# Lecture 5: Generalized (co)homology

9/16/25

## 1 Cofiber sequences and the Puppe sequence

If $f : X \to Y$ is a map of CW complexes, the reduced *mapping cone* is the space $Y \cup_f CX = (Y \coprod X \wedge [0,1])/\sim$, where $(x,1) \sim f(x)$, $(x,0) \sim *$ and $(*_x, 1) \sim *$. If we vary $f$ by a (base-point preserving) homotopy, $Y \cup_f CX$ changes by a homotopy equivalence.

We may likewise form the reduced mapping cone of a map $f : X \to Y$ in the stable homotopy category. $f$ is represented by a function of spectra $f : X' \to Y$, where $X'$ is a cofinal subspectrum of $X$. By replacing $f$ by a homotopic map, we may assume that $f'_n : X'_n \to Y_n$ is a cellular map. Then $Y \cup_f CX$ is the spectrum whose $n$th space is $Y_n \cup_{f_n} CX'_n$ and whose structure maps are induced from those of $X$ and $Y$. This is well-defined up to isomorphism, because varying $f$ by a homotopy does not change the isomorphism class of $Y \cup_f CX$.

If $i : A \to X$ is the inclusion of a closed subspectrum, then define $X/A$ be the spectrum whose $n$th space is $X_n/A_n$ and whose structure maps are those induced from the structure maps of $X$.

**Exercise 1.1.** *Show the evident map $X \cup_i CA \to X/A$ is an isomorphism in the stable homotopy category because on the level of spaces we have homotopy equivalences.*

**Definition 1.2.** *A* cofiber sequence *is any sequence equivalent to a sequence of the form $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$*

**Proposition 1.3.** *([A, III Prop 3.9]) For each $Z$, the sequence $[Y \cup_f CX, Z] \to [Y, Z] \to [X, Z]$ is exact.*

*Proof.* Since the composite $X \to Y \cup_f CX$ is null, we have that the image of $[Y \cup_f CX, Z] \to [Y, Z]$ is indeed in the kernel of $[Y, Z] \to [X, Z]$. Suppose

$g : Y' \to Z$ is a function such that $Y'$ is a cofinal subspectrum of $Y$ and such that the associated morphism is null in $[X, Z]$. We wish to construct a map $Y \cup_f CX \to Z$ extending the map $g : Y \to Z$. To do this, choose a cofinal subspectrum $X'$ of $X$ such that $gf$ is defined as a function on $X'$. . We checked that we may choose a cofinal subspectrum $Y''$ of $Y$ containing the image of $X'$ . By hypothesis, there is a cofinal subspectrum $W$ of $\mathrm{Cyl}(X)$ such that there is a function $H : W \to Z$ giving a homotopy between $gf$ and the constant map. Since $X' \wedge [0,1]_+$ is also a cofinal subspectrum of $\mathrm{Cyl}(X)$, by passing to the intersection, we may assume that $W$ is a subspectrum of $X' \wedge [0,1]_+$. Thus we may form the cofinal subspectrum $(Y'' \cup_f W)/((x,0) \sim *, (*_x, 1) \sim *)$ of $Y \cup_f CX$ and $H$ and $g$ determine a function $(Y'' \cup_f W)/((x,0) \sim *, (*_x, 1) \sim *) \to Z$.

□

Any map can be extended to a cofiber sequence. In particular, we can extend cofiber sequences to the right

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to (Y \cup_f CX) \cup_i CY \to$$

**Exercise 1.4.** $(Y \cup_f CX) \cup_i CY \cong \Sigma X$

From Exercise 1.4, we get that

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to \Sigma X \to \Sigma Y \to \Sigma(Y \cup_f CX) \to \Sigma\Sigma X \tag{1}$$

has all three term sequences cofiber sequences.

Note that desuspensions and suspensions of cofiber sequences are cofiber sequences. Applying $\Sigma^{-1}$ to the cofiber sequence $Y \cup_f CX \to \Sigma X \to \Sigma Y$, we have that $\Sigma^{-1} Y \cup_f CX \to X \to Y$ is a cofiber sequence. Thus we may continue (1) to the left.

