

Lecture 2: Spectra

9/6/25

We are going to study stable phenomena in homotopy theory, so we build a category of *Spectra* where homotopy theory of spaces makes sense and where suspension is forced to be an equivalence. There are a number of equivalent points of view on spectra. Here is the one in Adams [A, Part III. 2].

1 Category of Spectra

Definition 1.1. A spectrum E is a sequence of spaces E_n with a basepoint together with maps

$$\epsilon_n : S^1 \wedge E_n = \Sigma E_n \rightarrow E_{n+1}.$$

The index n may vary over \mathbb{Z} or $\mathbb{Z}_{\geq 0}$. Let's have n vary over \mathbb{Z} to be definite.

For a based space Y , let $\Omega Y = \text{Map}_*(S^1, Y)$ denote the topological space of based maps $S^1 \rightarrow Y$. Note that there is a natural equivalence

$$\text{Map}_*(\Sigma X, Y) \cong \text{Map}(X, \Omega Y)$$

for any based spaces X and Y . Thus the maps ϵ_n are equivalent to the data of maps

$$\epsilon'_n : E_n \rightarrow \Omega E_{n+1}.$$

If E_n is connected, then the image of ϵ'_n will automatically lie inside the connected component of the base point in ΩE_{n+1} . Let $\Omega_0 E_{n+1}$ denote the connected component of the base point. If all our spaces are connected, we could then consider ϵ'_n as a map to $\Omega_0 E_{n+1}$.

Definition 1.2. A spectrum E is an Omega spectrum if the maps ϵ'_n are weak equivalences for all n .

Definition 1.3. A map (“function of degree 0” in $[A]$) $E \rightarrow F$ of spectra is a sequence of maps $f_n : E_n \rightarrow F_n$ such that

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\epsilon_n} & E_{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ \Sigma F_n & \xrightarrow{\epsilon_n} & F_{n+1} \end{array}$$

commutes.

This diagram is required to commute strictly, as opposed to commuting up to homotopy.

We do not yet have a notion of homotopy classes of maps or a good way to do homotopy theory on this category. We will start to do that on Friday.

It’s useful to also have a notion of a *function of degree r* $E \rightarrow F$. This is a sequence of maps $f_n : E_n \rightarrow F_{n-r}$ such that the analogous diagram commutes.

1.1 Examples

1.2. Suspension spectra. Given a space X , the suspension spectrum $\Sigma^\infty X$ of X is the spectrum with $(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X \cong S^n \wedge X & \text{if } n \geq 0 \\ * & \text{if } n < 0. \end{cases}$

1.3. Eilenberg-MacLane spectra. An Eilenberg-MacLane space of type (π, n) is a space $K(\pi, n)$ with a base point such that

$$\pi_*(K(\pi, n)) = \begin{cases} \pi & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

For any space X , the sequence

$$0 \rightarrow \text{Ext}(\mathbb{H}_{n-1}(X), \pi) \rightarrow \mathbb{H}^n(X, \pi) \rightarrow \text{Hom}(\mathbb{H}_n(X), \pi) \rightarrow 0$$

is exact by the universal coefficient theorem for cohomology. Thus if X is $(n-1)$ -connected, $\mathbb{H}^n(X, \pi) \cong \text{Hom}(\mathbb{H}_n(X), \pi)$. When $\pi_n(X) = \pi$, the inverse of the Hurewicz homomorphism is an element of the right hand side, whence we have an element of $\mathbb{H}^n(X, \pi)$. Call this element ι .

Theorem 1.4. (*Representability of cohomology*) For any Y , the map $[Y, K(\pi, n)] \rightarrow H^n(Y, \pi)$ defined by $[f] \mapsto f^* \iota$ is a bijection.

This is a nice illustration of obstruction theory. See, for example, [MT, Theorem 1, chapter 1] or [H, Theorem 4.57].

Corollary 1.5. *There is a canonical homotopy equivalence between any two $K(\pi, n)$ spaces.*

Since $\Omega K(\pi, n+1)$ is a $K(\pi, n)$ -space, we therefore have map $\epsilon'_n : K(\pi, n) \rightarrow \Omega K(\pi, n+1)$.

