresponding to the boundedness of the particle concentration. The existence result depends on certain bounds on the flux coefficients, particularly on the degeneracy in the ψ-diffusion term as $\psi \to 0$, that are consistent with the asymptotic results obtained for the physical system. The result established here may be useful in future study of this system, for example in developing numerical methods that preserve the bounds on the solution as done for other equations from lubrication theory [20] or in studying the growth of singularities and long-time behavior of advancing fronts.

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