Lecture 7: Alexander Duality

10/3/25

Let X be a finite CW complex embedded cellularly into S^n, i.e. $X \hookrightarrow S^n$. Let $A \subset S^n - X$ be a finite CW complex such that A is homotopy equivalent to $S^n - X$.

Example 1.1. $X = S^{p-1}$, $A = S^{q-1}$, $n = p + q - 1$.

The *join* of X and A is $X \star A = X \times [0,1] \times A / \sim$ where $(x, 0, a_1) \sim (x, 0, a_2)$ and $(x_1, 1, a) \sim (x_2, 1, a)$. It can be expressed as the colimit or push-out

$$\begin{array}{ccc} X \times A & \longrightarrow & \mathcal{C}X \times A \\ \downarrow & & \downarrow \\ X \times \mathcal{C}A & \longrightarrow & X \star A \end{array},$$

where $\mathcal{C}X$ denotes the non-reduced cone on X. (Push-out means a colimit of this shape.) There is a homotopy equivalence $X \star A \to \Sigma(X \wedge A)$. You can see this by hand, or alternatively note that the columns of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \vee A & \longrightarrow & A \\ \downarrow{=} & & \downarrow & & \downarrow{=} \\ X & \longleftarrow & X \times A & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \wedge A & \longrightarrow & * \end{array}$$

are cofiber sequences. Take the homotopy push-out of all rows to obtain the cofiber sequence

$$* \to ? \to \Sigma(X \wedge A),$$

where ? is the push-out of

$$X \leftarrow X \times A \to A.$$

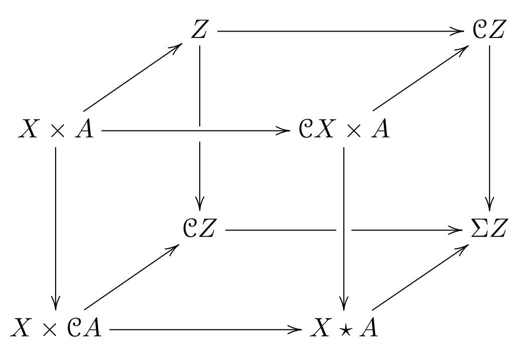
This gives that ? is $\Sigma(X \wedge A)$. On the other hand, ? is homotopy equivalent to the push-out of

$$X \times \mathcal{C}A \leftarrow X \times A \rightarrow \mathcal{C}X \times A,$$

which is $X \star A$.

Assume for the moment that no point of X is antipodal to a point of A. Then we have a map $X \star A \to S^n$ given by $(x, t, a) \mapsto \gamma_{x,a}(t)$, where $\gamma_{x,a} : [0,1] \to S^n$ is the geodesic of length one from x to a. Thus we have a map $\Sigma X \wedge A \to S^n$. In the stable homotopy category, we therefore have a map $\Sigma^{-(n-1)} A \wedge X \to S^0$. It follows that we have a natural map $\Sigma^{-(n-1)} A \to F(X, S^0) = DX$.

If there is a pair of antipodal points, one in X and one in A, we still have an analogous map $\Sigma^{-(n-1)} A \to DX$. Namely, wiggle A so there is a point of S^n that is not in $X \cup A$. Fix a point p not in $X \cup A$. Let $\mathrm{St} : S^n - \{p\} \to \mathbb{R}^n$ denote the stereographic projection. We obtain a map $\mu : X \times A \to S^{n-1}$ by $\mu(x, a) = (\mathrm{St}(x) - \mathrm{St}(a))/|\mathrm{St}(x) - \mathrm{St}(a)|$. Given a map $X \times A \to Z$, the *Hopf construction* is the map $X \star A \to \Sigma Z$ obtained by the morphism of pushout diagrams from the push-out diagram formed by the front face to the push-out formed by the back face



There are nice pictures in [M, p. 54].

Applying the Hopf construction to to μ we obtain the desired map $X \star A \to \Sigma S^{n-1}$, whence $\Sigma^{-(n-1)} A \to DX$.

We will show that $\Sigma^{-(n-1)} A \to DX$ is an isomorphism in the stable homotopy category. We will use the following facts.

