Lecture 9: Cobordism

10/22/25

1 Cobordism groups

We follow [At]. Let's work in a category of topological spaces satisfying the conditions for the classification of vector bundles. For a smooth manifold M of dimension n, let $\tau_M = \wedge^n TM$ be the orientation bundle. If M is a manifold with boundary N, there is a canonical isomorphism of $\tau_M|_N \cong \tau_N$. Let $\mathcal{M}_n$ denote the set of pairs (M, τ) where M is a compact smooth manifold (with boundary) of dimension n and τ is the orientation bundle. Let $\mathcal{M}_n^0$ denote the subset consisting of pairs where M has empty boundary. We have a boundary map

$$d : \mathcal{M}_n \to \mathcal{M}_{n-1}^0.$$

Let X be a topological space and α be a line bundle on X. Define a map $(M, \tau) \to (X, \alpha)$ to be a continuous map $f : M \to X$ together with an isomorphism $f^*\alpha \cong \tau$. Let

$$C_n(X, \alpha) = \{(M, \tau), F\}$$

be the set of tuples $(M, \tau), F$, where $(M, \tau) \in \mathcal{M}_n^o$ and $F : (M, \tau) \to (X, \alpha)$ is a map.

Say that $(M_1, \tau_1), F_1$ and $(M_2, \tau_2), F_2$ are cobordant if there exists $(N, \tau) \in \mathcal{M}_{n+1}$ and a map $F : (N, \tau) \to (X, \alpha)$ such that

1. $d((N, \tau)) \cong (M_1, \tau_1) \coprod (M_2, \tau_2)$.
2. $F|_{M_1} = F_1$ and $F|_{M_2} = -F_2$.

Note that cobordism determines an equivalence relation.

Definition 1.1. *The n-dimension oriented bordism group of X with coefficients in α, denoted $M\,\mathrm{SO}_n(X, \alpha)$ is defined to be*

$$\Omega_n^{\mathrm{SO}}(X, \alpha) = C_n(X, \alpha)/\sim,$$

where $\sim$ denotes the cobordism equivalence relation.

The disjoint union gives $\Omega_n^{\mathrm{SO}}(X, \alpha)$ the structure of an abelian group.

An almost complex manifold of dimension n is a $2n$-dimensional manifold with a reduction of the structure group of the tangent bundle from $O(2n)$ to $U(n)$. Using almost complex manifolds and complex line bundles, one can similarly define $\Omega^{\mathrm{SU}}(X, \alpha)$. Forgetting the complex or real line bundle defines $\Omega_n^U(X)$ and $\Omega_n^O(X)$, respectively.

The framed version of cobordism groups are defined as follows [M]. A framing of a closed smooth (real) manifold M is an embedding $i : M \to \mathbb{R}^N$ and a trivialization of the normal bundle N_i. A framed manifold is *null-bordant* if it is the boundary of a framed manifold with boundary. Two framed manifolds are *cobordant* if their difference is null-bordant.

Definition 1.2. *Let $\Omega_n^{\mathrm{fr}}(X)$ denote the group of cobordism classes of framed n-manifolds equipped with a map to X.*

2 Thom's theorem

It is not clear that the problem of classifying manifolds (with various structures) up to cobordism is even in homotopy theory, but it is. René Thom introduced the Thom complex $MSO(m)$, which defines a CW-spectrum MSO whose mth space is $MSO(m)$. The oriented cobordism groups are the associated generalized homology groups.

Theorem 2.1. *Let $\mathcal{O}$ denote the trivial bundle on X.*

$$\begin{aligned}\Omega_n^{\mathrm{SO}}(X, \mathcal{O}) &\cong \varinjlim_m \pi_{m+n}(MSO(m) \wedge X_+) \\ &\cong MSO_n(X).\end{aligned}$$

The proof uses the *Thom collapse map*: Let $i : M \to \mathbb{R}^N$ be an embedding of an n-manifold. Let N_i denote the normal bundle. Then the Thom collapse map is the induced map

$$\mathrm{Th}(i) : S^N \cong (\mathbb{R}^N)^+ \to (\mathbb{R}^N)^+/((\mathbb{R}^N)^+ - i(M)) \simeq D(N_i)/S(N_i) \simeq \mathrm{Th}(N_i).$$

There is also a *Thom diagonal map*: For any vector bundle V over a space X, the composition $D(V) \to D(V) \times D(V) \to D(V) \times X$ of the diagonal map with the projection induces a map

$$\Delta : \mathrm{Th}(V) \simeq D(V)/S(V) \to (D(V) \times X)/(S(V) \times X) \simeq \mathrm{Th}(V) \wedge X_+.$$