Let $H\pi$ be the spectrum whose n th space is $K(\pi, n)$ together with the maps ϵ'_n . This is the Eilenberg-MacLane spectrum for the group π .

In particular, $H\pi$ is an omega spectrum.

1.4. Thom spectra: MO , MU , MSO etc..

We first define Thom spaces. For a vector bundle $E \rightarrow X$, we can choose a continuously varying norm on the fibers. Then we have the associated disk bundle $D(E) = \{e \in E : |e| \leq 1\}$ and the associated sphere bundle $S(E) = \{e \in E : |e| = 1\}$

Definition 1.6. Let $\text{Th}(E) = D(E)/S(E)$ be the Thom space of E .

If X is a compact CW-complex, then $\text{Th}(E)$ is homotopy equivalent to E^+ the one point compactification of E . (Proof: $D(E)$ is compact because it is a fiber bundle with compact base and fiber, thus $\text{Th}(E)$ is compact. E is homotopy equivalent to $D(E) - S(E)$ using a homotopy equivalence of \mathbb{R}^n with the open ball. Note that $D(E) - S(E)$ is a subspace of $\text{Th}(E)$ whose complement is a point. To see that the topology is right, one could proceed as follows: E^+ is the terminal compactification, so there is a continuous map $\text{Th}(E) \rightarrow E^+$. We have just seen that this is a bijection. Since E^+ is Hausdorff and $\text{Th}(E)$ is compact, $\text{Th}(E) \rightarrow E^+$ is therefore a homeomorphism.)

Let $\underline{1}$ be the trivial real vector bundle of rank 1 on X . By choosing a symmetric, we can have $D(\underline{1} \oplus E) \cong [0, 1] \times D(E)$, and $S(\underline{1} \oplus E) \cong \partial D(\underline{1} \oplus E) = \{0, 1\} \times D(E) \cup [0, 1] \times S(E)$. Thus $\text{Th}(\underline{1} \oplus E) \cong$

$$\frac{[0, 1] \times D(E)}{\{0, 1\} \times D(E) \cup [0, 1] \times S(E)} \cong \frac{S^1 \times D(E)}{* \times D(E) \cup S^1 \times S(E)} \cong \frac{S^1 \times \text{Th}(E)}{* \times \text{Th}(E) \cup S^1 \times *} = S^1 \wedge \text{Th}(E).$$

Exercise 1.7. If $p : E \rightarrow X$ and $p' : E' \rightarrow Y$ are two sphere bundles, we can define the fiber wise join

$$E \hat{*} E' = \frac{E \times [0, 1] \times E'}{\sim},$$

where $(x, 1, y) \sim (x, 1, y')$ if $p'y = p'y'$ and $(x, 0, y) \sim (x', 0, y)$ if $px = px'$. You can check that $E \hat{*} E'$ is a sphere bundle over $X \times Y$. The Thom space of a sphere bundle $E \rightarrow B$ is its mapping cone. Show that

1. $\text{Th}(S(V)) \cong \text{Th} V$ for a vector bundle V
2. $\text{Th}(X \times S^{n-1}) \cong \Sigma^n X_+$, where $X_+ = X \amalg *$
3. $\text{Th}(E \hat{*} E') \cong \text{Th}(E) \wedge \text{Th}(E')$.

Example 1.8. Let $X = \mathbb{R}P^n = S^n/(v \sim -v)$ and let $\mathbb{L} \rightarrow \mathbb{R}P^n$ denote the tautological bundle, which is the vector bundle whose fiber over $v \in S^n$ is $\ell = \mathbb{R}v$. We can identify $\text{Th}(\mathbb{L}^*)^k$ with a truncated projective space for any k as follows. Any linear function from $\ell \in \mathbb{R}^{n+1}$ to \mathbb{R}^k determines a graph in $\mathbb{R}^k \times \mathbb{R}^{n+1}$, which in turn determines an element of $(\mathbb{R}^k \times \mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \cong \mathbb{R}P^n + k$. The only elements of $\mathbb{R}P^n + k$ not equal to such a graph, are lines in \mathbb{R}^k . Thus we have a vector bundle

$$V = \mathbb{R}P^{n+k} - \mathbb{R}P^{k-1} \rightarrow \mathbb{R}P^n$$

whose fiber over $\ell \in \mathbb{R}P^n$ is $\text{Hom}(\ell, \mathbb{R}^k)$. Thus $V = (\mathbb{L}^*)^k$.

