Math 260: Python programming in math

Fall 2020

Least squares, gradient descent

Gompertz model

Earlier, we looked at the **logistic model** for population growth,

$$y' = ry(1 - y/K).$$

The **Gompertz model** instead uses the equation

$$y' = -ay \log(y/K)$$

where a and K are constants.

- Used to describe growth of tumors ($y(t) =$ tumor size)
- Growth rate per y (i.e. $a \log(y/K)$) non-linear in y

The solution to this ODE is

$$y(t) = K \exp(-ae^{-bt}) \tag{M}$$

Suppose we have an experiment measuring the tumor size at N times:

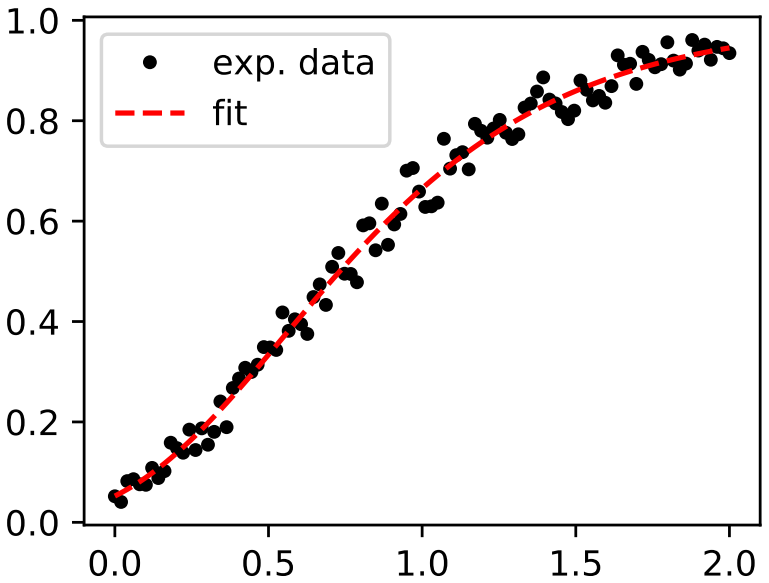
$$(\hat{t}_k, \hat{y}_k), \qquad k = 1, \cdots, N. \tag{D}$$

Gompertz model

$$y(t) = K \exp(-ae^{-bt}) \tag{M}$$

$$(\hat{t}_k, \hat{y}_k), \qquad k = 1, \cdots, N. \tag{D}$$

The goal: estimate **parameters** K, a and b by **fitting** (M) to the data (D).



(data here generated from exact solution plus some noise...)

Gompertz model

$$y(t) = K \exp(-ae^{-bt}) \tag{M}$$

$$(\hat{t}_k, \hat{y}_k), \qquad k = 1, \cdots, N. \tag{D}$$

To define the 'best fit', consider the **least-squared error**

$$E(\mathbf{r}) = \sum_{k=1}^{N} (y(\hat{t}_k; \mathbf{r}) - \hat{y}_k)^2$$

where $\mathbf{r} = (r_1, r_2)$ denotes the parameters (a, b).
(The $\mathbf{r}$ vector is more convenient for computation).

- The function E should have a minimum at $\mathbf{r}^* = (a^*, b^*)$ - these are the best fit parameters
- Since the number of parameter is small, E cannot be made zero
- How do we find this minimum?