**Corollary 1.5.** *If $X \to Y \to Z$ is a cofiber sequence in the stable homotopy category, then for any $W$, the sequence*

$$\ldots \to [X, W]_{n+1} \to [Z, W]_n \to [Y, W]_n \to [X, W]_n \to \ldots$$

*is exact.*

*Proof.* The sequence

$$\ldots \to [\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to \ldots$$

is exact by the above chain of cofiber sequences and Proposition 1.3. Since we have identified desuspension with a shift, suspension may also be identified with a shift. Thus $[\Sigma^n Y, W] = [Y, W]_n$. □

## 2 Fiber sequences are cofiber sequences stably

**Proposition 2.1.** *If* $X \xrightarrow{f} Y \xrightarrow{i} Z$ *is a cofiber sequence in the stable homotopy category, then for any* $W$*, the sequence*

$$\ldots \to [W,X]_n \to [W,Y]_n \to [W,Z]_n \to [W,X]_{n-1} \to \ldots$$

*is exact.*

*Proof.* As above, it suffices to show that

$$[W,X] \to [W,Y] \to [W,Z]$$

is exact. Since the composite $X \to Z$ is null, we have that the composite $[W,X] \to [W,Z]$ is 0. Let $g : W \to Y$ be a map such that $ig$ is nullhomotopic. The choice of a null-homotopy gives the morphism $h : CW \to Z$ from cone on $W$ to $Z$. We then obtain $j$ and $k$ in the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & Z \arrow[r] & \Sigma X \arrow[r, "-f"] & \Sigma Y \\
W \arrow[r] & W \arrow[u, "g"] \arrow[r] & CW \arrow[u, "h"] \arrow[r] & \Sigma W \arrow[u, "j"] \arrow[r, "-1"] & \Sigma W \arrow[u, "\Sigma g"]
\end{tikzcd}$$

Since suspension is an equivalence, we have that the image of $\Sigma^{-1} j$ under $[W,X] \to [W,Y]$ is $g$. □

One could define a fiber sequence to be $X \to Y \to Z$ such that the composite $X \to Z$ was the constant map and such that the sequence satisfies the conclusion of Proposition 2.1. Dually, one could also define a cofiber sequence to be $X \to Y \to Z$ such that $X \to Z$ is null and satisfying the conclusion of 1.5. This is the same as the above (exercise: use a natural map and the 5 lemma to show it induces an isomorphism on $\pi_*$). Proposition 2.1 can therefore be stated by saying that fiber sequences and cofiber sequences are the same in the stable homotopy category.

## 3 Smash product

For $X$ and $Y$ objects of the stable homotopy category, one can construct a commutative, associative smash product with unit such that $X \wedge (-)$ preserves cofiber sequences and wedge sums. Let's not construct this. We also assume the existence for each $X$, $Y$ of a spectrum $F(X,Y)$ such that

$$[W \wedge X, Y] = [W, F(X,Y)]. \tag{2}$$

# 4 Generalized (co)homology

**Definition 4.1.** *Let $E$ be a spectrum. The generalized $E$-homology of degree $r$ is the functor from the stable homotopy category to abelian groups*

$$X \mapsto E_r(X) := \pi_r(E \wedge X).$$

*The generalized $E$-(co)homology of degree $r$ is*

$$X \mapsto E^r(X) := \pi_{-r}F(X, E).$$

Let $A$ be an abelian group and $HA$ denote the Eilenberg-MacLane spectrum associated to $A$.

**Proposition 4.2.** *There is a natural isomorphism between the singular reduced homology with coefficients in $A$ and the generalized homology of $HA$,*

$$HA_*X \cong \tilde{\mathrm{H}}_*(X, A).$$

On the left hand side, we may choose any 0 simplex of $X$ as the base point to take the suspension spectrum.

