Atiyah Duality: M manifold

$$\mathbb{D}(\Sigma^\infty_+ M) \simeq M^{-TM}$$

view M in stable htpy theory

Thom space of spectra

$V \to M$ vector bundle

Thom space M^V or $Th(V)$ $\quad M^V = \dfrac{\text{Disk bundle}}{\text{sphere bundle}}$

$$M^V \simeq \frac{\mathbb{P}(V \oplus 1)}{\mathbb{P}V}$$

trivial bundle

For M compact, $M^V = \underset{\text{compactification}}{\text{one pt}}(V)$

Ex: $M^{1^n} \simeq \Sigma^n M$

Ex: $Th(\mathbb{P}^n, m\mathcal{O}(1)) \simeq \mathbb{P}^{m+n}/\mathbb{P}^{n-1}$

V oriented for H $\Rightarrow$ Thom isomorphism theorem: $H^n(M^V) \cong H^{n-rk}(M)$

"M^V and M have the same cells"

- $V \in KO^0(X)$ $\quad V \cong V_1 \ominus V_2 \cong V' \ominus 1^m$

group completion of $\mathbb{R}$-vector bundles

$V_2 \xhookrightarrow{\text{summand}} 1^m$

$$M^V := \Sigma^{-m} M^{V'}$$

$\underline{Ex}$: $(\mathbb{P}^1)^{-TM} \simeq (\mathbb{P}^1)^{-\mathcal{O}(2)} \simeq (\mathbb{P}^1)^{\mathcal{O}(-2)-1^2} \simeq \Sigma^{-2}(\mathbb{P}^1)^{\mathcal{O}(-2)}$

$$0 \to \mathcal{O}(-2) \xrightarrow[(x^2, y^2)]{} \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(2) \to 0$$

$\simeq \Sigma^{-2} \dfrac{\mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O})}{\mathbb{P}(\mathcal{O}(-2))}$

Hirzebruch surface F_2

- Atiyah duality $\Rightarrow$ Poincaré duality

$$1 \to M_+ \wedge M^{-TM}$$

Apply $\tilde{H}^*$, Künneth

$$H^*(M) \otimes \tilde{H}^*(M^{-TM}) \to H^*(1) \cong \mathbb{Z}$$

perfect pairing

$$H^q(M) \otimes \tilde{H}^{-q}(M^{-TM}) \to \mathbb{Z}$$

Thom isomorphism (TM oriented) $\tilde{H}^{-q}(M^{-TM}) \cong$

$H^{-q+n}(M)$

Sign of shift:

$H^*(M) \cong H^{*+1}(\Sigma M)$

$\cong H^{*+r}(\Sigma^r M)$

$\Rightarrow H^*(Th) \cong H^{*-\mathrm{rank}}(M)$

$\Rightarrow$ perfect pairing

$$H^q(M) \otimes H^{n-q}(M) \to \mathbb{Z}$$

Poincaré duality

- The fundamental class $[M]$ is described using the maps in the definition of the dual $\mathbb{D}M$

$M \hookrightarrow \mathbb{R}^k$ embedding

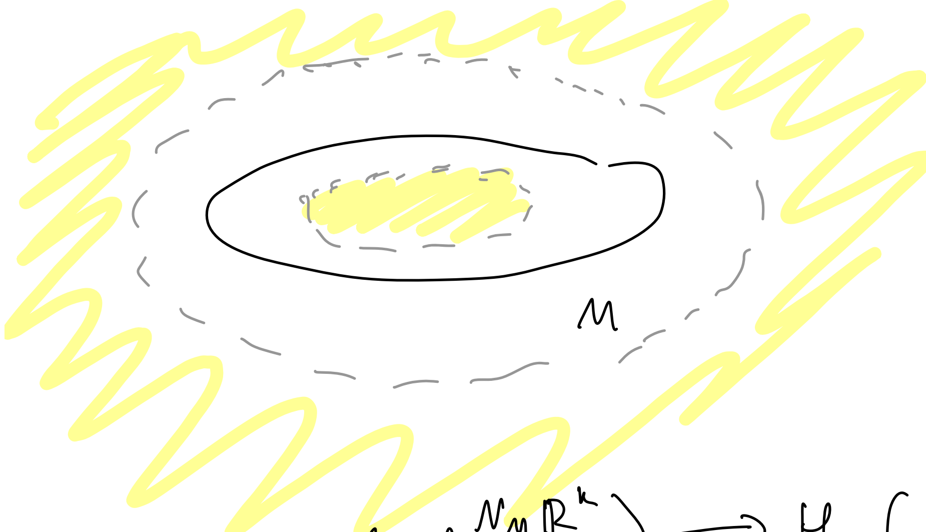
$TM \oplus N_M \mathbb{R}^k \simeq 1^k$

$\Rightarrow -TM = N_M \mathbb{R}^k - 1^k$

$\Rightarrow M^{-TM} \simeq \Sigma^{-k} M^{N_M \mathbb{R}^k}$

Thom collapse map

$c : S^k = \mathbb{R}^k \cup \{\infty\} \longrightarrow M^{N_M \mathbb{R}^k} \simeq S^k / S^k - \text{tubular nbhd } M$



