NOTES FOR THE TALK: ON THE SECTION CONJECTURE

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1. INTRODUCTION: ANABELIAN CONJECTURES

Want: understand solutions to polynomial equations, maps between schemes.

Try: control maps between schemes using topological spaces.

$$X \mapsto \mathrm{Et}(X)$$

$\mathrm{Et}(X)$ denotes the étale homotopy type of Artin - Mazur, Friedlander. $\mathrm{Et}(X)$ is a pro-object in simplicial sets. View Et as a functor to the homotopy category of pro-simplicial sets.

1.1. *Example.*

- Let X be a finite type scheme over $\mathbb{C}$. Then $\mathrm{Et}(X)$ is equivalent to the profinite completion of the complex points of X, denoted $X(\mathbb{C})^{\wedge}$.
- Let k be a field and $G_k = \mathrm{Gal}(k^s/k)$ denote the absolute Galois group of k. Then $\mathrm{Et}(\mathrm{Spec}\, k) \cong BG_k$.

1.2. Question. For a,b in $\mathbb{Q}^*$, when is $G_{\mathbb{Q}[\sqrt{a}]} \cong G_{\mathbb{Q}[\sqrt{b}]}$?

Answer: if and only if $\mathbb{Q}[\sqrt{a}] \cong \mathbb{Q}[\sqrt{b}]$, i.e. if and only if $a = b$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

1.3. *Theorem. —* (Neukirch, Uchida) *Let k_1 and k_2 be finite extensions of $\mathbb{Q}$. Then the natural map*

$$\mathrm{Iso}(k_2, k_1) \to \mathrm{Iso}(G_{k_1}, G_{k_2})$$

is a bijection.

Conclude: sometimes Et reflects isomorphisms.

1.4. *Remark.* Et is not fully faithful. For instance

$$\mathrm{Map}(X_{\mathbb{C}}, \mathbb{A}^1_{\mathbb{C}}) = \mathcal{O}(X_{\mathbb{C}}) \neq \mathrm{Map}(\mathrm{Et}(X_{\mathbb{C}}), \mathrm{Et}(\mathbb{A}^1_{\mathbb{C}})) = *.$$

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1.5. Definition. Let X be a pointed, normal scheme. The étale fundamental group $\pi_1(X)$ of X is $\pi_1(\mathrm{Et}(X))$.

1.6. *Example.* Let X/k be a smooth curve over a number field. Suppose that the Euler characteristic χ of X is < 0. Then

$$\mathrm{Et}(X) = B\pi_1(X).$$

1.7. *Anabelian Conjectures.* — (Grothendieck)*Let* k *be a number field. There is a full subcategory* An_k *of finite type smooth schemes over* k *including* $\mathrm{Spec}\, k$*, smooth curves with* $\chi < 0$*, and total spaces of vibrations with base and fiber in* An_k*, such that for all* X_1*,* X_2 *in* An_k*, the natural map*

$$\mathrm{Map}^{\textit{dominant}}_{\textit{Scheme}}(X_1, X_2) \to \mathrm{Map}^{\textit{open,out}}_{G_k}(\pi_1 X_1, \pi_1 X_2) \tag{1}$$

from dominant scheme maps to open maps of profinite groups over G_k *up to equivalence by conjugation by elements of* $\pi_1(X_2)$*, is a bijection.*

1.8. *Theorem.* — (Mochizuki) *The bijection* (1) *holds for* k *a number field, or more generally* k *any subfield of a finitely generated extension of* $\mathbb{Q}_p$ *– called* sub-p-adic fields*,* X_1 *any smooth scheme over* k*,* X_2 *any smooth curve over* k *of negative Euler characteristic.*

1.9. *Section Conjecture.* — (Grothendieck) *Let* X *be a smooth, compact algebraic curve of genus* ≥ 2 *defined over a number field* k*. Then*

$$\mathrm{Map}(\mathrm{Spec}\, k, X) \to \mathrm{Map}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X))$$

is a bijection.