*Proof.* Let $X$ be a pointed CW-complex with $n$-skeleton $X^{(n)}$. Let $C_n$ be the set of $n$-cells of $X$, i.e. $X^{(n)}/X^{(n-1)} \cong \vee_{C_n} S^n$. We show that $HA_*X$ can be computed with the reduced cellular chain complex associated to $A$.

Note that $HA_*(S^n) \cong HA_{*-n} \cong \begin{cases} 0 & \text{if } * \neq n \\ \mathbb{Z} & \text{if } * = n. \end{cases}$. Thus by Proposition 4.5, $HA_*(X^{(n)}/X^{(n-1)}) \cong \begin{cases} \oplus_{C_n} A & \text{if } * = n \\ 0 & \text{if } * \neq n. \end{cases}$.

Considering the cofiber sequences

$$X^{(n-1)} \to X^{(n)} \to X^{(n)}/X^{(n-1)}$$

and the associated long exact sequences in $HA_*$ (Proposition 4.5 ), we have that $\mathrm{colim}_{m\to\infty} HA_n X^{(m)} \cong HA_n X^{(n+1)}$. Since $S^n$ is compact, we can show similarly to the argument above that $\mathrm{colim}_{m\to\infty} HA_n X^{(m)} \cong HA_n X$. It follows that $HA_*X$ can be computed with the chain complex

$$\ldots \to HA_n(X^{(n)}/X^{(n-1)}) \to HA_{n-1}(X^{(n-1)}/X^{(n-2)}) \to \ldots$$

where the differential is the composite of the boundary map $HA_n(X^{(n)}/X^{(n-1)}) \to HA_{n-1}X^{(n-1)}$ with the map induced from the quotient map $X^{(n-1)} \to X^{(n-1)}/X^{(n-2)}$.

To identify this complex with the cellular complex computing $\tilde{\mathrm{H}}_*(X, A)$ it remains to show that the degree $m$ map $S^n \to S^n$ induces multiplication by $m : A \to A$ on $HA_n$. We may show this for the degree $m$ map given as an $m$-fold pinch map followed by $\vee_{i=1}^m i$

$$S^n \to \vee_{i=1}^m S^n \overset{\vee_{i=1}^m 1}{\to} S^n.$$

The result follows by applying $HA_n$ and using the isomorphism $HA_n(\vee_{i=1}^m S^n) \cong \oplus_{i=1}^m A$ and the fact that 1 induces the identity.

$\square$

**Definition 4.3.** *A* generalized reduced homology theory *on based CW-complexes is a sequence of functors* $\tilde{h}_* : \mathrm{ho}\,\mathbf{CW}_* \to \mathbf{Ab}$ *from the homotopy category of CW-complexes equipped with a base point to abelian groups for* $* \in \mathbb{Z}$ *such that*

1. *There are natural boundary maps* $\tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(Y)$ *such that*

   $$\ldots \to \tilde{h}_*(Y) \to \tilde{h}_*(X) \to \tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(A) \to \ldots$$

   *is exact for each CW pair* $(X, Y)$.

2. *The map* $\oplus_{a \in A} \tilde{h}_*(X_a) \to \tilde{h}_*(\vee_{a \in A} X_a)$ *is an isomorphism.*

**Remark 4.4.** *A generalized reduced homology theory* $\tilde{h}_*$ *on based CW-complexes can be extended to a functor on non-based CW-complexes by setting* $\tilde{h}_*(X) = \mathrm{Ker}(\tilde{h}_*(X_+) \to \tilde{h}_*(S^0))$. *A choice of base point of a CW complex* $X$ *gives a map* $S^0 \to X_+$, *which in turn produces a direct sum decomposition* $\tilde{h}_*(X_+) \cong \tilde{h}_*(X) \oplus \tilde{h}_*(S^0)$. *Thus for a based CW complex, the new definition is canonically isomorphic to the old.*

**Proposition 4.5.** *The functors* $\mathrm{ho}\,\mathbf{CW}_* \to \mathbf{Ab}$ *defined* $X \mapsto E_*(\Sigma^\infty X)$ *defines a generalized reduced homology theory for any object* $E$ *of the stable homotopy category.*

*Proof.* For a CW-pair, there is a homotopy equivalence $X \cup CY \cong X/Y$, whence

$$Y \to X \to X/Y$$

is a cofiber sequence. Since $E \wedge (-)$ preserves cofiber sequences, we have a cofiber sequence

$$E \wedge Y \to E \wedge X \to E \wedge (X/Y).$$

The long exact sequence in homotopy groups associated to a cofiber sequence gives 1.