Least squares: gradient descent

Now, ignoring the context, what we really want is to

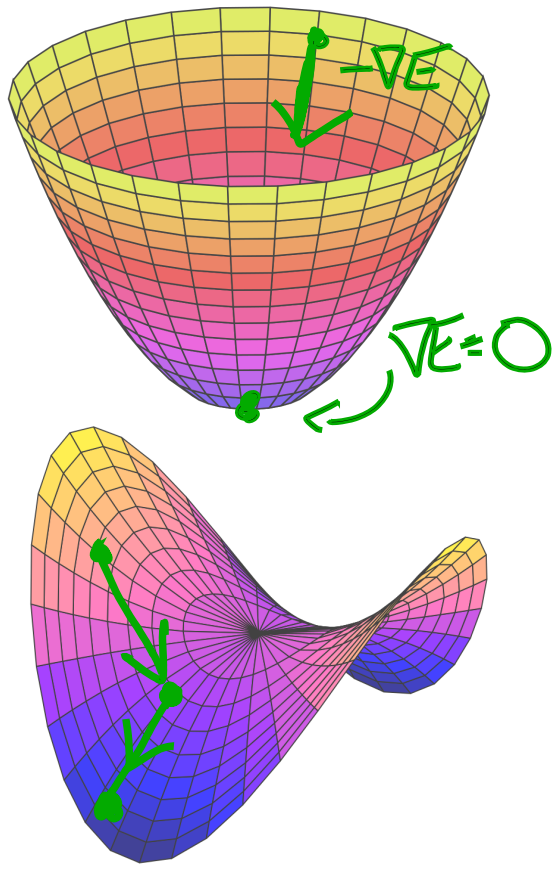
$$\text{find a minimum of } E(\mathbf{r}), \qquad (E = \text{some function of } \mathbf{r} = (r_1, r_2))$$

Key facts (calculus!)

- At a minimum of E, the gradient is zero:

$$\mathbf{r}^* \text{is a minimum of } E \implies \nabla E = 0 \text{ at } \mathbf{r}^*$$

  where $\nabla E = (\frac{\partial E}{\partial r_1}, \frac{\partial E}{\partial r_2})$.
- At any point $\mathbf{r}$, the function E is decreasing fastest in the $-\nabla E$ direction.



The latter holds since

$$(\text{rate of change of } E \text{ in dir. } \mathbf{v}) = \mathbf{v} \cdot \nabla E$$

which is most negative when $\mathbf{v} = -\nabla E$.

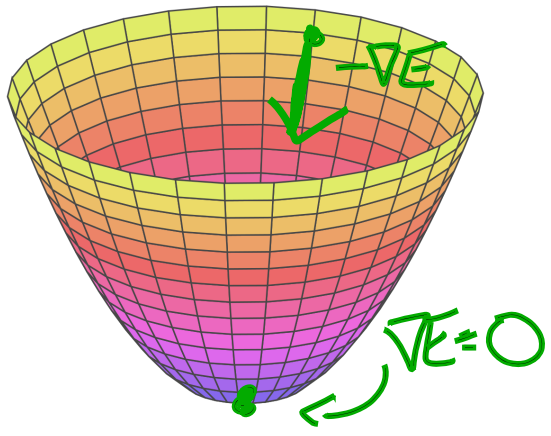
We call $\mathbf{v}$ the direction of **steepest descent**.
This fact suggests that we can 'follow' the gradient down to look for a min. of E...

Least squares: gradient descent

To search for a possible minimum $\mathbf{x}^*$ of the function

$$F(\mathbf{x}), \qquad F : \mathbb{R}^n \to \mathbb{R}$$

we can iteratively 'follow the gradient.'



Algorithm sketch: gradient descent

Pick a point $\mathbf{x}_k$. Then,

- Calculate the steepest descent direction

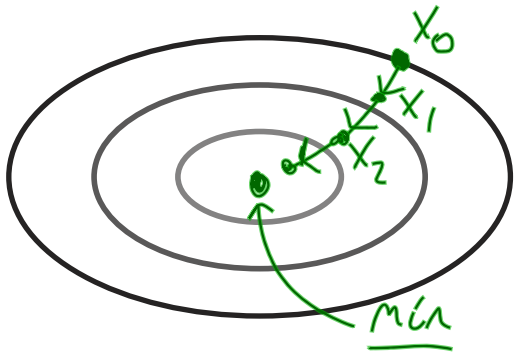
$$\mathbf{v} = -\nabla F(\mathbf{x}_k).$$

- Search along $\mathbf{v}$, i.e. along the half-line

$$q(\alpha) = \mathbf{x}_k + \alpha \mathbf{v}$$

to find a new point with a smaller F value.

- Set this to be the 'next' point $\mathbf{x}_{k+1}$ and repeat.