The section conjecture is analogous to an equivalence between fixed points and homotopy fixed points:

- $\mathrm{Map}(\mathrm{Spec}\, k, X) = X(k) = \pi_0(X(k)) = \pi_0(X(\bar{k})^{G_k})$
- Suppose that X has a fixed base point.

  $$\mathrm{Map}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X)) = H^1(G_k, \pi_1(X_{\bar{k}})) = \pi_0(\mathrm{Et}(X_{\bar{k}})^{hG_k}),$$

  where $\mathrm{Et}(X_{\bar{k}})^{hG_k}$ denotes the homotopy fixed points of $\mathrm{Et}(X_{\bar{k}})$. (Here

  $$\mathrm{Map}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X))$$

  means maps in the homotopy category of pro-simplicial sets over BG_k.)

1.10. *Sullivan Conjecture.* — (Carlsson, Lannes, Miller, Dwyer-Miller-Neisendorfer) G *finite* p*-group.* X *finite* G*-CW-complex*

$$(X^G)^{\wedge}_p \to (X^{\wedge}_p)^{hG},$$

is a weak equivalence, where $(-)^{\wedge}_p$ *denotes Bousfield-Kan* p*-completion.*

- This implies the section conjecture over $\mathbb{R}$ which says that the natural map

$$\pi_0(X(\mathbb{R})) \to \mathrm{Map}_{\mathrm{Et}(R)}(\mathrm{Et}(R), \mathrm{Et}(X))$$

is a bijection.
- The existence of the Selmer group shows that the Section Conjecture does not hold for X an elliptic curve over a number field: let S denote the Selmer group. There are natural inclusions

$$\mathrm{Map}(\mathrm{Spec}\, k, X) \subseteq S \subseteq \mathrm{Map}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X))$$

with S not always equal to $\mathrm{Map}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X))$.
- The Section Conjecture is unknown. The only cases for which it is verified are for certain curves without sections and without points. These were constructed independently by T. Szamuely and J. Stix.
- There is a modification of the Section Conjecture applying to smooth algebraic curves which are not assumed to be compact.

2. TOWARDS THE SECTION CONJECTURE

We study

$$\mathrm{Map}_{\mathrm{Et}(\mathrm{Spec}\, k)}(\mathrm{Et}(\mathrm{Spec}\, k), \mathrm{Et}(X)) = \mathrm{Map}_{BG_k}(BG_k, B\pi_1 X)$$

by approximating

$$\mathrm{Et}(X) \to \mathrm{Et}(\mathrm{Spec}\, k)$$

which is equivalent to approximating

$$EG_k \times_{G_k} B\pi \to BG_k$$

where

$$\pi = \pi_1(X_{\overline{k}}).$$

We will use the lower central series, denoted as follows:

$$\overline{[\pi, \pi]} = [\pi]_2 \supseteq [\pi]_3 \supseteq [\pi]_4 \supseteq \dots$$

$$[\pi]_n = \overline{[\pi, [\pi]_{n-1}]}.$$

This gives a tower approximating X

(2)

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & B\pi_1(X)/[\pi]_n \arrow[d] \\
 & \vdots \arrow[d] \\
 & B\pi_1(X)/[\pi]_3 \arrow[d] \\
B\pi_1(X) \arrow[r] \arrow[ru] \arrow[ruuu] & B\pi_1(X)/[\pi]_2
\end{tikzcd}$$

(2) is similar to the Goodwillie tower of $B\pi$ as a G_k-space.