When $A$ is finite, 2 follows from the fact that the long exact sequence in $\pi_*$ associated to the cofiber sequence

$$E \wedge X \to (E \wedge X) \vee (E \wedge Y) \to (E \wedge Y)$$

splits into short exact sequences using the map $(E \wedge Y) \to (E \wedge X) \vee (E \wedge Y)$.

In general, we have $E \wedge (\vee_{a \in A} X_a) \cong \vee_{a \in A}(E \wedge X_a)$. Then $E_*(\vee_{a \in A} X_a) \cong \pi_*(\vee_{a \in A}(E \wedge X_a))$. Recall that $\pi_*(\vee_{a \in A}(E \wedge X_a)) \cong \operatorname{colim}_{n \to \infty} \pi_{n+*}(\vee_{a \in A}(E \wedge X_a)_n)$. Since $S^{n+*}$ is compact, any element of $\pi_{n+*}(\vee_{a \in A}(E \wedge X_a)_n)$ is in the image of a $\pi_{n+*}(\vee_{a \in A'}(E \wedge X_a)_n)$ with $A'$ finite. Since $S^{n+*} \times [0,1]$ is also compact, any homotopy between two maps $S^{n+*} \to \vee_{a \in A}(E \wedge X_a)_n$ also factors through some $(\vee_{a \in A'}(E \wedge X_a)_n)$ with $A'$ finite. Thus $\pi_{n+*}(\vee_{a \in A}(E \wedge X_a)_n) \cong \operatorname{colim}_{A' \subset A} \pi_{n+*}(\vee_{a \in A'}(E \wedge X_a)_n)$ where the colimit is taken over finite subsets $A'$ of $A$. Since colimits commute, we have that

$$E_*(\vee_{a \in A} X_a) \cong \operatorname*{colim}_{A' \subset A} \operatorname*{colim}_{n \to \infty} \pi_{n+*}(\vee_{a \in A'}(E \wedge X_a)_n) \cong \operatorname*{colim}_{A' \subset A} \oplus_{A'} E_*(X_a) \cong \oplus_A E_*(X_a).$$

□

The converse to Proposition 4.5 holds as well. In fact, Adams showed something better:

**Theorem 4.6.** *([A2]) Let $E$ be a spectrum and let $\tilde{h}_*$ be a generalized reduced homology theory. Suppose we have a map of homology theories $f : E_* \to \tilde{h}_*$. Then there exists a spectrum $F$ together with an isomorphism $F_* \cong \tilde{h}_*$ and a map $E \to F$ such that the induced map $E_* \to F_* \cong \tilde{h}_*$ is $f$.*

We'll show a version in cohomology. See [H, Section 4.E]

**Definition 4.7.** *A* generalized reduced cohomology theory *on based CW-complexes is a sequence of contravariant functors $\tilde{h}^* : \mathrm{ho}\,\mathbf{CW}_*^{op} \to \mathbf{Ab}$ from the homotopy category of CW-complexes equipped with a base point to abelian groups for $* \in \mathbb{Z}$ such that*

1. *There are natural boundary maps $\tilde{h}^*(X/Y) \leftarrow \tilde{h}_{*-1}(Y)$ such that*

$$\ldots \leftarrow \tilde{h}^*(Y) \leftarrow \tilde{h}_*(X) \leftarrow \tilde{h}_*(X/Y) \leftarrow \tilde{h}_{*-1}(A) \leftarrow \ldots$$