- This generates a sequence $\mathbf{x}_0, \mathbf{x}_1, \cdots$, of points with

$$F(\mathbf{x}_0) > F(\mathbf{x}_1) > F(\mathbf{x}_2) > \cdots \to F(\mathbf{x}^*) \quad \text{(if all goes well...)}$$

- When to stop? When $|F(\mathbf{x}_{k+1}) - F(\mathbf{x}_k)|$ is small enough, or $|\mathbf{x}_{k+1} - \mathbf{x}_k|$...

Gradient descent - backtracking

A scheme is required to search for a minimum along the (half) line

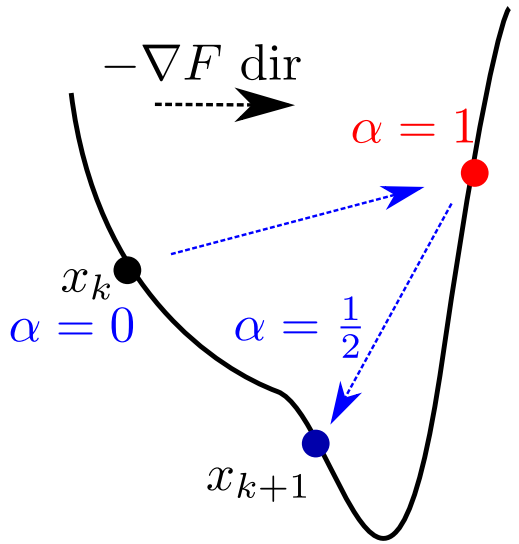
$$q(\alpha) = \mathbf{x}_k + \alpha \mathbf{v}, \quad \alpha > 0 \tag{L}$$

This step (the **line search**) can be done in several ways.

Line search strategy: Backtracking

After identifying the line (L),

- Guess $\alpha = 1$
- If $F(q(\alpha))$ is less than $F(\mathbf{x}_k)$ by enough, accept the step
- If not, set α to $\alpha/2$ and repeat.



- Since f is decreasing along $\mathbf{v}$ close to $\mathbf{x}_k$, backtracking will always find an α
- Simplest 'enough' criterion: just use

$$F(q(\alpha)) < F(\mathbf{x}_k).$$

(Better choices $\rightarrow$ more progress per step).

More sophisticated approaches exist... but this will at least guarantee progress!

Gradient descent: example

function: $F(x, y) = (x-1)^2 + 4y^2$

gradient: $\nabla F = (2(x-1), 8y)$

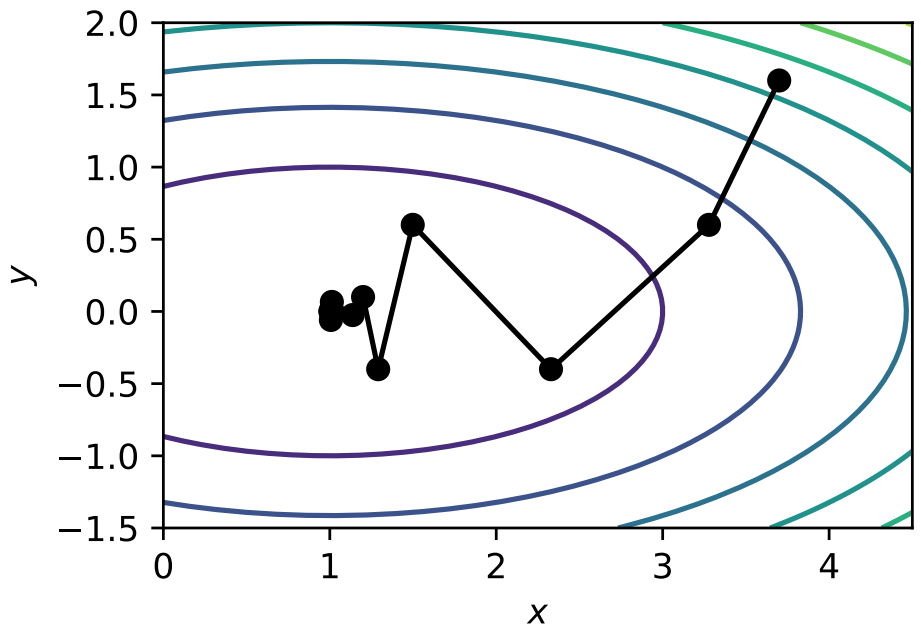
Gradient descent steps ($\mathbf{x} = (x, y)$):

