Please read collegiality statement and fill out questionnaire

L-theory, bilinear forms, and surgery

L-theory, topological and algebraic L-theory cohomology theories

L is the letter after K

For context we begin with remarks on K-theory

algebraic K-theory   topological K-theory

Topological K_0

X top space

$\mathrm{Vect}_{\mathbb{C}}(X)$ = isomorphism classes of $\mathbb{C}$-vector bundles on X, $\oplus$ "monoid"

There is no $\ominus$, but can be added formally

The Grothendieck group or group completion of a monoid M is

$$K_0(M) = M \times M \big/ \sim$$

$(a_+, a_-) \sim (b_+, b_-) \iff \exists\, k \in M$ s.t.

$$a_+ + b_- + k = b_+ + b_- + k$$

"$(a_+, a_-) = a_+ - a_-$"

Def:
$K^0(X) :=$ group completion $(\mathrm{Vect}_{\mathbb{C}}(X))$

equivalenty (although needs proof)

$$K^0(X) = \frac{\{\text{isomorphism classes of } \mathbb{C}\text{-vector bundles on } X\}}{[P] \sim [P_0] \oplus [P_1]}$$

when $0 \to P_0 \to P \to P_1 \to 0$

is an exact sequence of

vector bundles on X

Ex: $K^0(pt) \xrightarrow[\text{rank}]{\cong} \mathbb{Z}$

Ex: $\mathcal{O}(1) \to \mathbb{P}^1 \cong S^2$

ref: Hatcher
Vector Bundles K-thy
Chap 2.1

$$K^0(S^2) \cong \mathbb{Z}[\mathcal{O}(1)] / (\mathcal{O}(1) - 1)^2$$

Ex: $K^0(X) = \tilde{K}^0(X) \oplus \mathbb{Z}$ $\qquad \tilde{K}^0(X) = \ker(K^0(X) \to K(pt))$

Bott periodicity

$$\tilde{K}(X) \to \tilde{K}(\Sigma^2 X) \qquad \Sigma X = \frac{S^1 \times X}{S^1 \times * \cup * \times X}$$

<u>Algebraic K-theory</u> (Grothendieck 1950's)

R ring

<u>Def</u>: $K_0(R) :=$ group completion of monoid of isomorphism classes of finite projective R-modules

$$\cong \frac{\{\text{isomorphism classes of finite projective } R\text{-modules } P\}}{[P] \sim [P_0] \oplus [P_1]}$$

when

$$0 \to P_0 \to P \to P_1 \to 0$$

is a short exact sequence

Rmk: A projective module over a local ring is free, so projective modules are locally free

Ex: F field $\quad K_0(F) \xrightarrow{\cong} \mathbb{Z}$

$$[P] \mapsto \dim P$$

ref. (Morrow PCMI)

Exercise: R ring, $I \subseteq R$ ideal contained in the Jacobson radical, Then

$$K_0(R) \xrightarrow{\cong} K_0(R/I)$$

In particular, for any local ring R,

$$K_0(R) \cong \mathbb{Z}$$

Ex: $\mathcal{O}$ Dedekind domain. The class group

$$cl(\mathcal{O}) := \text{isom classes of non-zero ideals } I \text{ of } \mathcal{O}$$

with group operation $I + J = IJ$ (ideal generated by ab, for $a \in I$, $b \in J$) Recall each I is a finite, projective, $\mathcal{O}$-module so we have $[I] \in K_0(\mathcal{O})$

There is a short exact sequence

$$0 \longrightarrow Cl(\mathcal{O}) \longrightarrow K_0(\mathcal{O}) \longrightarrow \mathbb{Z} \to 0$$

$$[I] \mapsto [I]-[\mathcal{O}]$$

$$[P] \longmapsto \operatorname{rank}_{\mathcal{O}} P$$

(Morrow PCMI)

Exercise: Check this short exact sequence.

