

Additional exercises

12/3/25

Here are exercises for some of the lectures which don't have associated notes.

1 Symmetric monoidal categories

Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category as in [DP, Section 1]

Definition 1.1. *A in \mathcal{C} is dualizable if there exists A^* in \mathcal{C} and $\epsilon : A^* \otimes A \rightarrow 1$ and $\eta : 1 \rightarrow A \otimes A^*$ such that*

$$A \rightarrow A \otimes A^* \otimes A \rightarrow A \quad A^* \rightarrow A^* \otimes A \otimes A^* \rightarrow A^*$$

are the identity morphisms on A and A^ respectively.*

Exercise 1.2. *Suppose A and B are dualizable. Describe a natural bijection $\mathcal{C}(A, B) = \mathcal{C}(B^*, A^*)$*

Exercise 1.3. *Suppose that \mathcal{C} admits an internal $\text{Hom} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ adjoint to \times in the sense that there are natural bijections*

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, \text{Hom}(B, C))$$

If A is dualizable, show that $A^ \cong \text{Hom}(A, 1)$.*

Exercise 1.4. *Let R be a commutative ring with unit. Let M be an R -module. Show that M is dualizable if and only if it is finitely generated and projective.*

2 Limits and colimits

Exercise 2.1. *Show suspension is an equivalence of categories from the stable homotopy category to itself.*

3 Chern classes, Pontrjagin classes, Oriented cobordism

Exercise 3.1. (*Bezout's theorem*) Compute the Chern classes of $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2)$ on $\mathbb{C}\mathbb{P}^2$

Exercise 3.2. (*Lines on a cubic surface*) Let $\text{Gr}_{\mathbb{C}}(2, 4)$ denote the Grassmannian of 2-dimensional linear subspaces of \mathbb{C}^4 . Let γ denote the rank 2 tautological bundle. Compute the Chern classes of $\text{Sym}^3 \gamma^*$ the third symmetric power of the dual tautological bundle. The top Chern class's pushforward to the point is the number of lines on a smooth cubic surface.

Exercise 3.3. [MS, 16-E]

Exercise 3.4. [MS, 16-F]

References

- [A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.
- [DP] A. Dold and D. Puppe *Duality, Trace, and Transfer* Proceedings of the Steklov Institute of Math., 1984.
- [MS] J.W. Milnor and J.D. Stasheff *Characteristic Classes* 1965, 2022.