   *is exact for each CW pair $(X, Y)$.*

2. *The map $\tilde{h}^*(\vee_{a \in A} X_a) \to \prod_{a \in A} \tilde{h}^*(X_a)$ is an isomorphism.*

**Exercise 4.8.** *Show any generalized reduced cohomology theory satisfies the Mayer-Vietoris axiom: For $X$ the union of subcomplexes $A$ and $B$ containing the basepoint and $a \in h(A)$ and $b \in h(B)$ which restrict to the same element of $h(A \cap B)$, there exists an element $x \in h(X)$ whose restrictions to $A$ and $B$ are the given elements $a$ and $b$.*

**Theorem 4.9.** *(Brown Representability) Every reduced cohomology theory on based CW-complexes is the generalized* $E$*-cohomology associated to an* $\Omega$*-spectrum* $E$*.*

**Theorem 4.10.** *Let* $h : \mathrm{ho}\,\mathbf{CW}_*^{op} \to \mathbf{Ab}$ *be a contravariant functor satisfying the Mayer–Vietoris axiom and thee wedge axion. Then there is a connected CW-complex* $K$ *and* $u \in h(K)$ *representing* $h$ *in the sense that that the natural transformation*

$$T_u : [X, K]_* \to h(X)$$

$$T_u(f) = f^*(u)$$

*is a bijection for all* $X$*.*

**Exercise 4.11.** *Use* $X \vee * = X$ *to show that* $h(*)$ *has one element, whence* $h(*) = 0$*.*

**Exercise 4.12.** *For* $h$ *as in Theorem 4.10, the sequence*

$$h(A) \leftarrow h(X) \leftarrow h(X/A)$$

*is exact for any* $A \subset X$ *pointed sub-CW-complex.*

Following [H], we break the proof into two lemmas. We refer to Hatcher for the proofs.

**Lemma 4.13.** *[H:Lemma 4E.3] Given any pair* $(Z, z)$ *with* $Z$ *a connected CW complex and* $z \in h(Z)$*, there exists a based connected CW-complex* $K$ *and* $u \in h(K)$ *such that* $T_u$ *is bijective for* $X$ *any sphere.*

**Lemma 4.14.** *[H:Lemma 4E.4] Suppose* $(K, u)$ *is a pair where* $K$ *is a based connected CW-complex and* $u \in h(K)$ *is such that* $T_u$ *is bijective for* $X$ *any sphere. Then for any based CW pair* $(X, A)$*, any* $x \in h(X)$*, and any* $f : A \to K$ *with* $f^*u = x|_A$ *there exists a map* $g : X \to K$ *extending* $f$ *with* $g^*(u) = x$*.*

*Proof.* (of Theorem 4.10) By Lemma 4.13, there exists $(K, u)$ such that $T_u$ is bijective for $X$ any sphere. Now take $X$ an arbitrary based CW complex. We show that $T_u$ is bijective. It follows immediately from Lemma 4.14 applied to $(X, *)$ that $T_u$ is surjective. To show injectivity, let $f_0, f_1 \in [X, K]_*$ be such that $T_u(f_0) = T_u(f_1)$. We need to construct a homotopy between $f_0, f_1$. Apply Lemma 4.14 to $(X \wedge [0,1]_+, X \vee X)$ and $x \in h(X \wedge [0,1]_+) \cong h(X)$ equal to $f_0^*u = f_1^*u$. This produces the needed homotopy between $f_0$ and $f_1$ □

*Proof.* (of Theorem 4.9) Let $h^*$ be a generalized reduced cohomology theory on based CW-complexes. By Exercise 4.8 and Theorem 4.10, there is a connected CW-complex $K_i$ and $u_i$ in $h^i(K_i)$ such that $T_{u_i} : [X, K_i]_* \to h^i(X)$ defined

$T_{u_i}(f) = f^*u_i$ is a natural isomorphism. We construct a canonical equivalence $\Omega K_{i+1} \simeq K_i$.