Why is the map $Cl(\mathcal{O}) \to K_0(\mathcal{O})$ a well-defined group homomorphism

X smooth scheme (variety) over a field

Def: $K_0(X) = \underset{\text{completion}}{\text{group}} \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{locally free} \\ \quad \mathcal{O}_X\text{-modules} \end{array} \right\}$

These are the vector bundles on X

$Z \hookrightarrow X$ irreducible closed subscheme

$\mathcal{O}_Z$ = functions on Z = structure sheaf of Z

There exists a resolution (needs proof)

$$0 \leftarrow \mathcal{O}_Z \leftarrow P_0 \leftarrow \cdots \leftarrow P_d \leftarrow 0$$

P_i vector bundle on X

$$[Z] := \sum_i (-1)^i [P_i] \in K_0(X)$$

The <u>topological filtration</u> (niveau) on $K_0(X)$ is defined

$d = \dim X$

$$K_0 \supseteq \mathrm{Fil}_d \supseteq \mathrm{Fil}_{d-1} \supseteq \cdots \supseteq \mathrm{Fil}_0$$

$$\mathrm{Fil}_j K_0(X) := \text{subgroup generated by } \left\{ [Z] : \begin{array}{c} Z \hookrightarrow X \\ \text{irred closed} \\ \text{subscheme} \\ \dim \leq j \end{array} \right\}$$

Part of the Riemann–Roch theorem states that

$$\begin{aligned} CH^j(X) &\longrightarrow \mathrm{Fil}_j K_0(X) / \mathrm{Fil}_{j-1} K_0(X) \\ Z &\longmapsto [Z] \end{aligned}$$

is surjective and has kernel killed by $(j-1)!$

ref. Fulton "Intersection theory" Example 15.1.5, 15.2.16

Bott periodicity: $\tilde{K}^0(\mathbb{P}^1 \wedge X) \cong \tilde{K}^0(X)$

$\mathbb{A}^1$-htpy theory

Table from Lurie L2 minus Hermitian K-theory

"modulo cobordism"

	K-theory	Hermitian K-theory $\to$	L-theory
input:	Projective module	Projective module with quadratic or bilinear form	Projective module with quadratic or bilinear form
for $\mathbb{R}$-vector spaces:	dimension	dimension and signature	signature
classical version:	K^0	Grothendieck-Witt group GW	Witt group W
invariants of manifolds:	Euler characteristic	Euler char & Signature	Signature
local to global principle:	Gauss-Bonnet thm		Hirzebruch Signature theorem

Why is this mini course happening now?

9 authors Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nikolaus, Steimle

inspired by Lurie

motivations

ref L1 Lurie

Question 1: let X be a CW complex. When does there exist a smooth compact manifold M and a homotopy equivalence

$$X \simeq M \ ?$$

or: what distinguishes the homotopy type of a compact smooth manifold from the htpy type of other CW complexes?

A1: compact manifolds satisfy Poincaré duality

M dim n, smooth, compact oriented

$$H^q(M;\mathbb{Z}) \times H^{n-q}(M;\mathbb{Z}) \to H^n(M,\mathbb{Z}) \xrightarrow{\cap[M]} H^0(M,\mathbb{Z}) \cong \mathbb{Z}$$

non-degenerate, graded symmetric bilinear pairing

$\rightsquigarrow$ elt of an L group

<u>Def</u>: Let X be a simply connected CW complex. X is a <u>simply connected Poincaré complex of dim n</u> if $\exists$ $[X] \in H_n(X;\mathbb{Z})$ s.t. $\forall q$

$$H^q(X;\mathbb{Z}) \xrightarrow[\cong]{\cap [X]} H_{n-q}(X;\mathbb{Z})$$

- $[X]$ is called a "fundamental class"

- $q=0$ $\quad \mathbb{Z} \cong H^0(X;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})$

$$1 \longmapsto [X]$$

$\subset$ well-defined up to sign

- $q>n$ $\quad H^q(X;\mathbb{Z}) \longrightarrow H_{n-q}(X;\mathbb{Z}) = 0$

$\Rightarrow$ n determined by X

- If X is htpy equivalent to an oriented, compact, smooth mfld, then X must be a Poincaré complex

<u>Question2</u>: Let X be a simply connected Poincaré complex of dim n. When does

there exist a homotopy equivalence

$$X \simeq M$$

M smooth, oriented, compact manifold dim n?

A2: M has tangent bundle TM closely related to Poincaré duality.