We study $\mathrm{Map}_{\mathrm{Et\,Spec\,k}}(\mathrm{Et\,Spec\,k}, \mathrm{Et\,X})$ by its approximations

$$\mathrm{Map}_{\mathrm{Et\,Spec\,k}}(\mathrm{Et\,Spec\,k}, B\pi_1(X)/[\pi]_n).$$

For notational convenience, given two objects Y and Z in the homotopy category of pro-spaces over $\mathrm{Et\,Spec\,k} \cong BG_k$, let $Z(Y) = \mathrm{Map}_{\mathrm{Et\,Spec\,k}}(Y, Z)$. This notation is meant to recall that the k points of X are denoted $X(k)$, so it's natural to denote the maps

$$\mathrm{Et\,Spec\,k} \to \mathrm{Et\,X}$$

by $\mathrm{Et\,X}(\mathrm{Et\,Spec\,k})$. Abbreviate $\mathrm{Et\,X}(\mathrm{Et\,Spec\,k})$ by $\mathrm{Et\,X}(\mathrm{Et\,k})$.

It is also convenient to define

$$(B\pi_1(X)/[\pi]_n)(\mathrm{Et\,k})^{ab} = \mathrm{Image}(B\pi_1(X)/[\pi]_n(\mathrm{Et\,k}) \to B\pi_1(X)/[\pi]_2(\mathrm{Et\,k}))$$

$$\mathrm{Et\,X}(\mathrm{Et\,k})^{ab} = \mathrm{Image}(\mathrm{Et\,X}(\mathrm{Et\,k}) \to B\pi_1(X)/[\pi]_2(\mathrm{Et\,k})).$$

The lifting problem

$$\begin{tikzcd}
 & B\pi_1(X)/[\pi]_{n+1} \arrow[d] \\
BG_k \arrow[r] \arrow[ru, dashed] & B\pi_1(X)/[\pi]_n
\end{tikzcd}$$

gives rise to obstructions of Jordan Ellenberg.

For $k = \mathbb{R}$, the first of these obstructions is enough to determine $\pi_0(X(\mathbb{R}))$, strengthening the section conjecture over $\mathbb{R}$. Moreover, replacing $\pi_1(X)$ by the absolute Galois group of the function field $G_{\mathbb{R}(X)}$ and defining the analogous obstructions determines $X(\mathbb{R})$ itself:

2.1. *Theorem.* — (W.) *For an algebraic curve X over $\mathbb{R}$ such that each component has a real point*

$$\pi_0(X(\mathbb{R})) = (B\pi_1(X)/[\pi]_3)(\mathrm{Et}\,\mathbb{R})^{ab}.$$

2.2. *Corollary.* — (W.) *Let* X *be a smooth, compact, algebraic curve over* $\mathbb{R}$ *with a real point. Let* $X(\mathbb{R})^{\pm}$ *denote the real points of* X *equipped with a real tangent direction. Then*

$$X(\mathbb{R})^{\pm} = (BG_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_3)(\mathrm{Et}\,\mathbb{R})^{ab}.$$

The approximations $(B\pi_1(X)/[\pi]_n)(\mathrm{Et}\,k)$ all suffer from a defect which we will correct. First, we explain the problem:

2.3. *Remark.* Nilpotent groups are products of p-groups, so

$$\pi/[\pi]_n = \prod_p (\pi/[\pi]_n)_p$$

and

$$B\pi/[\pi]_n = \prod_p B(\pi/[\pi]_n)_p.$$

Maps

$$f_1, f_2 : \mathrm{Et}\,k \to \mathrm{Et}\,X = EG_k \times_{G_k} \pi$$

over $\mathrm{Et}\,k$ give rise, and are equivalent to, G_k-equivariant maps $f_1, f_2 : EG_k \to B\pi$. We obtain maps $(f_i)_p : EG_k \to B(\pi/[\pi]_n)_p$ by composition. Then construct the map

$$(f_1)_q \times \prod_{p \neq q} (f_2)_p : EG_k \to \prod_p B(\pi/[\pi]_n)_p = B(\pi/[\pi]_n)$$

and the corresponding map

$$(f_1)_q \times \prod_{p \neq q} (f_2)_p : \mathrm{Et}\,k \to EG_k \times_{G_k} \pi/[\pi]_n = B\pi_1(X)/[\pi]_n.$$

This map does not necessarily come from a map

$$\mathrm{Et}\,k \to \mathrm{Et}\,X,$$

so we should eliminate maps like $(f_1)_q \times \prod_{p \neq q}(f_2)_p$ from approximations of $\mathrm{Et}\,X(\mathrm{Et}\,k)$. (When G_k is a p-group, all maps $\mathrm{Et}\,k \to EG_k \times_{G_k} (\pi/[\pi]_n)_q$ for $q \neq p$ are null homotopic, so we did not run into this problem for $k = \mathbb{R}$.)