$c \in \pi_k(M^{N_M \mathbb{R}^k}) \longrightarrow H_k(M^{N_M \mathbb{R}^k})$

Thom isomorphism theorem $\cong$ orientation $H_{k-(k-n)}(M)$

$\cong H_n(M)$

image of c is $[M] \in H_n(M)$

Thus: Let M be a simply connected, oriented, smooth manifold of dimension n. Then there exists a vector bundle ζ on M of dimension n (namely the tangent bundle) and a class $\eta_M (= \Sigma_{-k} c)$ in $\pi_0 M^{-\zeta}$ such that the image of η_M under the composition

$$\pi_0(M^{-\zeta}) \longrightarrow H_0(M^{-\zeta}; \mathbb{Z}) \cong H_n(M; \mathbb{Z})$$

Thom iso for orientation

is a fundamental class of M

Question 3: Let X be a simply connected Poincaré complex of dimension n. Suppose we are given a vector bundle ζ of dimension n on X and a homotopy class $\eta \in \pi_0 X^{-\zeta}$ whose image in $H_n(X; \mathbb{Z})$ is a fundamental class for X.

Does there exist a smooth manifold M of dim n and a htpy equivalence

$$f : M \xrightarrow{\sim} X$$

s.t. $f^*\xi = T_M$ in $KO^0(X)$ and $f^*\eta = \eta_M \in M^{-T_M}$?

$\underbrace{\text{An} \Leftrightarrow \text{answer}}$:

Assume $n = 4k$

non-deg symmetric bilinear form

$$\langle\,,\,\rangle : H^{2k}(X;\mathbb{R}) \times H^{2k}(X;\mathbb{R}) \to H^{4k}(X;\mathbb{R}) \xrightarrow{[X]} \mathbb{R}$$

from Poincaré duality

Sylvester's thm (we'll see generalization next time)

$\exists$ basis $\{x_1, \ldots, x_a, x_{a+1}, \ldots, x_{b+a}\}$ for $H^{2k}(X;\mathbb{R})$

s.t. $\langle x_i, x_j \rangle =$

$$\begin{array}{c} \overbrace{\hphantom{1 \; \cdots \; 1}}^{a} \; \overbrace{\hphantom{-1 \; \cdots \; -1}}^{b} \\ \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix} \end{array}$$

$$\text{Signature}(\langle\ ,\ \rangle) := a-b$$

Let $\sigma_M = \text{Signature}(\langle\ ,\ \rangle)$

Hirzebruch signature formula:

Let $p_i(TM)$ $i=1,\dots,k$ denote the Pontryagin classes of TM

$$p_i(TM) = (-1)^i c_{2i}(TM \otimes \mathbb{C}) \in H^{4i}(M;\mathbb{Z})$$

Chern classes

$$\sigma_M = L(p_1(TM),\dots,p_k(TM))[M]$$

L is some polynomial

For example

$n=4$ $\quad L = \dfrac{p_1(TM)}{3}[M]$

$n=8$ $\quad L = \dfrac{7p_2(TM) - p_1(TM)^2}{45}[M]$

<u>Remark</u> (Lurie L1 12) For manifold M, Poincaré duality satisfied for a local reason, so one might expect a "local" formula for σ_M

(Lurie L24-L25) Signature formula results from 2 orientations on $L \wedge H\mathbb{Q}$

cohomology thy representing $X \mapsto H^*(X;\mathbb{Q})$

surgery theory gives converse

<u>Thm</u> (Browder, Novikov?) Let X be a simply connected, poincaré complex of dimension $4k > 4$, let ζ be an oriented vector bundle on X of rk $4k$ let $\eta \in \pi_0 X^{-\zeta}$ be such that the image of η in $H_{4k}(X;\mathbb{Z})$ is a fundamental class

