

Lecture 1: stable homotopy theory

8/27/25

1 Stability

A phenomenon in homotopy theory is *stable* if it occurs in all sufficiently large dimensions in essentially the same way. Otherwise, it is *unstable*. The meaning of sufficiently large usually depends on the connectivity of the spaces involved. There is some confusing terminology associated with connectivity, so let's recall the following definitions. Write $[X, Y]$ for the set of base-point preserving homotopy classes of maps $X \rightarrow Y$, where X and Y are based CW-complexes.

Definition 1.1. X is n -connected if for all base points and all $k \leq n$ the homotopy group $\pi_k(X) := [S^k, X]$ is trivial.

For example, if X is a CW-complex of dimension $\leq n$ and Y is n -connected, then any map $X \rightarrow Y$ is null-homotopic. The wedge product $X \vee Y$ of spaces X and Y is the coproduct in based spaces and is defined by taking the disjoint union and identifying the base points. The smash product $X \wedge Y$ of X and Y is defined to be

$$X \wedge Y = X \times Y / (X \vee Y),$$

but it isn't the categorical product in based spaces; that's still $X \times Y$. The (reduced) suspension ΣX of X is defined $\Sigma X = S^1 \wedge X$. The suspension of a map $f : X \rightarrow Y$ is $1 \wedge f : S^1 \wedge X \rightarrow S^1 \wedge Y$, defining a function Σ

$$\Sigma : [X, Y] \rightarrow [\Sigma X, \Sigma Y].$$

Theorem 1.2. If Y is $n-1$ connected, then E is surjective if $\dim X \leq 2n-1$ and bijective if $\dim X < 2n-1$.

When suspension induces an isomorphism, maps are called stable.

Example 1.3. $\Sigma : [S^3, S^2] \cong \mathbb{Z}\eta \rightarrow [S^4, S^3] \cong \mathbb{Z}/2$ is surjective. The map $\eta : S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \rightarrow S^2 = \mathbb{C}\mathbb{P}^1$ defined $\eta(z_1, z_2) = z_1/z_2$ is the Hopf map. $\Sigma : [S^4, S^3] \cong \mathbb{Z}/2 \rightarrow [S^5, S^4] \cong \mathbb{Z}/2$ is an isomorphism. (You can use this example to remember the $n-1, 2n-1, <$ etc.)

It follows from the Hurewicz theorem (see for example [H, 4.2 p. 366]) that the suspension of an n -connected space, is $n+1$ -connected. Thus when $\dim X$ is finite, $2n-1-\dim X$ increases by one under stabilization Σ . So, all maps become stable after sufficiently many suspensions. Stable homotopy classes of maps are easier to compute because they form generalized (co)homology theories. They are the maps in the homotopy category of the stable homotopy theory. Stable homotopy theory has the advantages that

- all generalized (co)homology theories are representable.
- finite CW complexes (e.g. manifolds) are dualizable.

We'll define and discuss these in a few lectures. Today we'll discuss some applications of stable homotopy theory.

2 A few applications

Question 2.1. *What is the maximum number of linearly independent vector fields on S^{n-1} ?*

This question was solved by Adams: S^{n-1} has $\rho(n)-1$ linearly independence vector fields, but not $\rho(n)$, where $\rho(n)$ is the n th Radon-Hurwitz number. $\rho(n)$ is computed by expressing n as $n = (2a+1)2^b$ and $b = c+4d$ with $0 \leq c < 4$, then setting $\rho(n) = 2^c + 8d$.

Problem 2.2. *Classify compact oriented smooth n -manifolds up to a certain equivalence relation called cobordism.*

Denote the resulting group by Ω_n . It is not clear that this problem is even in homotopy theory, but it is, and moreover it turns out to be stable: René Thom introduced the Thom complex $MSO(m)$ and gave an isomorphism

$$\Omega_n \cong \pi_{m+n}(MSO(m))$$

for $m > n+1$. See [At].

Question 2.3. *For which n is S^{n-1} an H -space?*

For example, S^3 is an H -space, but S^5 is not. The answer is “yes” if and only if $n = 1, 2, 4, 8$, and was also solved by Adams. This problem is unstable, but it can be solved with stable homotopy theory. There exist natural transformations

$$\text{Sq}^n : H^m(X, \mathbb{Z}/2) \rightarrow H^{m+n}(X, \mathbb{Z}/2)$$

called *Steenrod operations* which are examples of *stable cohomology operations*. The question can be reduced to the following problem. Suppose that $m \geq n$. Is there a CW complex $X = S^m \cup e^{m+n}$ such that Sq^n is non-zero? This question is also equivalent to the Hopf invariant 1 problem, which asks for which n does there exist a map $f : S^{2n-1} \rightarrow S^n$ with Hopf invariant one? (Answer: $n=2,4,8$). The Hopf invariant $H(f)$ of f is defined as follows. Form the complex $X = S^n \cup e^{2n}$ where the attaching map is f . Equivalently,

$$S^{2n-1} \xrightarrow{f} S^n \rightarrow X$$

is a cofiber sequence. Then $H^*(X) = \mathbb{Z}$ or 0 with \mathbb{Z} exactly when $*$ = $0, n, 2n$. Let x be a generator in dimension n and y be a generator in dimension $2n$. Then $x^2 = H(f)y$. There are pretty pictures of this invariant. For example, for a map $S^3 \rightarrow S^2$, $H(f)$ can be computed as the linking number of the two 1-manifolds given by inverse images of two chosen points in S^2 . See [S]. This question can also be phrased in terms of the $s = 1$ line of the Adams spectral sequence for the stable homotopy groups of the sphere.

Question 2.4. *For which n , does there exist a smooth, stably framed n -manifold with Kervaire invariant one?*

Although this question dates from the 1960s, it wasn't fully solved until December of 2024 [LWX], with a major breakthrough in 2009, when Hill Hopkins and Raveland solved the question except in dimension 126. Suppose that n is congruent to $2 \pmod{4}$ and that M is a smooth stably framed n -manifold. Then there is a quadratic refinement $q : H^{n/2}(M, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ of the intersection pairing, i.e.,

$$q(x + y) = q(x) + q(y) + \langle x \cup y, [M] \rangle,$$

coming from a calculation that the n th stable homotopy group of $K(\mathbb{Z}/2, n/2)$ is $\mathbb{Z}/2$. The Arf invariant of such a quadratic form is 0 if and only if q takes the value 0 more often than it takes the value 1. Otherwise the Arf invariant is 1. The Kervaire invariant of M is the Arf invariant of q . The answer to the question is "yes" if and only if $n = 2, 6, 14, 30, 62$. It is due to Hill, Hopkins, and Raveland for $n > 126$. The $n = 126$ case was recently solved in [LWX]. This question can also be phrased in terms of the $s = 2$ line of the Adams spectral sequence for the stable homotopy groups of the sphere. See [M].

3 Exercises

Exercise 3.1. *If Y is n connected, and X is a CW complex of dimension $d \leq n$, show that $[X, Y] = *$.*

Exercise 3.2. *Show $(D^n \times X)/(S^{n-1} \times X) \simeq S^n \wedge X_+$, where $X_+ = X \amalg *$.*

Exercise 3.3. Show $H^{n+1}(\Sigma X) \cong H^n(X)$.

Exercise 3.4. (Extra credit) Compute $q : H^{n/2}(M, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ for $(S^1)^2$. See [M]. Does this quadratic form relate to those of Gross-Harris Real Algebraic Curves? Or Benedetti-Silhol Spin and Pin^{-1} structures, immersed and embedded surfaces...?

References

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