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \nabla F(\mathbf{x}_k)$$

stopping condition: $||\mathbf{x}_{k+1} - \mathbf{x}_k|| < \texttt{tol}$

```
def func(x): # F
    return (x[0] - 1)**2 + 4*x[1]**2

def dfunc(x):  # gradient of F
    dx = 2*(x[0] - 1)
    dy = 8*(x[1])
    return np.array((dx, dy))
```

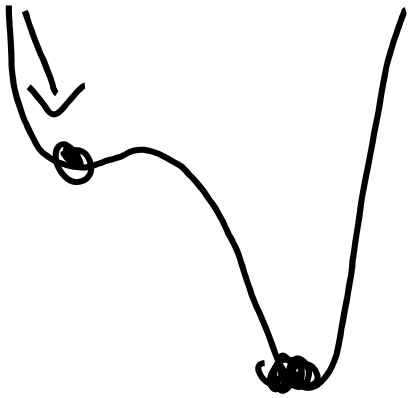


With $\texttt{tol} = 10^{-3}$, completes in 10 steps (not great, but works!).

Gradient descent: limitations

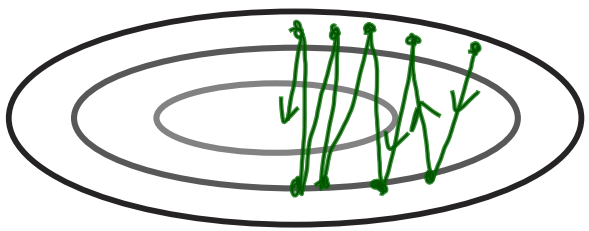
Issue 1: Local minima can be a problem...

- We can only hope for a 'local minimum' - gradient descent can get stuck in local valleys
- Finding the global minimum of F is harder!
- A good initial guess helps, but requires prior knowledge of F.



Issue 2: The gradient gives limited information...

- $-\nabla F$ is not always the best direction to search for a minimum...
- Can be slow, give bad information when far from the minimum, etc.
- Many ways to improve it by using a better searching scheme!



Some examples for reference: Levenberg-Marquardt, Gauss-Newton, BFGS...

Gradient descent: applied to least squares

Now back to non-linear least squares... we need to minimize

$$E(\mathbf{r}) = \sum_{k=1}^{N} (y(\hat{t}_k; \mathbf{r}) - \hat{y}_k)^2$$

where $\mathbf{r} = (r_1, r_2)$ (the parameters to fit). To find ∇E, compute

$$\frac{\partial}{\partial r_j}\left[(y(\hat{t}_k, \mathbf{r}) - \hat{y}_k)^2\right] = 2(y(\hat{t}_k, \mathbf{r}) - \hat{y}_k) \cdot \frac{\partial f}{\partial r_j}(\hat{t}_k, \mathbf{r})$$

by the chain rule. Then

$$\frac{\partial E}{\partial r_j} = 2\sum_{k=1}^{N}\left[(y(\hat{t}_k, \mathbf{r}) - \hat{y}_k)\frac{\partial y}{\partial r_j}(\hat{t}_k, \mathbf{r})\right] \text{ for } j = 1, 2$$

We then guess initial parameters $\mathbf{r}_0 = (a_0, b_0)$.
The gradient descent loop computes