Question 3 answer is yes

$\Longleftrightarrow$

X satisfies the Hirzebruch signature theorem

Algebraic motivation for L-theory

k field

X smooth, proper scheme $/k$ dim n

analogue compact

Ω_X = Kähler differentials $X/k \cong T^*X$

cotangent bundle

$\omega_X = \det \Omega_X$

$V \to X$ vector bundle rank r

Grothendieck-Serre duality $\mathrm{Tr}: H^n(X, \omega_X) \to k$

perfect pairing $H^q(X, F) \otimes H^{n-q}(X, \mathcal{H}om(F, \omega_X)) \to k$

locally free

Def: V is oriented by the data (L, e)

$L \to X$ line bundle and

$\omega_X \otimes \det V \overset{e}{\cong} L^{\otimes 2}$

$V \underset{\sigma}{\leftrightarrows} X$ σ section of V

Def: The Koszul complex $\mathrm{Kos}(V, \sigma)$

is $\mathrm{Kos}(V, \sigma)$

deg n deg 0

$$0 \to \Lambda^r V^* \xrightarrow{d_r} \Lambda^{r-1} V^* \xrightarrow{d_{r-1}} \cdots \to \Lambda^2 V^* \to V^* \xrightarrow{d_1} \mathcal{O} \to 0$$

$$d_k(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} (-1)^{i+1} \sigma(e_i)\, e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k$$

- There is a pairing

$$\mathrm{Kos}(V,\sigma) \otimes \mathrm{Kos}(V,\sigma) \longrightarrow \Lambda^r V^* [n]$$

i.e. for the duality

$$\mathbb{D} : \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(X)$$

$$\mathbb{D}(C_*) = R\mathcal{H}om(-, \det V^*[n])$$

$$\det V^* = \Lambda^r V^*$$

$\mathrm{Kos}(V,\sigma)$ is self dual

- Suppose V is oriented and $n = r$

Then $\big(\mathrm{Kos}(V,\sigma) \otimes \mathcal{L}\big) \otimes \big(\mathrm{Kos}(V \otimes \mathcal{L})\big) \longrightarrow \det V^* \otimes \mathcal{L}^{\otimes 2}[n]$

$$\wr\wr\, e$$

$$\omega_X[n]$$

Perfect pairing

$$H^q(X, \Lambda^q V^* \otimes \mathcal{L}) \otimes H^{n-q}(X, \Lambda^{n-q} V^* \otimes \mathcal{L})$$

$$\downarrow$$

$$H^n(X, \omega_X)$$

$$\downarrow$$

$$R$$

$\rightsquigarrow$ elt of L groups of fields

This is an Euler class in $\mathbb{A}^1$-htpy thy (Bachmann-W, Levine-Raksit for tgt bundles, Hopkins, Serre)

Witt groups

R commutative ring, M line bundle on Spec R

i.e. Projective R module rank 1

$\sigma : M \to M$ $\quad \sigma^2 = 1_M$ $\quad$ ex: $\sigma = \pm 1_M$

$\mathrm{Proj}(R)$ = finitely generated projective R-modules

<u>Def</u>: • $D_M : \mathrm{Proj}(R)^{op} \xrightarrow[\simeq]{R \text{ linear}} \mathrm{Proj}(R)$

$P \longmapsto \mathrm{Hom}_R(P, M)$

is the duality associated to M

<u>Ex</u>: When M is the symmetric monoidal unit $D_M = \mathbb{D}$

• $\mathrm{Hom}_R(P \otimes P, M)$ = M-valued bilinear forms on P

$\circlearrowleft C_2 = \{1, \tau\}$ τ acts by $\sigma \circ$ flip

• $\mathrm{Hom}_R(P \otimes P, M)_{C_2}$ = M-valued quadratic forms

$$\Big\downarrow \mathrm{Sym} \qquad x \mapsto x + \tau x$$

$\mathrm{Hom}_R(P \otimes P, M)^{C_2}$ = M-valued symmetric bilinear forms

<u>Exercise</u> Land L1: Show that M-valued quadratic forms are equivalently described by

a) $b: P \otimes P \longrightarrow M$ symmetric bilinear

b) $q: P \longrightarrow M_{C_2}$ s.t.