1. Let Z be a spectrum and let Y be a finite spectra. Then $DY \wedge Z \cong F(Y, Z)$.

   Proof. By the definition of $F(Y, Z)$, what we need to show is that for any spectrum X, we have $[X, DY \wedge Z] \cong [X \wedge Y, Z]$. This was Exercise 1.3 (1) and (2) from L8. □

2. $\pi_r F(Y,Z) = [Y,Z]_r$.

   Proof. By Lemma 1.2 of Lecture 4, which is [A, III Prop 2.8], we have $\pi_r F(Y,Z) \cong [\Sigma^\infty S^r, F(Y,Z)]$. By the definition of $F(Y,Z)$ we have $[\Sigma^\infty S^r, F(Y,Z)] \cong [S^r \wedge Y, Z]$. By our identification of desuspension with a shift, we also have suspension identified with a shift. It follows that $[S^r \wedge Y, Z] \cong [Y,Z]_r$. □

3. Let Y be a finite spectrum. Suppose $[Y, H\mathbb{Z}]_r = 0$ for all $r \in \mathbb{Z}$. Then $Y \simeq *$.

   Proof. Choose n such that Y_n contains representatives for all the stable cells. There is a finite subcomplex $K \subset Y_n$ containing all the chosen representatives. By replacing K by its suspension $\Sigma K \subset Y_{n+1}$, we may assume that K is simply connected. This inclusion induces a map $\Sigma^{-n} K \to Y$ which is the inclusion of a cofinal subspectrum, and therefore an isomorphism in the stable homotopy category. Thus $0 = [\Sigma^{-n} K, H\mathbb{Z}]_r \cong [K, H\mathbb{Z}]_{r-n} \cong \tilde{\mathrm{H}}^{n-r}(K, \mathbb{Z})$. This implies that K is null-homotopic by the Hurewicz theorem and the universal coefficient theorem. □

4. If X is a CW complex, $\pi_r(X \wedge H\mathbb{Z}) \cong \tilde{H}_r(X, \mathbb{Z})$. A proof has been added to the notes on generalized homology.

5. [H;Prop 2B.1] For an embedding $h : D^k \to S^n$, we have $\tilde{H}_*(S^n - h(D^k)) = 0$ for all $*$. There is a short clear proof in Hatcher. It does not use machinery. It is a compactness argument, using the definition of singular chains.

Theorem 1.2. *(Alexander Dualty) Let X be a finite CW complex embedded into S^n, such that $S^n - X$ is homotopy equivalent to a finite CW complex. Then*

$$DX \cong \Sigma^{-(n-1)}(S^n - X)$$

is an isomorphism in the stable homotopy category. Furthermore, for any spectrum E,

$$E^r X \cong E_{n-r-1}(S^n - X) \tag{1}$$

$$E_r X \cong E^{n-1-r}(S^n - X). \tag{2}$$

For example, $\tilde{\mathrm{H}}_r(X, \mathbb{Z}) \to \tilde{\mathrm{H}}^{n-1-r}(S^n - X, \mathbb{Z})$ is an isomorphism.

Proof. The assertion $DX \cong \Sigma^{-(n-1)}(S^n - X)$ implies the other assertions of the theorem as follows. We have a map $DX \to \Sigma^{-(n-1)}(S^n - X)$, which we are assuming to be an isomorphism. Smashing with E and taking π_{-r} shows (1) using (1). Changing the roles of $S^n - X$ and X, as well as changing r to $n-1-r$, shows (2).

We show that $\Sigma^{-(n-1)}A \to F(X, S^0)$ is an isomorphism in the stable homotopy category by induction on the number of cells of X. If X has one cell, X is a point and the claim holds.

Now note that if $X \subset S^n$ is an embedding, and if we then embed $S^n \subset S^{n+1}$ along the equator, we get a new embedding $X \subset S^{n+1}$ such that $S^{n+1} - X \cong \Sigma(S^n - X)$. If we prove that $DX \cong \Sigma^{-((n+1)-1)}(S^{n+1} - X)$, we then have $DX \cong \Sigma^{-((n+1)-1)}\Sigma(S^n - X) \cong \Sigma^{-(n-1)}(S^n - X)$, which is what we wanted to show. In other words, given an embedding $X \subset S^n$, we may assume that n is as large as we wish.

