Lecture 3: the stable homotopy category

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The *stable homotopy category* is the category of spectra where the morphisms, instead of being maps of spectra, are something like homotopy classes of maps of spectra.

There are many constructions of the stable homotopy category. Here is the one in Adams [A, Part III. 2]. This construction starts from a category of spectra, and then changes the maps (which he calls "functions") to something like homotopy classes of maps (which he calls "morphisms"). The reason that there are other constructions is that there is a problem with this one. It is important for the stable homotopy category to have an associative and commutative smash product denoted $\wedge$ (or sometimes $\otimes$ because of similarities with the tensor product). For example, this smash product appears in the definition of the generalized homology theory associated to a spectrum, as well as in the definition of the cup product, and in duality theorems. The problem with the category of spectra given here is that it does not have an associative and commutative smash product, and then the resulting construction of an associative and commutative smash product on the stable homotopy category is problematic (but it exists).

1 Homotopy category of spectra

We follow [A].

Let E be a spectrum with spaces E_n and structure maps $\epsilon_n : \Sigma E_n \to E_{n+1}$. The homotopy groups of E are defined

$$\pi_r(E) := \operatorname*{colim}_{n\to\infty} \pi_{r+n} E_n$$

where the maps implicit in this colimit take the class represented by $\gamma : S^{n+r} \to E_n$ to the class of $\epsilon_n \circ \Sigma\gamma$.

Last time, we defined a *function of degree* r of spectra $E \to F$ to be a sequence of maps $f_n : E_n \to F_{n-r}$ such that the appropriate diagram commutes diagram commutes. If F is an Ω spectrum, this is a good definition of a function. However, in general, we're missing some maps we'd like this way.

Example 1.1.

- *Let* $\mathbb{S}$ *denote the sphere spectrum* $\mathbb{S} = \Sigma^\infty_+ *$. *Let* $\eta : S^3 \to S^2$ *be the Hopf map. We should have a function* $\mathbb{S} \to \mathbb{S}$ *of degree* 1. *However, we do not have appropriate maps maps* $S^2 \to S^1$, $S^1 \to S^0$.
- *There are interesting stable homotopy classes of maps in* $\pi_{4k-1}\mathbb{S} = \mathrm{colim}_n[S^{n+4k-1}, S^n]$ *generating the image of* J *in* $\pi_{4k-1}\mathbb{S}$. *Let* $E_n = S^{n+3} \vee S^{n+7} \vee S^{n+11} \vee \dots$ *Let* $F_n = S^n$. *We don't have a function* $E \to F$ *whose component on* S^{n+4k-1} *is a generator for the image of* J.

Definition 1.2. *E is a CW-spectrum if*

- *for each n, E_n is a CW-complex with base point.*
- *each map* $\epsilon_n : \Sigma E_n \to E_{n+1}$ *is an isomorphism from* ΣE_n *to a sub complex of* E_{n+1}.

Remark 1.3. *When we have a definition of weak equivalence, we will see that any spectrum is weakly equivalent to a CW-spectrm. See [A, Part III, Exercise after 3.12]. So it is okay to restrict attention to CW-spectra.*

A *subspectrum* A of a CW-spectrum E is a spectrum with $A_n \subset E_n$ a subcomplex.

A is said to be *cofinal* in E if for each n and each finite subcomplex $K \subset E_n$ there is an m such that $\Sigma^m K$ maps into A_{m+n} under the canonical map

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\epsilon_n} \Sigma^{m-1} E_{n+1} \xrightarrow{\Sigma^{m-2}\epsilon_{n+1}} \dots \xrightarrow{\epsilon_{n+m-1}} E_{m+n}$$

If E and E' are two cofinal subspectra of E, and $f' : E' \to F$ and $f'' : E'' \to F$ are two functions, say that f' and f'' are *equivalent* if there is a cofinal sub spectrum E'' contained in E' and E'' such that the restrictions of f' and f'' to E''' are equal. Since the intersection of two cofinal subspectra is a cofinal subspectra, this defines an equivalence relation.

Adams calls the set of equivalence classes of functions from all cofinal subspectra of E to F the "maps" from E to F.

