

Lecture 8: Atiyah Duality

10/8/25

1 Thom spectra

A *pointed space over X* will mean a map $Y \rightarrow X$ with a section (or basepoint) $X \rightarrow Y$. The *smash product over X* is defined

$$Y_1 \wedge_X Y_2 := \operatorname{colim}(Y_1 \times_X Y_2 \leftarrow (X \times_X Y_2 \cup Y_1 \times_X X) \rightarrow X)$$

For a vector bundle V on X , recall that the Thom space $\operatorname{Th}(V)$ of V is $\operatorname{Th}(V) \cong \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V)$

Exercise 1.1. *Show show the Thom space of the trivial bundle is $\operatorname{Th}(\mathcal{O}) \cong (X_+) \wedge \mathbb{P}^1$.*

Exercise 1.2. *Let V_1 and V_2 be two vector bundles on X . Construct a weak equivalence $\operatorname{Th}(V_1 \oplus V_2) \simeq \operatorname{Th}(V_1) \wedge_X \operatorname{Th}(V_2)$*

Let $K_0(X)$ be the group completion of the monoid of vector bundles on X under direct sum. Any element of $K_0(X)$ can be expressed as a formal difference $[V_1] - [V_2]$ of vector bundles. We may choose a bundle V_3 on X such that $V_2 \oplus V_3 \cong \mathcal{O}^n$ is a trivial bundle. Let $\operatorname{Th}([V_1] - [V_2]) := (\mathbb{P}^1)^{\wedge -n} \operatorname{Th}(V_1 \oplus V_3)$.

Exercise 1.3. *Show this construction induces a well defined map Th from $K_0(X)$ to isomorphism classes of objects in the stable homotopy category.*

2 Atiyah Duality

Let X be a finite CW complex. We saw last time that the Spanier-Whitehead dual of X is equivalent to $\Sigma^{-(n-1)}(S^n - X)$ where $X \rightarrow S^n$ is any non-surjective

embedding of X into a sphere whose complement is equivalent to a finite CW complex. Moreover, such an embedding always exists.

Such an embedding gives an embedding of X in \mathbb{R}^n by removing a point of S^n . Adding this point to X , we have the formula

$$\mathbb{D}(X_+) \cong \Sigma^{-(n-1)}(\mathbb{R}^n - X). \quad (1)$$

We may also replace \mathbb{R}^n by the homeomorphic open disk D^n .

Note that

$$(\mathbb{R}^n - X) \rightarrow \mathbb{R}^n \rightarrow \Sigma(\mathbb{R}^n - X)$$

is a cofiber sequence, because \mathbb{R}^n is contractible. Thus we may rewrite (1) as $\mathbb{D}(X_+) \cong \Sigma^{-n}(\mathbb{R}^n/(\mathbb{R}^n - X))$. For any neighborhood N of X in \mathbb{R}^n , we thus also have $\mathbb{D}(X_+) \cong \Sigma^{-n}N/(N - X)$

A sufficiently small neighborhood N deformation retracts back to X . Furthermore, if X is a manifold, N can be chosen so as to identify with the disk bundle $D(N_X\mathbb{R}^n)$ of the normal bundle $N_X\mathbb{R}^n \rightarrow X$ with ∂N identified with the sphere bundle $S(N_X\mathbb{R}^n)$. Such an N is a *tubular neighborhood*. We then have $N/(N - X) \cong N/\partial N \cong D(N_X\mathbb{R}^n)/S(N_X\mathbb{R}^n) \cong \text{Th}(N_X\mathbb{R}^n)$. Combining with the previous we obtain

$$\mathbb{D}(X_+) \cong \Sigma^{-n} \text{Th}(N_X\mathbb{R}^n) \quad (2)$$

Recall that if we add a trivial bundle $\underline{\mathbb{R}} \rightarrow X$ to a vector bundle $V \rightarrow X$, the resulting thom space $\text{Th}(V \oplus \underline{\mathbb{R}}) \cong \Sigma \text{Th}(V)$. This leads to the definition $\text{Th}(V \oplus \mathbb{R}^n) \cong \Sigma^n \text{Th}(V)$ for n positive or negative. Note that the Whitney sum of the tangent bundle and the normal bundle of $X \rightarrow \mathbb{R}^n$ is the trivial bundle of rank n , i.e. $N_X\mathbb{R}^n \oplus T_X \cong \underline{\mathbb{R}}^n$. Rearranging terms, we have:

Theorem 2.1. *If X is a compact manifold without boundary, $\mathbb{D}(X_+) \cong \text{Th}(-T_X)$.*