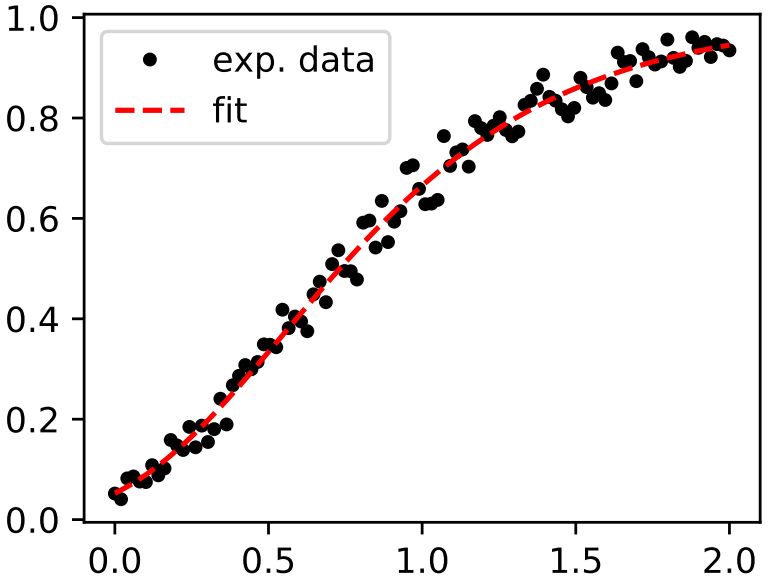
$$\mathbf{v}_k = -\nabla E \text{ at } \mathbf{r}_k$$
$$\alpha_k = \text{result of the search step}$$
$$\mathbf{r}_{k+1} = \mathbf{r}_k + \alpha_k \mathbf{v}_k$$

Note that convergence to the 'right' minimum may require a good $\mathbf{r}_0$.

Gradient descent: applied to least squares

$$y(t) = K \exp(-ae^{-bt}) \tag{M}$$

$$(\hat{t}_k, \hat{y}_k), \qquad k = 1, \cdots, N. \tag{D}$$



(data here generated from $(a, b) = (3, 2)$ some noise...)

Result: $(a, b) \approx (2.980, 1.979)$, after 43 iterations.

Aside: Non-dimensionalization

For numerics and analysis, it's important to put the problem in the right form!

Consider, for example, a model for cell growth with a 'rapid growth' term

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) + sp^2, \quad p(0) = p_0.$$

modeling a population $p(t)$ of cells that an grow abnormally fast.

- This problem has **three parameters** (plus an initial condition p_0).
- Understanding qualitative behavior means exploring a 4d space!

We can reduce the number of parameters using **non-dimensionalization**.

- The idea: rescaling a variable by a constant ($p = C\hat{p}$) does not change how the system behaves (just 'changing the units')
- The process: scale all the variables, plug into the model equations.

  Then choose scales that simplify the eqs. (and/or that are meaningful).

Aside: Non-dimensionalization

Model equation:

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) + sp^2, \quad p(0) = p_0.$$

$p(t) =$ population (number of cells) and $t =$ time in minutes.

There are **two** variables to scale here. Define 'non-dimensional' variables

$$\hat{p} = p/P, \quad \hat{t} = t/T.$$

where P_0 (cells) and T_0 (minutes) are constants ('scales') to be found.

Plug into the model to get

$$\frac{P}{T}\frac{d\hat{p}}{d\hat{t}} = rP\hat{p}(1 - \frac{P}{K}\hat{p}) + sP^2\hat{p}^2, \quad \hat{p}(0) = p_0/P.$$

Now multiply through by T/P to get a non-dimnensional equation (all terms have no units):

$$\frac{d\hat{p}}{d\hat{t}} = rT\hat{p}(1 - \frac{P}{K}\hat{p}) + sTP\hat{p}^2, \quad \hat{p}(0) = p_0/P.$$

Now simplifying the logistic term to cancel out r, K suggests the scales

$$T = \frac{1}{r}, \qquad P = K.$$

Aside: Non-dimensionalization

Now simplifying the logistic term to cancel out r, K suggests the scales

$$T = \frac{1}{r}, \qquad P = K.$$

This gives

$$\frac{d\hat{p}}{d\hat{t}} = \hat{p}(1-\hat{p}) + \frac{sK}{r}\hat{p}^2, \quad \hat{p}(0) = p_0/K.$$

Finally, we see the underlying parameters that matter, which are

$$\sigma = \frac{sK}{r}, \quad \hat{p}_0 = p_0/K.$$

Relabeling $y = \hat{p}$ and $\tau = \hat{t}$ we then have an ODE for $y(\tau)$,

$$\frac{dy}{d\tau} = y(1-y) + \sigma y^2, \quad y(0) = y_0.$$

which has a single parameter σ (plus initial condition).