Definition 1.4. *An equivalence class of a map from a cofinal subspectrum of E to F will be called a* map *from E to F.*

Definition 1.5. *For a spectrum E define the suspension ΣE to be the spectrum given by $E_2, E_3, \dots$ and the desuspension $\Sigma^{-1}E$ to be $*, E_1, E_2, \dots$.*

Example 1.6. *Let $\mathbb{S}$ denote the* sphere spectrum *meaning the suspension spectrum of S^0. Let $K \subset \mathbb{S}$ be the subspectrum with $K_n = *$ for $n \leq 2$ and $K_n = \mathbb{S}^n$ for $n \geq 3$. Let $\eta : S^3 \to S^2$ be the Hopf map. $\Sigma^n \eta$ defines a map $K_{n+3} \to \mathbb{S}_{n+2}$ for $n \geq 0$. For $n < 0$, $K_{n+3} = *$ is a point and there is a unique map $K_{n+3} \to \mathbb{S}_{n+2}$. These maps $K_{n+3} \to \mathbb{S}_{n+2}$ are the data of a pmap of degree 1 from $\mathbb{S}$ to $\mathbb{S}$. Equivalently, this data gives a pmap $\mathbb{S} \to \Sigma^{-1}\mathbb{S}$.*

Proposition 1.7. *Composition of functions of spectra determines a well-defined composition of maps.*

Proof. Let F' be a cofinal subspectrum of a spectrum F and $E \to F$ a function of spectra (or a function of degree $r \neq 0$). We claim that there is a cofinal subspectrum of E which maps into F'. Note that there is a largest CW subcomplex E'_n of E_n which is mapped to F', i.e. a cell is included precisely if it and all the lower dimensional cells required by attaching maps are mapped into F'. By the definition of a function of spectra, and the fact that F' is a subspectrum, we must have $\epsilon_n(\Sigma E'_n) \subset E'_{n+1}$, where $\epsilon_n : \Sigma E_n \to E_{n+1}$ is the structure map in E. Thus there is a largest CW subspectrum E' of E mapping to F'. We claim that this subspectrum is cofinal. Let K be a finite sub complex of E_n. The image of K is compact, and therefore is contained in a finite sub complex K_F of F_n. See for example [H, Prop A.1 p 520]. Since F' is cofinal, there is an m as above for K_F. Since $\Sigma^m K$ maps into $\Sigma^m K_F \subset F'$, we have that $\Sigma^m K$ maps into F', from which it follows that $\Sigma^m K$ is in E', showing E' is cofinal.

Given two maps $f : E \to F$ and $g : F \to H$, we have cofinal sub spectra F' and E' of F and E on which g and f are defined respectively. By the above, there is a cofinal subspectrum E'' of E' mapping into F' under f. It is straightforward to check that a cofinal subspectrum of a cofinal subspectrum is cofinal. Thus E'' is a cofinal subspectrm of E. We may define $g \circ f$ on E''. ☐

Now we wish to define the homotopy class of a map. For a space X, let $X_+ = X \coprod *$. Let $\mathrm{Cyl}(E)$ denote the spectrum whose nth space is $[0,1]_+ \wedge E_n$ and with structure maps given by the flip $S^1 \wedge [0,1]_+ \wedge E_n \to [0,1]_+ \wedge S^1 \wedge E_n$ composed with $[0,1]_+ \wedge \epsilon_n$. The maps $\{0\}_+ \to [0,1]_+$ and $\{1\}_+ \to [0,1]_+$ allow us to define two maps $E \to \mathrm{Cyl}(E)$. Two maps $f, g : E \to F$ are *homotopic* if there is a map $H : \mathrm{Cyl}(E) \to F$ such that H precomposed with our maps $E \to \mathrm{Cyl}(E)$ produces f and g. The standard proof that homotopy is an equivalence relation applies.

Definition 1.8. *A morphism $E \to F$ in the stable homotopy category from a CW-spectrum E to F is a homotopy class of maps. A morphism of degree r is a homotopy class of a degree r pmap. Let $[E, F]$ denote the morphisms from E*

to F in the stable homotopy category, and let $[E, F]_r$ denote the morphisms of degree r.