This implies Poincaré duality [H, Theorem 3.30], but to see this, we need to take as given the Thom isomorphism theorem, which says that if $V \rightarrow X$ is an orientable vector bundle of rank n , then there is a natural isomorphism $\tilde{H}^{*+n}(\text{Th}(V), \mathbb{Z}) \cong H^*(X, \mathbb{Z})$. One way to think about this is that orientable is a condition which says that V behaves like a trivial bundle after taking cohomology, so $\tilde{H}^*(\text{Th}(V), \mathbb{Z}) \cong \tilde{H}^*(\text{Th}(\mathbb{R}^n), \mathbb{Z})$. Since $\text{Th}(\mathbb{R}^n) \cong S^n \wedge (X_+)$, we have $\tilde{H}^*(\text{Th}(V), \mathbb{Z}) \cong \tilde{H}^*(\Sigma^n X_+, \mathbb{Z}) \cong H^{*-n}(X, \mathbb{Z})$. Furthermore, for any abelian group A , there is a notion of A -orientable, and a natural isomorphism $\tilde{H}^{*+n}(\text{Th}(V), A) \cong H^*(X, A)$ when V is A -orientable. Furthermore, for E a spectrum, there is a notion of E -orientable, and a natural isomorphism $E^{*+n} \text{Th}(V) \cong E^*(X_+)$ when V is E -orientable. We'll discuss this later.

Corollary 2.2. (*Poincaré duality*) Suppose that X is a compact n -manifold without boundary and E is a spectrum such that the tangent space T_X of X is E -orientable. Then

$$E_*(X_+) \cong E^{n-*}(X_+).$$

In particular, if X is orientable, then $H_*(X, \mathbb{Z}) \cong H^{n-*}(X, \mathbb{Z})$.

Proof. Since $\mathbb{D}(X_+) \cong \text{Th}(-T_X)$, we have that $[\mathbb{D}(X_+), E]_* \cong [\text{Th}(-T_X), E]_*$. The right hand side is $E^{-*} \text{Th}(-T_X)$. By the Thom isomorphism theorem,

$$E^{-*} \text{Th}(-T_X) \cong E^{-* - \text{rank}(-T_X)}(X_+) \cong E^{n-*} X_+.$$

The left hand side is $[\mathbb{D}(X_+), E]_* \cong [S^0, (X_+) \wedge E]_* \cong E_*(X_+)$. \square

Via the formula $\text{Th}(R^n) \cong S^n \wedge X_+$, X_+ can be interpreted as the Thom space of the trivial 0-dimensional bundle. We can generalize Theorem 2.1 with a formula, due to Atiyah [At], for the dual of any Thom space over X . For a compact n -manifold with boundary, view the tangent bundle as a rank n vector bundle together with a distinguished sub-bundle of rank $n - 1$ when restricted to the boundary.

Theorem 2.3. (*Atiyah duality*) If X is a compact manifold with boundary ∂X then

$$\mathbb{D}(X/\partial X) \cong \text{Th}(-T_X).$$

If X is a compact manifold without boundary and V is a smooth vector bundle, then

$$D(\text{Th } V) \cong \text{Th}(-T_X - V).$$

Proof. Embed X into the Euclidean n -disk D^n , so that ∂X is embedded into $\partial D^n = S^{n-1}$. Assume these embeddings are cellular and that X is transverse to S^{n-1} . Choose a tubular neighborhood N of X such that $N \cap S^{n-1}$ is a tubular neighborhood of ∂X .

$X/\partial X$ is homotopy equivalent to $Y = X \cup \mathcal{C}\partial X \subset D^n \cup \mathcal{C}S^{n-1} \cong S^n$, where as before \mathcal{C} denotes the (unreduced) cone. Applying Alexander duality to $Y \subset S^n$, we have

$$\mathbb{D}(X/\partial X) \cong \Sigma^{-(n-1)}(S^n - Y).$$

Note that $\mathcal{N} = N \cup \mathcal{C}(N \cap S^{n-1})$ is a neighborhood of Y , which deformation retracts onto Y . Thus $S^n - Y \cong (S^n - \mathcal{N})$. Since the cone point is not in $(S^n - \mathcal{N})$, we have that $(S^n - \mathcal{N}) \cong D^n - N$.