Suppose X is a sphere. By the previous, we may assume that $X = S^m$ is embedded into S^n for $n > m$. If X is embedded as a lower dimensional equator by $i : S^m \to S^n$, then we can see directly that the complement $A \cong S^n - X$ is as in Example 1.1. Let $g : S^m \to S^{n'}$ be an arbitrary embedding of S^m into $S^{n'}$ with image X' and $n' > m$. Then there is a canonical embedding of

$$\theta : S^m \times [0,1] \to S^n \star S^{n'}$$

such that we have the commutative diagram

$$\begin{array}{ccccc} S^n \times \{0\} & \longrightarrow & S^n \star S^{n'} & \longleftarrow & S^{n'} \times \{1\} \\ {\scriptstyle i}\uparrow & & \uparrow & & \uparrow{\scriptstyle g} \\ S^m \times \{0\} & \longrightarrow & S^m \times [0,1] & \longleftarrow & S^m \times \{1\} \end{array}$$

We obtain maps $S^n \star S^{n'} - \theta(S^m \times [0,1]) \to S^n \star S^{n'} - (g(S^m) \times \{1\}) \to \Sigma^{n+n'+1-1}DX$. It follows that it suffices to show that

$$f : S^n \star S^{n'} - \theta(S^m \times [0,1]) \to S^n \star S^{n'} - (g(S^m) \times \{1\})$$

is a homotopy equivalence. We may assume both spaces are simply connected by increasing n and n'. It thus suffices by the Hurewicz theorem to show that $H_*(f;\mathbb{Z})$ is an isomorphism for all $*$. For any embedding of $S^m \times [0,1]$ into S^n, we show that $S^N - (S^m \times [0,1]) \to S^N - (S^m \times \{0\})$ induces an isomorphism on homology by induction on m.

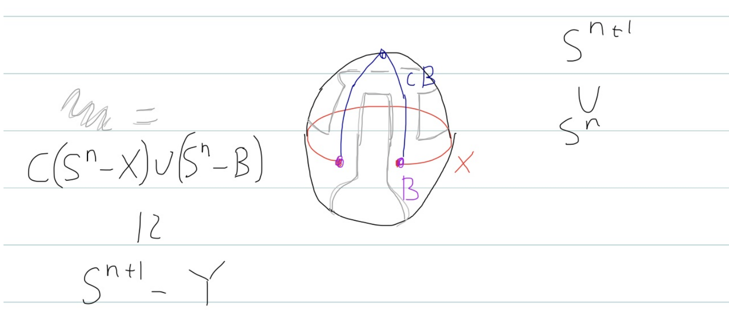
If $m = 0$, then $S^N - (S^0 \times \{0\})$ is the intersection of open subsets A_i for $i = 1,2$ of the form $S^N - \{*\}$ whose union is S^N. Similarly, $S^N - S^0 \times [0,1]$ is the intersection of open subsets B_i for $i = 1,2$ of the form $S^N - [0,1]$ whose union is S^N. Applying Mayer-Vietoris, we get a commutative diagram

$$\begin{array}{ccccccc} \ldots \tilde{H}_i(S^N - (S^0 \times \{0\})) & \longrightarrow & \tilde{H}_i(A_1) \oplus \tilde{H}_i(A_2) & \longrightarrow & \tilde{H}_i(S^N) & \longrightarrow & \ldots \\ \uparrow & & \uparrow & & \uparrow & & \\ \ldots \tilde{H}_i(S^N - (S^0 \times [0,1])) & \longrightarrow & \tilde{H}_i(B_1) \oplus \tilde{H}_i(B_2) & \longrightarrow & \tilde{H}_i(S^N) & \longrightarrow & \ldots \end{array}$$

By [H;Prop 2B.1], $\tilde{H}_i(A_j) \cong \tilde{H}_i(A_j) \cong 0$ from which the result follows by the 5 lemma.

Suppose inductively that for any embedding of $S^{m-1} \times [0,1]$ into S^n, we show that $S^N - (S^{m-1} \times [0,1]) \to S^N - (S^{m-1} \times \{0\})$ induces an isomorphism on homology. Take an embedding $S^m \times [0,1]$ into S^n. Breaking S^m into two closed hemispheres, we see that $S^n - (S^m \times \{0\})$ is the intersection of open subsets A_i for $i = 1, 2$ of the form $S^n - D^m$ whose union is $S^n - S^{m-1}$. Similarly, express $S^m \times [0,1]$ as the union of two copies of $D^m \times [0,1]$. Then $S^n - (S^{m-1} \times [0,1])$ is the intersection of two open subsets B_i for $i = 1, 2$ of the form $S^n - (D^m \times [0,1])$ whose union is $S^n - (S^{m-1} \times [0,1])$. The same Mayer-Vietoris argument combined with the induction hypothesis and [H;Prop 2B.1] completes the proof for all m. Thus f is a homotopy equivalence and the theorem holds for a sphere. So, we have the case where X has one cell, and the case where X is a sphere.