Lemma 1.9. *([A, III Prop 2.8]) Let F be any spectrum. For X a finite CW-complex there is a natural identification $[\Sigma^\infty X, F]_r = \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$*

On the right hand side the colimit is taken over maps $[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, F_{n+1}]$ which are the composition of the suspension $[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, \Sigma F_n]$ with the map $[\Sigma^{n+r+1}X, \Sigma F_n] \to [\Sigma^{n+r+1}X, F_{n+1}]$ induced by the structure map of F $\Sigma F_n \to F_{n+1}$.

Proof. For a map $f_{n+r} : \Sigma^{n+r}X \to F_n$, there is a pmap of degree r of spectra $\Sigma^\infty X \to F$ defined on the cofinal subspectrum whose mth space is $\Sigma^m X$ for $m \geq n+r$ and $*$ for $m < n+r$. This pmap is given by $\Sigma^{m-n-r}f_{n+r}$ for $m \geq n+r$ and is the unique map from $*$ for $m < n+r$. Moreover, if $f_{n+r}, f'_{n+r} : \Sigma^{n+r}X \to F_n$ are homotopic, we may likewise construct a pmap $\operatorname{Cyl}(\Sigma^\infty X) \to F$ of degree r. Thus there is a well defined map $[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$.

Note that, by construction, the image of f_{n+r} under

$$[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, F_{n+1}] \to [\Sigma^\infty X, F]_r$$

equals the image of f_{n+r} under

$$[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$$

when restricted to the cofinal subspectrum of $\Sigma^\infty X$ whose mth space is $\Sigma^m X$ for $m \geq n+r+1$ and $*$ for $m < n+r-1$. In particular, these images are equal as pmaps of degree r. Thus we have a well defined function

$$\theta : \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r.$$

θ is surjective as follows. Let K be a cofinal subspectrum of $\Sigma^\infty X$ and let $g : K \to F$ be a function of spectra of degree r. Since K is cofinal and X is a finite complex, there exists an m such that $\Sigma^m X \subset K_m$. Thus g is in the image of $[\Sigma^m X, F_{m-r}]$, showing surjectivity.

θ is injective by applying the surjectivity of θ for the pmaps of degree r from $\operatorname{Cyl}(\Sigma^\infty X) = \Sigma^\infty \operatorname{Cyl}(X)$ to F.

□

Proposition 1.10. *For X a finite CW-complex, there is a natural isomorphism $[\Sigma^\infty X, H\mathbb{Z}]_{-r} \cong \mathrm{H}^r(X, \mathbb{Z})$.*

The assumption that X is a finite CW-complex is not necessary, but here is a proof in this case. We use the following Lemma.

Proof. By Lemma 1.9, $[\Sigma^\infty X, H\mathbb{Z}]_{-r} = \operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, K(\mathbb{Z}, n)]$. The adjunction between Σ and Ω gives an equivalence of topological spaces $\operatorname{Map}(\Sigma\Sigma^{n-r}X, K(\mathbb{Z}, n+1)) \cong \operatorname{Map}(\Sigma^{n-r}X, \Omega K(\mathbb{Z}, n+1))$. Applying π_0, we have a bijection $[\Sigma^{n-r+1}X, K(\mathbb{Z}, n+1)] = [\Sigma^{n-r}X, \Omega K(\mathbb{Z}, n+1)]$. Since $\Omega K(\mathbb{Z}, n+1) \cong K(\mathbb{Z}, n)$, we have a bijection $[\Sigma^{n-r+1}X, H\mathbb{Z}_{n+1}] = [\Sigma^{n-r}, H\mathbb{Z}_n]$. Unwinding definitions, we see that this bijection is compatible with the transition maps of the colimit $\operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, H\mathbb{Z}_n]$. Thus $\operatorname{colim}_{n\to\infty}[\Sigma^{n-r}X, H\mathbb{Z}_n] = [X, H\mathbb{Z}_r] \cong \mathrm{H}^r(X, \mathbb{Z})$ where the last isomorphism is the representability of cohomology discussed before. □

2 Exercises

Exercise 2.1. *Construct a spectrum* E *and* k *such that* $\pi_k E$ *is not attained as any* $[S^{k+n}, E_k]$.

Exercise 2.2. *Show that there are canonical isomorphisms* $[E, F]_r \cong [E, \Sigma^{-r}F] \cong [\Sigma^r E, F]$ *and* $\pi_n E \cong [\Sigma^\infty_+ S^n, E]$.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*