Thus, we need maps to satisfy some sort of compatibility across different primes. Here is one way to do this: consider the Abel-Jacobi map $X \to \mathrm{Pic}^1 X$ from the smooth curve X to its Picard scheme. There is a commutative diagram

$$\begin{tikzcd}
X(k) \arrow[r] \arrow[d] & \mathrm{Et}\,X(\mathrm{Et}\,k) \arrow[d] \\
\mathrm{Pic}^1 X(k) \arrow[r] & \mathrm{Et}\,\mathrm{Pic}^1 X(\mathrm{Et}\,k)
\end{tikzcd}$$

We study $\mathrm{Et}\,X(\mathrm{Et}\,k)$ by approximating the set

$$\mathrm{Pic}^1 X(k) \cap \mathrm{Image}(\mathrm{Et}\,X(\mathrm{Et}\,k) \to \mathrm{Et}\,\mathrm{Pic}^1 X(\mathrm{Et}\,k))$$

by

$$\mathrm{Pic}^1 X(k) \cap \mathrm{Image}(B\pi_1(X)/[\pi]_n(\mathrm{Et}\, k) \to \mathrm{Et}\,\mathrm{Pic}^1 X(\mathrm{Et}\, k))$$

It follows from Poincaré duality and the fact that $\mathrm{Pic}^1 X$ is a semi-abelian variety that the map

$$\mathrm{Et}\, X \to \mathrm{Et}\,\mathrm{Pic}^1 X$$

is naturally identified with

$$B\pi_1 X \to B\pi_1 X/[\pi]_2.$$

Thus, in total we study $\mathrm{Et}\, X(\mathrm{Et}\, k)$ by approximating the set

$$\mathrm{Pic}^1 X(k) \cap \mathrm{Et}\, X(\mathrm{Et}\, k)^{ab}$$

by

$$\mathrm{Pic}^1 X(k) \cap (B\pi_1(X)/[\pi]_n)(\mathrm{Et}\, k)^{ab}.$$

We can also consider obstructions coming from unipotent representations of the fundamental group. These are very similar to Ellenberg's.

Let U_n denote the subgroup of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ consisting of those matrices with diagonal entries 1 and whose entries below the diagonal are 0, i.e. $n \times n$-matrices (a_{ij}) such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i > j$. Let $Z \subset U_n$ be the center. There is a corresponding central extension

$$Z \subset U_n \twoheadrightarrow U_n/Z.$$

2.4. Obstruction construction. Define an action of G_k on U_n and a G_k-equivariant map

$$\pi \to U_n.$$

From this, consider the lifting problem

$$\begin{tikzcd}
 & EG_k \times_{G_k} U_n \arrow[d] & \\
BG_k \arrow[ur, dashed] \arrow[r] & EG_k \times_{G_k} (U_n/Z) & \mathrm{Et}(X) \arrow[l] \arrow[ul]
\end{tikzcd}$$

which gives rise to an obstruction.

We will compute some of these obstructions in the case $X = \mathbb{P}^1_k - \{0, 1, \infty\}$.[1] Then $X \to \mathrm{Pic}^0 X$ can be identified with[2]

$$\mathbb{P}^1_k - \{0,1,\infty\} \to \mathbb{G}_{m,k} \times \mathbb{G}_{m,k}$$
$$x \mapsto (x, 1-x).$$

Applying $H^*(G_k, -)$ to the Kummer sequence

$$1 \to \mu_{\ell^m} \to \mathbb{G}_m \overset{z \mapsto z^{\ell^m}}{\to} \mathbb{G}_m \to 1$$

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[1] $\mathbb{P}^1_k - \{0, 1, \infty\}$ is not compact, but there is a modification of the section conjecture for non-compact smooth curves.