Let Y be a finite CW complex. We can express Y as the mapping cone of $B \to X$ where $B \cong S^m$ and X is a finite CW complex with one fewer cell. Choose any embedding of X into S^n. Embedding S^n into S^{n+1} along the equator, the cone on B can be embedded canonically so the cone point is the north pole of S^{n+1}, giving an embedding of Y.



From the picture, we see that

$$(S^n - X) \to (S^n - B) \to (S^{n+1} - Y)$$

is a cofiber sequence. Note that this also shows that any finite CW complex can be embedded in some sphere. We obtain a cofiber sequence

$$(S^{n+1} - Y) \to \Sigma(S^n - X) \simeq S^{n+1} - X \to S^{n+1} - B.$$

Since taking duals preserves cofiber sequences,

$$DY \to DX \to DB$$

is a cofiber sequence. By the inductive hypothesis, $DX \cong \Sigma^{-(n-1)}(S^n - X)$ and $DB \cong \Sigma^{-(n-1)}(S^n - B)$ and it follows that the theorem holds for this particular embedding of Y.

Now, we really want to start with $g : Y \subset S^n$ for some n satisfying the hypothesis and show the claim for this embedding. For any such g, take again the equatorial embedding $S^n \to S^{n+1}$ and replace g by $g : Y \to S^{n+1}$. There is is an embedding $H : X \cup_B (D^n \times [0,1])/(S^m \times [0,1]) \to S^{n+1}$ which moves the center of D^n viewed in Y up to the north pole along a geodesic. Any other interior point of D^n is on a straight line from the center to the boundary. Move these points up by the appropriate convex combination along a geodesic. This keeps $B \cong S^m$ fixed, so determines a well-defined embedding H. Note that our favorite embedding of Y and g both factor through H. Let $\overline{H}$ denote the image of H. It suffices to show that $S^{n+1} - \overline{H} \to S^{n+1} - Y$ is a homotopy equivalence. By suspending, we may again assume that both spaces are simply connected, and show the map is a homology equivalence. $S^{n+1} - \overline{H}$ is the intersection of $S^{n+1} - Y$ and $S^{n+1} - H((D^n \times [0,1])/(S^m \times [0,1]))$. Since $(D^n \times [0,1])/(S^m \times [0,1])$ is homeomorphic to a disk, we may apply [H;Prop 2B.1] to conclude that $\tilde{H}_*(S^{n+1} - H((D^n \times [0,1])/(S^m \times [0,1]))) \cong 0$. The union of $S^{n+1} - Y$ and $S^{n+1} - H((D^n \times [0,1])/(S^m \times [0,1]))$ is $S^{n+1} - D^{m+1}$ where the D^{m+1} is the final cell we attached to Y. Thus the reduced homology of the intersection is trivial. Running the Mayer-Vietoris sequence

$$\to \tilde{H}_*(S^{n+1} - \overline{H}) \to \tilde{H}_*(S^{n+1} - Y) \oplus \tilde{H}_*(S^{n+1} - H((D^n \times [0,1])/(S^m \times [0,1]))) \to \tilde{H}_*(S^{n+1} - D^{m+1}) \to$$

shows that $S^{n+1} - \overline{H} \to S^{n+1} - Y$ is a homology equivalence as claimed.

□

2 Exercises

Exercise 2.1. *Show the suspension of X is the homotopy pushout of*

$$* \leftarrow X \to *$$

.

Exercise 2.2. *Let $F : X \to Y$ be a map of simply connected topological spaces such that $\tilde{H}_*(F)$ is an isomorphism for all $*$. Show that F is a homotopy equivalence.*

Exercise 2.3. *Suppose we have a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

where the horizontal rows are cofiber sequences. Show that if two of f, g, and h are isomorphisms in the stable homtopy category, then so is the third.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[H] Allen Hatcher, *Algebraic Topology.*

[M] Haynes Miller, *Vector Fields on spheres, etc. (course notes).*