Thus

$$\begin{aligned}
\mathbb{D}(X/\partial X) &\cong \Sigma^{-(n-1)}(S^n - Y) \\
&\cong \Sigma^{-n}\Sigma(S^n - Y) \\
&\cong \Sigma^{-n}\Sigma(S^n - \mathcal{N}) \\
&\cong \Sigma^{-n}\Sigma(D^n - N) \\
&\cong \Sigma^{-n} \text{Th}(N_X D^n) \\
&\cong \text{Th}(-T_X)
\end{aligned}$$

The second assertion follows from the first. Let $D(V)$ and $S(V)$ denote the disk and sphere bundles of V respectively. Then $D(V)$ is a compact manifold with boundary $S(V)$, so $\mathbb{D} \text{Th}(V) \cong \text{Th}(-T_{D(V)})$. The kernel of $T_V \rightarrow T_X$ is the fiber-wise tangent bundle, which is V . Let $v : D(V) \rightarrow X$ denote the restriction of $V \rightarrow X$ to $D(V)$. Then there is a short exact sequence $v^*V \rightarrow T_{D(V)} \rightarrow v^*T_X$. We say in section 1 that up to stable homotopy, the Thom space of a vector bundle only depends on the bundle's class in K -theory. Thus $\text{Th}(-T_{D(V)}) \cong \text{Th}(-v^*(V \oplus T_X))$. Since v is a homotopy equivalence, $\text{Th}(-v^*(V \oplus T_X)) \cong \text{Th}(-(V \oplus T_X))$, as claimed. \square

Example 2.4. Truncated projective space Recall that $X = \mathbb{R}\mathbb{P}^n = S^n / (v \sim -v)$ and that the tautological line bundle $\mathbb{L} \rightarrow \mathbb{R}\mathbb{P}^n$ is the vector bundle whose fiber over $v \in S^n$ is $\ell = \mathbb{R}v$.

Put the standard inner product on \mathbb{R}^{n+1} , and let ℓ^\perp denote the n -dimensional subspace of \mathbb{R}^{n+1} of those vectors perpendicular to ℓ . Then $T_{[v]}\mathbb{R}\mathbb{P}^n \cong \text{Hom}(\ell, \ell^\perp)$. Since $\ell \oplus \ell^\perp \cong \mathbb{R}^{n+1}$, we have

$$0 \rightarrow \mathbb{R} = \text{Hom}(\ell, \ell) \rightarrow \text{Hom}(\ell, \mathbb{R}^n) \rightarrow \text{Hom}(\ell, \ell^\perp) \rightarrow 0.$$

This globalizes to an exact sequence

$$0 \rightarrow \underline{R} \rightarrow (\mathbb{L}^*)^{n+1} \rightarrow T_{\mathbb{R}\mathbb{P}^n} \rightarrow 0$$

of vector bundles on $\mathbb{R}\mathbb{P}^n$.

It turns out that up to homotopy the Thom space of V only depends on the class of V in K -theory. Then it follows that

$$\mathbb{D}\mathbb{R}\mathbb{P}^n \cong \text{Th}(-T_{\mathbb{R}\mathbb{P}^n}) \cong \text{Th}(\underline{R} - (\mathbb{L}^*)^{n+1}).$$

We can identify $\text{Th}(\mathbb{L}^*)^k$ with a truncated projective space for any k as follows. Any linear function from $\ell \in \mathbb{R}^{n+1}$ to \mathbb{R}^k determines a graph in $\mathbb{R}^k \times \mathbb{R}^{n+1}$, which in turn determines an element of $(\mathbb{R}^k \times \mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \cong \mathbb{R}\mathbb{P}^n + k$.

The only elements of $\mathbb{R}P^{n+k}$ not equal to such a graph, are lines in \mathbb{R}^k . Thus we have a vector bundle

$$V = \mathbb{R}P^{n+k} - \mathbb{R}P^{k-1} \rightarrow \mathbb{R}P^n$$

whose fiber over $\ell \in \mathbb{R}P^n$ is $\text{Hom}(\ell, \mathbb{R}^k)$. Thus $V = (\mathbb{L}^*)^k$.

$\text{Th}((\mathbb{L}^*)^k)$ is the one point compactification of $(\mathbb{L}^*)^k$ because X is compact. Thus

$$\text{Th}((\mathbb{L}^*)^k) \cong \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}.$$

There is notation for $\mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$. Define $\mathbb{R}P_k^{n+k} = \mathbb{R}P^{n+k}/\mathbb{R}P^{k-1}$.

In total, we obtain

$$\mathbb{D}(\mathbb{R}P_+^n) \cong \Sigma \mathbb{R}P_{-(n+1)}^{-1}.$$

3 Exercises

Exercise 3.1. Show that $\text{Th}(\mathbb{P}^1, \mathcal{O}(2)) \simeq \text{Th}(\mathbb{P}^1, \mathcal{O}(-2)) \simeq (\mathbb{P}^1)^{\wedge 2} \vee \mathbb{P}^1$.

References

- [At] M. Atiyah, *Thom Complexes*, Proc. London Math. Soc (3) 11, 1961, 291-310.
- [H] Allen Hatcher, *Algebraic Topology*.
- [M] Haynes Miller, *Vector Fields on spheres, etc. (course notes)*.