Duality: V finite dim'l vector space over k

$$V^* = \mathrm{Hom}(V, k)$$

$$\mathrm{Hom}(V, W) \cong V^* \otimes W$$

$V \otimes V^* \longrightarrow k$ Given $V^* \cong V$ one obtains

nondeg symm bilinear form $V \otimes V \to k$

Def: A symmetric monoidal category is

$(\mathcal{C}, \otimes, I, \tau)$ where $\mathcal{C}$ is a category

with a tensor or smash product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

with a unit I s.t. $I \otimes C \cong C$ and

$C \otimes I \simeq C$ and $\tau : C \otimes D \xrightarrow{\cong} D \otimes C$

is a symmetry isomorphism.

ex: vector spaces

modules over a comm ring

chain cplxs of modules

stable htpy cat

$\mathbb{A}^1$-stable htpy cat

finite dim'l replaced by...

$A \in Ob(\mathcal{C})$ is dualizable if $\exists$ $B \in Ob(\mathcal{C})$

and $\mathbb{1} \xrightarrow{\eta} A \otimes B$, $B \otimes A \xrightarrow{\varepsilon} \mathbb{1}$

s.t.

$$\begin{tikzcd} A \arrow[r, "\eta \otimes 1"] \arrow[rr, bend right, "1"'] & A \otimes B \otimes A \arrow[r, "1 \otimes \varepsilon"] & A \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "1"'] & B \otimes A \otimes B \arrow[r] & B \end{tikzcd}$$

Notation: $\mathbb{D}A = B$

Exercise: $\mathrm{Hom}(A, C) \simeq \mathrm{Hom}(1, \mathbb{D}A \otimes B)$

Rmk In a closed symmetric monoidal category
$\mathrm{Hom}(A, C) \simeq \mathbb{D}A \otimes B$ in $\mathcal{C}$

ex: • f.g. projective modules are dualizable objects in modules over a commutative ring

• f.g. chain cplxs, projective in each dimension are dualizable objects in chain complexes

• Smooth, compact manifolds M are dualizable in the stable homotopy category

Homotopy theory $\longrightarrow$ Stable htpy theory

here cohomology theories are representable and finite CW complexes X have $\mathbb{D}X \simeq \mathrm{Hom}(X,)$

Atiyah Duality: $\mathbb{D}M \simeq M^{-TM}$

Thom space of minus the tgt bundle

$V \to M$ vector bundle

Thom space M^V or $Th(V)$ $\quad M^V = \dfrac{\text{Disk bundle}}{\text{sphere bundle}}$

$$M^V \simeq \frac{\mathbb{P}(V \oplus 1)}{\mathbb{P}V}$$ trivial bundle

For M compact, $M^V = $ one pt compactification (V)

Ex: $M^{1^n} \simeq \Sigma^n M$

Ex: $Th(\mathbb{P}^n, m\mathcal{O}(1)) \simeq \mathbb{P}^{m+n} / \mathbb{P}^{n-1}$

$$(M_+) \wedge M^{-TM} \longrightarrow 1$$

Apply H^*, Künneth

$$H^*(M) \otimes H^*(M^{-TM}) \longrightarrow H^*(1) \cong \mathbb{Z}$$

perfect pairing

$$H^q(M) \otimes H^{-q}(M^{-TM}) \longrightarrow \mathbb{Z}$$

Thom isomorphism (TM oriented) $H^{-q}(M^{-TM}) \cong$

$$H^{-q+n}(M)$$

Sign of shift:

$$H^*(M) \cong H^{*+1}(\Sigma M)$$

$$\cong H^{*+r}(\Sigma^r M)$$

$$\Rightarrow H^*(Th) \cong H^{*-\text{rank}}(M)$$

$\Rightarrow$ perfect pairing

$$H^q(M) \otimes H^{n-q}(M) \longrightarrow \mathbb{Z}$$ Poincaré duality

The fundamental class is described by the structure maps in $1 \longrightarrow M_+ \wedge M^{-TM}$ as follows:

$M \hookrightarrow \mathbb{R}^k$ embedding

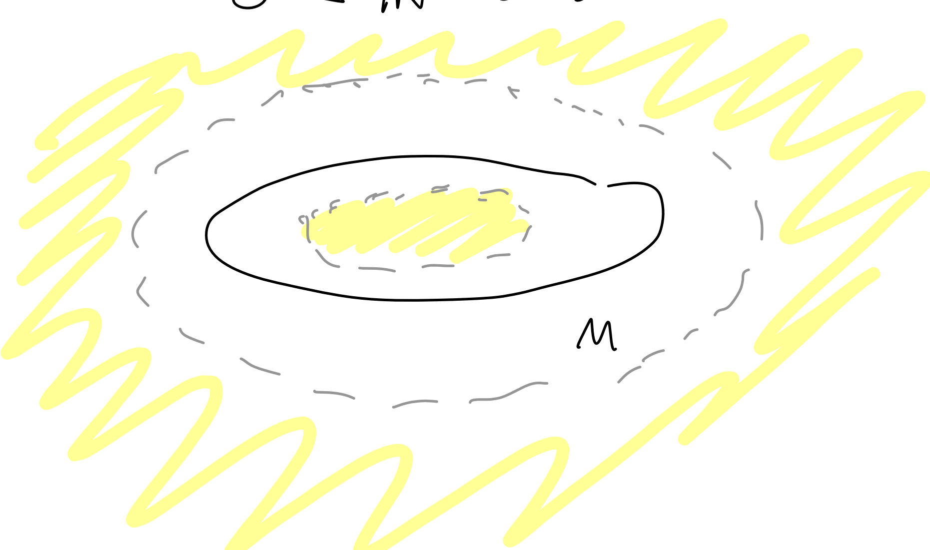
$TM + N_M\mathbb{R}^k \simeq 1^k$

$\Rightarrow -TM = N_M\mathbb{R}^k - 1^k$

$\Rightarrow M^{-TM} \simeq \Sigma^{-k} M^{N_M\mathbb{R}^k}$

Thom collapse map

$$c : S^k = \mathbb{R}^k \cup \{\infty\} \longrightarrow M^{N_M\mathbb{R}^k} \simeq \frac{S^k}{S^k - \text{tubular nbhd } M}$$



$$c \in \pi_k\left(M^{N_M\mathbb{R}^k}\right) \longrightarrow H_k\left(M^{N_M\mathbb{R}^k}\right) \underset{\text{orientation}}{\simeq} H_{k-(k-n)}(M)$$

$$\cong H_n(M)$$

image of c is $[M] \in H_n(M)$

Thus: Let M be a simply connected, oriented, smooth manifold of dimension n. Then there exists a vector bundle ζ on M of dimension n (namely the tangent bundle) and a class $\eta_M (= \Sigma_{-k} c)$ in $\pi_0 M^{-\zeta}$ such that the image of η_M under the composition

$$\pi_0(M^{-\zeta}) \longrightarrow H_0(M^{-\zeta}; \mathbb{Z}) \cong H_n(M; \mathbb{Z})$$

Thom iso for orientation

is a fundamental class of M

Question 3: Let X be a simply connected Poincaré complex of dimension n. Suppose we are given a vector bundle ζ of dimension n on X and a homotopy class $\eta \in \pi_0 X^{-\zeta}$ whose image in $H_n(X;\mathbb{Z})$ is a fundamental class for X.

Does there exist a smooth manifold M of dim n and a htpy equivalence

$$f : M \xrightarrow{\sim} X$$

s.t. $f^* \zeta = T_M$ in $KO^0(X)$ and $f^*\eta = \eta_M \in M^{-T_M}$?

k-thy of $\mathbb{R}$-vector bundles

An $\Leftrightarrow$ answer:

Assume $n = 4k$

non-deg symmetric bilinear form

$$\langle \, , \, \rangle : H^{2k}(X;\mathbb{R}) \times H^{2k}(X;\mathbb{R}) \to H^{4k}(X;\mathbb{R}) \xrightarrow{[X]} \mathbb{R}$$

from Poincaré duality

Sylvester's thm (we'll see generalization next time)

$\exists$ basis $\{x_1, \dots, x_a, x_{a+1}, \dots, x_{b+a}\}$ for $H^{2k}(X;\mathbb{R})$

s.t. $$\langle x_i, x_j \rangle = \begin{array}{c} \overbrace{\hphantom{1 \; 1 \; \cdots \; 1}}^{a} \; \overbrace{\hphantom{-1 \; \cdots \; -1}}^{b} \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & \ddots & \\ & & & & & & -1 \end{bmatrix} \end{array}$$

$$\text{Signature}(\langle\,,\,\rangle) := a - b$$

Let $\sigma_M = \text{Signature}(\langle\,,\,\rangle)$

Hirzebruch signature formula:

Let $p_i(TM)$ $i = 1,..,k$ denote the Pontryagin classes of TM

$$p_i(TM) = (-1)^i c_{2i}(TM \otimes \mathbb{C}) \in H^{4i}(M; \mathbb{Z})$$

Chern classes

$$\sigma_M = L(p_1(TM), \ldots, p_k(TM))[M]$$

L is some polynomial

For example

$n=4$ $$L = \frac{p_1(TM)}{3}[M]$$

$n=8$ $$L = \frac{7p_2(TM) - p_1(TM)^2}{45}[M]$$

__Remark__ (Lurie L1 12) For manifold M, Poincaré duality satisfied for a local reason, so one might expect a "local" formula for σ_M

(Lurie L24–L25) Signature formula results from 2 orientations on $L \wedge H\mathbb{Q}$

cohomology thy representing $X \mapsto H^*(X;\mathbb{Q})$

Surgery theory gives converse

__Thm__ (Browder, Novikov?) Let X be a simply connected, Poincaré complex of dimension $4k > 4$, let ζ be an oriented vector bundle on X of rk $4k$, let $\eta \in \pi_0 X^{-\zeta}$ be such that the image of η in $H_{4k}(X;\mathbb{Z})$ is a fundamental class

Then:

Question 3 answer is yes

$\Longleftrightarrow$

X satisfies the Hirzebruch signature theorem

Algebraic motivation

k field

analogue compact

X smooth, proper scheme $/k$ dim n

$\Omega_X =$ Kähler differentials $X/k \cong T^*X$

cotangent bundle

$\omega_X = \det \Omega_X$

$V \to X$ vector bundle rank r

Grothendieck-Serre duality $\mathrm{Tr} : H^n(X, \omega_X) \to k$

perfect pairing $H^q(X, F) \otimes H^{n-q}(X, \mathcal{H}om(F, \omega_X)) \to k$

locally free

<u>Def</u>: V is <u>oriented</u> by the data (L, e)

$L \to X$ line bundle and

$\omega_X \otimes \det V \overset{e}{\cong} L^{\otimes 2}$

$V \rightleftarrows X$ (σ) $\quad \sigma$ section of V

<u>Def</u>: The <u>Koszul complex</u> $\mathrm{Kos}(V, \sigma)$

is $\mathrm{Kos}(V, \sigma)$

deg n $\quad$ deg 0

$$0 \to \Lambda^r V^* \xrightarrow{d_r} \Lambda^{r-1} V^* \xrightarrow{d_{r-1}} \cdots \to \Lambda^2 V^* \to V^* \xrightarrow{d_1} \mathcal{O} \to 0$$

$$d_k(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} (-1)^{i+1} \sigma(e_i)\, e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_k$$

- There is a pairing

$$\mathrm{Kos}(V,\sigma) \otimes \mathrm{Kos}(V,\sigma) \longrightarrow \Lambda^r V^* [n]$$

i.e. for the duality

$$\mathbb{D} : \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(X)$$

$$\mathbb{D}(C_*) = R\mathcal{H}om(-, \det V^*[n])$$

$$\det V^* = \Lambda^r V^*$$

$\mathrm{Kos}(V,\sigma)$ is self dual

- Suppose V is oriented and $n = r$

Then $\left(\mathrm{Kos}(V,\sigma) \otimes \mathcal{L}\right) \otimes \left(\mathrm{Kos}(V \otimes \mathcal{L}\right) \longrightarrow \det V^* \otimes \mathcal{L}^{\otimes 2}[n]$

$$\det V^* \otimes \mathcal{L}^{\otimes 2}[n] \cong \omega_X[n]$$

Perfect pairing

$$H^q(X, \Lambda^q V^* \otimes \mathcal{L}) \otimes H^{n-q}(X, \Lambda^{n-q} V^* \otimes \mathcal{L})$$

$$
\begin{array}{c}
\downarrow \\
H^n(X, \omega_X) \\
\downarrow \\
R
\end{array}
$$

$\rightsquigarrow$ elt of L groups of fields

This is an Euler class in $\mathbb{A}^1$-htpy thy
(Bachmann-W, Levine-Raksit for tgt bundles,
Hopkins, Serre)

<u>references</u>

Lurie L-thy and surgery L1, L2

Morrow PCMI L1

W 8803 L8, L10