[2] We translate by a chosen base point so the Abel-Jacobi map is $X \to \mathrm{Pic}^0 X$ instead of $X \to \mathrm{Pic}^1 X$.

gives

$$k^* \to H^1(G_k, \mathbb{Z}_\ell(1)).$$

2.5. *Notation.*

- x in k^* also denotes the corresponding element of H^1.
- $\langle x_1, x_2, \ldots, x_n \rangle$ denotes the n-fold Massey product.
- One can define a certain class f_n in $H^1(G_k, \mathbb{Z}_\ell(n-1))$. For k a number field, f_n can be expressed in terms of Deligne-Soulé classes.

Note that $\mathrm{Pic}^0\, \mathbb{P}^1_k - \{0, 1, \infty\}(k) = (\mathbb{G}_{m,k} \times \mathbb{G}_{m,k})(k) = k^* \times k^*$.

2.6. *Theorem (W.). — Let* $X = \mathbb{P}^1_k - \{0, 1, \infty\}$*. There are unipotent representations of* $\pi_1(X_{\bar{k}})$ *which show that for all*

$$(x, y) \in \mathrm{Pic}^0 X(k) \cap \mathrm{Et}\, X(\mathrm{Et}\, k),$$

we have

$$\langle x, x, \ldots, x, y, x, \ldots, x, x \rangle = 0 \tag{3}$$

$$\langle y, x, x, \ldots, x, x, y \rangle = f_n \cup y \tag{4}$$

where the Massey products on the left hand side are order n*, the* y *can appear in any position in* (3)*, and* $n = 2, 3, \ldots$.

Furthermore, for $n = 2, 3$ *the subset of* $\mathrm{Pic}^0 X(k)$ *determined by* (3) *and* (4) *is*

$$\mathrm{Pic}^0(k) \cap B\pi_1(X)/[\pi]_{n+1}(\mathrm{Et}\, k)^{ab}.$$

We have been careless about defining systems of Massey products. For $n < 4$ this is simply an artifact of the talk. For higher n, there is more to be understood.

The representations in Theorem 2.6 come from the failure of the section conjecture for $\mathbb{G}_m$.[3]

Since we know the rational points of $\mathbb{P}^1_k - \{0, 1, \infty\}$, the purpose of Theorem 2.6 is to study $\mathrm{Et}\, X(\mathrm{Et}\, k)$. For instance, the section conjecture for $\mathbb{P}^1_k - \{0, 1, \infty\}$ implies that the section conjecture holds for an open subset of any curve defined over a number field.

From the opposite point of view, since we know the rational points of $\mathbb{P}^1_k - \{0, 1, \infty\}$, Theorem 2.6 places restrictions on the differential graded algebra $C^*(\mathrm{Spec}\, k)$ of étale cochains of $\mathrm{Spec}\, k$.

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[3] Since $\mathbb{G}_m$ does not have negative Euler characteristic, it is not one of the curves discussed by the section conjecture.

2.7. *Corollay (W.).* — *For* $x \in k^* - \{1\}$,

$$\langle x, \ldots, x, 1-x, x, \ldots x, \rangle = 0.$$
$$\langle 1-x, x, \ldots, x, 1-x \rangle = f_n \cup (1-x)$$

Because of the existence of tangential base points, the same method shows that

$$\langle x, \ldots, x, -x, x, \ldots x \rangle$$
$$\langle -x, x, \ldots, x, -x \rangle = f_n \cup (-x).$$

2.8. *Remark.* Guillou and Sharifi have results which overlap with this corollary by completely different methods from each other and me.

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