As a sanity check, σ should be **non-dimensional** (no units):

$$\sigma = s\frac{K}{r} \sim \frac{1}{\text{cells} \cdot t}\frac{\text{cells}}{1/t}.$$

Non-dimensionalization

In summary, we converted the dimensional equation

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) + sp^2, \quad p(0) = p_0. \tag{D}$$

into **non-dimensional form**

$$\frac{dy}{d\tau} = y(1 - y) + \sigma y^2, \quad y(0) = y_0 \tag{ND}$$

with $y = p/K$ and $\tau = rt$ and $\sigma = sK/r$. This tells us:

- The key parameter is σ, a ratio of the rapid to base growth rates.
- The behavior of this ODE really depends on one parameter, not three
- For numerics, we should use (ND): it leads to simpler code, *and* we can more easily use properties of the equation to improve the solver.
- For example - p can asymptote at a time t^* due to the rapid growth. It is easier to estimate this with (ND), since only σ is involved.

Remember: the code may need to know how to convert back e,g,

$$\text{times } \tau = 1, 2, 3, \cdots \iff \text{times } t = 1/r, 2/r, \cdots \text{ minutes .}$$

Example:
parameter estimation for ODEs

ODEs: parameter estimation

Here's a real example of this sort of problem,
where we 'fit' an ODE model to experimental data.

In fluid dynamics, we need to measure the **surface tension** γ of a fluid-
how much energy per area is required to 'stretch' the surface of a fluid.

One method to do so is a **pendant drop** experiment, where you create a droplet that hangs from a source (like a drop from a faucet).

Then, we fit a theoretical model to this data using **least-squares**,
where γ, the surface tension, is one of the parameters.

Reference: Berry, Joseph D., et al. "Measurement of surface and interfacial tension using pendant drop tensiometry."

Journal of colloid and interface science 454 (2015): 226-237.

Pendant drop

Surface tension in a round drop is described by the **Young-Laplace** formula:

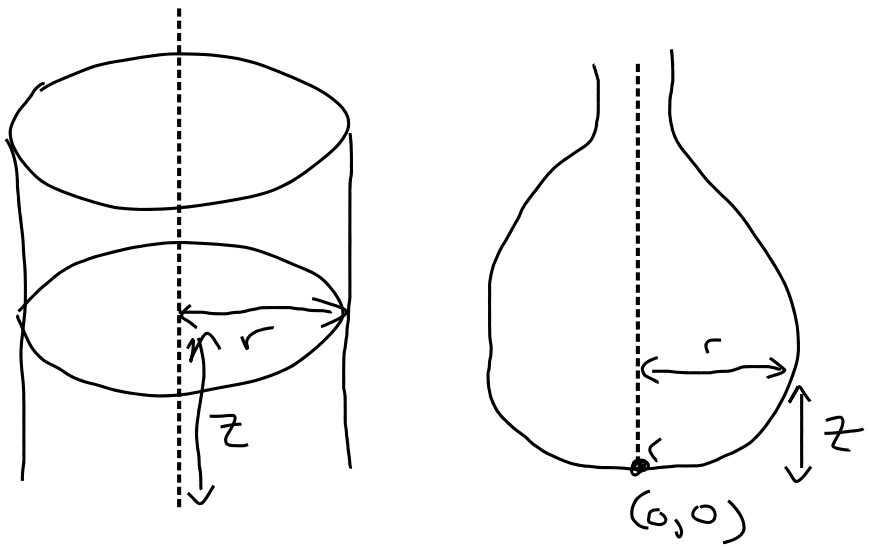
$$\text{pressure from s.t.} = \Delta p = -\gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1, R_2 are the radii of curvature.
Gravity creates **hydrostatic pressure** proportional to depth:

$$\text{pressure from gravity} = -Cz$$

(e.g. the weight of water creating pressure in deep ocean).