$\text{Th}((\mathbb{L}^*)^k)$ is the one point compactification of $(\mathbb{L}^*)^k$ because X is compact. Thus

$$\text{Th}((\mathbb{L}^*)^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}.$$

There is notation for $\mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$. Define $\mathbb{R}P_k^{n+k} = \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$.

A model for $BO(n)$ is the Grassmannian of n -planes in \mathbb{R}^∞ , meaning $BO(n) = EO(n)/O(n)$ where $EO(n)$ is the colimit $m \rightarrow \infty$ of the space of orthogonal n -tuples in \mathbb{R}^m . Let $\zeta_n \rightarrow BO(n)$ be the vector bundle whose fiber over a point of $BO(n)$ is the vector space spanned by the orthogonal n -tuples. ζ_n is called the *tautological bundle* over the classifying space of $BO(n)$, or the *universal* real vector bundle of rank n . Any rank n real vector bundle over a space Y is the pull-back of ζ_n under some map $Y \rightarrow BO(n)$. See [H2, Theorem 1.16]. Thus the $n+1$ bundle $\underline{1} \oplus \zeta_n$ maps to ζ_{n+1} . Applying Th , we have $\epsilon_n : \Sigma \text{Th}(\zeta_n) = \text{Th}(\underline{1} \oplus \zeta_n) \rightarrow \text{Th}(\zeta_{n+1})$.

Definition 1.9. MO is the spectrum whose n th space is $\text{Th}(\zeta_n)$ and structure maps ϵ_n

Definition 1.10. MU is the spectrum whose $2n$ th space is the Thom space of the tautological bundle over $BU(n)$, and whose $(2n+1)$ st space is the suspension of the $(2n)$ th space.

Alternatively, we could construct MU as follows, once Σ is an equivalence and we have colimits. Then it makes sense to define a spectrum $MU(n) = \Sigma^{-2n} \text{Th}(\zeta_n)$ where ζ_n now denotes the tautological bundle over $BU(n)$. There is a canonical map

$$MU(n-1) \cong \Sigma^{2-2n} \text{Th}(\zeta_{n-1}) \cong \Sigma^{-2n} \text{Th}(\zeta_{n-1} \oplus \underline{1}) \rightarrow \Sigma^{-2n} \text{Th}(\zeta_n) \cong MU(n).$$

Then $MU = \text{colim}_{n \rightarrow \infty} MU(n)$.

2 Additional exercises

Exercise 2.1. For a pointed space X , prove an adjunction between $(-)\wedge X$ and $\text{maps}_*(X, (-))$ under appropriate topological hypothesis. See [ST].

Exercise 2.2. Let $Z \subseteq Y$ be a subcomplex. Show $(Y \times X)/(Z \times X) \simeq (Y/Z) \wedge X_+$.

Exercise 2.3. Let $V \rightarrow X$ be a vector bundle. Show there are homotopy equivalences $\text{Th}(V) \simeq \mathbb{P}(V \oplus \underline{1})/\mathbb{P}(V) \simeq V/(V - 0_X)$ where 0_X denotes the 0 section of V .

Exercise 2.4. Let \mathbb{L} denote the tautological bundle on $\mathbb{R}\mathbb{P}^2$. Compute the map $\text{Th}(\mathbb{L}) \rightarrow \text{Th}(\mathbb{L}^2)$ in terms of more familiar spaces.

References

- [A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.
- [H] Allen Hatcher, *Algebraic Topology*.
- [H2] Allen Hatcher, *Vector bundles and K-theory*.
- [MT] Robert Mosher and Martin Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Dover, 1968, 2008.
- [ST] N.E. Steenrod *A Convenient Category of Topological Spaces*.