Proof. Let $(M, \tau), F$ be a representative of an element of $\Omega_n^{\mathrm{SO}}(X, \mathcal{O})$. Then τ is equipped with a trivialization. By the Whitney embedding theorem, we may choose an embedding $i : M \to \mathbb{R}^N$. Let $\mathrm{Th}(i) : S^N \to \mathrm{Th}(N_i)$ denote the associated Thom collapse map. Since $\tau \cong \det TM$ is trivialized and so is $\det T\mathbb{R}^N$, the normal bundle N_i inherits an orientation, giving rise to a classifying map

$$b(N_i) : M \to BSO(N - n)$$

together with an isomorphism $b(N_i)^*\gamma_{N-n} \cong N_i$ of oriented vector bundles on M. Taking the associated map of Thom spaces gives

$$\mathrm{Th}(b(N_i)) : \mathrm{Th}(N_i) \to MSO(N - n)$$

We obtain the Pontrjagin–Thom construction

$$S^N \xrightarrow{\mathrm{Th}(i)} \mathrm{Th}(N_i) \xrightarrow{\Delta} \mathrm{Th}(N_i) \wedge M_+ \xrightarrow{\mathrm{Th}(b(N_i))\wedge 1} MSO(N - n) \wedge X_+ \qquad (1)$$

which determines an element of $\varinjlim_m \pi_{m+n}(MSO(m) \wedge X_+)$. Since the map $S^N \xrightarrow{\mathrm{Th}(i)} \mathrm{Th}(N_i) \xrightarrow{\Delta} \mathrm{Th}(N_i) \wedge M_+$ is the Nth suspension counit in the duality between M_+ and $\mathrm{Th}(-TM)$, its homotopy class is independent of the choice of embedding. Including $\mathbb{R}^N$ into $\mathbb{R}^{n+1}$ suspends $\mathrm{Th}(i)$ whence (1) defines the same element of the colimit. Moreover, after sufficiently many inclusions, all embeddings become isotopic by a result of Wu giving a homotopy between $b(N_i)$ and $b(N_{i'})$. Thus the homotopy class of (1) is independent of the choice of embedding.

We claim that if $(M_1, \tau_1), F_1$ and $(M_2, \tau_2), F_2$ are cobordant, they give rise to homotopic maps (1). Let N be a cobordism. Embed N into the Euclidean m-disk D^m, so that $dN = M_1 \coprod M_2$ is embedded into $\partial D^m = S^{m-1}$. Assume these embeddings are cellular and that N is transverse to S^{m-1}. There is a tubular neighborhood of N whose intersection with S^{m-1} is a tubular neighborhood of ∂N. The map

$$D^m \to D^m/(D^m - N) \simeq \mathrm{Th}(N_N D^m) \to \mathrm{Th}(N_N D^m) \wedge N_+ \to MSO(m - (n+1)) \wedge X_+$$

is a null-homotopy between the difference of the Pontrjagin–Thom constructions (1) associated to $(M_i, \tau_i), F_i$ for $i = 1, 2$.

We therefore have a well-defined map

$$\psi : \Omega_n^{\mathrm{SO}}(X, \mathcal{O}) \to \varinjlim_m \pi_{m+n}(MSO(m) \wedge X_+)$$

We show ψ is surjective in Proposition 2.7. Injectivity follows from a similar argument producing a cobordism from a null-homotopy of the difference.

For any CW complex X and CW spectrum E, we have $E_n(X) \cong \pi_n(E \wedge \Sigma_+^\infty X) \cong \varinjlim_m \pi_{m+n}(E_m \wedge X_+)$. ☐

Theorem 2.2. *(Steenrod Approximation Theorem) Let* $\pi : E \to B$ *be a smooth vector bundle with a Euclidean metric. Let* σ *be a continuous section of* π*. For every* ϵ*, there is a smooth section* s *such that* $|s(b) - \sigma(b)| < \epsilon$ *for all* $b \in B$*.*

Corollary 2.3. *Let* $f : M \to N$ *be a continuous map between smooth manifolds and let* $K \subset M$ *be a closed subset and* $U \subset M$ *an open set whose closure is compact and disjoint from* K*. Let* $Z \subset N$ *be a closed subset such that* U *is an open neighborhood of* $f^{-1}(Z)$ *and* $f^{-1}(Z)$ *is compact. Then there is a continuous map* $g : M \to N$ *which is homotopic to* f *such that* $g|_U$ *is smooth,* $g^{-1}(Z) \subset U$*, and* $g|_K = f|_K$*.*

This is a modification of [M, L16 Cor 1.9].