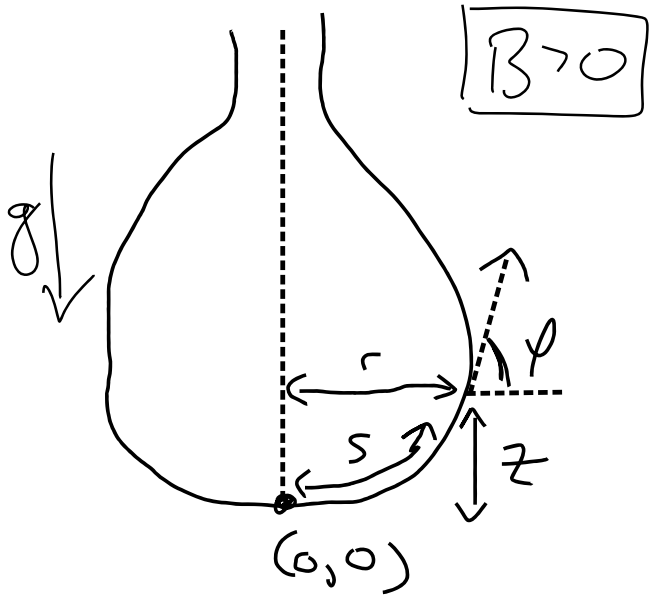
Now we assume the drop is circular around the vertical axis (cylindrical symmetry) and consider coordinates (r, z) (cylindrical).



Pendant drop

... we obtain the following system of ODEs:

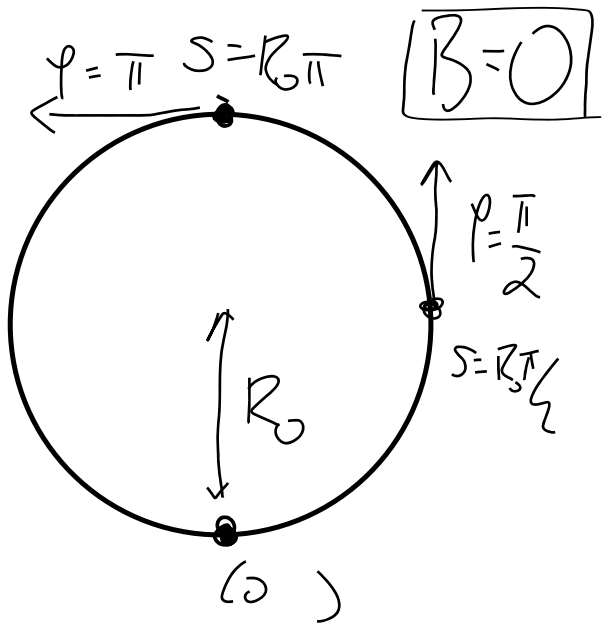
$$\frac{d\phi}{ds} = \frac{2}{R_0} - \frac{B}{R_0^2} z - \frac{\sin\phi}{r}$$
$$\frac{dz}{ds} = \cos\phi$$
$$\frac{dr}{ds} = \sin\phi$$



- The independent variable s is the arc length.
- The parameters are R_0 (a base 'radius') and $B = \rho g R_0^2/\gamma$, the **Bond number**.
- Given B, R_0 we can compute surface tension!

The ODE only has an analytical solution with no gravity - just a spherical drop:

$$\phi = s/R_0,$$
$$r = R_0 \sin(\phi), \quad z = R_0(\cos(\phi) - 1),$$



Pendant drop

Letting $\mathbf{y} = (\phi, z, r)$ we have a system of the form

$$\mathbf{y}'(s) = F(\mathbf{y})$$

subject to the initial conditions

$$z = r = \phi = 0 \text{ at } s = 0.$$

In python:

```
def pendant(s, y, b, r0):
    dp = 2/r0 - b*y[1]/r0**2 - sin(y[0])/r
    dz = cos(y[0])
    dr = sin(y[0])
    return np.array([dp, dz, dr])
```

We can then solve it using an ode solver...