$$q(rx) = r^2 q(x)$$

$$q(x+y) - q(x) - q(y) = [b(x,y)]$$

$$b(x,x) = q(x) + \sigma q(x)$$

<u>pf</u>: Take $\tilde{b} \in \mathrm{Hom}_R(P \otimes P, M)_{C_2}$

Define $q: P \longrightarrow M_{C_2}$ by $q(x) = \tilde{b}(x,x)$ in $M/(1-\sigma)M$

well-defined because

$\sigma \tilde{b}(\text{flip})(x,x) = \sigma \tilde{b}(x,x)$ and

$\tilde{b}(x,x) = \sigma \tilde{b}(x,x)$ in $M/(1-\sigma)M$

$$q(rx) = \tilde{b}(rx, rx) = r^2 \tilde{b}(x) \text{ in } M_{C_2}$$

$$q(x+y)-q(x)-q(y)=\tilde{b}(x,y)+\tilde{b}(y,x)$$

Define: $b = \mathrm{Sym}\,\tilde{b}$

$$b(x,y)=\tilde{b}(x,y)+\sigma\tilde{b}(y,x)$$

In M_{C_2}, $\sigma\tilde{b}(y,x)=\tilde{b}(y,x) \Rightarrow$

$$q(x+y)-q(x)-q(y)=b(x,y) \quad \text{in } M_{C_2}$$

- $b(x,x) = \tilde{b}(x,x)+\sigma\tilde{b}(x,x)$

$$= q(x)+\sigma q(x)$$

in M

<u>other direction</u>

Take b, q as above.

$$\tilde{b}(x,x)=q(x) \quad \text{in } M_{C_2}$$

$$\tilde{b}(x,y)+\sigma\tilde{b}(y,x)=b(x,y)=\sigma b(y,x) \text{ in } M \quad (\ast)$$

Suppose P is free. Let $\{e_1,\dots,e_n\}$ be a basis

For $i<j$ let $\tilde{b}(e_i,e_j)=b(e_i,e_j)$

and $\tilde{b}(e_j, e_i) = 0$

Choose lifts $\widetilde{q(e_i)}$ in M of $q(e_i)$ in M_{C_2}

Let $\tilde{b}(e_i, e_i) = \widetilde{q(e_i)}$

extend $\tilde{b}$ to be bilinear.

We satisfy $b = \mathrm{Sym}\, \tilde{b}$ by $(\star)$

and bilinearity.

Q: Is this really well defined in $\mathrm{Hom}_R(P \otimes P, M)_{C_2}$?

What about the case with P not free?

$$(x,y) \mapsto \tilde{b}(x,y) \quad = \quad (x,y) \mapsto \sigma \tilde{b}(y,x) \qquad \text{in } \operatorname{Hom}_R(P \otimes P, M)_{C_2}$$

$\operatorname{Im}(\operatorname{Sym})$ = M-valued even forms

$$\operatorname{Hom}_R(P \otimes P, M) \xrightarrow{\cong} \operatorname{Hom}_R(P, \operatorname{Hom}_R(P, M))$$

$$b \longmapsto \tilde{b} : P \to D_M P$$

- An M-valued form b is perfect or unimodular if the associated map $P \xrightarrow{\cong} D_M P$ is an isomorphism

- For $P \in \mathrm{Proj}(R)$ define

$$hyp(P) = (P \oplus D_M P, ev)$$

$$(P \oplus D_M P) \otimes (P \oplus D_M P) \xrightarrow{(0, ev, ev, 0)} M$$

is a symmetric, unimodular M-valued form

Land LI
Exercise: Show $hyp(P)$ is canonically a quadratic form

- A symmetric unimodular form b on P is called metabolic if $\exists$ a Lagrangian

i.e. $L \subseteq P$ $L \in \mathrm{Proj}(R)$ s.t.

$$0 \to L \to P \underset{\cong}{\overset{\tilde{b}}{\to}} D_M P \to D_M L \to 0$$

is exact $(\Rightarrow b|_L = 0)$

- if (b, q) is quadr and L a

Lagrangian for b, we say that L is a quad Lagrangian if in addition

$q|_L = 0$

Exercise: Let (b, q) be a quadr form on P

Land LI

show: if $\exists$ a quadr Lagrangian L, then $(b, q) \cong hyp(L)$

I owe you hint

- Show that not every sym. metabolic form is hyperbolic

Def: $\cdot W^s_\sigma(R; M) = \left\{ \begin{array}{l} \text{isom classes} \\ \text{of unimod} \\ M\text{-valued} \\ \text{sym forms} \end{array}, \oplus \right\} \Big/ \begin{array}{l} \text{metabolic} \\ \text{forms} \end{array}$

- $$W^{q}_{\nabla}(R;M) = \left\{ \begin{array}{l} \text{isom classes} \\ \text{of unimod} \\ M\text{-valued quadr} \\ \text{forms} \end{array} , \oplus \right\} \Big/ \left\langle \begin{array}{l} \text{quad} \\ \text{metabolic} \\ \text{forms} \end{array} \right\rangle$$

references

- Lurie L|
- Land L|