Theorem 2.4. *(Thom Transversality) Let* U, M*, and* Z *be smooth manifolds. Let* $i : Z \to M$ *be a smooth embedding. Then the topological space of smooth maps* $U \to M$ *which are transverse to* i *is dense in the topological space of smooth maps* $U \to M$*.*

Theorem 2.5. *(Functor of points of the Thom space) Let* $V \to X$ *be a smooth vector bundle on a smooth compact manifold* X*. Let* M *be a smooth compact manifold. Then there is a natural bijection*

$$\frac{\{(Y \hookrightarrow M, i : Y \to X, \theta : N_Y M \cong i^*V\}}{\sim} \cong [M, \mathrm{Th}(V)]$$

given by sending (Y, i, θ) *to*

$$M \to M/(M - Y) \simeq \mathrm{Th}(N_Y M) \overset{\theta}{\simeq} \mathrm{Th}(i^*V) \xrightarrow{i} \mathrm{Th}(V)$$

Proof. We may put a metric on the bundle $V \to X$ and define the unit disk bundle $D(V) \to V \to \mathbb{P}(V \oplus \mathcal{O}) \to \mathrm{Th}(V)$. Let $e : M \to \mathrm{Th}(V)$ be a map.

Apply Corollary 2.3 to the map $f : M' \to V$ where $M' = e^{-1}(V)$ and $f = e|_{M'}$. Note that both $M' \subset M$ and V are smooth manifolds, in contrast to the Thom space $\mathrm{Th}(V)$ itself. Let $D(V)^0$ denote the open disk bundle and let $U = f^{-1}(D(V)^0)$. Note that U is open and its closure is a closed subset of M, whence compact. Let $D(V)^{2,0} \subset V$ denote the open disk bundle of radius 2. Let $K \subset M'$ denote $K = f^{-1}(V \setminus D(V)^{2,0})$ the inverse image of the complement of the open disk of radius 2. Since f is continuous, K is closed and is disjoint from the closure of U as required by Corollary 2.3. Let Z be the zero section of V. Applying Corollary 2.3, we obtain a homotopic map g such that $g|_K = f|_K$ and g is smooth on U and $U \supset g^{-1}(Z)$. Since $g|_K = f|_K$, we may extend g to a map on all of M by setting $g|M - M' = g|_{M-M'}$ with g homotopic to e. Using Thom transversality, we may assume that g is transverse to the 0 section. We recover the manifold Y mapping to e by the pullback

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d] & Z \arrow[d] \\
M \arrow[r, "g"] & \mathrm{Th}(V \to X)
\end{tikzcd} \tag{2}$$

Since g in transverse to g, Y is indeed a manifold. The commutative diagram (2) induces an isomorphism $N_Y M \cong i^*V$. $\square$

Exercise 2.6. *In the proof Theorem 2.5, check that the map in the statement of Theorem 2.5 associated to* (Y, i, θ) *is homotopic to* g.

Proposition 2.7. *The map* $\psi : \Omega_n^{\mathrm{SO}}(X, \mathcal{O}) \to \varinjlim_m \pi_{m+n}(MSO(m) \wedge X_+)$ *is surjective.*

Proof. An arbitrary element of $\pi_{m+n}(MSO(m) \wedge X_+)$ can be represented by a map $c : S^N \to MSO(N-n) \wedge X_+$. Composing with $X_+ \to S^0$, we obtain a map

$$C : S^N \to MSO(N-n) \simeq \mathrm{Th}(\gamma_{N-n}).$$

$B\,\mathrm{SO}(i) \simeq \mathrm{colim}_m \mathrm{Gr}^{\mathrm{SO}}(i,m)$ and $M\,\mathrm{SO}(i) \simeq \mathrm{colim}\,\mathrm{Th}(\gamma_{i,m})$ where $\mathrm{Gr}^{\mathrm{SO}}(i,m)$ denotes the Grassmannian

$$\mathrm{Gr}^{\mathrm{SO}}(i,m) = \{L : \mathbb{R}^i \hookrightarrow \mathbb{R}^m\}/SO(i)$$

of oriented i planes in m-space and $\gamma_{i,m}$ denotes the tautological bundle. Since S^N is compact, we may express C as the composition of

$$e : S^N \to \mathrm{Th}(\gamma_{N-n,m} \to \mathrm{Gr}^{\mathrm{SO}}(N-n)) \cong \mathbb{P}(\gamma_{N-n,m} + \mathcal{O})/\mathbb{P}(\gamma_{N-n,m})$$

with the inclusion $\mathrm{Th}(\gamma_{N-n,m}) \to \mathrm{Th}(\gamma_{N-n})$.

$\square$

Exercise 2.8. *State and sketch the proof of the analogous theorem giving a geometric interpretation for* $\Omega_n^{\mathrm{SO}}(X, \alpha)$ *for a line bundle* α *on* X.

Theorem 2.9. $\Omega_n^{\mathrm{fr}}(X) \cong \pi_n \Sigma_+^\infty X$.

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[At] M.F. Atiyah *Bordism and cobordism* 1960.

[H] Allen Hatcher, *Algebraic Topology.*

[M] Mark Mclean *Lecture notes Mat566* `https://www.math.stonybrook.edu/~markmclean/MAT566/lecture16.pdf`.

[M] Haynes Miller, *Kervaire Invariant One (after M.A. Hill, M.J. Hopkins, and D.C. Ravenel*, Séminarie BOURBAKI, 63éme année, 2010-2011, n. 1029.