Lecture 4: stable homotopy theory

9/12/25

We defined the *stable homotopy category* as the category with objects CW-spectra and morphisms homotopy classes of maps of spectra. We would like a stable homotopy theory, in the sense of a model category or ∞-category of spectra. Here's why:

1 Motiviation

Let **Top** denote the category of topological spaces with continuous maps. Define **HoTop** as the category of topological spaces with homotopy classes of continuous maps.

Many invariants of **Top** pass through **HoTop**:

$$\mathbf{HoTop} \to \mathbf{Ab}.$$

However, **HoTop** does not have many colimits. Given a category $\mathcal{C}$ and a functor $F\colon I \to \mathcal{C}$ (a diagram), a *colimit* of F is an object $\operatorname{colim} F$ equipped with maps

$$\iota_i\colon F(i) \to \operatorname{colim} F$$

satisfying the universal property: for any compatible collection of maps $\{F(i) \to C\}$, there exists a unique map

$$\operatorname{colim} F \to C.$$

The natural guess for a colimit in **HoTop** will usually fail. Instead of requiring diagrams that commute strictly up to homotopy, considering instead *homotopy coherent* diagrams, allows one tp define a homotopy theory of spaces or spectra, which will have (co)limits.

Additionally, there is a resulting definition of the stable homotopy theory which is also appealingly intuitive: Let R be a commutative ring and x be an element of r. For any R-module M, we can formally invert multiplication by r by forming

$$M[r^{-1}] \cong \operatorname{colim}(M \xrightarrow{r} M \xrightarrow{r} M \dots)$$

With the appropriate setup, one can define spectra $\mathbf{Sp}$ to be the homotopy theory

$$\mathbf{Sp} \simeq \mathbf{Top}_*[(S^1)^{\wedge -1}] \simeq \operatorname{colim}_{\mathbf{Pr}^L}(\mathbf{Top}_* \xrightarrow{\wedge S^1} \mathbf{Top}_* \xrightarrow{\wedge S^1} \dots)$$

2 ∞-categories

See `https://www.math.ias.edu/~lurie/287xnotes/Lecture2.pdf` page 4-5

See `https://www.math.ias.edu/~lurie/287xnotes/Lecture3.pdf` page 1-2

There is an infinity category $\mathbf{Cat}_\infty$ of infinity categories and an equivalence

$$\mathbf{Sp} \simeq \lim_{\mathbf{Cat}_\infty}(\dots \xrightarrow{\Omega} \mathbf{Top}_* \xrightarrow{\Omega} \mathbf{Top}_*)$$

3 Exercises

Exercise 3.1. *Show suspension is an equivalence of categories from the stable homotopy category to itself.*

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[I] R. Iwasa *Motivic Stabe Homotopy Theory* IHES lecture July 18, 2023.

[SAG] J. Lurie, *SAG Appendix C.*