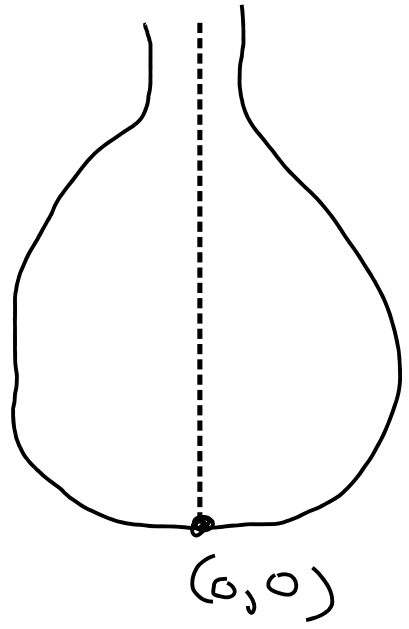
```
ic = np.zeros(3)  # initial condition
b = 1
r0 = 1
s, y = ode_solver(lambda s,y: pendant(s,y,b,r0), [0, 10], ic, h)
```

Assume that the ODE solver takes a function $f(s, y)$...
The lambda function lets us 'capture' b and r_0.
This makes `pendant` into a function of (s, y) only.

Pendant drop

Next, we need to know where to stop.

- The pendant shape ends where the drop is attached to the device.
- Assume this occurs when $\phi = \pi/2$
- $\phi = \pi/2$ happens once before, too
- $\implies$ stop at second point with $\phi = \pi/2$



Now let's assume we have measured data

$$(\hat{s}_k, \hat{r}_k, \hat{z}_k), \quad k = 0, \cdots N$$

where $s_k = kh$ for a uniform spacing h, stopping with the criterion above.

Let $r(s; B, R_0)$ (etc.) be the numerical solution. Then the LS error is

$$E(B, R_0) = \sum_{k=0}^{N} \left[(r(\hat{s}_k; B, R_0) - \hat{r}_k)^2 + (z(\hat{s}_k; B, R_0) - \hat{z}_k)^2 \right]$$

Pendant drop

Finally, we have stated a least-squares problem - to minimize

$$E(B, R_0) = \sum_{k=0}^{N} \left[(r(\hat{s}_k; B, R_0) - \hat{r}_k)^2 + (z(\hat{s}_k; B, R_0) - \hat{z}_k)^2 \right]$$

the least squares error between the **measured** and **numerical** solutions.

This can be solved using gradient descent. Finally, the surface tension is computed from B and R_0 using

$$\gamma = \frac{B}{\rho g R_0^2}.$$

All that is left is to apply gradient descent. We need:

- A function to compute $E(B, R_0)$ (which calls the ODE solver)
- A function to compute ∇E (more work required!)

Non-dimensionalization

For our fluid droplet example - in dimensional form,

$$\gamma\left(\frac{d\phi}{ds}+\frac{\sin\phi}{r}\right)=2\gamma R_0-\rho g z,\ \frac{dz}{ds}=\cos\phi,\qquad \frac{dr}{ds}=\sin\phi$$

where $2\gamma R_0$ is a 'reference pressure' $\rho =$ fluid density.

We can use R_0, the droplet 'radius' as a scale for all length variables,

$$s=R_0\hat{s},\quad z=R_0\hat{z},\quad r=R_0\hat{r},$$

and ϕ is already non-dimensional, leading to

$$\frac{\gamma}{R_0}\left(\frac{d\phi}{d\hat{s}}+\frac{\sin\phi}{\hat{r}}\right)=2\gamma R_0-\frac{\rho g}{R}\hat{z}.$$

Multiply by the factor on the left to get the non-dimensional form

$$\frac{d\phi}{d\hat{s}}+\frac{\sin\phi}{\hat{r}}=2-\frac{\rho g R_0^2}{\gamma}.$$

Non-dimensionalization

We then get the non-dimensional system

$$\frac{d\phi}{ds} + \frac{\sin\phi}{r} = 2 - Bz$$

$$\frac{d\hat{z}}{d\hat{s}} = \cos\phi, \qquad \frac{d\hat{r}}{d\hat{s}} = \sin\phi$$

where

$$B = \frac{\rho g R_0^2}{\gamma} \quad \text{('Bond number')}.$$

The Bond number can be interpreted as

$$B = \frac{\text{strength of gravity forces}}{\text{strength of surface tension forces}}.$$

The non-dimensionalization tells us that:

- The system's behavior really depends only on B

  e.g. a water droplet and a liquid gallium (like mercury) droplet - about 7 times as dense and 7 times higher surface tension - behave the same way.

- We really only need one parameter for the ODE