For any CW complex $A$, the cofiber sequence

$$A \to * \to \Sigma A$$

defines an isomorphism $\delta_{A,i} : h^i(A) \xrightarrow{\cong} h^{i+1}(\Sigma A)$.

We have a canonical evaluation map $\mathrm{ev} : \Sigma\Omega A \to A$.

Let $a : [X, \Omega Y] \to [\Sigma X, Y]$ be the adjunction taking $f$ to $af = \mathrm{ev} \circ \Sigma f$.

The composition of isomorphisms

$$[K_i, K_i] \xrightarrow{T_{u_i}} h^i(K_i) \xrightarrow{\delta_{K_i}} h^{i+1}(\Sigma K_i) \xleftarrow{T_{u_{i+1}}} [\Sigma K_i, K_{i+1}] \xrightarrow{a} [K_i, \Omega K_{i+1}]$$

and the identity map $1_i : K_i \to K_i$ defines

$$f = a^{-1}T^{-1}_{u_{i+1}}\delta_{K_i}u_i : K_i \to \Omega K_{i+1}.$$

Note that $T_{u_{i+1}}(af) = \delta_{K_i}u_i$ or in other words

$$(\mathrm{ev} \circ \Sigma f)^* u_{i+1} = \delta_{K_i}u_i. \tag{3}$$

The composition of isomorphisms

$$[\Omega K_{i+1}, \Omega K_{i+1}] \xrightarrow{a} [\Sigma\Omega K_{i+1}, K_{i+1}] \xrightarrow{T_{u_{i+1}}} h^{i+1}(\Sigma\Omega K_{i+1}) \xleftarrow{\delta_{\Omega K_{i+1}}} h^i(\Omega K_{i+1}) \xleftarrow{T_{u_i}} [\Omega K_{i+1}, K_i]$$

and the identity morphism $1_{\Omega K_{i+1}} : \Omega K_{i+1}, \Omega K_{i+1}$ defines

$$g = T^{-1}_{u_i}(\delta_{\Omega K_{i+1}})^{-1}(\mathrm{ev}^* u_{i+1}) : \Omega K_{i+1} \to K_i.$$

Note that

$$\delta_{\Omega K_{i+1}} g^* u_i = \mathrm{ev}^* u_{i+1}. \tag{4}$$

We claim that $f$ and $g$ are the claimed homotopy equivalences. To show that $gf \sim 1_{K_i}$, it suffices to show $(gf)^*u_i = u_i$ because $T_{u_i}$ is a natural isomorphism. Applying $(\Sigma f)^*$ to (4) and then combining with (3) gives

$$(\Sigma f)^* \delta_{\Omega K_{i+1}} g^* u_i = (\Sigma f)^* \mathrm{ev}^* u_{i+1} = \delta_{K_i} u_i.$$

The commutative diagram

$$\begin{tikzcd}
h^i(K_i) \arrow[r, "\delta_{K_i}"] & h^{i+1}(\Sigma K_i) \\
h^i(\Omega K_{i+1}) \arrow[u, "f^*"] \arrow[r, "\delta_{\Omega K_{i+1}}"] & h^{i+1}\Sigma\Omega K_{i+1} \arrow[u, "(\Sigma f)^*"']
\end{tikzcd}$$

then implies that

$$\delta_{K_i} f^* g^* u_i = \delta_{K_i} u_i.$$

Since $\delta_{K_i}$ is an isomorphism, it follows that $f^* g^* u_i = u_i$ as claimed.

We leave the verification that $fg \sim 1_{\Omega K_{i+1}}$ as an exercise.

□

**Exercise 4.15.** *Show that that* $fg \sim 1_{\Omega K_{i+1}}$ *in the proof of Theorem 4.9.*

## 5 Additional exercises

**Exercise 5.1.** *Show any generalized homology theory as defined by Hatcher satisfies the Mayer-Vietoris sequence.*

## References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[A2] J.F. Adams, *A variant of E.H. Brown's Representability Theorem*, Topology, vol 10, pp 185-198, 1971.

[H] Allen Hatcher, *Algebraic Topology.*