Lecture 8: Spanier-Whitehead Duality

1/23/15

We work in the stable homotopy category. For X and Y objects of the stable homotopy category, we are assuming the existence of a smash product $X \wedge Y$ and a spectrum $F(X,Y)$ such that

$$[W \wedge X, Y] = [W, F(X,Y)]. \tag{1}$$

For notational simplicity, based spaces will be identified with their suspension spectra. So, S^0 also denotes $\Sigma^\infty S^0 = \mathbb{S}$ the sphere spectrum. Here are the things we need to assume:

- For any X (meaning that X is an object of the stable homotopy category), the smash product $X \wedge S^0$ and $S^0 \wedge X$ are canonically isomorphic to X. Note that for a space X, this follows from the fact that the smash products in spaces $S^0 \wedge X$ and $X \wedge S^0$ are homeomorphic to X, and that the smash product of suspension spectra is the suspension spectra of the smash product.
- Smash product is associative and commutative, i.e. for any objects X and Y, there is a canonical isomorphism $X \wedge Y \cong Y \wedge X$.
- $\wedge$ preserves cofiber sequences.

Such a smash product and function spectrum satisfying the above really do exist (see for example [MSS]), but we will discuss this existence after getting some applications.

Definition 1.1. *A* dual *of X is an object Y equipped with maps*

$$e : X \wedge Y \to S^0$$

$$\eta : S^0 \to Y \wedge X$$

such that the compositions

$$X \wedge S^0 \overset{1\wedge\eta}{\to} X \wedge Y \wedge X \overset{e\wedge 1}{\to} X \wedge S^0$$

and

$$S^0 \wedge Y \stackrel{\eta\wedge 1}{\to} Y \wedge X \wedge Y \stackrel{1\eta e}{\to} Y \wedge S^0$$

are the canonical isomorphisms.

These canonical isomorphisms are sometimes just called "the identity."

Remark 1.2. *This definition makes sense in any category with a smash product $\wedge$, and an object S^0 such that $X \wedge S^0 \cong X$ for all X. Such categories are called* monoidal. *Examples include are complexes of R-modules where $\wedge = \otimes$ and $S^0 = R$.*

Exercise 1.3.
1. *Show that if Y is dual to X, then for all W and Z, the map*
$$[Z \wedge Y, W] \stackrel{\wedge X}{\to} [Z \wedge Y \wedge X, W \wedge X] \stackrel{\eta^*}{\to} [Z, W \wedge X]$$
is an isomorphism.
2. *If Y is a dual of X, then X is a dual of Y.*
3. *Show that adjoint functors from a category to itself is the same data as dual objects in the category of functors from a category to itself, where the smash product is composition.*

Proposition 1.4. *If the dual of X exists, it is $F(X, S^0)$. In particular, duals are unique.*

Proof. By Exercise 1.3 2, if X has a dual Y, then X is itself the dual of Y. Applying Exercise 1.3 1 with the roles of X and Y switched, we therefore have $[Z \wedge X, S^0] \cong [Z, Y]$ for all Z. By Yoneda's lemma and the definition of $F(X, S^0)$, we therefore have $Y = F(X, S^0)$. □

It therefore makes sense to write DX for Y. DX is called the *Spanier-Whitehead dual* of X. By 2 of the above exercise, $D^2 X \cong X$.

Proposition 1.5. *D is a contravariant functor on the subcategory of dualizable objects.*

Proof. Suppose $f : X_1 \to X_2$ is a morphism in the stable homotopy category, and that X_1 and X_2 are dualizable. Then

$$[DX_2, DX_1] = [S^0, X_2 \wedge DX_1] = [X_1, X_2].$$

Thus there is a function in $[DX_2, DX_1]$ corresponding to f. □

Say that a CW spectrum is *finite* if it has a finite number of stable cells.

Theorem 1.6. *Let X be a finite CW spectrum. The dual of X exists.*

We first prove a couple lemmas. Suppose $A \subset X$ is an inclusion of CW spectra. The map $A \to X$ induces a map of cohomology theories $[X \wedge (-), S^0] \to [A \wedge (-), S^0]$, whence a natural map $F(X, S^0) \to F(A, S^0)$.

Lemma 1.7. $F(X/A, S^0) \cong \Sigma^{-1}F(A, S^0)/F(X, S^0)$

Proof. We wish to show that the cohomology theory $W \mapsto [W \wedge X/A, S^0]$ is represented by $\Sigma^{-1}F(A, S^0)/F(X, S^0)$.

The fiber sequence

$$\Sigma^{-1}F(A, S^0)/F(X, S^0) \to F(X, S^0) \to F(A, S^0)$$

and the map $F(X/A) \to F(X, S^0)$ whose image in $F(A, S^0)$ is null gives a natural map $F(X/A, S^0) \to \Sigma^{-1}F(A, S^0)/F(X, S^0)$.

By the long exact sequence associated to a cofiber sequence, we have an exact sequence

$$[W \wedge \Sigma X, S^0] \to [W \wedge \Sigma A, S^0] \to [W \wedge X/A, S^0] \to [W \wedge X, S^0] \to [W \wedge A, S^0].$$

This exact sequence can be rewritten

$$[W, \Sigma^{-1}F(X, S^0)] \to [W, \Sigma^{-1}F(A, S^0)] \to [W, F(X/A, S^0)] \to [W, F(X, S^0)] \to [W, F(A, S^0)].$$

This latter exact sequence maps to the exact sequence

$$[W, \Sigma^{-1}F(X, S^0)] \to [W, \Sigma^{-1}F(A, S^0)] \to [W, \Sigma^{-1}F(A, S^0)/F(X, S^0)] \to [W, F(X, S^0)] \to [W, F(A, S^0)].$$

The result then follows by the 5-Lemma and the long exact sequence associated to a fiber sequence. □

Remark 1.8. *By* (1)*, with $Y = S^0$ and $W = F(X, S^0)$, there is a canonical $\eta : F(X, S^0) \wedge X \to S^0$, i.e. η in $[F(X, S^0) \wedge X, S^0]$, corresponding to the identity in $[F(X, S^0), F(X, S^0)]$.*

Exercise 1.9. *Show that if*

$$A \to X \to Y = X \cup CA$$

is a cofiber sequence such that A and X are dualizable, then Y is dualizable and the sequence

$$DY \to DX \to DA$$

is a fiber (or equivalently cofiber) sequence.

Proof. (of Theorem 1.6) Note that the dual of S^n is S^{-n} by taking the obvious maps for η and ϵ. Since finite CW complexes can be built by attaching spheres along attaching maps (and this process is the same as taking the cofiber), we have by induction that the duals of finite CW complexes exist. Then note that $D\Sigma X = \Sigma^{-1} DX$. It then follows that the duals of finite spectra exist. □

References

[A] J.F. Adams, *Stable Homotopy and Generalized Homology* Chicago Lectures in Mathematics, The University of Chicago Press, 1974.

[MSS] M.A. Mandell, J.P. May, S. Schwede, and B. Shipley *Model Categories of Diagram Spectra*, 1999.